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Relevant Decision Problems and Value of Information

Lily Ling Yang [1]

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[1] University of Mannheim, lily.yang@uni-mannheim.de

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Relevant Decision Problems and Value of Information*

Lily Ling Yang†

Department of Economics

University of Mannheim

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Abstract

In this paper, we employ a novel approach to study the value of information in games. A decision problem is *relevant* to another if the optimal decision rule of the former, when applied to the latter, is better than making a decision without any information. In a game, if the problem originally faced by a player is relevant to the problem induced by a change, the player benefits more from her own information after the change. Using the notion of relevance, we study the value of information in various games, even when a closed form solution is not available.

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†University of Mannheim, lily.yang@uni-mannheim.de

1 Introduction

In recent years, information choice has been introduced to models in many subfields of economics, including industrial organization, organizational economics, political economy, monetary economics and financial economics.[1] The new approach offers new insights by endogenizing information structures in standard models and bridging the gap between rational and behavioral approaches. One of the questions that has been asked repeatedly in the literature is:

Question #1: When is information acquisition strategic complement/substitute?

Question #1 is important because complementarity and substitutability in information acquisition often have very different implications on economic observables. For example, in monetary economics, firms' incentives to coordinate on price levels together with complementarity in information acquisition generates a mechanism for price stickiness (Reis, 2006). In contrast, in financial economics, substitutability in information acquisition in models of investment choice causes investors to learn more about their home-country assets and provides an explanation for the equity home bias puzzle (Van Nieuwerburgh and Veldkamp, 2009). Moreover, strategic complementarity and substitutability in information acquisition also have important consequences for welfare. For example, when information acquisition is substitute, individuals may lack the incentives to acquire a sufficient amount of information for it to aggregate properly (e.g., Grossman and Stiglitz, 1980; Matthews, 1984; Martinelli, 2006).

More generally, we can consider the following question:[2]

Question #2: How does a player's value of information change when the opponents' strategies change?

[1] See Veldkamp (2011) for an overview of the literature.

[2] *Question #2* is in a sense more general than *Question #1.* This is because information acquired by the opponents only has an effect on a player if the opponents' strategies change as a result.

In this paper, we develop an approach to tackle *Questions #1&2*. Most of the studies that explore the value of information in games focus on tractable but specific models, e.g., the normal-quadratic setting. This approach provides interesting and powerful insights in many economic environments, but the need to solve the model explicitly and arrive at a closed form solution could be an insurmountable obstacle for generalization. To circumvent the need of a closed form solution, we introduce in this paper a notion called relevance, and show that the relevance relation is closely related to the value of information in individual decision problems. Then we transform games into individual decision problems, and apply the notion of relevance. By this novel approach, we generalize the existing results in quadratic games and extend them to information structures that are not well studied in the literature due to the lack of closed form solutions, and we also study a global game and a multi-expert persuasion game. These applications show that sometimes we could draw interesting conclusions about how value of information changes based on properties of the equilibrium strategies alone.

In an individual decision-making environment, we establish a partial order on payoff functions for a given information structure. Consider two decision problems, called A and B, with different payoff functions but the same information structure. In problem A, if the use of problem B's optimal decision rule generates a higher payoff than making a decision without any information, then we say problem B is relevant to problem A. The implication of problem B being relevant to problem A is that the value of information in a grand problem, which combines A and B, is certainly higher than that in problem B (Proposition 1). Moreover, under an additional condition, the lack of relevance of the grand problem to problem A implies that the value of information in the grand problem is lower than that in problem B (Proposition 2).

To apply the notion of relevance in games, we transform the best-responding problems in games into individual decision problems, and therefore the comparison of the value of information across different equilibria to a comparison of different payoff functions for a given information structure. We apply this construction to quadratic games, a global game, and a multi-expert persuasion game. Based on the relevance relation, we identify conditions under which a player's benefit from having information about the state of the world is higher in one equilibrium than in the other. In quadratic games, we find conditions on information structures under which we are able to draw an unambiguous conclusion on whether information is complement. We first establish our results for the monotone information structures, with which a closed form solution is in general not available and therefore has not received much attention in the existing literature. We find that complementarity in actions translates into complementarity in information, while the substitutability in actions does not necessarily translate into substitutability in information (Proposition 3). Then, we consider affine information structures, which are assumed by the majority of the literature (Proposition 4) and show that the linearity of the equilibrium strategies allows us to obtain additional results about complementarity inheritance when actions are substitutes (Proposition 5). In a global game setting, we discuss how to compare the value of information in different market environments. We show that investors find their information less valuable when the original market condition is good (bad) and the costs of investment decrease (increase) (Proposition 6). In a multi-expert persuasion game, we study how the presence of another strategic expert changes the existing expert's value of information. We find that information is substitute when the experts have opposite extreme biases (Proposition 7).

Finally, we suggest that the notion of relevance can also be viewed as a decision making heuristic for boundedly rational agents. When a decision maker has limited "computational power", it may not be feasible for her to identify the optimal decision rule for the problem at hand. Instead, she may find the following heuristic useful.

1. Compute the expected payoff of adopting a decision rule that proves to be useful *for another problem* in the past.

2. If the result is better than acting without extra information, adopt it.

A decision maker who adopts this decision-making heuristic implicitly invokes the notion of relevance. As argued in Gilboa and Schmeidler (1995), this kind of heuristics seems to resemble daily decision making more closely than rational theory. This paper contributes by providing a perspective on it.

1.1 Related Literature

A sizable literature studies the value of information in individual decision problems. The literature focuses on the ordering of information structures for a family of payoff functions (i.e., Blackwell, 1951, 1953; Lehman, 1988; Athey and Levin, 2000). The present paper differs from this strand of literature in the objects the ordering is on. In this paper, we study how to rank two payoff functions for the same information structure in terms of how much the decision maker benefits from having the corresponding information structure.

The majority of the literature on the value of information in games setting focuses on the linear-quadratic model due to its tractability. The strand of literature that is most relevant to this paper is the study on complementarity/substitutability in information acquisition. Hellwig and Veldkamp (2009) consider a model with a continuum of players and find that players have complementarity (substitutability) in information when actions are strategic complements (substitutes). Jiménez-Martínez (2014) considers a two-player version of their model with a general payoff function and finds that this complementarity inheritance result does not always hold. Gendron-Saulnier and Gordon (2015) study information choice in a class of games more general than the linear-quadratic framework. In Section 4.3, we provide results on complementarity inheritance in a general linear-quadratic model (Propositions 4 and 5) as well as quadratic games with monotone information structures, which has not been studied in the literature (Proposition 3).[3] Another strand of literature studies the social value of information and finds mixed results. Notable works include Morris and Shin

[3] Although Gendron-Saulnier and Gordon (2015) also study quadratic games with monotone information structure, they restrict the players to covertly choose only the dependence between their signals but not the informativeness of the signal about the true state. In this paper, we do not impose such a restriction.

(2002), Angeletos and Pavan (2004, 2007), Myatt and Wallace (2015), and Ui and Yoshizawa (2015).

Aside from quadratic games, Szkup and Trevino (2015) consider a global game and establish that complementarity in actions does not necessarily lead to complementarity in information acquisition. In Section 4.4, we also look into a global game and study the value of information when a full characterization of the equilibrium is not available.

In the persuasion game literature, Kartik, Lee, and Suen (2017) show that extreme-biased experts' have substitutability in information when the experts have linear preferences over the decision maker's beliefs. In Section 4.5, we introduce information choice to Bhattacharya and Mukherjee (2013) and show that information acquisition is always substitute for oppositely biased experts.

Finally, this paper is also related to decision making under bounded rationality. Gilboa and Schmeidler (1995) provide a model in which the merit of an act for the problem at hand is evaluated by average utility levels that resulted from using it in similar situations in the past. In their model, similarity is subjective and derived from preferences. In this paper, we take the basics of expected utility theory as granted and offer a prospective on similarity (i.e., relevance). In our view, failure to adhere to rational theory could be a result of limited cognitive (computational) power.

The rest of the paper is organized as follows. Section 2 proposes the framework. Section 3 introduces the notion of relevance, and discusses the relationship between relevance and value of information. Section 4 contains applications. Section 5 concludes. Some of the proofs are relegated to the Appendix.

2 The Decision Problem

The stochastic environment consists of an unknown state of the world Θ, with realization $\theta \in \Theta$, and a signal X with realization $x \in X$.[4] Given a prior $H \in \triangle(\Theta)$, the signal

[4] By abuse of notations, we use Θ and X to denote both the random variables and the sets of realizations.

distribution induces a joint distribution over states and signals, $F : \Theta \times X \to [0,1]$. We call F an *information structure*. Let $F_X(\cdot|\theta)$ be the signal distribution conditional on $\Theta = \theta$ and $F_X(\cdot)$ be the marginal signal distribution, i.e., $F_X(\cdot) = E_\Theta[F_X(\cdot|\theta)]$. Let $F_\Theta(\cdot|x)$ be the conditional distribution of Θ given $X = x$. The posterior is consistent with the prior, i.e., for all $\theta \in \Theta$, $E_X[F_\Theta(\theta|x)] = H(\theta)$. A *null information structure* F_ϕ, or interchangeably ϕ, is an information structure that satisfies $(F_\phi)_\Theta(\theta|x) = (F_\phi)_\Theta(\theta|x') = H(\theta)$ for all $x, x' \in X$.

After observing the signal realization, a decision maker (DM) chooses an action $a \in A \subseteq \mathbb{R}$. His payoff function is $u : A \times \Theta \to \mathbb{R}$. A payoff function u and an information structure F constitute a *decision problem* $\langle F, u \rangle$. A *decision rule* $\sigma : X \to \triangle(A)$ assigns a distribution over actions to every signal realization. The value of a decision rule σ in the decision problem $\langle F, u \rangle$[5] is

$$V(\sigma, F, u) = E_\Theta\left[\int_X u(\sigma(x), \theta)\, dF_X(x|\theta)\right].$$

We call $\sigma^*_{F,u}$ an *optimal decision rule* for the decision problem $\langle F, u \rangle$, if $\sigma^*_{F,u}$ is optimal for every $x \in X$, i.e.,

$$\sigma^*_{F,u}(x) \in \arg\max_{s \in \triangle(A)} \int_\Theta u(s, \theta)\, dF_\Theta(\theta|x).$$

The (ex ante) value of the decision problem $\langle F, u \rangle$ is

$$V(F, u) = V\left(\sigma^*_{F,u}, F, u\right).$$

Similarly, we call $a^*_{\phi,u}$ an optimal default action, or simply *default action*, for the decision problem $\langle F, u \rangle$, if $a^*_{\phi,u}$ is optimal given the prior H. We further denote the set of optimal decision rules for decision problem $\langle F, u \rangle$ by $\sum^*_{F,u}$ and the set of optimal default actions for the decision problem $\langle F, u \rangle$ by $\sum^*_{\phi,u}$.[6]

We assume throughout this paper that optimal decision rule and default action exist.

[5] By abuse of notations, we use u to denote both the payoff when the assigned action is degenerate and the *expected* payoff when the decision rule assigns a distribution over actions.

[6] Note that $a^*_{\phi,u}$ only depends on the prior, so the set of default actions for the decision problem $\langle F, u \rangle$ is the same as that for the decision problem $\langle F', u \rangle$.

This can be ensured by, for example, assuming that the action space A is compact and u is continuous in a. However, we do not impose such restrictions formally, as an optimal decision rule may exist in applications even if these assumptions are violated.

The *value of information structure* F in the decision problem $\langle F, u\rangle$ is defined as

$$V(F,u) - V(\phi,u).$$

The value of information is the payoff difference between an optimal decision rule based on the information structure F and a default action based on the prior.

3 Relevance

In this section, we introduce the notion of relevance. Consider a DM facing a decision problem. The DM finds a decision rule favorable when it is better than her default action. If this favorable decision rule happens to be an optimal decision rule for another decision problem, then we say that the second decision problem is *relevant* to the DM's first decision problem. Formally,

Definition 1 (Relevance) *$\langle F, u\rangle$ is relevant to $\langle F, v\rangle$ if and only if there exists $\sigma^*_{F,u} \in \sum^*_{F,u}$ such that*

$$V\left(\sigma^*_{F,u}, F, v\right) \geq V\left(a^*_{\phi,v}, F, v\right).$$

As discussed in the introduction, we can interpret relevance as a heuristic for decision rule adoption. If $\langle F, u\rangle$ is relevant to $\langle F, v\rangle$, then it is profitable for a boundedly rational agent to adopt $\sigma^*_{F,u}$ for $\langle F, v\rangle$.

We say that $\langle F, u\rangle$ is *strongly relevant* to $\langle F, v\rangle$, when the inequality is strict.[7] We denote a relevance relation by $\overset{R}{\Rightarrow}$, and a strong relevance relation by $\overset{R}{\rightarrow}$. Conversely, $\langle F, u\rangle$ is *not*

[7]The relevance relation is reflexive for any information structure F, but the strong relevance relation is not necessarily reflexive. Consider a constant payoff function u, we have, $V\left(\sigma^*_{F,u}, F, u\right) = V\left(a^*_{\phi,u}, F, u\right)$.

relevant to $\langle F, v\rangle$ if and only if for all $\sigma^*_{F,u} \in \sum^*_{F,u}$, $V\left(\sigma^*_{F,u}, F, v\right) < V\left(a^*_{\phi,v}, F, v\right)$. We denote this by $\langle F, u\rangle \overset{R}{\nRightarrow} \langle F, v\rangle$. $\overset{R}{\nrightarrow}$ is defined analogously. In the online appendix, we discuss some of the properties of the relevance relation. In particular, the relevance relation is in general not complete, symmetric or transitive.

3.1 Relevance and Value of Information

In this section, we investigate how the relevance relation relates to value of information in decision problems.

Even if $\langle F, u\rangle$ is relevant to $\langle F, v\rangle$, we still do not know how to compare $V(F, u)$ and $V(F, v)$. Instead, the following two simple propositions show that the notion of relevance can be employed to compare the value of information in $\langle F, u\rangle$ and $\langle F, u+v\rangle$, using $\langle F, v\rangle$ as a bridge between the two.

Proposition 1 *If $\langle F, u\rangle \overset{R}{\Rightarrow} \langle F, v\rangle$, then the value of information F is higher in $\langle F, u+v\rangle$ than in $\langle F, u\rangle$, i.e.,*

$$V(F, u+v) - V(\phi, u+v) \geq V(F, u) - V(\phi, u). \tag{1}$$

Proof. *Take $\sigma^*_{F,u} \in \sum^*_{F,u}$ such that $V\left(\sigma^*_{F,u}, F, v\right) \geq V(\phi, v)$. We have*

$$\begin{aligned}
& V(F, u+v) - V(\phi, u+v) \\
\geq\ & V\left(\sigma^*_{F,u}, F, u+v\right) - V(\phi, u+v) \\
=\ & \left[V\left(\sigma^*_{F,u}, F, u\right) - V\left(a^*_{\phi,u+v}, \phi, u\right)\right] + \left[V\left(\sigma^*_{F,u}, F, v\right) - V\left(a^*_{\phi,u+v}, \phi, v\right)\right] \\
\geq\ & \left[V\left(\sigma^*_{F,u}, F, u\right) - V(\phi, u)\right] + \left[V\left(\sigma^*_{F,u}, F, v\right) - V(\phi, v)\right] \\
\geq\ & V\left(\sigma^*_{F,u}, F, u\right) - V(\phi, u),
\end{aligned}$$

where the first inequality follows from the suboptimality of $\sigma^*_{F,u}$ in $\langle F, u+v\rangle$, the second inequality follows the suboptimality of $a^*_{\phi,u+v}$ in both $\langle \phi, u\rangle$ and $\langle \phi, v\rangle$, and the last inequality *follows from the assumption that* $\langle F,u\rangle \overset{R}{\Rightarrow} \langle F,v\rangle$*.* ■

We call $\langle F,u\rangle$, $\langle F,v\rangle$, and $\langle F,u+v\rangle$ the *original* problem, the *difference* problem, and the *new* problem, respectively. The value of information in the new problem is higher than that in the original problem if the original problem is relevant to the difference problem. The intuition is straightforward. If the original problem $\langle F,u\rangle$ is relevant to the difference problem $\langle F,v\rangle$, $\sigma^*_{F,u}$ has an advantage in $\langle F,v\rangle$ over $a^*_{\phi,v}$. This implies that the optimal decision rule must yield an even higher payoff in the new problem $\langle F,u+v\rangle$. It can be shown that the opposite direction of this statement is not true.

While $V(F,u+v) - V(\phi,u+v) > V(F,u) - V(\phi,u)$ does not imply the relevance of $\langle F,u\rangle$ to $\langle F,v\rangle$, the contrapositive of the next proposition shows that it implies the relevance of $\langle F,u+v\rangle$ to $\langle F,v\rangle$ under one extra assumption.

Proposition 2 *Suppose* $\Sigma^*_{\phi,u} \cap \Sigma^*_{\phi,v} \neq \phi$ *, if* $\langle F,u+v\rangle \overset{R}{\nrightarrow} \langle F,v\rangle$*, then the value of information* F *is lower in* $\langle F,u+v\rangle$ *than in* $\langle F,u\rangle$*, i.e.,*

$$V(F,u+v) - V(\phi,u+v) \leq V(F,u) - V(\phi,u). \tag{2}$$

Proof. *Since* $V(\phi,u+v) \leq V(\phi,u) + V(\phi,v)$*,* $a \in \Sigma^*_{\phi,v} \cap \Sigma^*_{\phi,u}$ *implies that* $a \in \Sigma^*_{\phi,u+v}$*. Take any* $\sigma^*_{F,u} \in \sum^*_{F,u}$*, we have*

$$\begin{aligned}
& V(F,u) - V(\phi,u) \\
\geq\ & V\left(\sigma^*_{F,u+v}, F, u\right) - V(\phi,u) \\
=\ & \left[V\left(\sigma^*_{F,u+v}, F, u+v\right) - V(\phi,u+v)\right] - \left[V\left(\sigma^*_{F,u+v}, F, v\right) - V(\phi,v)\right] \\
\geq\ & V(F,u+v) - V(\phi,u+v),
\end{aligned}$$

where the equality follows from the assumption that $\Sigma^*_{\phi,u} \cap \Sigma^*_{\phi,v} \neq \phi$*, and the last inequality follows from the assumption that* $\langle F,u+v\rangle \overset{R}{\nrightarrow} \langle F,v\rangle$*.* ■

The notion of relevance is quite intuitive, and it provides an intuitive way to rank payoff functions given an information structure, according to the value of information in the corresponding decision problems. It is a partial order, as is Blackwell's order. Blackwell's order starts from an information structure that is more valuable to all payoff functions to one that is less valuable by a garbling of the former. An ordering based on the relevance relation starts from a payoff function that values one information structure less to one that values more. Notice that the definition of relevance we used depends neither on the value $V\left(\sigma^*_{F,u+v}, F, u+v\right)$ nor $V\left(\sigma^*_{F,u}, F, u\right)$. Thus, it is the *direction of change*, rather than the *absolute magnitude*, of value of information that is of interest when we try to establish a relevance relation.

In Section 4, we will repeatedly use the relationship between a relevance relation and the value of information. Starting from two situations we would like to compare, we construct a difference problem and then apply Proposition 1 or 2 to infer the direction of change of the value of information.

4 Applications

In this section, we apply the notion of relevance to games and find conditions under which a player's value of information about the state of the world is unambiguously higher in one equilibrium than in another. As in an individual decision problem, we measure the value of information by how much a player benefits from having it. Two distinct assumptions can be made about the knowledge of the other players as a player's information changes in this calculation. One is to assume *overt* information acquisition, so that any change in a player's information becomes common knowledge. This approach is taken by Jiménez-Martínez (2014). Gendron-Saulnier and Gordon (2015) and Kartik, Lee, and Suen (2017), on the other hand, assume *covert* information acquisition. As a result, the strategies of the other players are *fixed at a particular equilibrium under a particular information structure* while a player's information changes. This is the approach that we take in this paper. For a sizable portion of the literature (e.g., Hellwig and Veldkamp, 2009; Myatt and Wallace, 2012; Szkup and Trevino, 2015), this distinction does not matter, as these models contain a continuum of players. As a result, a single player's action has no effect on the equilibrium strategies of the rest of the players. The incentive to acquire information is the same whether it is done overtly or covertly.

To apply the notion of relevance to games, we first define the *induced difference problem* in a game given two equilibria and then show how Propositions 1 and 2 can be applied to establish results on the value of information.

4.1 Games and the Induced Difference Problem

Consider a game with N players and let I denote the set of players. Player $i \in I$ receives a signal $x_i \in X_i \subseteq \mathbb{R}$. Denote $X = \times_{i \in I} X_i$. An information structure F for the game is a joint distribution over states and signals, $F : \Theta \times X \to [0, 1]$. Player i chooses an action $a_i \in A_i \subseteq \mathbb{R}$. Denote $A = \times_{i \in I} A_i$. Player i's payoff function is $u_i : A \times \Theta \to \mathbb{R}$. Player i's strategy is a mapping from the received signals to distributions over actions, $\sigma_i : X_i \to \triangle(A_i)$. Given the information structure F, let $F_X(\cdot|\theta)$ denote the signal distribution conditional on $\Theta = \theta$, and $F_{X_i}(\cdot|\theta)$ denote player i's signal distribution conditional on $\Theta = \theta$. The players have a common prior, thus $E_{X_i}[F_\Theta(\theta|x_i)] = E_{X_j}[F_\Theta(\theta|x_j)] = H(\theta)$ for all $i, j \in I$.

In order to consider the value of information in games, we need to incorporate the other players' strategies. Fixing the other players' strategies, player i faces an individual decision problem. We call the situation in which the other players use $\sigma_{-i}(x_{-i})$ and $\sigma'_{-i}(x_{-i})$ the original problem and the new problem, respectively, and define the original problem $\langle F_i, \widetilde{u}_i \rangle$ and the new problem $\langle F_i, \widetilde{w}_i \rangle$ correspondingly. $X_{-i} \times \Theta$ is now the new state space. Given any $\sigma_{-i}(x_{-i})$, we can define the corresponding decision problem $\langle F_i, \widetilde{u}_i \rangle$ and discuss the value of information for player i in $\langle F_i, \widetilde{u}_i \rangle$, which is equivalent to the value of information given the other players are using $\sigma_{-i}(x_{-i})$. Similar for $\sigma'_{-i}(x_{-i})$ and $\langle F_i, \widetilde{w}_i \rangle$.

Given $\sigma_{-i}(x_{-i})$, player i's payoff function is

$$\widetilde{u}_i(a_i, x_{-i}, \theta) := u_i(a_i, \sigma_{-i}(x_{-i}), \theta),$$

in the original problem $\langle F_i, \widetilde{u}_i \rangle$. Similarly, given $\sigma'_{-i}(x_{-i})$, player i's payoff function is

$$\widetilde{w}_i(a_i, x_{-i}, \theta) := u_i\left(a_i, \sigma'_{-i}(x_{-i}), \theta\right),$$

in the new problem $\langle F_i, \widetilde{w}_i \rangle$. Then, in the *induced* difference problem, player i has information structure F_i and the following payoff function,

$$\widetilde{v}_i(a_i, x_{-i}, \theta) := \widetilde{w}_i(a_i, x_{-i}, \theta) - \widetilde{u}_i(a_i, x_{-i}, \theta).$$

The construction of the difference problem depends on $\sigma_{-i}(x_{-i})$ and $\sigma'_{-i}(x_{-i})$, and $\widetilde{v}_i$ is the change in player i's payoff due to a change in other players' strategy, from $\sigma_{-i}(x_{-i})$ to $\sigma'_{-i}(x_{-i})$. Next, we will discuss how to compare the value of information across different equilibria based on the induced difference problem defined above.

4.2 Value of Information in Equilibrium

In games, the value of information depends on the equilibrium strategies. Given an information structure F, $\sigma^*(x)$ is an equilibrium strategy profile if and only if

$$\sigma_i^*(x_i) \in \arg\max_{s_i \in \triangle(A_i)} \int_{X_{-i} \times \Theta} u_i\left(s_i, \sigma_{-i}^*(x_{-i}), \theta\right) dF_{X_{-i} \times \Theta}(x_{-i}, \theta | x_i) \tag{3}$$

for all $i \in I$. In order to highlight the dependence of the equilibrium strategy profile $\sigma^*(x)$ on the information structure F, we denote it by $\sigma^*(x; F)$. Denote player i's payoff by playing $\sigma_i^*(x_i; F)$ when the other players are playing the equilibrium strategy $\sigma_{-i}^*(x_{-i}; F)$

by $V(F_i; F)$, i.e.,

$$V(F_i; F) = E_{X\times\Theta}\left[u_i\left(\sigma^*(x; F), \theta\right)\right],$$

and it is also player i's payoff in the equilibrium corresponding to $\sigma^*(x; F)$.

Given $\sigma^*_{-i}(x_{-i}; F)$, a_i^* is player i's (optimal) default action if and only if

$$a_i^* \in \arg\max_{a_i \in A_i} \int_{X_{-i}\times\Theta} u_i\left(a_i, \sigma^*_{-i}(x_{-i}; F), \theta\right) dF(x_{-i}, \theta).$$

Analogously, we denote it by $a_i^*(F)$. Note that the optimal default action a_i^* depends on the information F_i not because player i has information F_i, but *the other players believe that player i has it.* Denote player i's payoff by playing $a_i^*(F)$ when the other players are playing $\sigma^*_{-i}(x_{-i}; F)$ by $V(\phi; F)$, i.e.,

$$V(\phi; F) = E_{X_{-i}\times\Theta}\left[u_i\left(a_i^*(F), \sigma^*_{-i}(x_{-i}; F), \theta\right)\right].$$

Under the interpretation of *covert information acquisition*, $V(\phi; F)$ is the payoff player i can get by unilaterally and covertly deviating from F_i to ϕ in the equilibrium $\sigma^*(x; F)$.

Given an information structure F, we define the value of information F_i to player i in equilibrium by

$$V(F_i; F) - V(\phi; F). \tag{4}$$

The difference between $V(F_i; F)$ and $V(\phi; F)$ measures how much player i benefits from information F_i when holding the other's beliefs of player i's information constant at F_i. Thus, the value of information defined by (4) can be interpreted as a measure of the incentives of player i to deviate from acquiring the equilibrium level of information F_i in a game with covert information acquisition. The higher the value is, the more likely that the equilibrium with information F_i can be sustained given a fixed cost of information acquisition.[8] With (4) at hand, we can now define complementarity and substitutability in information in games.

[8]This notion of value of information is not the only notion that one could think about in this situation. Readers might also consider the value of information in the following sense,

For each player i, consider two information levels $\left\{\phi, \overline{F}_i\right\}$, we define,

Definition 2 *Given an information structure $\overline{F} = \left(\overline{F}_1, ..., \overline{F}_N\right)$, we say that players $-i$'s information is complementary (substitute) to player i's information if and only if for all equilibria $\sigma^*\left(x; \overline{F}_i, \overline{F}_{-i}\right)$ and $\sigma^*\left(x; \overline{F}_i, \phi^{I-1}\right)$,*

$$V\left(\overline{F}_i; \overline{F}_i, \overline{F}_{-i}\right) - V\left(\phi; \overline{F}_i, \overline{F}_{-i}\right) \geq (\leq) V\left(\overline{F}_i; \overline{F}_i, \phi^{I-1}\right) - V\left(\phi; \overline{F}_i, \phi^{I-1}\right). \tag{6}$$

In words, other players' information is complementary to a player's own information if the value of own information increases with the other players' information levels. To incorporate the possibility of multiple equilibria, our definition of complementarity/substitutability requires the inequality (6) to be satisfied for all combinations of equilibria under the information profiles $\left(\overline{F}_i, \overline{F}_{-i}\right)$ and $\left(\overline{F}_i, \phi^{I-1}\right)$. An equally legitimate definition requires the inequality (6) to be satisfied for at least a pair of equilibria. However, the equilibria involved are always unique when we apply these two concepts in this paper.[9] The distinction between the two definitions is thus immaterial.

Next, we apply the Propositions 1 and 2 to three different games. In Section 4.3, we consider quadratic games, and discuss how a change in the other players' information affects a player's value of information in equilibrium. In Section 4.4, we consider a global game, and discuss how value of information changes when costs of investment change. In Section 4.5, we consider a persuasion game, and discuss how the presence of another strategic expert changes the existing expert's value of information. Some of the proofs are relegated to the

$$V\left(F_i; F\right) - V\left(\phi; \phi, F_{-i}\right). \tag{5}$$

The main difference between these two notions is that, when player i switches to the default action, player i's opponents continue to use the strategies $\sigma^*_{-i}\left(x_{-i}; F\right)$ in (4), while player i's opponents switch to $\sigma^*_{-i}\left(x_{-i}; \phi, F_{-i}\right)$ in (5). As noted previously, the value of information defined by (5) is applicable when information acquisition is overt. The two notions (4) and (5) are equivalent when there is a continuum of players, as the equilibrium strategies of the other players do not depend on the information of the single player i, i.e., $\sigma^*_{-i}\left(x_{-i}; F\right) = \sigma^*_{-i}\left(x_{-i}; \phi, F_{-i}\right)$.

[9] Strictly speaking, the persuasion game we consider in Section *4.5* has more than one equilibrium, as the experts are free to send different messages when he is indifferent. Clearly, this multiplicity has no bearing on our calculations.

Appendix.

4.3 Quadratic Games

The quadratic payoff function has been employed extensively to study whether information is complement or substitute in the literature. Consider the *generalized* quadratic payoff function, under which player i's payoff from the action profile $a = (a_1, ..., a_N)$ when the state of the world is θ is given by

$$u_i(a, \theta) = -a_i^2 + 2\alpha a_i \sum_{j \neq i} a_j + 2\beta_i \theta a_i + f_i(a_{-i}, \theta), \tag{7}$$

where $\alpha, \beta_i \in \mathbb{R}$ are constants and $f_i : \mathbb{R}^N \to \mathbb{R}$ is a measurable function. In words, player i's payoff can be decomposed into two parts, a quadratic part that is quadratic in player i's own action a_i and a functional part that is independent of a_i. In the quadratic part, α is the coefficient measuring the effect of the interaction between player i's action a_i and the other players' aggregate action $\sum_{j \neq i} a_j$, and β_i is the coefficient measuring the effect of the interaction between player i's action a_i and the state of the world θ. As a result, player i's best response depends on both α and β_i, but not on the function f_i. Thus, the function f_i affects player i's payoff in a given equilibrium but not the set of equilibria. We assume that $(N-1)|\alpha| < 1$ and $\beta_i > 0$.

Given a generalized quadratic payoff function, the single parameter α characterizes the interaction between player i's action and the aggregate action. If $\alpha \geq (\leq) 0$, player i's action and the aggregate action are strategic complements (substitutes). How would the interaction of actions affects the interactions of information acquisitions? In the model of Hellwig and Veldkamp (2009) with a continuum of players, complementarity (substitutability) in actions translates nicely to complementarity (substitutability) in information. Jiménez-Martínez (2014) finds that this result does not always hold in a two-player model with overt information acquisition.

In this section, we investigate the issue of complementarity inheritance in our setting by exploring the difference problem induced by a change in the other players' strategies and applying the notion of relevance to it. For simplicity, we only consider a zero-one problem, i.e., player i has either no information ϕ or a fixed amount of information $\overline{F}_i$. Denote player i's information level by $F_i \in \left\{\phi, \overline{F}_i\right\}$. To simplify the notation, we only consider the case with two players, i.e., $N = 2$, the result can be easily extended to the multiple-player case.[10]

4.3.1 Monotone Information Structures

The main result of this section of the paper is extending the analysis of complementarity inheritance in the literature to monotone information structures (Proposition 3). The discussion on affine information structures, under which additional results can be derived, is postponed to Section 4.3.2.

Definition 3 (Monotone information structure) *An information structure is monotone if and only if*

M1 X_i *and* Θ *are compact and convex subsets of* $\mathbb{R}$.

M2 *For all* $i \in I$, $E(\theta|x_i)$ *is continuous and increasing in* x_i.

M3 *For all* $i, j \in I$, $i \neq j$, $F(x_j|x_i)$ *is continuous in* x_i *and* x_j *and decreasing in* x_i. *i.e., the conditional distribution of* x_j *given* x_i *can be ordered by first order stochastic dominance.*

Under a monotone information structure, closed form solution is in general not available. For this reason, to the best of my knowledge, with the sole exception of Gendron-Saulnier and Gordon (2015), who focus on players' choice of *information dependence* rather than *information level,* no paper has studied the value of information in a quadratic game with a monotone information structure. Applying the notion of relevance, we can show that,

[10] Notice that the multiplier α for the interaction between a_i and a_j is same for all $i, j \in I$.

Proposition 3 *In a two-player quadratic game with monotone information structure,*

1. *when actions are strategic complements, i.e.,* $\alpha \geq 0$*, information is strategic complement;*

2. *when actions are strategic substitutes, i.e.,* $\alpha \leq 0$*,*

   (a) *if* $\sigma_j^* \left(x_j; \overline{F}\right)$ *is decreasing in signal realization* x_j*, then player* j*'s information is complementary to player* i*'s information, and*

   (b) *if* $\sigma^* \left(x; \overline{F}\right)$ *is increasing in signal realizations, then the players' information is strategic substitute.*

Intuitively, when actions are strategic complements, information is not only useful in "matching" the state, but also "matching" the opponent's action. Thus, when the opponent's action becomes more responsive to the state, value of information increases. Similarly, when actions are strategic substitutes and the opponent's strategy after acquiring information is decreasing in his signal realization, information again has dual purposes–to "match" the state and to "dodge" the opponent's action. As a result, information also becomes more valuable after the opponent has acquired information. Finally, if the equilibrium $\sigma^* \left(x; \overline{F}\right)$ is increasing, acquired information helps "matching" the state but not "dodging" the opponent's action, so the value of information diminishes after the opponent has acquired information. Part (1) and part (2.b) of Proposition 3 generalize the complementarity inheritance results in Hellwig and Veldkamp (2009). Part (2.a) is absent in Hellwig and Veldkamp (2009), as the equilibrium strategies are never decreasing in their model. Further discussions of their results are relegated to the next subsection on affine information structures.

Next, we will demonstrate how Proposition 3 can be derived using the notion of relevance through Lemmas 1–4. Let $\sigma^* (x; F)$ be an equilibrium strategy profile defined in (3), then we have

Lemma 1 *Given any information structure F and any equilibrium strategy profile of the quadratic game $\sigma^*(x;F)$, for $i \in \{1,2\}$, $E(\sigma_i^*(x_i;F)) = \overline{a}_i$, where $(\overline{a}_1, \overline{a}_2)$ is the equilibrium action pair under the information structure (ϕ, ϕ).*

By Lemma 1, player j must take the action $\overline{a}_j$ in equilibrium under $(\overline{F}_i, \phi)$. Therefore, in the original problem, which corresponds to the equilibrium under $(\overline{F}_i, \phi)$, the payoff function is

$$\widetilde{u}_i(a_i, x_j, \theta) = -a_i^2 + 2\alpha a_i \overline{a}_j + 2\beta_i \theta a_i + f_i(\overline{a}_j, \theta).$$

In the new problem, which corresponds to the equilibrium under $\overline{F} = (\overline{F}_1, \overline{F}_2)$, the payoff function is

$$\widetilde{w}_i(a_i, x_j, \theta) = -a_i^2 + 2\alpha a_i \sigma_j^*(x_j; \overline{F}) + 2\beta_i \theta a_i + f_i(\sigma_j^*(x_j; \overline{F}), \theta).$$

Therefore, in the induced difference problem, the payoff function is

$$\widetilde{v}_i(a_i, x_j, \theta) = \underbrace{2\alpha a_i \left(\sigma_j^*(x_j; \overline{F}) - \overline{a}_j\right)}_{\text{interaction between } a_i \text{ and } \left(\sigma_j^*(x_j;\overline{F}) - \overline{a}_j\right)} + \underbrace{f_i\left(\sigma_j^*(x_j; \overline{F}), \theta\right) - f_i(\overline{a}_j, \theta).}_{\text{other terms independent of } a_i} \tag{8}$$

In the induced difference problem $\langle \overline{F}_i, \widetilde{v}_i \rangle$, the payoff function $\widetilde{v}_i$ captures the impact of a change in player j's strategy on player i's payoff. Intuitively, $\widetilde{v}_i$ is the incremental payoff change with respect to a change in player j's strategy due to a change in player j's information, i.e., from $\overline{a}_j$ to $\sigma_j^*(x_j; \overline{F})$. In the difference problem $\langle \overline{F}_i, \widetilde{v}_i \rangle$, we have,

Lemma 2 *$\overline{a}_i$ is an optimal default action in the difference problem $\langle \overline{F}_i, \widetilde{v}_i \rangle$.*

To prove Lemma 2, take unconditional expectation of (8) and apply Lemma 1 to conclude that the expected payoff in the difference problem $E(\widetilde{v}_i(a_i, x_j, \theta))$ is independent of a_i. Thus, we can simply take $\overline{a}_i$ to be the optimal default action in the difference problem.

By Lemma 2, the *ex ante* payoff difference between $\sigma_i^*\left(x_i;\overline{F}_i,\phi\right)$ and $\overline{a}_i$ in the difference problem $\left\langle \overline{F}_i,\widetilde{v}_i\right\rangle$ can be written as

$$
\begin{aligned}
& V\left(\sigma_i^*\left(x_i;\overline{F}_i,\phi\right),\overline{F}_i,\widetilde{v}_i\right)-V\left(\phi,\widetilde{v}_i\right) \\
= \; & 2\alpha E\left[\left(\sigma_i^*\left(x_i;\overline{F}_i,\phi\right)-\overline{a}_i\right)\left(\sigma_j^*\left(x_j;\overline{F}\right)-\overline{a}_j\right)\right] \\
= \; & 2\alpha Cov\left(\sigma_i^*\left(x_i;\overline{F}_i,\phi\right),\sigma_j^*\left(x_j;\overline{F}\right)\right),
\end{aligned}
$$

where the second equality follows from Lemma 1. Similarly, we have

$$
V\left(\sigma_i^*\left(x_i;\overline{F}\right),\overline{F}_i,\widetilde{v}_i\right)-V\left(\phi,\widetilde{v}_i\right)=2\alpha Cov\left(\sigma_i^*\left(x_i;\overline{F}\right),\sigma_j^*\left(x_j;\overline{F}\right)\right).
$$

Thus, we have,

Lemma 3 *The original problem is relevant to the difference problem if and only if*

$$
\alpha Cov(\sigma_i^*\left(x_i;\overline{F}_i,\phi\right),\sigma_j^*\left(x_j;\overline{F}\right))\geq 0, \tag{9}
$$

and the new problem is relevant to the difference problem if and only if

$$
\alpha Cov(\sigma_i^*\left(x_i;\overline{F}\right),\sigma_j^*\left(x_j;\overline{F}\right))\geq 0. \tag{10}
$$

Notice that Lemmas 1–3 do not require the information structure to be monotone. Thus, they apply to any information structure. By Lemmas 1 and 2, $\overline{a}_i$ is a common default action across the three problems. Thus, given Lemma 3, we need only to check (9) and (10) to apply Propositions 1 and 2. To do so, we make use of the properties of monotone information structures.

Lemma 4 *In a two-player quadratic game with monotone information structure, the equilibrium is unique and continuous. Moreover, if actions are strategic complements, the equilibrium is also increasing in signal realizations.*

The proof of Lemma 4 is simple. Given that our assumption that $|\alpha| < 1$, the best response mapping is a contraction. The contraction mapping theorem then implies that the game has a unique equilibrium.[11] When $\alpha \geq 0$, $(M2)$ and $(M3)$ ensure that the best response mapping preserves monotonicity, implying an increasing fixed point. Intuitively, $(M2)$ implies that the expected state of the world is higher conditional on a higher signal, and $(M3)$ implies that player j's expected action is also higher if player j's strategy is increasing. Complementarity then ensures that player i would also take a higher action given a higher signal in equilibrium.

We are now ready to prove Proposition 3. We will provide the arguments for the proof and relegate their verifications to the Appendix.

When actions are strategic complements, by Lemma 4, the unique equilibrium is increasing under both $\left(\overline{F}_i, \phi\right)$ and $\overline{F}$. $(M3)$ then ensures that the covariance between the strategies $\sigma_i^*\left(x_i; \overline{F}_i, \phi\right)$ and $\sigma_j^*\left(x_j; \overline{F}\right)$ must be positive. Therefore, the original problem $\left(\overline{F}_i, \widetilde{u}_i\right)$ is relevant to the difference problem $\left(\overline{F}_i, \widetilde{v}_i\right)$. By Proposition 1, the value of information is higher when the other player has more information. The generalization of part (1) of Proposition 3 to more than 2 players is straightforward. When $N > 2$ and $\alpha \geq 0$, there still exists a unique equilibrium that is increasing in signal realizations. By the same argument, complementarity in actions translates into complementarity in information.

When actions are strategic substitutes, there is no guarantee that an increasing equilibrium exists under $\overline{F}$. However, it is easy to see that $\sigma^*\left(x; \overline{F}_i, \phi\right)$ is increasing. Suppose further that $\sigma_j^*\left(x_j; \overline{F}\right)$ is decreasing in x_j. In this case, $(M3)$ again allows us to conclude that (9) holds. As before, the original problem $\left(\overline{F}_i, \widetilde{u}_i\right)$ is relevant to the difference problem $\left(\overline{F}_i, \widetilde{v}_i\right)$. Therefore, player i's value of information is higher when player j has more information. If there are more than 2 players, the generalization of part (2.a) of Proposition 4 requires that $\sigma_j^*\left(x_j; \overline{F}\right)$ is decreasing in x_j for all $j \neq i$. If $\sigma^*\left(x; \overline{F}\right)$ is increasing, by a similar argument, (10) does not hold; therefore, the new problem $\left\langle \overline{F}_i, \widetilde{w}_i \right\rangle$ is not strongly relevant

[11] The same approach is used in Mason and Valentinyi (2010) to establish the existence and uniqueness of monotone pure strategy equilibrium in Bayesian games under a different set of assumptions.

to the difference problem $\langle \overline{F}_i, \widetilde{v}_i \rangle$. Since, by Lemmas 1 and 2, $\overline{a}_i$ is a common default action in the original and difference problems, Proposition 2 implies that the value of information is lower when the other player has more information.

The monotone information structure goes beyond the normal-quadratic setting. Adhering to the quadratic payoff function greatly simplifies the induced difference problem. By Lemma 3, we only need to evaluate the sign of the covariance between two equilibrium strategies to apply either Proposition 1 or Proposition 2. Extending our results beyond the quadratic payoff functions, however, will be challenging, as Lemma 3 no long holds.

Next, we return to the more familiar affine information structure and show that the results in Proposition 3 apply as well. Moreover, affine information structure allows us to identify conditions under which the premise of (2.a) or (2.b) holds.

4.3.2 Affine Information Structures

The next class of information structures we consider is the affine information structure. This class of information structures includes the most familiar Normal environment, which the majority of papers in the literature assume. As pointed out by Vives (1988), the class of affine information structures includes many cases other than the Normal environment. The signals could distribute according to Binomial, Negative Binomial, Poisson, Gamma or Exponential distributions when natural conjugate priors are assigned.

Definition 4 (Affine information structure) *An information structure is affine if and only if for all $i, j \in I$,*

A1 $E(\theta|x_i) = \delta_i x_i + d_i$, *where* $0 < \delta_i < 1$ *and* $d_i \in \mathbb{R}$*;*

A2 $E(x_j|x_i) = E(x_j) + E(\theta|x_i) - E(\theta)$.

Consider the Normal environment as an example. Let θ distribute according to a Normal distribution with mean μ and finite variance σ^2. Player i receives a signal x_i such that

$x_i = \theta + \varepsilon_i$, where ε_i is a noise term independent of both θ and ε_j and distributed according to a Normal distribution with mean 0 and variance υ. Under these assumptions, $E(\theta|x_i) = \frac{\sigma^2}{\sigma^2+\upsilon}x_i + \frac{\upsilon}{\sigma^2+\upsilon}\mu$. $(A1)$ and $(A2)$ are satisfied.

Under these information structures, we establish the existence of a unique *linear* equilibrium in the following lemma.

Lemma 5 *In a two-player quadratic game with affine information structure, the equilibrium is unique and linear. Moreover, if actions are strategic complements, then the equilibrium is also increasing in signal realizations.*

Notice that even though monotone information structures seem to allow greater flexibility in equilibrium behaviors, the class of affine information structures is, strictly speaking, not a subset of the monotone information structures. As a result, we must establish the validity of Proposition 3 in this new environment.

Proposition 4 *In a two-player quadratic game with affine information structure,*

1. *when actions are strategic complements, i.e., $\alpha \geq 0$, information is strategic complement;*

2. *when actions are strategic substitutes, i.e., $\alpha \leq 0$,*

   (a) if $\sigma_j^*\left(x_j; \overline{F}\right)$ is decreasing in signal realization x_j, then player j's information is complementary to player i's information, and

   (b) if $\sigma^*\left(x; \overline{F}\right)$ is increasing in signal realizations, then the players' information is strategic substitute.

The proof of Proposition 4 through the notion of relevance is similar to that of Proposition 3 and is relegated to the Appendix. (Notice that Lemmas 1–3 remain valid for affine information structures.) Part 2 of Proposition 4 provides two conditions on equilibrium behaviors given which we can conclude whether information is strategic complement or substitute. For a two-player quadratic game with affine information structure, we can check these assumptions easily.

Lemma 6 *Consider a two-player quadratic game with affine information structure, in which actions are strict strategic substitutes, i.e.,* $\alpha < 0$*. Suppose* $\beta_i \geq \beta_j$*, then,* $\sigma_i^* \left(x_i; \overline{F}\right)$ *is increasing in signal realization* x_i*. Moreover,* $\sigma_j^* \left(x_j; \overline{F}\right)$ *is increasing (decreasing) in signal realization* x_j *if and only if* $\beta_j \geq (\leq) \left|\alpha\right| \delta_i \beta_i$*.*[12]

It is easy to see that, when $\alpha < 0$, it is impossible to have both players using decreasing strategies, as one of the players can switch to an increasing strategy and receive a higher payoff. Intuitively, the player whose preference is more sensitive to the state of the world, i.e., the player with a higher β, would use an increasing strategy in the unique equilibrium. Whether the remaining player would also use an increasing strategy depends on his preference intensity. If the incentive to "match" the state is high enough, he will still use an increasing strategy in equilibrium. Otherwise, the incentive to "dodge" the other player's action overwhelms the incentive to "match" the state of the world, the equilibrium strategy becomes decreasing. Applying Proposition 4, we conclude that,

Proposition 5 *Consider a two-player quadratic game with affine information structure, in which actions are strict strategic substitutes, i.e.,* $\alpha < 0$*, and suppose* $\beta_i \geq \beta_j$*. Then, if* $\beta_j \geq \left|\alpha\right| \delta_i \beta_i$*, the players' information is strategic substitute. If* $\beta_j \leq \left|\alpha\right| \delta_i \beta_i$*, player* j*'s information is complementary to player* i*'s information.*

Notice that applying Proposition 4 yields a prediction on the complementary and substitutability of information in all cases except for player j when $\beta_j \leq \left|\alpha\right| \delta_i \beta_i$. In that case, when player i is uninformed, player j employs an increasing strategy in equilibrium and

[12] In a N-player version of this game, if the players have identical β's, the equilibrium is always increasing. This is true even if the players have different information. This result is demonstrated by Jiménez-Martínez (2014) in the two-player case and Hellwig and Veldkamp (2009) in the case with a continuum of players.

uses his information to "match" the state of the world. When player i is informed, player j employs a decreasing strategy and uses his information to "dodge" player i's action. The comparison of the value of information between these two situations is in general ambiguous.

4.4 A Global Game

In this section, we apply the notion of relevance to obtain sufficient conditions for a decrease in the value of information in a global game with heterogeneous agents when there is a uniform increase/decrease in their investment costs.

Consider a global game with a continuum of investors indexed by $i \in [0,1]$. Investors choose simultaneously whether to invest (I) or not (N). The economic fundamental is characterized by $\theta \in \Theta \subseteq \mathbb{R}$, where $[0,1] \subseteq \Theta$. For each individual i, investment costs $c_i \geq 0$, where c is a continuously differentiable function of the index i. The return to a successful investment is 1 and an investment is successful if and only if the proportion of investors who chooses to invest is high enough, i.e., $p > 1 - \theta$. The payoff to no investment is always 0. To summarize, the payoff function for investor i is given by

$$u_i(I, p, \theta) = \begin{cases} 1 - c_i & \text{if } p > 1 - \theta, \\ -c_i & \text{o.w.,} \end{cases} \quad \text{and } u(N, p, \theta) = 0.$$

We impose the following assumptions on the information structure throughout this section on global games.

Definition 5 (Monotone information structure in global games) *An information structure F of a global game is monotone if and only if it satisfies the followings:*

MG1.1 *The support set Θ is convex, and the support set X_i is convex and compact,*

MG1.2 *The functions $f_\Theta(\theta)$, $f_{X_i}(x_i)$, $f_\Theta(\theta|x_i)$, $f_{X_i}(x_i|\theta)$, $\frac{dF_{X_i}(x_i|\theta)}{d\theta}$ and $\frac{\partial F_\Theta(\theta|x_i)}{\partial x_i}$ exist and are continuous,*

MG2 *Conditional on θ, x_i is i.i.d.,*

MG3 *For all $x_i \in X_i$ and $\theta \in \Theta$, $\frac{dF_{X_i}(x_i|\theta)}{d\theta} \leq 0$,*

MG4 *There exists $k > 0$ such that for all $x_i \in X_i$ and $\bar{\theta} \in [0, 1]$, $\frac{\partial F_\Theta(\bar{\theta}|x_i)}{\partial x_i} \leq -k$.*

$(MG1.1)$ and $(MG1.2)$ are technical assumptions for equilibrium existence. $(MG2)$ assumes that the investors are homogeneous in private information. $(MG3)$ and $(MG4)$ impose monotonicity on the information structure, which, together with the complementarity of own action with both the state of the world and the average action, guarantees the existence of a monotone equilibrium in cut-off strategies. Proposition 6 is the main result of this section.

Proposition 6 *Suppose*

1) no investment is originally an optimal default action for investor i in the equilibrium of the original global game and the investment cost for each investor increases, or

2) investment is originally an optimal default action for investor i in the equilibrium of the original global game and the investment cost for each investor decreases,

then, there exists an equilibrium after the change in the investment costs in which investor i's value of information decreases.

Intuitively, when the market sentiment is bad enough so that not investing is optimal in the absence of information, a further increase in the investment costs across the economy renders information even less useful, as the investor may as well withdraw from the market. Similar intuition applies to the opposite case when the market sentiment is already good and investment costs decrease.

The remainder of this section is dedicated to the proof of Proposition 6 using the notion of relevance. As before, we only provide the arguments and leave the details in the Appendix. We will focus on equilibrium in cut-off strategies. Investor i's strategy σ_i is a cut-off strategy if and only if there exists $\tau_i \in X_i$ such that

$$\sigma_i(x_i) = \begin{cases} I & \text{if } x_i > \tau_i, \\ N & \text{if } x_i < \tau_i. \end{cases}$$

With a slight abuse of notations, we also denote a cut-off strategy σ_i by its cut-off τ_i. Given that the investors' strategies, the average action p is a deterministic function of the state of the world θ. Moreover, by $(MG3)$, $p(\theta)$ is a continuous increasing function that is bounded between 0 and 1. Thus, there exists a unique cut-off $\overline{\theta} \in [0,1]$ such that

$$p\left(\overline{\theta}\right) + \overline{\theta} = 1. \tag{11}$$

As $\Pr(p(\theta) > 1 - \theta | x_i = \tau_i) = 1 - F_{\Theta}\left(\overline{\theta} | x_i = \tau_i\right)$, the cut-off $\overline{\theta}$ then implies a unique best response in cut-off strategies for each investor. An application of Glicksberg's fixed point theorem shows that there exists an equilibrium in cut-off strategies.

Lemma 7 *In a global game with monotone information structure, there exists an equilibrium in cut-off strategies.*

Now, consider two games with different cost functions, c and c'. In the equilibria in cut-off strategies considered, investor j's optimal cut-off strategies are τ_j and τ_j', and the corresponding cut-offs for the state are $\overline{\theta}$ and $\overline{\theta}'$, respectively.

Investor i's payoff function is

$$\widetilde{u}_i(I,\theta) = 1_{\left\{\theta > \overline{\theta}\right\}} - c_i, \text{ and } \widetilde{u}_i(N,\theta) = 0$$

in the original problem. Investor i's payoff function is

$$\widetilde{w}_i(I,\theta) = 1_{\{\theta > \overline{\theta}'\}} - c_i', \text{ and } \widetilde{w}_i(N,\theta) = 0$$

in the new problem. Therefore, investor i's payoff function is

$$\widetilde{v}_i\left(I,\theta\right)=1_{\{\theta>\overline{\theta}'\}}-1_{\{\theta>\overline{\theta}\}}+c_i-c_i', \text{ and } \widetilde{v}_i\left(N,\theta\right)=0$$

in the difference problem. Given the payoff function in the difference problem, we have,

Lemma 8 *In the difference problem, (no) investment is a dominant strategy and thus an optimal default action if* $\overline{\theta}\geq\overline{\theta}'$ *and* $c_i\geq c_i'$ *(*$\overline{\theta}\leq\overline{\theta}'$ *and* $c_i\leq c_i'$*).*

Intuitively, Lemma 8 says that investment becomes more profitable if the market sentiment improves and your own cost of investment decreases. Similarly, no investment would become more attractive if the opposites are true. When a default action is also a dominant strategy, the new problem can never be strongly relevant to the difference problem. Furthermore, if the default actions in the original and difference problems also happen to be identical, we can apply Proposition 2 and conclude that,

Lemma 9 *Given a change in the individual costs of investment in the economy, investor i's value of information decreases if either*

1) the cut-off $\overline{\theta}$ *in the new equilibrium is higher than in the original equilibrium, no investment is an optimal default action for investor i in the original equilibrium, and investor i's cost of investment increases, or*

2) the cut-off $\overline{\theta}$ *in the new equilibrium is lower than in the original equilibrium, investment is an optimal default action for investor i in the original equilibrium, and investor i's cost of investment decreases.*

Proposition 6 follows immediately from Lemma 9 if we can show that there exists an equilibrium in which $\overline{\theta}$ increases (decreases) after the costs of investment increase (decrease). This is guaranteed by $(MG3)$ and $(MG4)$. Notice that in addition to being an important intermediate step to prove Proposition 6, Lemma 9 also provides us some idea of how value of information changes when the change in the individual costs of investment is *not* uniform across the economy and Proposition 6 does not apply.

4.5 A Multi-expert Persuasion Game

In this section, we apply the notion of relevance to a multi-expert persuasion game studied in Bhattacharya and Mukherjee (2013). Some assumptions in Bhattacharya and Mukherjee (2013) that are unimportant for our purpose are relaxed. Kartik, Lee, and Suen (2017) prove a related result in a model where the experts have linear preferences over the decision maker's beliefs.

The model has a single DM and two experts. The players have a commonly known prior belief on the state of the world $\theta \in [0,1]$ that is distributed according to a probability density function f that is bounded and has full support. The DM's payoff $u^{DM}(y,\theta)$ depends on the state of the world θ and the action $y \in [0,1]$ she takes. The function u^{DM} is continuously differentiable. Moreover, given $\theta \in [0,1]$, $u^{DM}(\cdot,\theta)$ is strictly concave and is maximized at $y = \theta$. The payoff of expert $i \in \{1,2\}$ is given by the function $u_i(y,\theta)$. The two experts have opposite and extreme biases. That is, given $\theta \in [0,1]$, $u_1(\cdot,\theta)$ is strictly increasing and $u_2(\cdot,\theta)$ is strictly decreasing. If expert i acquires information, he receives signal $x_i = \theta$ with probability $p_i \in (0,1)$ and $x_i = \varphi$ with probability $1 - p_i$, independent of expert j's signal. With the null information structure, expert i always receives the null signal $x_i = \varphi$.

The sequence of events is as follows. First, each expert receives a signal privately and then simultaneously sends a message to the DM. The message m_i that expert i can send is restricted to $\{\varphi, x_i\}$. In other words, he can only choose to reveal or conceal his signal. In particular, expert i can only send $m_i = \varphi$ if $x_i = \varphi$. Next, the DM chooses y based on the messages received and the game ends.

Bhattacharya and Mukherjee (2013) show that the equilibrium of this game is characterized by a null action $\mathbf{y}^*$, which the DM takes when $m_1 = m_2 = \varphi$. Moreover, expert 1 (2) reveals his signal if $x_1 > \mathbf{y}^*$ ($x_2 < \mathbf{y}^*$) and hides his signal if $x_1 < \mathbf{y}^*$ ($x_2 > \mathbf{y}^*$). When the received signal is equal to $\mathbf{y}^*$, the expert is indifferent between sending $\mathbf{y}^*$ or φ and either message can be sent in equilibrium. When a nonempty message, i.e., $m_i \neq \varphi$, is received, the DM chooses the optimal action $y = m_i$.

We are interested in how the presence of expert 2 changes the value of expert 1's information. To answer this question, we consider two equilibria: in the first one, expert 2 has no information; and in the second one, expert 2 has information. We call the former the original equilibrium and the latter the new. Let $\mathbf{y}_n^*$ ($\mathbf{y}_o^*$) be the null action in the equilibrium in which expert 1 has information and expert 2 has information (no information). We have

Lemma 10 *The equilibrium null action shifts towards expert* 1*'s preferred action after expert* 2 *have acquired information, but it does not reach the upper bound. i.e.,* $\mathbf{y}_o^* < \mathbf{y}_n^* < 1$.

We construct the original and new problems corresponding to the original and new equilibria, respectively. Then we define the induced difference problem accordingly. Under the null information, expert 1 can only send $m_1 = \varphi$; therefore, expert 1's default actions are identical in all three problems. The payoff difference between $\mathbf{y}_n^*$ and the default action $m_1 = \varphi$ in the difference problem is

$$-p_1 \int_{\mathbf{y}_n^*}^{1} \{u_1(\mathbf{y}_n^*, \theta) - u_1(\mathbf{y}_o^*, \theta)\} f(\theta) \, d\theta. \tag{12}$$

(See the Appendix for details.) By Lemma 10, (12) is always negative. Thus, the new problem is not strongly relevant to the difference problem. Applying Proposition 2, we conclude that

Proposition 7 *When experts have opposite and extreme biases, information is substitute.*

This result can be understood intuitively. When an extra expert acquires information and reports strategically, the DM raises her null action from $\mathbf{y}_o^*$ to $\mathbf{y}_n^*$. This change has two effects on the value of expert 1's information. First, whenever expert 1 reveals his information in the new equilibrium, the DM's null action is lifted from $\mathbf{y}_n^*$ to θ instead of from $\mathbf{y}_o^*$. Since $\mathbf{y}_o^* < \mathbf{y}_n^*$, the gain from disclosure is smaller in the new equilibrium. The decrease in the expected gain from disclosure is captured by (12), which we obtain from the difference problem. Second, expert 1 is less likely to use his information in the new equilibrium. Since he only uses his information when the state is higher than the null action, $\mathbf{y}_o^* < \mathbf{y}_n^*$ implies that information is less likely to be useful in the new problem. Notice that the second effect does not appear in our calculation, because, by applying Proposition 2, we have bypassed it by noticing that it must be negative.[13]

One might be interested in knowing if the answer to our question changes when the experts have the *same* biases. In this case, applying the notion of relevance yields no prediction. Although the negative effects identified in the previous paragraph seem to turn positive when the DM's null action decreases, one must add to it the effect of expert 2's disclosure. Since the experts have the same ordinal preferences, whenever expert 1 would like to disclose, so does expert 2. Thus, expert 1's disclosure may be unnecessary and duplicate expert 2's. The duplication effect reduces the value of information.[14] Without imposing further restrictions, the overall effect is ambiguous in general, see Kartik, Lee, and Suen (2017) for a related discussion.

5 Conclusion

We introduce the notion of relevance in this paper. This notion speaks to a lot of decision making situations. Given a fixed information structure, the ranking of payoff functions, according to the value of information in the corresponding decision problem, is closely related to the idea of relevance. We apply the notion of relevance to quadratic games, a global game, and a persuasion game. In all the three games, we study the value of information without relying on a closed form solution. To establish a relevance relation or the lack of it in games, what matters most often is the structure of the equilibrium strategies, for instance,

[13]The negative effect arises from the suboptimality of the optimal strategy for the new problem in the original problem.

[14]The negative effect of *duplicated disclosure* on the value of information is best illustrated by the presence of a non-strategic expert 2, i.e., expert 2 sends $m_2 = \varphi$ only when $x_2 = \varphi$. In this case, a change from an equilibrium in which expert 2 has no information to an equilibrium in which expert 2 has information decreases the value of information to expert 1. This is because with some probability, the DM learns the true state from expert 2 and expert 1's disclosure does not make a difference. In the Appendix, we demonstrate this by showing that the new problem is not relevant to the difference problem.

monotonicity/linearity in quadratic games, cutoff strategies in the global game, and the null action in the persuasion game. A direction for future research to explore further the notion of relevance in individual decision problems and find useful conditions under which the relevance relation satisfies certain desirable properties such as symmetry and transitivity.

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6 Appendix

Proof of Proposition 1. Proof in text. ■

Proof of Proposition 2. Proof in text. ■

Proof of Lemma 1. Given any information structure F, take any equilibrium (σ_1^*, σ_2^*), the first order conditions of the players imply that for $i \in \{1,2\}$,

$$\sigma_i^*(x_i; F) = \alpha E\left(\sigma_j^*(x_j; F)|x_i\right) + \beta_i E(\theta|x_i).$$

Applying the law of iterated expectations, we have

$$E(\sigma_1^*(x_1; F)) = \alpha E(\sigma_2^*(x_2; F)) + \beta_1 E(\theta), \tag{13}$$

$$E(\sigma_2^*(x_2; F)) = \alpha E(\sigma_1^*(x_1; F)) + \beta_2 E(\theta). \tag{14}$$

Our assumption that $|\alpha| < 1$ implies that the solution to (13) and (14) is unique. Thus, for $i \in \{1,2\}$,

$$E(\sigma_i^*(x_i; F)) = \overline{a}_i.$$

■

Proof of Lemma 2. Proof in text. ■

Proof of Lemma 3. Proof in text. ■

Proof of Lemma 4. Given any strategy profile σ, consider the best response mapping $\mathbf{B}$ which takes strategy profile σ and returns the best response strategy profile $\mathbf{B}(\sigma)$, where, for each $i \in \{1,2\}$, the i-component of the best response mapping, $\mathbf{B}_i(\sigma)$ is player i's best response to player j's strategy σ_j. We have

$$\mathbf{B}_i(\sigma)(x_i) = \alpha E(\sigma_j|x_i) + \beta_i E(\theta|x_i).$$

By $(M1)$, given any bounded strategy profile σ, $\mathbf{B}_i(\sigma)$ is bounded and well-defined. By $(M2)$ and $(M3)$, $\mathbf{B}(\sigma)$ is also continuous. Thus, it is without loss of generality to focus on the space of bounded continuous functions from X to A, which we denote by S. Consider the metric space (S,d), where d is the metric associated with the supremum norm. (S,d) is a complete metric space. For all $\sigma,\sigma'\in S$,

$$
\begin{aligned}
& d\left(\mathbf{B}(\sigma),\mathbf{B}(\sigma')\right) \\
= \; & \left\|\mathbf{B}(\sigma)-\mathbf{B}(\sigma')\right\|_\infty \\
= \; & \sup_{(x_1,x_2)\in X_1\times X_2}\left\{\left|\alpha\left[E(\sigma_2-\sigma_2'|x_1)\right]\right|^p+\left|\alpha\left[E(\sigma_1-\sigma_1'|x_2)\right]\right|^p\right\}^{\frac{1}{p}} \\
= \; & |\alpha|\sup_{(x_1,x_2)\in X_1\times X_2}\left\{\left|E(\sigma_2-\sigma_2'|x_1)\right|^p+\left|E(\sigma_1-\sigma_1'|x_2)\right|^p\right\}^{\frac{1}{p}} \\
\leq \; & |\alpha|\left\{\sup_{x_2\in X_2}\left|\sigma_2(x_2)-\sigma_2'(x_2)\right|^p+\sup_{x_1\in X_1}\left|\sigma_1(x_1)-\sigma_1'(x_1)\right|^p\right\}^{\frac{1}{p}} \\
= \; & |\alpha|\sup_{(x_1,x_2)\in X_1\times X_2}\left\{\left|\sigma_2(x_2)-\sigma_2'(x_2)\right|^p+\left|\sigma_1(x_1)-\sigma_1'(x_1)\right|^p\right\}^{\frac{1}{p}} \\
= \; & |\alpha|\, d(\sigma,\sigma').
\end{aligned}
$$

Since, $|\alpha|<1$ by assumption, $\mathbf{B}$ is a contraction. By the contraction mapping theorem, e.g. Theorem 3.2 in Stokey, Lucas, and Prescott (1989), $\mathbf{B}$ has a unique fixed point.

Finally, suppose $\alpha\geq 0$, $(M2)$ and $(M3)$ imply that given any increasing strategy profile σ, $\mathbf{B}(\sigma)$ is also increasing. Thus, the fixed point of $\mathbf{B}$ must also be increasing. ■

Proof of Proposition 3. Part *(1)*: By Lemma 4, if $\alpha\geq 0$, the equilibrium strategies $\sigma_i^*\left(x_i;\overline{F}_i,\phi\right)$ and $\sigma_j^*\left(x_j;\overline{F}\right)$ are increasing in their signal realization. Since X_i is convex, Lemma 1 and continuity imply that there must be an $\hat{x}_i\in X_i$ such that

$$\int_{X_j}\sigma_j^*\left(x_j;\overline{F}\right)dF\left(x_j|\hat{x}_i\right)=\overline{a}_j.$$

Since $\sigma_j^*\left(x_j;\overline{F}\right)$ is an increasing function of x_j and the family $F(\cdot|x_i)$ is ordered by first order stochastic dominance, we must have

$$\int_{X_j} \sigma_j^* \left(x_j; \overline{F}\right) dF\left(x_j|x_i\right) \geq \overline{a}_j, \text{ if } x_i > \hat{x}_i,$$

and

$$\int_{X_j} \sigma_j^* \left(x_j; \overline{F}\right) dF\left(x_j|x_i\right) \leq \overline{a}_j, \text{ if } x_i < \hat{x}_i.$$

Thus,

$$\begin{aligned}
& Cov\left(\sigma_i^*\left(x_i; \overline{F}_i, \phi\right), \sigma_j^*\left(x_j; \overline{F}\right)\right) \\
= & \int_{x_i \leq \hat{x}_i} \left(\sigma_i^*\left(x_i; \overline{F}_i, \phi\right) - \overline{a}_i\right) \underbrace{\left(\int_{X_j} \sigma_j^*\left(x_j; \overline{F}\right) dF\left(x_j|x_i\right) - \overline{a}_j\right)}_{\leq 0} dF\left(x_i\right) \\
& + \int_{x_i > \hat{x}_i} \left(\sigma_i^*\left(x_i; \overline{F}_i, \phi\right) - \overline{a}_i\right) \underbrace{\left(\int_{X_j} \sigma_j^*\left(x_j; \overline{F}\right) dF\left(x_j|x_i\right) - \overline{a}_j\right)}_{\geq 0} dF\left(x_i\right) \\
\geq & \left(\sigma_i^*\left(\hat{x}_i; \overline{F}_i, \phi\right) - \overline{a}_i\right) \int_{X_i} \left(\int_{X_j} \sigma_j^*\left(x_j; \overline{F}\right) dF\left(x_j|x_i\right) - \overline{a}_j\right) dF\left(x_i\right) \\
= & \left(\sigma_i^*\left(\hat{x}_i; \overline{F}_i, \phi\right) - \overline{a}_i\right)\left(E\left[\sigma_j^*\left(x_j; \overline{F}\right)\right] - \overline{a}_j\right) \\
= & 0,
\end{aligned}$$

where the inequality follows from the fact that $\sigma_i^*\left(x_i; \overline{F}_i, \phi\right)$ is increasing in x_i and the last equality follows from Lemma 1.

Part (*2.a*): Consider the information structure $\left(\overline{F}_i, \phi\right)$, by Lemma 1, $\sigma_j^* = \overline{a}_j$. The first order condition of the player i becomes

$$\sigma_i^*\left(x_i; \overline{F}_i, \phi\right) = \alpha \overline{a}_j + \beta_i E\left(\theta|x_i\right).$$

By $(M2)$, $\sigma_i^*\left(x_i; \overline{F}_i, \phi\right)$ must be increasing in x_i. Part (*2.a*) will follow from an application of Lemma 3 and Proposition 1 if

$$Cov\left(\sigma_i^*\left(x_i;\overline{F}_i,\phi\right),\sigma_j^*\left(x_j;\overline{F}\right)\right)\leq 0.$$

Since X_i is convex, Lemma 1 and continuity imply that there must be an $\hat{x}_i \in X_i$ such that

$$\int_{X_j}\sigma_j^*\left(x_j;\overline{F}\right)dF\left(x_j|\hat{x}_i\right)=\overline{a}_j.$$

Since $\sigma_j^*\left(x_j;\overline{F}\right)$ is by assumption a decreasing function of x_j and the family $F\left(\cdot|x_i\right)$ is ordered by first order stochastic dominance, we must have

$$\begin{aligned}
& Cov\left(\sigma_i^*\left(x_i;\overline{F}_i,\phi\right),\sigma_j^*\left(x_j;\overline{F}\right)\right)\\
= & \int_{x_i\leq\hat{x}_i}\left(\sigma_i^*\left(x_i;\overline{F}_i,\phi\right)-\overline{a}_i\right)\underbrace{\left(\int_{X_j}\sigma_j^*\left(x_j;\overline{F}\right)dF\left(x_j|x_i\right)-\overline{a}_j\right)}_{\geq 0}dF\left(x_i\right)\\
& +\int_{x_i>\hat{x}_i}\left(\sigma_i^*\left(x_i;\overline{F}_i,\phi\right)-\overline{a}_i\right)\underbrace{\left(\int_{X_j}\sigma_j^*\left(x_j;\overline{F}\right)dF\left(x_j|x_i\right)-\overline{a}_j\right)}_{\leq 0}dF\left(x_i\right)\\
\leq & \left(\sigma_i^*\left(\hat{x}_i;\overline{F}_i,\phi\right)-\overline{a}_i\right)\int_{X_i}\left(\int_{X_j}\sigma_j^*\left(x_j;\overline{F}\right)dF\left(x_j|x_i\right)-\overline{a}_j\right)dF\left(x_i\right)\\
= & \left(\sigma_i^*\left(\hat{x}_i;\overline{F}_i,\phi\right)-\overline{a}_i\right)\left(E\left[\sigma_j^*\left(x_j;\overline{F}\right)\right]-\overline{a}_j\right)\\
= & 0,
\end{aligned}$$

where the inequality follows from the fact that $\sigma_i^*\left(x_i;\overline{F}_i,\phi\right)$ is increasing in x_i and the last equality follows from Lemma 1.

Part (*2.b*) follows from the fact that

$$Cov\left(\sigma_i^*\left(x_i;\overline{F}\right),\sigma_j^*\left(x_j;\overline{F}\right)\right)\geq 0,$$

the proof of which follows from replacing $\sigma_i^*\left(x_i;\overline{F}_i,\phi\right)$ with $\sigma_i^*\left(x_i;\overline{F}\right)$ in the proof of part (*1*), and an application of Lemma 3 and Proposition 2. ■

Proof of Lemma 5. The first order condition for equilibrium is

$$\sigma_i^*(x_i) = \alpha E\left(\sigma_j^*(x_j)|x_i\right) + \beta_i E(\theta|x_i).$$

Denote $E_i(\theta) = E(\theta|x_i)$, $E_iE_j(\theta) = E(E(\theta|x_j)|x_i)$, $E_{ij}^k(\theta) = E\left(E\left(E_{ij}^{k-1}(\theta)|x_j\right)|x_i\right)$ and $E_{ij}^0(\theta) = \theta$. Then,

$$\begin{aligned}
\sigma_i^*(x_i) &= \beta_i E(\theta|x_i) + \alpha E\left(\sigma_j^*(x_j)|x_i\right) \\
&= \beta_i E_i(\theta) + \alpha\beta_j E_iE_j(\theta) + \alpha^2 E\left(E\left(\sigma_i^*(x_i)|x_j\right)|x_i\right) \\
&= \ldots \\
&= \sum_{k=1}^{\infty}\left(\alpha^2\right)^{k-1}\left[\beta_i E_{ij}^{k-1}E_i(\theta) + \alpha\beta_j E_{ij}^k(\theta)\right]. \qquad (15)
\end{aligned}$$

By $(A1)$, there exist $0 < \delta_i, \delta_j < 1$ and $d_i, d_j \in \mathbb{R}$, such that $E(\theta|x_i) = \delta_i x_i + d_i$ and $E(\theta|x_j) = \delta_j x_j + d_j$. Then, by $(A2)$, we have, for all $k \geq 1$,

$$\begin{aligned}
E_{ij}^k E_i(\theta) &= E_{ij}^k(\delta_i x_i + d_i) \\
&= \delta_i E_{ij}^{k-1}E_iE_j(x_i) + d_i \\
&= \delta_i E_{ij}^{k-1}E_i\left(E(x_i) + E(\theta|x_j) - E(\theta)\right) + d_i \\
&= \delta_i E_{ij}^k(\theta) + \left[\delta_i E(x_i) - \delta_i E(\theta) + d_i\right].
\end{aligned}$$

Let $\delta_i E(x_i) - \delta_i E(\theta) + d_i = d_i'$, then $E_{ij}^k E_i(\theta) = \delta_i E_{ij}^k(\theta) + d_i'$. Similarly, all $k \geq 1$, $E_{ij}^k(\theta) = \delta_j E_{ij}^{k-1}E_i(\theta) + d_j'$, where $d_j' = \delta_j E(x_j) - \delta_j E(\theta) + d_j$. Therefore, $\sigma_i^*(x_i)$ is linear in x_i. Since $0 < \delta_i, \delta_j < 1$, our assumption that $|\alpha| < 1$ implies that the sum (15) must converge. Thus, a unique equilibrium exists and is linear. Finally, by $(A1)$, $\delta_i, \delta_j > 0$ and $\beta_i, \beta_j > 0$, if $\alpha \geq 0$, $\sigma_i^*(x_i)$ must be increasing in x_i. ■

Proof of Proposition 4. Part (1): By Lemma 5, (9) is equivalent to

$$Cov\left(x_i, x_j\right) \geq 0.$$

By $(A1)$, there exists $\delta_i > 0$ and $d_i \in \mathbb{R}$ such that

$$E\left(\theta | x_i\right) = \delta_i x_i + d_i. \tag{16}$$

Applying the law of iterated expectations to (16), we have $E\left(\theta\right) = \delta_i E\left(x_i\right) + d_i$. $(A2)$ implies that

$$E\left(x_j | x_i\right) - E\left(x_j\right) = E\left(\theta | x_i\right) - E\left(\theta\right) = \delta_i\left(x_i - E\left(x_i\right)\right).$$

Thus, $Cov\left(x_i, x_j\right) = E\left(\left(x_i - E\left(x_i\right)\right)\left(E\left(x_j | x_i\right) - E\left(x_j\right)\right)\right) = \delta_i E\left(\left(x_i - E\left(x_i\right)\right)^2\right) \geq 0$. The result follows from an application of Lemma 3 and Proposition 1.

Part (2.a): Consider the information structure $\left(\overline{F}_i, \phi\right)$, by Lemma 1, $\sigma_j^* = \overline{a}_j$. The first order condition of the player i becomes

$$\sigma_i^*\left(x_i; \overline{F}_i, \phi\right) = \alpha \overline{a}_j + \beta_i E\left(\theta | x_i\right).$$

By $(A1)$, $\sigma_i^*\left(x_i; \overline{F}_i, \phi\right)$ must be increasing in x_i. By Lemma 5, $\sigma_i^*\left(x_i; \overline{F}_i, \phi\right)$ and $\sigma_j^*\left(x_j; \overline{F}\right)$ are linear. By assumptions, $\alpha \leq 0$ and $\sigma_j^*\left(x_j; \overline{F}\right)$ is decreasing in x_j. By the proof of part (1), $Cov\left(x_i, x_j\right) \geq 0$, thus $\alpha Cov\left(\sigma_i^*\left(x_i; \overline{F}_i, \phi\right), \sigma_j^*\left(x_j; \overline{F}\right)\right) \geq 0$ and the claim follows from Lemma 3 and Proposition 1. Similarly, part (2.b) follows from the proof of part (1) and an application of Lemma 3 and Proposition 2. ■

Proof of Lemma 6. Suppose $\alpha < 0$, by (15),

$$\begin{aligned}
\sigma_i^*\left(x_i\right) &= \sum_{k=1}^{\infty}\left(\alpha^2\right)^{k-1} E_{ij}^{k-1}\left[\beta_i E_i\left(\theta\right) + \alpha \beta_j E_i E_j\left(\theta\right)\right] \\
&= \sum_{k=1}^{\infty}\left(\alpha^2\right)^{k-1} E_{ij}^{k-1}\left[\left(\beta_i + \alpha \beta_j \delta_j\right) \delta_i x_i\right] + \text{terms independent of } x_i.
\end{aligned}$$

Since $0 < \delta_i, \delta_j < 1$, $\beta_i \geq \beta_j$ implies $\sigma_i^*(x_i)$ is increasing in x_i. Similarly, $\sigma_j^*(x_j)$ is increasing (decreasing) in x_j if and only if $\beta_j \geq (\leq) |\alpha| \delta_i \beta_i$. ■

Proof of Proposition 5. The result follows immediately from Proposition 4 and Lemma 6. ■

Proof of Lemma 7. Suppose each investor $j \neq i$ is using a cut-off strategy τ_j, we need to show that investor i's best response is uniquely given by a cut-off strategy, so that one can restrict the strategy space to the set of cut-off strategies to prove the existence of an equilibrium in cut-off strategies. By $(MG2)$,

$$p(\theta) = \int_0^1 \left\{1 - F_{X_j}(\tau_j|\theta)\right\} dj. \tag{17}$$

By $(MG3)$,

$$p'(\theta) = -\int_0^1 \frac{\partial F_{X_i}(\tau_i|\theta)}{\partial \theta} di \geq 0.$$

Thus, there exists a unique $\overline{\theta} \in [0,1]$ such that $p(\theta) \lesseqgtr 1 - \theta$ if and only if $\theta \lesseqgtr \overline{\theta}$. By $(MG4)$, $\Pr(p(\theta) > 1 - \theta|x_i)$ is strictly increasing in x_i. Since investor i's expected payoff of investment is $\Pr(p(\theta) > 1 - \theta|x_i) - c_i$ and the expected payoff of no investment is 0, investor i's best response is uniquely given by a cut-off strategy. Suppose τ_i is in the interior of X_i, τ_i is uniquely pinned down by

$$1 - F_\Theta\left(\overline{\theta}|\tau_i\right) = c_i. \tag{18}$$

Differentiating (18), we have

$$\frac{d\tau_i}{di} = \frac{\frac{dc_i}{di}}{-\frac{\partial F_\Theta\left(\overline{\theta}|x_i\right)}{\partial x_i}\Big|_{x_i=\tau_i}}.$$

By $(MG4)$, $\left|\frac{d\tau_i}{di}\right|$ is uniformly bounded by some positive constant ξ. Let $T \subseteq C([0,1])$ be the set of all functions $\tau : [0,1] \to X_i$ satisfying

$$|\tau(i) - \tau(j)| \leq \xi |i - j|,$$

for all $i, j \in [0,1]$. T is nonempty and convex. Moreover, since T is closed, bounded, and equicontinuous, it is also compact. An equilibrium in cut-off strategies is thus an element $\tau \in T$ such that the cut-off strategy $\tau(i)$ is optimal for investor i. Consider the metric space (T, d), where d is the metric associated with the supremum norm and the best response mapping $\mathbf{B} : T \to T$ which takes an element $\tau \in T$ and returns the best response of investor i, $\mathbf{B}(\tau)(i)$, to τ. We would like to show that $\mathbf{B}$ is continuous and then apply Glicksberg's fixed point theorem to show that a fixed point of $\mathbf{B}$ exists. Notice that τ defines the function p, p defines the cutoff $\overline{\theta}$ and $\overline{\theta}$ uniquely pins down $\mathbf{B}(\tau)$. Thus, it suffices to show that each of these mappings is continuous. Let $\tau \in T$ and $\theta \in [0,1]$ and consider

$$p^{\tau}(\theta) = \int_0^1 \{1 - F_{X_i}(\tau(i)|\theta)\}\, di.$$

Since $f_{X_i}(x|\theta)$ is continuous, there exists $M > 0$ such that $|f_{X_i}(x|\theta)| < M$ for all $x \in X_i$ and $\theta \in [0,1]$. For all $\varepsilon > 0$, consider consider $\tau, \tau' \in T$ such that $\|\tau - \tau'\|_\infty < \frac{\varepsilon}{M}$, then

$$\begin{aligned}
& \left\| p^{\tau} - p^{\tau'} \right\|_\infty \\
= & \sup_{\theta \in [0,1]} \left| p^{\tau}(\theta) - p^{\tau'}(\theta) \right| \\
\leq & \sup_{\theta \in [0,1]} \int_0^1 \left| F_{X_i}(\tau'(i)|\theta) - F_{X_i}(\tau(i)|\theta) \right| di \\
\leq & \int_0^1 M \left| \tau'(i) - \tau(i) \right| di \\
< & \ \varepsilon.
\end{aligned}$$

Thus, the mapping from τ to p is continuous.

Next, for any increasing and continuous function $p : [0,1] \to [0,1]$, let $\overline{\theta}(p)$ be the unique solution to $p\left(\overline{\theta}(p)\right) + \overline{\theta}(p) = 1$. We would like to show that $\overline{\theta}$ is a continuous function. For all $\varepsilon > 0$, consider two increasing and continuous functions g and h that satisfy $\|g - h\|_\infty < \varepsilon$.

Suppose $\overline{\theta}(g) \geq \overline{\theta}(h)$, then

$$
\begin{aligned}
& \left|\overline{\theta}(g) - \overline{\theta}(h)\right| \\
= \; & h\left(\overline{\theta}(h)\right) - g\left(\overline{\theta}(g)\right) \\
= \; & h\left(\overline{\theta}(h)\right) - h\left(\overline{\theta}(g)\right) + h\left(\overline{\theta}(g)\right) - g\left(\overline{\theta}(g)\right) \\
\leq \; & \left|h\left(\overline{\theta}(g)\right) - g\left(\overline{\theta}(g)\right)\right| \\
\leq \; & \|g - h\|_{\infty} \\
< \; & \varepsilon.
\end{aligned}
$$

where the first inequality follows from $\overline{\theta}(g) \geq \overline{\theta}(h)$ and the fact that h is increasing. Similarly, $\left|\overline{\theta}(g) - \overline{\theta}(h)\right| < \varepsilon$ if $\overline{\theta}(g) < \overline{\theta}(h)$. Thus, the mapping from p to $\overline{\theta}$ is continuous.

Finally, we want to show that $\mathbf{B}(\tau)$ is continuous in $\overline{\theta}$. Fix $i \in [0,1]$, if $\mathbf{B}(\tau)(i)$ is interior, differentiate (18), we have,

$$
\frac{d\mathbf{B}(\tau)(i)}{d\overline{\theta}} = \frac{f_{\Theta}\left(\overline{\theta}|\mathbf{B}(\tau)(i)\right)}{-\frac{\partial F_{\Theta}\left(\overline{\theta}|x_i\right)}{\partial x_i}|_{x_i=\mathbf{B}(\tau)(i)}}.
$$

By $(MG1.2)$ and $(MG4)$, there exists $\Psi > 0$ such that $\left|\frac{f_{\Theta}\left(\overline{\theta}|x_i\right)}{\frac{\partial F_{\Theta}\left(\overline{\theta}|x_i\right)}{\partial x_i}}\right| < \Psi$ for any $x_i \in X_i$ and $\overline{\theta} \in [0,1]$. Thus,

$$
\begin{aligned}
& \|\mathbf{B}(\tau) - \mathbf{B}(\tau')\|_{\infty} \\
= \; & \sup_{i \in [0,1]} |\mathbf{B}(\tau)(i) - \mathbf{B}(\tau')(i)| \\
\leq \; & \Psi\left|\overline{\theta} - \overline{\theta}'\right|.
\end{aligned}
$$

$\mathbf{B}(\tau)$ is Lipschitz continuous in $\overline{\theta}$. Together with the previous results, this means that the best response mapping $\mathbf{B} : T \to T$ is continuous. The existence of equilibrium then follows from an application of Glicksberg's fixed point theorem. ■

Proof of Lemma 8. Proof in text. ■

Proof of Lemma 9. Proof in text. ■

Proof of Proposition 6. Suppose no investment is originally an optimal default action for investor i and the investment cost for each investor $j \in [0,1]$ increases from c_j to $c_j + \Delta_j$, where $\Delta_j \geq 0$ and τ^c is an equilibrium under the cost function c. Let

$$T_c := \{\tau \in T : \forall i \in [0,1], \tau(i) \geq \tau^c(i)\},$$

where T is defined in the proof of Lemma 7. First, we show that the new best response mapping $\mathbf{B}$ satisfy $\mathbf{B}(T_c) \subseteq T_c$ so that we can apply Glicksberg's fixed point theorem to the mapping $\mathbf{B} : T_c \to T_c$ and conclude that under the cost function $c + \Delta$, there exists an equilibrium in which the cutoff $\overline{\theta}$ is greater than its counterpart in the original equilibrium under the cost function c. To see that $\mathbf{B}(T_c) \subseteq T_c$, notice that, by (17), $p(\theta)$ decreases if for each $i \in [0,1]$, $\tau(i)$ increases. Moreover, by (11), if for each $\theta \in [0,1]$, $p(\theta)$ decreases, $\overline{\theta}$ increases. By (18), if c_i and $\overline{\theta}$ both increase, $\mathbf{B}(\tau)(i)$ must also increase. Thus, we must have $\mathbf{B}(\tau)(i) \geq \tau^c(i)$. Next, by the same argument as in the proof of Lemma 7, T_c is nonempty, convex and compact and $\mathbf{B} : T_c \to T_c$ is continuous. Applying Glicksberg's fixed point theorem, we conclude that when the cost function for the economy becomes $c + \Delta$, there exists an equilibrium $\tau^{c+\Delta} \in T_c$. Thus, for each $i \in [0,1]$, $\tau^{c+\Delta}(i) \geq \tau^c(i)$. By (11) and (17), $\overline{\theta}$ increases in the new equilibrium. Applying Lemma 9, we prove part (1) of Proposition 6. The proof for part (2) is similar. ■

Proof of Lemma 10. Suppose $\mathbf{y}_n^* = 1$, then the DM's first order condition implies that

$$(1 - p_2) \int_0^1 \frac{\partial u^{DM}(1,\theta)}{\partial y} f(\theta)\, d\theta \geq 0,$$

which is impossible since $\frac{\partial u^{DM}(1,\theta)}{\partial y} < 0$ for all $\theta < 1$. Similarly, suppose $\mathbf{y}_n^* = 0$, then the

DM's first order condition implies that

$$(1-p_1)\int_0^1 \frac{\partial u^{DM}(0,\theta)}{\partial y} f(\theta)\, d\theta \leq 0,$$

which is impossible since $\frac{\partial u^{DM}(0,\theta)}{\partial y} > 0$ for all $\theta > 0$. Thus, $\mathbf{y}_n^* \in (0,1)$ and satisfies

$$\begin{aligned} 0 &= p_1(1-p_2)\int_0^{\mathbf{y}_n^*} \frac{\partial u^{DM}(\mathbf{y}_n^*,\theta)}{\partial y} f(\theta)\, d\theta + p_2(1-p_1)\int_{\mathbf{y}_n^*}^1 \frac{\partial u^{DM}(\mathbf{y}_n^*,\theta)}{\partial y} f(\theta)\, d\theta \\ &\quad + (1-p_1)(1-p_2)\int_0^1 \frac{\partial u^{DM}(\mathbf{y}_n^*,\theta)}{\partial y} f(\theta)\, d\theta. \end{aligned}$$

Suppose $\mathbf{y}_n^* \leq \mathbf{y}_o^*$, then

$$\begin{aligned} 0 &= p_1(1-p_2)\int_0^{\mathbf{y}_n^*} \frac{\partial u^{DM}(\mathbf{y}_n^*,\theta)}{\partial y} f(\theta)\, d\theta + p_2(1-p_1)\int_{\mathbf{y}_n^*}^1 \frac{\partial u^{DM}(\mathbf{y}_n^*,\theta)}{\partial y} f(\theta)\, d\theta \\ &\quad + (1-p_1)(1-p_2)\int_0^1 \frac{\partial u^{DM}(\mathbf{y}_n^*,\theta)}{\partial y} f(\theta)\, d\theta \\ &> (1-p_2)\left(p_1\int_0^{\mathbf{y}_n^*} \frac{\partial u^{DM}(\mathbf{y}_n^*,\theta)}{\partial y} f(\theta)\, d\theta + (1-p_1)\int_0^1 \frac{\partial u^{DM}(\mathbf{y}_n^*,\theta)}{\partial y} f(\theta)\, d\theta\right) \\ &\geq (1-p_2)\left(p_1\int_0^{\mathbf{y}_o^*} \frac{\partial u^{DM}(\mathbf{y}_o^*,\theta)}{\partial y} f(\theta)\, d\theta + (1-p_1)\int_0^1 \frac{\partial u^{DM}(\mathbf{y}_o^*,\theta)}{\partial y} f(\theta)\, d\theta\right), \end{aligned}$$

where the first inequality follows from the facts that $\mathbf{y}_n^* < 1$ and $\frac{\partial u^{DM}(\mathbf{y}_n^*,\theta)}{\partial y} > 0$ for all $\theta > \mathbf{y}_n^*$ and the second inequality from the fact that the term inside the bracket is strictly decreasing in the null action $\mathbf{y}$. Thus, the DM's first order condition in the original game implies that $\mathbf{y}_o^* = 0$. But this is impossible since $0 < \mathbf{y}_n^* \leq \mathbf{y}_o^*$. Therefore, we must have $\mathbf{y}_o^* < \mathbf{y}_n^* < 1$. ■

Proof of Proposition 7. In the original equilibrium, expert 2 has no information and the DM's null action is $\mathbf{y}_o^*$. Expert 1's payoff given the quadruple (m_1, x_1, x_2, θ) in the original problem is given by

$$
\begin{aligned}
\tilde{u}_1(x_1, x_1, x_2, \theta) &= \begin{cases} u_1(\theta, \theta) & \text{if } x_1 = \theta, \\ u_1(\mathbf{y}_o^*, \theta) & \text{if } x_1 = \varphi, \end{cases} \\
\tilde{u}_1(\varphi, x_1, x_2, \theta) &= u_1(\mathbf{y}_o^*, \theta),
\end{aligned} \tag{19}
$$

In the new equilibrium, expert 2 has information and the DM's null action is $\mathbf{y}_n^*$. Expert 1's payoff given the quadruple (m_1, x_1, x_2, θ) in the new problem is given by

$$
\tilde{w}_1(x_1, x_1, x_2, \theta) = \begin{cases} u_1(\theta, \theta) & \text{if } x_1 = \theta, \text{ or} \\ & x_1 = \varphi,\ x_2 = \theta \text{ and } \theta \leq \mathbf{y}_n^*, \\ u_1(\mathbf{y}_n^*, \theta) & \text{if } x_1 = \varphi,\ x_2 = \theta \text{ and } \theta > \mathbf{y}_n^*, \text{ or} \\ & x_1 = x_2 = \varphi. \end{cases}
$$

and

$$
\tilde{w}_1(\varphi, x_1, x_2, \theta) = \begin{cases} u_1(\theta, \theta) & \text{if } x_2 = \theta \text{ and } \theta \leq \mathbf{y}_n^*, \\ u_1(\mathbf{y}_n^*, \theta) & \text{if } x_2 = \theta \text{ and } \theta > \mathbf{y}_n^*, \text{ or } x_2 = \varphi, \end{cases}
$$

as expert 2 only sends message x_2 when $x_2 \leq \mathbf{y}_n^*$ in equilibrium. The difference problem is thus

$$
\tilde{v}_1(x_1, x_1, x_2, \theta) = \begin{cases} 0 & \text{if } x_1 = \theta, \\ u_1(\theta, \theta) - u_1(\mathbf{y}_o^*, \theta) & \text{if } x_1 = \varphi,\ x_2 = \theta \text{ and } \theta \leq \mathbf{y}_n^*, \\ u_1(\mathbf{y}_n^*, \theta) - u_1(\mathbf{y}_o^*, \theta) & \text{if } x_1 = \varphi,\ x_2 = \theta \text{ and } \theta > \mathbf{y}_n^*, \text{ or} \\ & x_1 = x_2 = \varphi. \end{cases}
$$

and

$$
\tilde{v}_1(\varphi, x_1, x_2, \theta) = \begin{cases} u_1(\theta, \theta) - u_1(\mathbf{y}_o^*, \theta) & \text{if } x_2 = \theta \text{ and } \theta \leq \mathbf{y}_n^*, \\ u_1(\mathbf{y}_n^*, \theta) - u_1(\mathbf{y}_o^*, \theta) & \text{if } x_2 = \theta \text{ and } \theta > \mathbf{y}_n^*, \text{ or } x_2 = \varphi. \end{cases}
$$

Given null information, expert 1 can only send $m_1 = \varphi$, therefore, expert 1's default action is identical in all three problems. Applying expert 1's optimal decision rule for the new problem to the difference problem and comparing the payoff to that obtained by using the default action $m_1 = \varphi$, we have

$$
\begin{aligned}
& E\left[\Delta\widehat{v}_1\left(x_1, x_2, \theta\right)\right] \\
= \;& E\left[\tilde{v}_1\left(x_1, x_1, x_2, \theta\right) 1_{\{x_1 \geq \mathbf{y}_n^*\}} + \tilde{v}_1\left(\varphi, x_1, x_2, \theta\right) 1_{\{x_1 < \mathbf{y}_n^*, x_1 = \varphi\}} - \tilde{v}_1\left(\varphi, x_1, x_2, \theta\right)\right] \\
= \;& -p_1 \int_{\mathbf{y}_n^*}^{1} \left\{u_1\left(\mathbf{y}_n^*, \theta\right) - u_1\left(\mathbf{y}_o^*, \theta\right)\right\} f\left(\theta\right) d\theta \\
< \;& 0.
\end{aligned}
$$

The result then follows from Proposition 2. ■

Proof for Footnote 14. It is not difficult to show that $\mathbf{y}_o^* = \mathbf{y}_n^* = \mathbf{y}^*$. The original problem is the same as in the proof of Proposition 7. i.e.,

$$
\begin{aligned}
\tilde{u}_1\left(x_1, x_1, x_2, \theta\right) &= \begin{cases} u_1\left(\theta, \theta\right) & \text{if } x_1 = \theta, \\ u_1\left(\mathbf{y}_o^*, \theta\right) & \text{if } x_1 = \varphi, \end{cases} \\
\tilde{u}_1\left(\varphi, x_1, x_2, \theta\right) &= u_1\left(\mathbf{y}_o^*, \theta\right),
\end{aligned}
$$

With a non-strategic expert 2, in the new problem, expert 1's payoff given the quadruple (m_1, x_1, x_2, θ) is given by

$$
\tilde{w}_1\left(x_1, x_1, x_2, \theta\right) = \begin{cases} u_1\left(\theta, \theta\right) & \text{if } x_1 = \theta \text{ or } x_2 = \theta, \\ u_1\left(\mathbf{y}_n^*, \theta\right) & \text{if } x_1 = x_2 = \varphi, \end{cases}
$$

and

$$
\tilde{w}_1\left(\varphi, x_1, x_2, \theta\right) = \begin{cases} u_1\left(\theta, \theta\right) & \text{if } x_2 = \theta, \\ u_1\left(\mathbf{y}_n^*, \theta\right) & \text{if } x_2 = \varphi. \end{cases}
$$

In the difference problem, expert 1's payoff given the quadruple (m_1, x_1, x_2, θ) is given by

$$\widetilde{v}_1(x_1, x_1, x_2, \theta) = \begin{cases} 0 & \text{if } x_1 = \theta, \\ u_1(\theta, \theta) - u_1(\mathbf{y}_o^*, \theta) & \text{if } x_1 = \varphi, \text{ and } x_2 = \theta, \\ u_1(\mathbf{y}_n^*, \theta) - u_1(\mathbf{y}_o^*, \theta) & \text{if } x_1 = x_2 = \varphi, \end{cases}$$

and

$$\widetilde{v}_1(\varphi, x_1, x_2, \theta) = \begin{cases} u_1(\theta, \theta) - u_1(\mathbf{y}_o^*, \theta) & \text{if } x_2 = \theta, \\ u_1(\mathbf{y}_n^*, \theta) - u_1(\mathbf{y}_o^*, \theta) & \text{if } x_2 = \varphi. \end{cases}$$

Without any information, expert 1 can only send $m_1 = \varphi$. Therefore, expert 1's default action is identical in all three problems. Applying expert 1's optimal decision rule for the new problem to the difference problem and comparing the payoff to that obtained by using the default action $m_1 = \varphi$, we have

$$\begin{aligned} & E\left[\Delta \widehat{v}_1(x_1, x_2, \theta)\right] \\ = \; & E\left[\tilde{v}_1(x_1, x_1, x_2, \theta) 1_{\{x_1 \geq \mathbf{y}_n^*\}} + \tilde{v}_1(\varphi, x_1, x_2, \theta) 1_{\{x_1 < \mathbf{y}_n^*, x_1 = \varphi\}} - \tilde{v}_1(\varphi, x_1, x_2, \theta)\right] \\ = \; & -p_1 \int_{\mathbf{y}_n^*}^1 \left[(1 - p_2) u_1(\mathbf{y}_n^*, \theta) + p_2 u_1(\theta, \theta) - u_1(\mathbf{y}_o^*, \theta)\right] f(\theta) \, d\theta \\ = \; & -p_1 p_2 \int_{\mathbf{y}^*}^1 \left(u_1(\theta, \theta) - u_1(\mathbf{y}^*, \theta)\right) f(\theta) \, d\theta \\ < \; & 0. \end{aligned}$$

Thus, the new problem is not relevant to the difference problem. Applying Proposition 2, we obtain the desired conclusion. ■