A Theory of Auctions With Endogenous Valuations

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Abstract

We study the revenue maximizing allocation of m units among n symmetric agents that have unit demand and convex preferences over the probability of receiving an object. Such preferences are naturally induced by a game where the agents take costly actions that affect their values before participating in the mechanism. Both the uniform $m + 1$ price auction and the discriminatory pay-your-bid auction with reserve prices constitute symmetric revenue maximizing mechanisms. Contrasting the case with linear preferences, the optimal reserve price reacts to both demand and supply, i.e., it depends both on the number of objects m and on number of agents n. The main tool in our analysis is an integral inequality involving majorization, super-modularity and convexity due to Fan and Lorentz (1954).

1 Introduction

We derive the revenue maximizing symmetric mechanism in a multi-unit auction framework where the agents' utility functions are convex in the physical allocation. Such non-linear preferences naturally arise as a reduced form representation of the agents' behavior when these agents (endowed with standard expected utility preferences) undertake, prior to the auction, costly actions that are unobservable to the designer, and that influence their valuations at the subsequent auction. All other features are standard and correspond to the

symmetric, private independent values model, e.g., we keep the usual quasi-linear assumption with respect to the monetary transfers.

Because of the ex-ante investments, the agents' valuations become endogenous to the mechanism. Hence, in contrast to the standard environment where the mechanism affects individual utilities directly via the physical allocation and the transfers, here a mechanism also provides incentives for individual investments, and thus indirectly affects *the distribution of* valuations. Therefore, a seller interested in maximizing expected revenue must take these incentives into account: he can provide stronger ex-ante incentives by increasing the probability of allocating an object, but is constrained by the limited supply and by the usual monopolistic supply reduction incentives. The revenue maximizing mechanism must finely balance these conflicting forces.

While expected utility preferences over allocations are always linear in the probability of allocation, temporal convex preferences induced by actions that are taken prior to the resolution of uncertainty have been studied by, among others, Kreps and Porteus [1979] and Machina [1984].[1] Single-Object auctions with general non-expected utility preferences have been studied by Karni and Safra [1989] and by Neilson [1984]. These authors studied the effect of risky prizes and agents that are not risk neutral over transfers. These features, that are not present in our own model (we maintain the assumption of quasi-linear preferences), are also behind some of their results, e.g., non-revenue equivalence among standard auction formats.[2]

Our main results characterize the revenue maximizing allocation within the class of symmetric mechanisms, and show how it can be implemented via standard auction formats. Our new optimization approach throws a new light also on the classical results obtained in the linear case, masterfully analyzed by Myerson [1981] and Riley and Samuelson [1981] - that setting is a special case of our present framework.

The main technical difficulty is due to the non-linearity of the ex-ante valuations in the allocation: the revenue maximization exercise cannot be performed "realization by realization", and the direct subject of maximization becomes the expected probability of getting an object - the "reduced form auction" in the language of Maskin and Riley [1984], Matthews

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[1]Most of the following literature has been confined to one-person decision problems. Independently of such representations, non-linear, a-temporal preferences in the allocation have been suggested as a realistic representation of some human choices by Kahneman and Tversky [1979] in their famous Prospect Theory and Quiggin [1982]. Consequences of multi-agent equilibrium behavior were first analyzed by Crawford [1990]. A recent contribution is Dillenberger and Raymond [2017].

[2]Karni and Safra focus on dynamic inconsistency in dynamic auction format. This phenomenon is due to the non-expected utility assumption.

[1984] and Border [1991]. Therefore, besides the standard monotonicity constraint stemming from the incentive compatibility requirement, the most complex constraint in our setting is thus the resource constraint arising from the characterization of reduced form auctions. Roughly put, it characterizes all expected probability assignments (conditional on an agent's type, i.e. marginals) that can be obtained from a probability assignment that determines who gets what for each realization of types.

In order to incorporate the above constraint, we employ a novel optimization technique that combines a recent characterization of reduced form, multi-unit auctions due to Che, Kim and Mierendorff [2013] with insights gained from majorization theory (pioneered by Hardy, Littlewood and Polya, 1929). Our main analytical tool is an elegant integral inequality due to Fan and Lorentz [1954]. Their result deals with the maximization of a family of convex functionals over the (convex) set of integrable functions that form an *orbit* under majorization. Fan and Lorentz identify convexity and super-modularity as sufficient (and almost necessary) conditions for their functionals to achieve their maximum on a monotone extreme point that coincides with the majorizing function of the respective orbit. In our setting, the functional associates to each reduced form auction the expected revenue it generates, and the Fan-Lorentz conditions, convexity and super-modularity, must hold on a kind of "virtual utility" function that accounts for the convex preferences.[3] The optimization exercise yields then an expected probability assignment where the units are allocated to the agents with the highest types, conditional on these exceeding a critical cutoff.

We then explicitly derive the bidding equilibria of the uniform $(m + 1)-$ price auction and of the discriminatory pay-your-bid auction with a reserve price (where m is the number of supplied units) and show that these auctions can implement the revenue maximizing allocation. Because of the non-linearity in probability and because the endogenous valuations that depend on the number of agents and objects, the equilibria of these auctions are non-trivial, e.g., the equilibrium of the uniform price auction is not in dominant strategies.

We next offer comparative statics results with respect to the critical cutoff and to the optimal reserve price. These results turn out to be very different from the classical ones obtained in the standard framework. While in the linear setting the optimal reserve price coincides with the optimal cutoff and is neither changed by the number of agents nor by the number of units (i.e., they are not responsive to either demand or supply), here we find that these objects are distinct and often display comparative statics in opposite directions:

1. Under the same general conditions used for the maximization exercise described above,

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[3]In particular, this function need not be separable in the type and the probability of obtaining a unit.

the optimal cutoff increases in the number of agents and decreases in the number of objects.

2. Under some additional sufficient conditions, the optimal reserve price decreases in the number of agents and increases in the number of objects.

We also illustrate our results in a variety of settings that form special cases of our framework, such as: auctions where the endogenous values are influenced either additively or multiplicatively by costly actions, auctions with entry costs, crowd-sourcing contests, and contests where money is burnt. In particular, we show that various sufficient conditions for monotone comparative statics of the optimal reserve price can be applied in these cases.

Finally, we briefly discuss the case where the Fan-Lorentz super-modularity condition is not satisfied: then the (convex) revenue functionals may attain their maximum on an extreme point that need not be monotone, and hence need not satisfy incentive compatibility. In those cases, we show how our focus on majorization yields a characterization of the "ironed" extreme points that do satisfy monotonicity. Similarly to the classical result of Myerson [1981], the optimal symmetric mechanism in the "non-regular" case is given by an auction where intervals of types are pooled in order to ensure monotonicity of the allocation.

The paper is organized as follows: In Section 2 we describe the auction model and offer a micro-foundation for the assumed non-linear preferences. In Section 3 we derive the revenue maximization problem with a focus on the complex resource constraint. In Section 4 we introduce several concepts and results from majorization theory. In Section 5 we derive the revenue maximizing allocation in the "regular" case. In Section 6 we derive the equilibria of standard auction formats in the present framework with endogenous valuations and show how these implement the optimal allocation. In addition we show how the optimal reserve price is affected by demand and supply. In Section 7 we display several applications to specific economic settings where an auction is preceded by costly investments. Section 8 briefly looks at the cases where ironing becomes necessary. Section 9 concludes.

2   The Model

2.1   The Reduced-Form Preferences

There are $m \geq 1$ identical and indivisible objects, and $n \geq m$ ex-ante symmetric bidders. Each bidder $i \in \{1, \dots, n\} = N$ has a type $\theta_i \in \Theta = [\underline{\theta}, \overline{\theta}] \subseteq \mathbb{R}_+$ that is her private information, and demands at most one object. Types are distributed $I.I.D.$ according to a distribution $F : \Theta \to [0, 1]$, with positive density $f > 0$.

Denote by $p_i = p_i(\theta_i)$ the interim probability with which agent i receives an object in a given mechanism, conditional on being of type θ_i. The utility of agent i as a function of her type θ_i, the probability p_i and the transfer y_i she makes is given by

$$h(p_i, \theta_i) - y_i.$$

We assume that the function $h : [0,1] \times \Theta \to \mathbb{R}_+$ is increasing in both variables, supermodular in (p_i, θ_i) and convex in p_i. These assumptions reflect that the agent's utility from receiving an object is non-negative and that higher types derive higher utility from the object. Furthermore, we assume that receiving an object with probability zero yields a utility of zero, independent of the type θ_i, i.e. for all $\theta_i \in \Theta$

$$h(0, \theta_i) = 0.$$

These assumptions generalize (and include as a special case) the usual assumptions made in the auction theory literature where $h(p_i, \theta_i) = p_i \cdot \theta_i$. As expected utility is linear in probabilities, the non-linear preferences we consider here include preferences which do not admit an expected utility representation.

2.2   Mechanisms

A *mechanism* specifies a set of reports R_i for each agent i, and a mapping from reports to an allocation and transfers:

$$x : \prod_{i \in N} R_i \to X = [0,1]^n$$
$$y : \prod_{i \in N} R_i \to \mathbb{R}^n.$$

Given a mechanism (x, y) each agent i picks an optimal report r_i. Agent i's optimization problem given her type θ_i is thus given by

$$\max_{r_i \in R_i} h(\mathbb{E}_{r_{-i}} [x_i (r_i, r_{-i}) \mid \theta_i], \theta_i) - \mathbb{E} [y_i(r_i, r_{-i}) \mid \theta_i].$$

We restrict attention to *symmetric mechanisms* that are invariant to any permutation of the agents' names and where every agent uses the same reporting strategy.

2.3 A Micro-Foundation: Reduction to Non-Linear Preferences

We offer here a "micro-foundation" for the above preferences by considering a model where, before participating in the mechanism, agents take costly actions that affect their values. Consider again a setting with n agents as above, but where each agent i takes an action $a_i \in A$, where $A \subset \mathbb{R}$ is a compact set. The taken action is also private information to the agent (and therefore unobservable to the principal).

Depending on her type θ_i, agent i has preferences over her own action $a_i \in A$, her own allocation $x_i \in [0,1]$ and her own transfer y_i. We assume that these preferences are standard expected utility preferences that are quasi-linear in the transfer and separable in the cost of an action $c(a_i) \geq 0$. The preference can be thus represented by a utility function of the form:

$$x_i \, v(a_i, \theta_i) - y_i - c(a_i) \,.$$

where the value for a unit of the good, $v(a_i, \theta_i) \in \mathbb{R}_+$, is assumed to be increasing in her type θ_i and in her action a_i, super-modular and non-negative.[4] Furthermore, we assume that there exists a costless action $a_0 \in A$ such that $c(a_0) = 0$.[5]

2.3.1 Timing

In order to analyze how hidden actions influence the agents' behavior, we need to specify the timing of actions. The timing we consider is as follows (c.f. Figure 1):

1. Each agent privately observes her type;

2. The designer (i.e., seller in the auction interpretation) commits to a mechanism;

3. Each agent decides whether to participate in the mechanism and privately chooses an action;

4. Each agent sends a message to the mechanism;

5. Depending on the sent messages, an allocation and transfers are realized.

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[4]The assumption that the value is non-negative corresponds to free disposal.

[5]This is a normalization: we can always subtract $\min_{a \in A} c(a)$ from the agent's utility without changing her preferences over actions, allocations, or transfers.
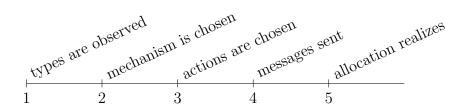
Figure 1: Timing of the game.

2.3.2 Reduction to Convex Preferences

Given a mechanism (x, y), an agent i can condition her action on the report she plans to send. Her reporting problem is equivalent to

$$\max_{r_i \in R_i} \left(\left\{ \max_{a_i \in A} \mathbb{E}\left[x_i(r) \mid \theta_i\right] \, v(a_i, \theta_i) - c(a_i) \right\} - \mathbb{E}\left[y_i(r) \mid \theta_i\right] \right). \tag{1}$$

We define $p_i(r_i, \theta_i) = \mathbb{E}\left[x_i(r) \mid \theta_i\right]$ to be the interim probability with which agent i receives an object in the mechanism and $h : [0, 1] \times \Theta \to \mathbb{R}$ to be the utility agent i receives when she takes the optimal action

$$h(p_i, \theta_i) = \max_{a_i \in A} p_i \, v(a_i, \theta_i) - c(a_i).^6 \tag{2}$$

Using these definitions, the reporting problem of agent i given in (1) becomes equivalent to the reduced form problem where agent i has non-linear preferences over her allocation probability p_i

$$\max_{r_i \in R_i} h(p_i(r_i, \theta_i), \theta_i) - \mathbb{E}\left[y_i(r) \mid \theta_i\right].$$

To show that this problem is indeed equivalent to the reduced problem introduced in Section 2.1 we need to verify that h as defined in (2) satisfies the assumptions of Section 2.1:

Lemma 1. $h(p_i, \theta_i)$ *is convex in* p_i, *increasing in* θ_i *and* p_i, *super-modular in* (p_i, θ_i) *and satisfies* $h(0, \theta_i) = 0$ *for all* $\theta_i \in \Theta$.

Proof. h is convex in p as it is the maximum over functions that are linear in p. It is increasing in p and θ as $p\,v(a, \theta) - c(a)$ is increasing in p and θ. As by assumption there exists an action a with $c(a) = 0$ and the cost is non-negative we have that $h(0, \theta) = \max_{a \in A_i} -c(a) = 0$. Finally,

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[6]We assume that such maximum always exists, which is satisfies for example if $\frac{\partial^2 v}{\partial a^2} \leq 0$ for all θ_i and $\frac{\partial^2 c}{\partial a^2} > 0$.

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let $a^*(p_i, \theta_i)$ be an optimal selection for the problem given in (2). As $p_i\, v(a_i, \theta_i) - c(a_i)$ is super-modular in (a_i, θ_i) we can always pick an optimal selection $a^*(p_i, \theta_i)$ which is non-decreasing in θ_i (Theorem 4 in Milgrom and Shannon [1994]). By the Envelope Theorem we have that h is absolutely continuous in p with (weak) derivative

$$\frac{\partial}{\partial p_i} h(p_i, \theta_i) = v(a^*(p_i, \theta_i), \theta_i). \tag{3}$$

As $a^*(p_i, \theta_i)$ is non-decreasing in θ_i and v is non-decreasing in a_i and θ_i the marginal utility associated with an increase in the probability of receiving the object $\frac{\partial}{\partial p_i} h$ is non-decreasing in θ_i and thus h is super-modular in (p_i, θ_i) □

We have thus obtained the model we started with: since h is defined independently of the agent's action, the mechanism design problem where the agent chooses an action before participating in the mechanism is equivalent to the mechanism design problem where each agent has a convex valuation given by $h_i(p_i, \theta_i)$, but chooses no costly action.

3   The Revenue Maximization Problem

We restrict attention to direct mechanisms where each agent reports his type θ_i, and where the mechanism specifies allocations and monetary transfers to all agents depending on the reported types. As $h(p_i, \theta_i)$ is super-modular, it is well-known that the incentive compatibility of a (symmetric) direct mechanism is equivalent to the standard monotonicity of the equilibrium expected allocation[7]

$$p(\theta_i) = p_i(\theta_i) = p_i(\theta_i, \theta_i) := \mathbb{E}[x_i(\theta_1, ..., \theta_n) \mid \theta_i]$$

together with an envelope condition determining the expected interim transfer (see for example Guesnerie and Laffont, [1984], Corollary 2.1). The envelope condition yields the following standard revenue equivalence result:

Proposition 1 (Revenue Equivalence). *The expected revenue in any symmetric, incentive compatible mechanism where the participation constraint is binding for the lowest type (i.e. the lowest type obtains zero utility) is given by*

$$n \int_\Theta H(p(\theta), \theta)\, f(\theta)\, d\theta, \tag{4}$$

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[7]As the mechanism is symmetric the definition of p is independent of the agent's identity i .

where the "virtual utility" $H : [0,1] \times [\underline{\theta}, \overline{\theta}] \to \mathbb{R}$ is defined by

$$H(p, \theta) := h(p, \theta) - \frac{\partial h(p, \theta)}{\partial \theta} \times \frac{1 - F(\theta)}{f(\theta)} . \tag{5}$$

In order to simplify the notation, it will often be convenient to reformulate the problem in terms of the quantile $t = F(\theta)$ associated with a type θ.[8] For quantile t we define the probability $q(t) = p(F^{-1}(t))$ and the associated virtual utility

$$G(q, t) := H(q, F^{-1}(t)) . \tag{6}$$

As $\frac{dt}{d\theta} = \frac{1}{f \circ F^{-1}(t)}$, integration by substitution of (4) yields the following equivalent characterization of expected revenue.

Corollary 1. *The expected revenue in any symmetric, incentive compatible mechanism where the participation constraint is binding for the lowest type is given by*

$$n \int_0^1 G(q(t), t) dt . \tag{7}$$

Note, that the expression given in (7) equals (4) if types are uniformly distributed on $[0, 1]$. Intuitively, Corollary 1 states that it is without loss of generality to restrict attention to uniformly distributed types $t \sim U([0, 1])$ as long as one allows for a general virtual utility G. This transformation greatly simplifies the analysis since it implies that, in order to characterize the optimal mechanism for arbitrarily distributed types, it is sufficient to determine the set of feasible interim allocations for uniformly distributed types.

3.1   The Resource Constraint

The non-linearity of the individual expected revenue G in the allocation probability q does not allow the use of a standard point-by-point maximization approach. Therefore, the crucial constraint becomes the feasibility (or resource) constraint: what expected interim allocations functions can be obtained (i.e. are marginals) of symmetric mechanisms? This question has been initially addressed (in the framework of a single object auction) by Matthews [1984], Maskin and Riley [1984], and Border [1991].[9] The following, more recent result characterizes

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[8]Since F has a strictly positive density, F^{-1} is well defined.

[9]These papers were written without connections to the earlier mathematical literature on the existence of measures with given marginals, e.g. Lorenz [1949], Gale [1957], Ryser [1957] and Strassen [1965].

the set of feasible interim allocation rules for multi-unit auctions with one-dimensional types:

Proposition 2 (Che, Kim & Mierendorff [2013], Corollary 4). *Consider an auction with n bidders where m units are available. A symmetric, monotone interim allocation rule $q : [0, 1] \to [0, 1]$ is a reduced form of a feasible allocation rule where no agent obtains more than k objects if and only if, for each $t \in [0, 1]$ it holds that*

$$n \int_t^1 q(s)ds \le \sum_{i=0}^n \min\{i \cdot k, m\} \binom{n}{i} (1 - t)^i \, t^{n-i} . \tag{8}$$

In order to use the above result, we need an additional Lemma reformulating the Che, Kim & Mierendorff characterization into an integral form that will be later related to the concept of (weak) majorization:[10]

Lemma 2. *It holds that*

$$\sum_{i=0}^n \min\{i, m\} \binom{n}{i} t^{n-i} (1 - t)^i = n \int_t^1 \phi_{m,n}(t)dt$$

where $\phi_{m,n}(t)$ is the probability that at most $m - 1$ out of $n - 1$ agents have a type larger than the type associated with the quantile t

$$\phi_{m,n}(t) := \sum_{i=0}^{m-1} \binom{n-1}{i} t^{n-1-i} (1 - t)^i . \tag{9}$$

The function $\phi_{m,n}$ is increasing in t and in m and decreasing in n. Furthermore, $\int_0^1 \phi_{m,n}(t)dt = \frac{m}{n}$.

As a consequence of the characterization of implementable interim allocation rules given in Proposition 2, Lemma 2 and Corollary 1, our revenue maximization problem is equivalent to the following problem:

Proposition 3 (Characterization of Revenue Maximizing Mechanisms). *A symmetric mechanism is revenue maximizing if and only if the induced interim probability of receiving an*

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[10]For a characterization of a single object reduced form auction in terms of second-order stochastic dominance see Hart and Reny [2015].

object q(t) solves

$$\max_q \ n \int_0^1 G(q(t), t)\mathrm{d}t \tag{10}$$

subject to:

$$q(t) \in [0, 1] \ \textit{for all } t \in [0, 1] \tag{11}$$

$$q \ \textit{non-decreasing} \tag{12}$$

$$\int_t^1 q(z)\mathrm{d}z \le \int_t^1 \phi_{m,n}(z)\mathrm{d}z \ \ \textit{for all } t \in [0, 1] \,, \tag{13}$$

where $t = F^{-1}(\theta)$, G is defined in (6) and $\phi_{m,n}$ is defined in (9).

4 Majorization and the Fan-Lorentz Inequality

In order to deal with the resource constraint identified above in (13), we first recall several concepts and results from the theory of majorization, pioneered by Hardy, Littlewood and Polya [1929]. For non-decreasing $q, \bar{q} \in L^1(0, 1)$ we say that $\bar{q}$ *majorizes* q, denoted by $q \prec \bar{q}$ if the following two conditions hold:

$$\int_t^1 q(v)dv \ \le \ \int_t^1 \bar{q}(v)dv \ \ \text{for all } t \tag{14}$$

$$\int_0^1 q(t)dt \ = \ \int_0^1 \bar{q}(t)dt.$$

We say that $\bar{q}$ *weakly majorizes* q, denoted by $q \prec_w \bar{q}$ if the first condition above holds (but not necessarily the second). If $q \prec_w \bar{q}$ it is easily seen that there exists $q' \le \bar{q}$ such that $q \prec q'$.

The above definitions can also be applied to any (possibly non-monotonic) functions $q, \bar{q} \in L^1(0, 1)$ if $q, \bar{q}$ in (14) are replaced by their non-decreasing re-arrangements. For such arbitrary functions, it is well known that $q \prec \bar{q}$ if and only if $q = T \bar{q}$ where T is a doubly stochastic operator. The set

$$\Omega(\bar{q}) := \{q : q \prec \bar{q}\}$$

called the *orbit* of $\bar{q}$, is weakly compact and convex. In particular, by Bauer's Maximum Principle (1958) a continuous, convex functional on $\Omega(\bar{q})$ attains its maximum on an extreme point of $\Omega(\bar{q})$. Ryff (1967) has shown that $q \in \Omega(\bar{q})$ is an extreme point of this set if and

only if $q = \bar{q} \circ \Psi$ where Ψ is a measure preserving transformation of $[0, 1]$ into itself.[11]

The next elegant result, due to Ky Fan and G.G. Lorentz [1954] identifies a very interesting set of convex functionals such that **all** of them attain their maximum on $\Omega(\bar{q})$ precisely at $q = \bar{q} \circ Id$. Note, that the Identity is of course a monotonic transformation (in fact, this is the only measure preserving transformation with this property).

Proposition 4 (Fan-Lorentz Theorem). *Let $L : [0, 1] \times [0, 1] \to \mathbb{R}$ be a function such that $L(q, t)$ is convex in q and super-modular in (q, t). Let $q, \bar{q} : [0, 1] \to [0, 1]$ be two non-decreasing functions such that $q \prec \bar{q}$. Then*

$$\int_0^1 L(q(t), t)dt \le \int_0^1 L(\bar{q}(t), t)dt \, .$$

5   The Revenue Maximizing Mechanism

Due to Revenue Equivalence (Proposition 1) we can characterize the optimal direct mechanism in terms of the implied allocation. Proposition 3 shows that an allocation is feasible if and only if it is weakly majorized by $\phi_{m,n}$. To maximize over feasible allocations we will use the Fan-Lorentz Theorem, and for this we need to introduce a regularity assumption ensuring that their conditions are satisfied by our expected revenue functional:

Definition 1 (Convex-Supermodularity). *We say that the environment is "convex super-modular" (CSM) if the virtual utility $H(p, \theta)$ defined in (5) is convex in p and super-modular in (p, θ).*

It is instructive to consider the meaning of the above conditions in the standard model with linear preferences (Myerson, [1981]) where

$$H(p, \theta) = p \left(\theta - \frac{1 - F(\theta)}{f(\theta)} \right) \, .$$

As H is linear in p, it is (weakly) convex in p. It is super-modular in (p, θ) if and only if the standard virtual value

$$J(\theta) = \theta - \frac{1 - F(\theta)}{f(\theta)}$$

is non-decreasing. Thus, for the special case of linear preferences, our definition of a CSM environment reduces to the definition of a "regular" environment given by Myerson [1981].

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[11]This is analogous to the discrete case, where the above result is a corollary of the Birkoff-von Neumann Theorem about the extremity of permutation matrices within the set of double stochastic matrices.

In the Appendix we give sufficient conditions on the valuation function $h(p, \theta)$ and on the distribution function F for the generalized virtual utility function $H(p, \theta)$ to satisfy CSM. The treatment is similar to the one in Guesnerie and Laffont [1984] and Fudenberg and Tirole [1991] (Chapter 7) who studied the one-person contracting setting with a valuation function that is **concave** in the allocation.[12]

To simplify notation we denote by

$$\psi_{m,n}(\theta) := \phi_{m,n}(F(\theta))$$

the interim probability with which an agent of type θ receives an object when m objects are allocated efficiently among n agents. We now derive the optimal symmetric mechanism using the Fan-Lorentz Theorem:

Theorem 1 (Revenue Maximizing Allocation). *Suppose the environment is convex super-modular. Then, the revenue maximizing mechanism allocates the m objects to the agents with the highest types, conditional on these exceeding a threshold θ^*, where θ^* is the unique solution to* [13]

$$H(\psi_{m,n}(\theta^*), \theta^*) = 0.$$

Moreover, in the optimal mechanism the expected utility of the lowest type, $\underline{\theta}$ is zero.

The intuition for the proof of Theorem 1 (see Appendix) is as follows. First, we argue that in a CSM environment the virtual value $G(q, t)$ expressed as a function of the quantile t is convex and super-modular in (q, t). As G is convex in q, the functional $q \mapsto \int_0^1 G(q(t), t)dt$ is convex. By Bauer's maximum principle this functional must attain a maximum on an extreme point of the set of monotone functions that satisfy the resource constraint given by inequality (13). This constraint is equivalent to q being weakly majorized by the allocation probability $\phi_{m,n}$ of being among the m highest types. We then show that this implies that q is majorized by a function of the form $t \mapsto 1_{\{t \geq t^*\}} \times \phi_{m,n}(t)$. Hence, by the Fan-Lorentz inequality, an interim probability of the form $t \mapsto 1_{\{t \geq t^*\}} \times \phi_{m,n}(t)$ maximizes revenue. The optimality of the cutoff type θ^* is proven by arguing that the expected revenue is quasi-concave in the cutoff θ^* and that $H(\psi_{m,n}(\theta^*), \theta^*) = 0$ is the relevant first-order condition.

To better understand the intuition behind the optimality of the above described allocation, let us compare the micro-foundation scenario with the standard case analyzed by

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[12]In the one person case they studied the resource constraint is not complex.

[13]If $H(\psi_{m,n}(\theta), \theta) > 0$ for any $\theta \in [\underline{\theta}, \bar{\theta}]$ we set $\theta^* = \underline{\theta}$, while if $H(\psi_{m,n}(\theta), \theta) < 0$ for any $\theta \in [\underline{\theta}, \bar{\theta}]$ set $\theta^* = \bar{\theta}$.

Myerson:

1. In the standard case, the seller is able to extract only the virtual value from every buyer. Under the regularity assumption, the virtual value is monotone in type, and hence the optimal mechanism allocates the units to the agents with the highest types, conditional on their virtual value being non-negative.

2. In the present environment, the probability of getting a unit affects the investment incentives, and hence indirectly influences the virtual value of every agent. As we shall see in the applications below, the increase in virtual value (due to an increase in the probability of obtaining an unit the object) is more substantial for an agent with a higher type. This increase the advantage of allocating the units to agents with higher types, conditional on their virtual values (which depend on the allocation probability) being non-negative.

5.1 Comparative Statics

In this subsection we provide some comparative statics of the optimal cutoff type with respect to the number of agents n and the number of objects m. In particular, our result shows that, in sharp contrast to the standard (linear) auction setting, the optimal cutoff type is influenced both by the number of agents and objects. Recall that

$$\phi_{m,n}(t) = \sum_{i=0}^{m-1} \binom{n-1}{i} t^{n-1-i}(1-t)^i$$

is the probability that an agent of type $\theta = F^{-1}(t)$ is among the m highest types of the n agents. Recall also that, from the definition of $\psi_{m,n}(\theta) = \phi_{m,n}(F(\theta))$ we know that $\psi_{m,n}(\theta)$ is increasing in m and θ and decreasing in n - these three basic properties account for our main result here showing that cutoff type θ^* increases in the number of agents n and decreases in the number of objects m.

Proposition 5 (Comparative Statics for the Cutoff Type). *Assume that the environment is convex super-modular (CSM). Then the optimal cutoff type θ^* increases in the number of agents n and decreases in the number of objects m.*

To get an intuition for this result, consider the case where the non-linearity of h in p stems from an investment decision made by the agents before participating in the auction,

14

as described in the micro-foundation part (Section 2.3). An increase in the number of agents decreases the individual chance of each agent to receive an object. This, in turn, reduces the individual incentives to invest, the resulting values, and the revenue that can be obtained from each type. In particular, the type with zero virtual utility - that determines the optimal cutoff type θ^* - must be larger than the analogous type when there are less bidders.[14] In the micro foundation agents with types below θ^* choose the action with zero cost.

While the expected revenue increases with the number of objects, as more objects relax the feasibility constraint, the monotonicity with respect to the number of agents is less obvious. In Example 1 below we show that indeed this monotonicity is not guaranteed in the present environment.

Weakly Convex Super-modular Environments

While the regularity of the standard virtual valuation J in the linear case is sufficient for an auction with reserve price to be revenue maximizing, it is not necessary. To see this, consider the case where J decreases for some types to which J assigns a negative virtual value. As agents with those types are excluded through the reserve price anyhow, their virtual value plays no role in determining the optimal allocation. In a similar way, we can relax the CSM requirement in our setup - this will turn out to be useful for some of the applications treated below.

Definition 2 (Weak Convex Supermodularity). *We say that the environment is "weakly convex super-modular" (wCSM) if the positive part of the virtual value $\max\{H(p,\theta),0\}$ is convex in p and super-modular in (p,θ).*

In the Appendix we argue that it is possible in our environment to exclude types with negative virtual utility, leading to the following generalization of Theorem 1.

Corollary 2. *Suppose the environment is **weakly convex super-modular**. Then, the revenue maximizing allocation allocates m objects to the agents with the highest types, conditional on these exceeding θ^* where θ^* is given by*

$$H(\psi_{m,n}(\theta^*), \theta^*) = 0.$$

Moreover, in the optimal mechanism the expected utility of the lowest type, $\underline{\theta}$ is zero.

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[14]An inverse effect is at work when the number of objects increases.

6 Implementation via Auctions

In this section we analyze two widely used, standard auction formats that can be used in our setting to implement the revenue maximizing allocation.

6.1 The Uniform Price Auction

We first look at the $(m+1)-$ uniform price auction with a reserve price. The equilibrium we describe below is **not in dominant strategies!** Moreover, in stark contrast to the linear case where the bid is constant and equal to the value for a unit, here the equilibrium bid responds both to demand (the number of bidders) and supply (the number of objects).

Proposition 6 (Equilibrium of the uniform price auction)**.** *Assume that the seller uses a uniform $(m+1)-$price auction with reserve price $\mathcal{R}$, and define $\theta'_{m,n}$ to be the type that solves:*

$$\frac{h\left(\psi_{m,n}(\theta'_{m,n}), \theta'_{m,n}\right)}{\psi_{m,n}(\theta')} = \mathcal{R} \tag{15}$$

1. The profile of bidding strategies

$$b_i(\theta) = b_{m,n}(\theta) = \begin{cases} \frac{\partial h(\psi_{m,n}(\theta),\theta)}{\partial p} & \theta \geq \theta'_{m,n} \\ 0 & \theta < \theta'_{m,n} \end{cases}$$

constitutes a symmetric, pure strategy Nash equilibrium in the uniform price auction among n bidders for m units.

2. The equilibrium bid $b_{m,n}(\theta)$ increases in θ and in the number of objects m and decreases in the number of bidders n.

Remark. Note that the equilibrium above displays a "jump" in the bidding function at the reserve price: the type $\theta'_{m,n}$ given in (15) submits a bid which exceeds the reserve price $\mathcal{R}$, or differently put, there exists an interval of types $[\theta_1, \theta_2]$ such that $\frac{\partial h(\psi_{m,n}(\theta),\theta)}{\partial p} > \mathcal{R}$ for $\theta \in [\theta_1, \theta_2]$, yet these types refrain from submitting bids above the reserve. This feature is mainly due to the convexity of h which implies that

$$\frac{\partial h\left(\psi_{m,n}(\theta), \theta\right)}{\partial p} \geq \frac{h\left(\psi_{m,n}(\theta), \theta\right)}{\psi_{m,n}(\theta)}$$

For an intuition, recall that, in the "micro-foundation" scenario, the bid $\frac{\partial h(\psi_{m,n}(\theta),\theta)}{\partial p}$ represents the post-investment value of the object to a bider with type θ (the investment is

already sunk at this stage!). However, for a type with post-investment value just above the reserve price, it is not profitable to invest and participate at the auction, since, in the best scenario where his type is among the m highest, he will get zero utility from winning and hence he looses the initial investment and thus has a negative utility.

Corollary 3 (Optimality of the Uniform Price Auction). *The uniform price auction with reserve price $\mathcal{R}^*$ is a symmetric revenue maximizing mechanism where $\mathcal{R}^*$ is given by*

$$\mathcal{R}^* = \frac{h\left(\psi_{m,n}(\theta^*), \theta^*\right)}{\psi_{m,n}(\theta^*)}$$

and θ^* solves

$$H(\psi_{m,n}(\theta^*), \theta^*) = 0.$$

Equilibrium and the Micro-foundation   In the general, abstract model preferences are non-standard. But, in our micro-foundation, agents choose an optimal action a^* given by

$$a^*\left(\theta\right) = \begin{cases} \arg\max_{a_i \in A} \psi_{m,n}(\theta)\, v(a_i, \theta) - c(a_i) & \text{if } \theta \geq \theta^* \\ 0 & \text{else} \end{cases},$$

and then submit a bid equal to the post-investment value $v(a^*\left(\theta\right), \theta) = \frac{\partial h(\psi_{m,n}(\theta), \theta)}{\partial p}$ whenever this value exceeds the reserve price. This profile of strategies (actions+bids) constitutes a standard Bayes-Nash equilibrium of this game.[15]

6.2   The Pay-Your-Bid Auction

In this section we show that the discriminatory Pay-Your-Bid Auction with a properly chosen reserve price also implements the revenue maximizing allocation. We first explicitly derive the equilibrium bidding strategies in our environment. Recall that in the discriminatory Pay-Your-Bid Auction with reserve price m agents that submit the highest bids conditional on these bids being above the reserve price get the objects and pay their bids. Other agents pay nothing.

Proposition 7 (Equilibrium of the Discriminatory Auction). *Assume that the seller uses a*

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[15]This profile is not in dominant strategies: conditional on the other bids, an agent knows whether he wins or not and may prefer to change his investment decision.

Pay-Your-Bid Auction with reserve price $\mathcal{R}$, *and define* $\theta'_{m,n}$ *to be the type that solves:*

$$\frac{h\left(\psi_{m,n}(\theta'),\theta'\right)}{\psi_{m,n}(\theta')} = \mathcal{R}.$$

The profile of bidding strategies

$$\beta(\theta) = \begin{cases} \mathcal{R}\frac{\psi_{m,n}(\theta')}{\psi_{m,n}(\theta)} + \frac{1}{\psi_{m,n}(\theta')}\int_{\theta'}^{\theta_i}\psi'_{m,n}(z)\frac{\partial h(\psi_{m,n}(z),z)}{\partial p}dz & \theta \geq \theta' \\ 0 & \theta < \theta' \end{cases}$$

constitutes a symmetric pure strategy Nash equilibrium in the Pay-Your-Bid auction among n *bidders for* m *units.*

Corollary 4 (Optimality of the Pay-Your-Bid Auction)**.** *The Pay-Your-Bid Auction with reserve price* $\mathcal{R}^*$ *is a symmetric revenue maximizing mechanism where the optimal reserve price is given by*

$$\mathcal{R}^* = \frac{h\left(\psi_{m,n}(\theta^*),\theta^*\right)}{\psi_{m,n}(\theta^*)},$$

and θ^* *solves*

$$H(\psi_{m,n}(\theta^*),\theta^*) = 0.$$

6.3 Comparative Statics with respect to the Optimal Reserve Price

Note that both auction formats (Uniform price and discriminatory Pay-Your-Bid auctions) apply the same optimal reserve price. Proposition 5 gives a clear-cut result regarding the effect of the change in the number of objects and agents on the optimal cutoff type. It is important to note that the above result and intuition **do not** carry over to the optimal reserve price ! There are two opposite effects at work here: on the one hand, from Proposition 5 it follows that an increase in the number of agents increases the cutoff type θ^*; but, on the other hand, it also decreases the probability of getting the object of the cutoff quantile type, $\psi_{m,n}(\theta^*)$. Since $\frac{h(p,\theta)}{p}$ is increasing in θ and increasing in p (due to convexity) the overall effect of increase in the number of agent on the optimal reserve price is ambiguous. The next proposition gives sufficient conditions for the second effect to dominate. Analogous effects are also at work for the dependence on the number of objects m. In the Applications section below we offer several illustrations.

Proposition 8 (Comparative Statics of the Optimal Reserve Price)**.** *Assume that the environment is convex super-modular (CSM). Assume further that h is concave in* θ *and that F*

is convex with a differentiable density function. Then the optimal reserve price $\mathcal{R}^*$ decreases in the number of agents n and increases in the number of objects m.

The above result yields some empirically testable predictions about the changes in the reserve price as a result of changes in demand (the number of bidders) or in supply (the number of units). This should be compared to the standard "knife-edge" result obtained in the linear case, whereby the reserve price is non-responsive to such changes: since in that case optimal cutoff type and optimal reserve price coincide, this non-response is indeed suggested by our results that display comparative statics in opposite directions for the general model where these two objects do not coincide.

7 Illustrations

In this Section we offer several applications of the above main results to particular micro-founded settings where investments affect values. These applications both extend and unify previous observations that were obtained in the literature by "ad-hoc" methods.

7.1 Additively Separable Investments

Assume in our micro-foundation that the costly action $a \in A \subseteq \mathbb{R}_+$ additively increases the agent's value θ for a unit of the good[16]

$$v(a, \theta) = a + \theta.$$

Note that we do not impose any restriction on the set of actions A, and allow it be either finite or infinite. The linear separability implies that the non-linear utility h induced by the environment where the agent takes an action is of the form

$$h(p, \theta) = \max_{a \in A} p\,(a + \theta) - c(a).$$

Take an arbitrary optimal selection $a^*(p) \in \arg\max_{a \in A} p\,a - c(a)$. Then, h is given by

$$h(p, \theta) = p\,\theta + p\,a^*(p) - c(a^*(p)) = p\,\theta + g(p),$$

---

[16]The two-agent, one object, additive case with a quadratic cost is treated in Zhang (2017). She also studies asymmetric mechanisms in that framework (that are not covered here).

where $g(p) := p\, a^*(p) - c(a^*(p))$. As h is convex and increasing in p, the function g is convex and increasing. Furthermore, $g(0) = 0$ due to the existence of the costless action $a = 0$. By plugging in (5) we obtain that the virtual utility H is given by

$$H(p,\theta) = p\, J(\theta) + g(p)$$

where $J(\theta) = \theta - \frac{1-F(\theta)}{f(\theta)}$ is the standard virtual value (a la Myerson). As g is convex, it follows immediately that H is convex in p. Furthermore, if J is increasing then H is super-modular, and thus the environment is CSM (Definition 1). Hence, Theorem 1 and Proposition 5 imply the following characterization of the optimal mechanism:

Corollary 5. *Assume that the standard virtual value* $J(\theta) = \theta - \frac{1-F(\theta)}{f(\theta)}$ *is increasing.*

1. *Then the environment with additively separable investments is convex super-modular, and both the Uniform-Price and the Pay-Your-Bid auction with reserve price*

$$\mathcal{R}^* = \frac{1 - F(\theta^*)}{f(\theta^*)}$$

   where θ^* solves

$$\theta^* + \frac{g(\psi_{m,n}(\theta^*))}{\psi_{m,n}(\theta^*)} = \frac{1 - F(\theta^*)}{f(\theta^*)}$$

   are symmetric, revenue maximizing mechanisms.

2. *If the hazard rate $\frac{f(\theta)}{1-F(\theta)}$ is increasing, then the optimal reserve price decreases in the number of objects and increases in the number of agents.*

The second part of the corollary immediately follows from the above characterization of the reserve price together with Proposition 5.

Next we show that the optimal revenue in the additively separable environment may not be monotone in the number of agents. Intuitively, an increase in the number of agents decreases the incentive to invest in the micro-foundation, and hence decreases the individual revenues from any type. On the other hands, the number of agents increases, which increases the total expected revenue. In the next example we show that the first effect may dominate, and the expected revenue may decrease in the number of agents.

Example 1. *Consider additively separable environments with quadratic costs and $c(a) = \frac{a^2}{2\beta}$ for some $\beta > 0$. This implies that $g(p) = \beta p^2$ and $H(p,\theta) = J(\theta)p + \beta p^2$. Plotting the*

revenue in the optimal mechanism when values are uniformly distributed $\theta \sim U([0,1])$ shows
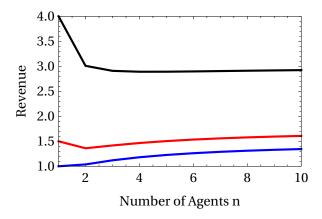*that the revenue might be decreasing in the number of agents (c.f. Figure 2).*

Figure 2: Revenue for the additively separable environments with quadratic costs $c(a) = \frac{a^2}{2\beta}$ and $\theta \sim U([0,1])$ in the optimal mechanism for $\beta = 2$ (blue), $\beta = 3$ (red), $\beta = 8$ (black).

A similar calculation to the one of Example 1 shows that also in the auction without reserve price the revenue need not to be monotone in the number of bidders. This implies that the celebrated result by Bulow and Klemperer [1996] need not to hold in our environment. That result states that, in the case without investments, the seller is always better off if he attracts an additional bidder to the auction and sets a zero reserve price, compared to the situation where he sets the optimal reserve price, but one less bidder participates. In our framework with investments the seller might be worse of if an additional bidder participates, and the result does not hold.

7.2   Multiplicative Separable Investments

In the micro-foundation we now assume that the action $a \in A = \mathbb{R}_+$ increases the agent's value multiplicatively[17]

$$v(a, \theta) = a\,\theta$$

and that the cost function is of the form

$$c(a) = b\frac{a^l}{l} \text{ where } l \geq 2 \text{ and } b > 0.$$

---

[17]While formally the set of actions is not compact here this is inconsequential: this assumption is only used to ensure that h is well defined, which we verify explicitly in this application.

This implies that the induced non-linear valuation h is of the form

$$h(p, \theta) = \max_{a \in A} p\, a\, \theta - c(a)\,.$$

The first order condition yields that the optimal investment as a function of the agent's type θ and her probability of receiving the object is given by

$$a_i^*(p, \theta) = \left(\frac{\theta\, p}{b}\right)^{\frac{1}{l-1}}\,.$$

Thus, the agent's utility h is given by

$$h(p, \theta) = p\,\theta\, a^*(p, \theta) - c(a^*(p, \theta)) = \frac{l-1}{l}\left(\frac{p^l \theta^l}{b}\right)^{\frac{1}{l-1}}\,.$$

It follows from plugging in (5) that the virtual utility H is given here by

$$
\begin{aligned}
H(p, \theta) &= h(p, \theta) - \frac{\partial h(p, \theta)}{\partial p} \times \frac{1 - F(\theta)}{f(\theta)} \\
&= p^{\frac{l}{l-1}}\left(\frac{\theta}{b}\right)^{\frac{1}{l-1}}\left(\frac{l-1}{l}\theta - \frac{1 - F(\theta)}{f(\theta)}\right)\,.
\end{aligned}
$$

Assume now that the function

$$K(\theta) := \frac{l-1}{l}\theta - \frac{1 - F(\theta)}{f(\theta)}$$

is increasing (a sufficient condition is the standard monotone hazard rate condition).[18] Note that

$$H(p, \theta) = p^{\frac{l}{l-1}}\left(\frac{\theta}{b}\right)^{\frac{1}{l-1}} K(\theta)$$

and thus assuming $\Theta = [0, \bar{\theta}]$ we obtain that $K(0) = -\frac{1}{f(0)} < 0$ and that $H(p, \theta) \geq 0$ implies that $K(\theta) \geq 0$. Hence, $\max\{H(p, \theta), 0\}$ is convex in p and super-modular in (p, θ), i.e. the environment is weakly convex super-modular (see Definition 2). Corollary 2 thus implies the following characterization of the optimal mechanism:

Corollary 6. *Assume that the function $K(\theta) = \frac{l-1}{l}\theta - \frac{1-F(\theta)}{f(\theta)}$ is increasing. Then, the environment with multiplicative separable investments and monomial cost is weakly convex*

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[18]For example, assuming an uniform distribution of types, we get $K(\theta) = \frac{2l-1}{l}\theta - 1$.

super-modular.

1. *Both the Uniform Price and the Pay-Your-Bid auctions with reserve price*

$$\mathcal{R}^* = \frac{l-1}{l} \left(\frac{\psi_{m,n}\left(\theta^*\right)\left(\theta^*\right)^l}{b} \right)^{\frac{1}{l-1}}$$

where θ^* solves $K(\theta^*) = 0$ are symmetric revenue maximizing mechanisms.

2. *The optimal reserve price is decreasing in the number of bidders n and increasing in the number of objects m.*

Remark: The second part of the above corollary easily follows because the critical cutoff θ^* is here independent of both n and m. The same observation and comparative statics results apply to any valuation function that is multiplicatively separable $h(p,\theta) = a(p)b(\theta)$ and that yields a (weakly) convex super-modular environment.

We next show that contrary to the additively separable investment case, in the multiplicative case with a single object, under relatively general assumptions, the expected revenue is monotone increasing in the number of agents.

Lemma 3. *Assume that $m = 1$. Assume further that $K(\theta) = \frac{l-1}{l}\theta - \frac{1-F(\theta)}{f(\theta)}$ is increasing and that F is concave. Then, the expected revenue increases in the number of agents n.*

7.3   Auctions with a Fixed Entry Cost[19]

This is also a multiplicative case, but there are only two actions: Enter $(a = 1)$ and Stay Out $(a = 0)$. Values are given by $v(a_i, \theta_i) = a_i\theta_i$ and $c(a_i) = ca_i$ where $c > 0$ is his entry cost. For the utility h function we obtain here

$$h(p,\theta) = \max\{p\theta - c, 0\}.$$

The marginal utility is given by

$$\frac{\partial h(p,\theta)}{\partial \theta} = \begin{cases} 0 \text{ if } p\theta < c \\ p \text{ if } p\theta > c \end{cases}$$

---

[19]Such auctions are studies by Menezes and Monteiro [2000] and by Celik and Ylankaya [2009]- the latter authors also study asymmetric mechanisms

and thus that the virtual utility is given by

$$H(p,\theta) = h(p,\theta) - \frac{\partial h(p,\theta)}{\partial \theta} \times \frac{1-F(\theta)}{f(\theta)} = \begin{cases} 0 & \text{if } p\theta < c \\ p \times \left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) - c & \text{if } p\theta > c \end{cases}.$$

Assuming an increasing virtual value $J(\theta) = \theta - \frac{1-F(\theta)}{f(\theta)}$, note that $pJ(\theta) - c \geq 0$ implies that $p\theta \geq c$ and thus

$$\max\{0, H(p,\theta)\} = \max\left\{0, p \times \left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) - c\right\}$$

which is convex in p and super-modular in (p,θ) and the environment is weakly convex super-modular. Thus Corollary 2 implies that the Uniform price and Pay-Your-Bid auction are optimal:

Corollary 7. *Assume that the hazard rate $\frac{f(\theta)}{1-F(\theta)}$ is increasing. Then, the environment with entry cost is weakly convex super-modular and both the Uniform price and the Pay-Your-Bid auctions with reserve price*

$$\mathcal{R}^* = \frac{1-F(\theta^*)}{f(\theta^*)}$$

where θ^* solves

$$\psi_{m,n}(\theta^*)\left(\theta^* - \frac{1-F(\theta^*)}{f(\theta^*)}\right) = c$$

are symmetric revenue maximizing mechanisms. The optimal reserve price is decreasing in the number of agents n and increasing in the number of objects m.

7.4 Optimal Crowd-Sourcing Contests

Our insights can be used to identify the optimal mechanisms in situations where the goal is different from revenue maximization. For example, Chawla, Hartline and Sivan [2015] study an auction for an indivisible object where the goal of the designer is to maximize the highest bid (rather than the sum of bids as in the optimal auction a la Myerson). They show that, when restricted to symmetric mechanisms, their maximization problem can be written as

$$n \int_{\Theta_i} p(\theta) N(\theta) f(\theta) d\theta$$

where the "virtual utility" $N : [0, \bar{\theta}] \to \mathbb{R}$ is defined by

$$N(\theta) := \theta F(\theta)^{n-1} - \frac{1 - F^n(\theta)}{n f(\theta)}$$

Because of the linear, separable form in p, it is again clear that the Fan-Lorentz conditions are satisfied if N is non-decreasing, and that the optimal mechanism is a standard auction with a reserve price. Note that, the optimal cutoff $N^{-1}(0)$ is depending on the number of bidders n. For example, taking the uniform distribution on $[0, 1]$ yields

$$N(\theta) = \theta^n \left(1 + \frac{1}{n} \right) - \frac{1}{n}$$

and the optimal cutoff is

$$\theta^* = \left(\frac{1}{n+1} \right)^{-\frac{1}{n}}$$

which is decreasing here in n.

8 Ironing

If the supermodularity condition which is necessary to apply the Fan-Lorentz theorem $\frac{\partial H}{\partial \theta \partial p} \geq 0$ does not hold everywhere, we may get other extreme points as revenue maximizers, and the monotonicity constraint may bind.[20] In that case, we need to get a deeper insight into the set of extreme points of the set

$$\Omega_{mon}(\phi) = \left\{ p : p \prec_w \phi = \sum_{i=0}^{m-1} \binom{n-1}{i} t^{n-1-i}(1-t)^i \wedge p \text{ non-decreasing} \right\}.$$

The set of extreme points of $\Omega_{mon}(\phi)$ contains additional elements compared to the set of extreme points of the orbit $\Omega(\phi)$ that were characterized by Ryff (1967). Two general observations can be made:

1. If p is an extreme point of $\Omega_{mon}(\phi_{m,n})$ and p is not constant on a certain interval, then p must coincide with $\phi_{m,n}$ on this interval (in particular, the allocation must be efficient there). This is Lemma 4 below.

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[20]For ironing in the single-agent case with a utility that is **concave** in the allocation, see Toikka [2011].

2. Assume that $p \in \Omega_{mon}(\phi_{m,n})$ equals a constant κ on an interval $[t_1, t_2]$ while $p(t) < \kappa$ for $t < t_1$ and $p(t) > \kappa$ for $t > t_2$ (recall that we consider only monotone p). By majorization, we must have $p(t_2) < \phi_{m,n}(t_2)$. If $p < \phi_{m,n}$ on the entire interval $[t_1, t_2]$, then p cannot be an extreme point since the majorization condition is not tight on this interval. Assume then that $p(t_3) = q(t_3)$ for an interior point $t_3 \in (t_1, t_2)$.[21] For p to be an extreme point, the majorization constraint is binding:

$$\int_{t_1}^{t_3} (\kappa - \phi_{m,n}(t))d\theta = \int_{t_3}^{t_2} (\phi_{m,n}(t) - \kappa)d\theta \Leftrightarrow$$

$$\kappa = \frac{\int_{t_1}^{t_2} \phi_{m,n}(t)dt}{t_2 - t_1}.$$

In other words, the "ironed" value κ is completely determined by the interval where p is constant and by the efficient allocation function $\phi_{m,n}$.[22]

Lemma 4 (Ironing). *Assume that the function p has a finite number of discontinuities on interval $[\underline{t}, \bar{t}]$. If the function p is an extreme point of $\Omega_{mon}(\phi_{m,n})$, and if it is continuous and strictly increasing on an interval $[t', t'']$, then $p(t) = \phi_{m,n}(t)$ for $t \in [t', t'']$.*

Remark: When comparing different extreme points, the seller essentially chooses between the efficient allocation and a constant probability of allocation over an interval of types: by switching from the efficient allocation to a constant probability the seller decreases the probability of allocating an unit to the higher types and increases the probability of allocation to the lower types. The super-modularity condition $\frac{\partial^2 H}{\partial p \partial \theta} \geq 0$ implies that the marginal revenue $\frac{\partial H}{\partial p}$ is an increasing function of agent's type: in other words, it is more beneficial to increase the winning probability of the higher than of the lower type. Therefore, the optimal extreme point is the efficient allocation. The same intuition implies that, if H is sub-modular $\frac{\partial^2 H}{\partial p \partial \theta} < 0$, the optimal mechanism reduces to an overall constant probability of obtaining the object. The following example shows that such a situation might arise naturally in the context of costly signalling:

Example 2 (Hartline and Roughgarden 2008). *Consider a "money burning" environment where costly signals are not beneficial to the mechanism designer: she seeks to maximize the*

---

[21]If p is not continuous the argument needs to be slightly adjusted. A monotonic function is not continuous at at most a countable set of points.

[22]Analogous results (without using majorization) were obtained in a one-object, discrete type setting by Vohra (2011) and for step allocation functions by Manelli and Vincent (2010).

utility from the allocation less the cost of individual signals. Assuming linear cost of signals the designer's expected utility is given by

$$n \, \mathbb{E} \left[p_i \left(\theta \right) \frac{1 - F\left(\theta \right)}{f\left(\theta \right)} \right].$$

Therefore, assuming a monotone hazard rate, we obtain $\frac{\partial^2 H}{\partial p \partial \theta} < 0$, and the optimal mechanism allocates all objects via a lottery (constant probability of obtaining the object). This is indeed Corollary 2.11 in Hartline and Roughgarden [2008].

9 Conclusion

We have analyzed revenue maximization in a multi-unit auction framework where the agents' utility functions are convex in the physical allocation. We have shown that such preferences naturally arise as a reduced form in cases where the agents undertake, prior to the auction, costly actions that affect their valuations. A revenue maximizing seller can provide stronger ex-ante incentives by increasing the probability of allocating an object, but is constrained by the limited supply and by the usual monopolistic supply reduction incentives: values become here endogenous to the mechanism.

Our main results employed a novel combination of techniques, focused on a majorization inequality. They identified the revenue maximizing allocation within the class of symmetric mechanisms, and illustrated how it can be implemented via standard auction formats. Finally, we displayed novel comparative statics pertaining to the optimal reserve price, and illustrated our results in several specific environments with ex-ante investments.

10 Appendix

Proof of Proposition 1: Denote by $p(\theta_i)$ the interim probability with which agent i receives an object when she is of type θ_i and by

$$u(\theta_i, \tilde{\theta}_i) = h(p(\tilde{\theta}_i), \theta_i) - \mathbb{E} \left[y(\tilde{\theta}_i, \theta_{-i}) \right]$$

the interim utility of agent i if she is of type θ_i and misrepresent her type as $\tilde{\theta}_i$. Recall that we only consider symmetric mechanisms, and thus the functions p and u are independent of the agents identity- we drop the agent's subindex i for the remainder of the proof. Let $U(\theta)$

be the indirect utility of an agent of type θ who reports truthfully

$$U(\theta) = \max_{\tilde{\theta}} u(\theta, \tilde{\theta}).$$

As it is optimal to report truthfully in a direct mechanism, the envelope theorem yields that

$$\frac{\partial U}{\partial \theta} = \frac{\partial u}{\partial \theta} = \frac{\partial h}{\partial \theta}.$$

This implies that we can represent an agent's interim utility as

$$U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \frac{\partial h(p(v), v)}{\partial \theta} dv = \int_{\underline{\theta}}^{\theta} \frac{\partial h(p(v), v)}{\partial \theta} dv$$

where the last equality follows from the binding participation constraint.

The surplus of the designer (revenue) from each type is then given by

$$h(p(\theta), \theta) - U(\theta) = h(p(\theta), \theta) - \int_{\underline{\theta}}^{\theta} \frac{\partial h(p(v), v)}{\partial \theta} dv$$

and thus the objective becomes

$$\max_{p} \int_{\underline{\theta}}^{\overline{\theta}} \left(h(p(\theta), \theta) - \int_{\underline{\theta}}^{\theta} \frac{\partial h(p(v), v)}{\partial \theta} dv \right) f(\theta) d\theta$$

Using integration by parts, we obtain:

$$\int_{\underline{\theta}}^{\overline{\theta}} \left(\int_{\underline{\theta}}^{\theta} \frac{\partial h(p(v), v)}{\partial \theta} dv \right) f(\theta) d\theta = \int_{\underline{\theta}}^{\overline{\theta}} \frac{\partial h(p(\theta), \theta)}{\partial \theta} [1 - F(\theta)] d\theta$$

so that the expected revenue from any agent is only a function of the interim probability p with which an agent receives an object

$$\int_{\underline{\theta}}^{\overline{\theta}} \left(h(p(\theta), \theta) - \frac{\partial h(p(\theta), \theta)}{\partial \theta} \times \frac{1 - F(\theta)}{f(\theta)} \right) f(\theta) d\theta.$$

Multiplying by the number of agents n yields the result.

Proof of Lemma 2:

$$\frac{\partial}{\partial t}\left(\sum_{i=0}^{n}\min\{i,m\}\binom{n}{i}t^{n-i}(1-t)^{i}\right)$$

$$=\frac{\partial}{\partial t}\left(m-\sum_{i=0}^{m-1}(m-i)\binom{n}{i}t^{n-i}(1-t)^{i}\right)$$

$$=-\sum_{i=0}^{m-1}(m-i)\binom{n}{i}\left[(n-i)t^{n-i-1}(1-t)^{i}-it^{n-i}(1-t)^{i-1}\right]$$

$$=\sum_{i=0}^{m-1}\left[(m-i)\binom{n}{i}it^{n-i}(1-t)^{i-1}\right]-\left[(m-i)\binom{n}{i}(n-i)t^{n-i-1}(1-t)^{i}\right]$$

$$=\sum_{i=0}^{m-1}\left[(m-i-1)\binom{n}{i+1}(i+1)t^{n-i-1}(1-t)^{i}\right]-\left[(m-i)\binom{n}{i}(n-i)t^{n-i-1}(1-t)^{i}\right]$$

$$=\sum_{i=0}^{m-1}t^{n-i-1}(1-t)^{i}\left[(m-i-1)\binom{n}{i+1}(i+1)-(m-i)\binom{n}{i}(n-i)\right]$$

$$=\sum_{i=0}^{m-1}t^{n-i-1}(1-t)^{i}\left[(m-i-1)\binom{n}{i}(n-i)-(m-i)\binom{n}{i}(n-i)\right]$$

$$=-\sum_{i=0}^{m-1}t^{n-i-1}(1-t)^{i}\binom{n}{i}[n-i]=-n\sum_{i=0}^{m-1}t^{n-i-1}(1-t)^{i}\binom{n-1}{i}=-n\,\phi(t)\,.$$

Finally, we obtain that $\int_0^1 \phi(t)dt$ is given by

$$n\int_0^1 \phi(t)dt = \sum_{i=0}^{n}\min\{i,m\}\binom{n}{i}t^{n-i}(1-t)^{i}\Big|_{t=0}=m\,.$$

Lemma 5. $G(0,t)=0$ *for all* $t \in [0,1]$. *Furthermore, if the environment is CSM then* $G(q,t)$, *expressed as a function of the quantile, is convex in* q *and super-modular in* (q,t).

Proof of Lemma 5: First, $G(0,t)=H(0,F^{-1}(t))=h(0,F^{-1}(t))-h_\theta(0,F^{-1}(t))\times\frac{1-F(\theta)}{f(\theta)}$. As $h(0,\theta)=0$ by assumption, it follows that $h_\theta(0,\theta)=0$ and thus $G(0,t)=0$. In a CSM environment $G(q,t)=H(q,F^{-1}(t))$ is convex in q since $H(p,\theta)$ is convex in p. Similarly, because H is super-modular, because F^{-1} is strictly increasing and because super-modularity is preserved under strictly monotone transformations, we obtain that G is super-modular.

Proof of Theorem 1: As the number of agents and objects is fixed throughout the proof and there is thus no risk of confusion we will drop the subindices and write ϕ for $\phi_{m,n}$

throughout the proof. Let $q^*(t)$ be the interim probability with which an agent of type $\theta = F^{-1}(t)$ receives an object in an optimal symmetric mechanism. As no agent can receive the object with an ex-ante probability greater than m/n in a symmetric mechanism, we have that

$$\int_0^1 q^*(t)dt \leq \frac{m}{n} = \int_0^1 \phi(t)dt \,.$$

Consequently, there exists a quantile $t^* \in [0,1]$ such that the ex-ante probability with which an agent receives an object in the optimal mechanism equals the probability with which an agent with a type higher than $F^{-1}(t^*)$ would receive an object under the interim allocation probability ϕ

$$\int_{t^*}^1 \phi(t)dt = \int_0^1 q^*(t)dt \,.$$

Let $\theta^* = F^{-1}(t^*)$ be the type corresponding to the quantile t^* and let $q(t)$ be the interim allocation probability that corresponds to the allocation rule that assigns a unit to an agent if and only if her type is above θ^* and at most $m-1$ have a valuation higher than herself, i.e.

$$q(t) = \mathbf{1}_{\{t \geq t^*\}}\phi(t) \,.$$

In the next step we will argue that q majorizes q^* , and thus the Fan-Lorentz Theorem implies that $q^* = q$. By Condition (13) of Proposition 3 we have that for all $t \in [0,1]$

$$\int_t^1 q^*(z)\mathrm{d}z \leq \int_t^1 \phi(z)\mathrm{d}z \,. \tag{16}$$

By the definition of the critical quantile t^* we obtain that

$$\int_t^1 q^*(z)\mathrm{d}z \leq \int_0^1 q^*(z)\mathrm{d}z = \int_{t^*}^1 \phi(z)\mathrm{d}z \,. \tag{17}$$

Combining, (16) and (17) that for all $t \in [0,1]$

$$\int_t^1 q^*(z)\mathrm{d}z \leq \min\left\{\int_t^1 \phi(z)\mathrm{d}z, \int_{t^*}^1 \phi(z)\mathrm{d}z\right\} = \int_t^1 \mathbf{1}_{t \geq \hat{t}}\phi(z)\mathrm{d}z = \int_t^1 q(z)\mathrm{d}z \,.$$

As $\int_0^1 q(z)dz = \int_0^1 q^*(z)dz$ by the definitions of t^* and q we obtain that q majorizes q^*. Note that, as the environment is CSM, Lemma 5 implies that G satisfies the conditions of

Proposition 4. By the Fan-Lorentz Theorem (Proposition 4) we thus have that

$$\int_0^1 G(q^*(t), t)dt \le \int_0^1 G(q(t), t)dt \,.$$

But, as q^* is the revenue maximizing interim probability, the above equation must hold with equality. Consequently, every mechanism which implements the interim probability $q(t) = \mathbf{1}_{\{t \ge t^*\}}\phi(t)$ is revenue maximizing.

Finally, observe that, by Lemma 5, $G(0, \cdot) = 0$. Thus, the expected revenue as a function of t^* is given by

$$t^* \mapsto \int_0^1 G\left(\mathbf{1}_{\{t \ge t^*\}}\phi(t), t\right) dt = \int_{t^*}^1 G\left(\phi(t), t\right) dt \,. \tag{18}$$

We show now that $\theta \mapsto H(\psi(\theta), \theta)$ changes sign at most once, from negative to positive. Assume that it crosses zero at $\theta^* \ne \underline{\theta}$, that is $H(\psi(\theta^*), \theta^*) = 0$. Then for any $\theta > \theta^*$ we have

$$H(\psi(\theta), \theta) - H(\psi(\theta^*), \theta^*)$$
$$= H(\psi(\theta), \theta) - H(\psi(\theta), \theta^*) + H(\psi(\theta), \theta^*) - H(\psi(\theta^*), \theta^*) \ge 0$$

where the last inequality follows from (a) convexity of H which implies that

$$\frac{H(\psi(\theta), \theta^*)}{\psi(\theta)} \ge \frac{H(\psi(\theta^*), \theta^*)}{\psi(\theta^*)} = 0$$

and hence

$$H(\psi(\theta), \theta^*) - H(\psi(\theta^*), \theta^*) \ge 0 \,.$$

and (b) supermodularity of H which implies that

$$H(\psi(\theta), \theta) - H(0, \theta) \ge H(\psi(\theta), \theta^*) - H(0, \theta^*) \iff$$
$$H(\psi(\theta), \theta) - H(\psi(\theta), \theta^*) \ge H(0, \theta) - H(0, \theta^*) = 0.$$

Since $\theta \mapsto H(\psi(\theta), \theta)$ changes sign at most once, from negative to positive, so does $t \mapsto G(\phi(t), t)$ since F is a monotone transformation. Since $t \mapsto G(\phi(t), t)$ changes its sign at most once, from negative to positive, (18) this function is quasi-concave, and thus the optimal quantile t^* satisfies the first order condition $G(\phi(t^*), t^*) = 0$. The result follows since

$$G(\phi(t^*), t^*) = H(\phi(t^*), F^{-1}(t^*)) = H((\phi \circ F)(\theta^*), \theta^*) = H(\psi_{m,n}(\theta^*), \theta^*).$$

Proof of Proposition 5: Let $\psi, \psi' : (0,1) \to (0,\infty)$ be continuous functions with $\psi(q) \leq \psi'(q)$ for all $q \in [0,1]$. Recall that $\theta \mapsto H(\psi(\theta), \theta)$, and $\theta \mapsto H(\psi'(\theta), \theta)$ change sign only once[23], from negative to positive, and define θ^*, θ^{**} implicitly by

$$0 = H(\psi(\theta^*), \theta^*)$$
$$0 = H(\psi'(\theta^{**})), \theta^{**}).$$

We have that

$$0 = H(\psi(\theta^*), \theta^*) = \frac{H(\psi(\theta^*), \theta^*)}{\psi(\theta^*)} \leq \frac{H(\psi'(\theta^*), \theta^*)}{\psi'(\theta^*)},$$

where the last inequality follows from the convexity of H, and from the fact that $H(0, \cdot) = 0$. Last inequality implies that $0 \leq H(\psi'(\theta), \theta)$ and, since $\theta \mapsto H(\psi'(\theta), \theta)$ changes sign only once, it follows that $\theta^* \geq \theta^{**}$. As $\phi_{m,n}$ decreases in n and increases in m point-wise, the result follows from the above by setting $\psi = \phi_{m,n} \circ F$.

Proof of Corollary 2: By Proposition 1 the revenue in any symmetric incentive compatible mechanism is given by

$$n \int_\Theta H(p(\theta), \theta) f(\theta)\, d\theta.$$

Let $\tilde{H}(p, \theta) := \max\{H(p, \theta), 0\}$. Trivially, an upper bound on the optimal revenue is thus given by

$$\max_p n \int_\Theta \max\{H(p(\theta), \theta), 0\} f(\theta)\, d\theta = \max_p n \int_\Theta \tilde{H}(p(\theta), \theta) f(\theta)\, d\theta. \tag{19}$$

As this environment is CSM it follows from Theorem 1 that the revenue maximizing mechanism allocates according $\psi = \phi \circ F$, conditional on being above a critical type θ^* that is the unique solution of $H((\phi \circ F)(\theta^*), \theta^*) = 0$. As $H(0, \theta) = 0$ for all θ by Lemma 5, we obtain that this achieves the upper bound given in (19) and thus is an optimal mechanism.

Proof of Proposition 6: 1. Assume that all bidders other than i use a strictly increasing strategy $\beta(\theta)$. Since agent i's utility is linear in money, her expected utility if she submits

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[23]If $H(\psi(\theta), \theta)$ and $H(\psi'(\theta), \theta)$ do not change sign, define θ^* and θ^{**} to be the relevant boundary of the interval $[\underline{\theta}, \bar{\theta}]$.

the bid $\beta(\hat{\theta})$ that is optimal for the type $\hat{\theta}$ is given by

$$h\left(\psi_{m,n}(\hat{\theta}),\theta_i\right) - \int_{\underline{\theta}}^{\hat{\theta}} \max\left\{\beta\left(\theta_j\right),\mathcal{R}\right\} f_{m:n-1}\left(\theta_j\right) d\theta_j \tag{20}$$

where $f_{m:n-1}$ is the density of the $m-$highest order statistic out of the $n-1$ competitors of i. Therefore, i chooses her bid b in order to maximize the value given in expression (20).

Taking the derivative of the last expression with respect to $\hat{\theta}$, we obtain

$$\frac{\partial h\left(\psi_{m,n}(\hat{\theta}),\theta_i\right)}{\partial p}\frac{\partial \psi_{m,n}(\hat{\theta})}{\partial \theta} - f_{m:n-1}(\hat{\theta})\max\{\beta(\hat{\theta}),\mathcal{R}\} \tag{21}$$

If β is a symmetric equilibrium strategy, then agent i with type θ_i prefers to bid $b = \beta\left(\theta_i\right)$. The first order condition for any type θ_i who submits a bid greater than the reserve price $\mathcal{R}$ is thus given by

$$\frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p}\frac{\partial \psi_{m,n}(\theta_i)}{\partial \theta} - \beta\left(\theta_i\right) f_{m:n-1}\left(\theta_i\right) = 0$$

Noting that

$$\frac{\partial \psi_{m,n}(\theta_i)}{\partial \theta} = f_{m:n-1}\left(\theta_i\right)$$

we obtain the candidate equilibrium bid

$$\beta\left(\theta_i\right) = \frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p}.$$

Now set θ' to be the solution to the equation

$$\frac{h\left(\psi_{m,n}(\theta'),\theta'\right)}{\psi_{m,n}(\theta^*)} = \mathcal{R}.$$

We complete the equilibrium bidding strategy:

$$\beta(\theta_i) = \begin{cases} \frac{\partial h(\psi_{m,n}(\theta_i),\theta_i)}{\partial p} & \text{for all } \theta_i \in [\theta',\overline{\theta}] \\ 0 & \text{for all } \theta_i \in [\underline{\theta},\theta']. \end{cases}$$

We still need to verify that bidding $\beta(\theta_i)$ is optimal for an agent of type θ_i. First, observe that the agent of type θ' has a utility of zero in equilibrium by the definition of the reserve price. As any other type $\theta_i \in [\theta',\overline{\theta}]$ receives a weakly higher utility, no higher type wants to deviate by bidding zero. To verify that the bid $\beta(\theta_i)$ is optimal we show that the agent's

objective (20) is concave. Plugging the definition of β in the derivative of (20) we obtain

$$\frac{\partial h\left(\psi_{m,n}(\hat{\theta}),\theta_i\right)}{\partial p}\psi'_{m,n}(\hat{\theta}) - f_{m:n-1}(\hat{\theta})\max\{\beta(\hat{\theta}),R\}$$

$$= \psi'_{m,n}(\hat{\theta})\left[\frac{\partial h\left(\psi_{m,n}(\hat{\theta}),\theta_i\right)}{\partial p} - \frac{\partial h\left(\psi_{m,n}(\hat{\theta}),\hat{\theta}\right)}{\partial p}\right]. \tag{22}$$

As h is super-modular and as $\psi'_{m,n}$ is positive, the right-hand side of (22) changes its sign once from positive to negative at $\hat{\theta} = \theta_i$. Consequently, the agent's objective (20) is quasi-concave and maximized by bidding truthfully $\hat{\theta} = \theta_i$. It is immediate that it can never be optimal for the agent to make a bid higher than $\beta(\bar{\theta})$ since a bid of $\beta(\bar{\theta})$ would already ensure that she wins the object and pays strictly less.

To verify that no type $\theta_i \in [\underline{\theta}, \theta']$ wants to deviate by making a non-zero bid observe that, by deviating to any bid in $[\beta(\theta'), \beta(\bar{\theta})]$, the agent would get a utility which is lower than the utility the type θ' would get from making this bid. But, by construction the optimal bid of the type θ' yields her a utility of zero, which implies that the utility from any bid greater than zero must be less than zero for all lower types.

2. The proof of this part follows by the monotonicity and convexity of h in p and from the observation that, by definition of $\psi_{m,n}$, the critical type $\theta'_{m,n}$ associated to a fixed reserve price $\mathcal{R}$ increases in the number of bidders n and decreases in the number of objects m. $\square$

Proof of Proposition 7: Agent i's utility in a symmetric equilibrium where all but bidder i use the bidding strategy β and i submits a bid b is given by

$$h\left(\psi_{m,n}(\beta^{-1}(b)),\theta_i\right) - \psi_{m,n}(\beta^{-1}(b))b. \tag{23}$$

Taking the derivative with respect to b yields that

$$\left[\frac{\partial h\left(\psi_{m,n}(\beta^{-1}(b)),\theta_i\right)}{\partial p} - b\right]\frac{\partial\psi(\beta^{-1}(b))}{\partial\theta}\frac{\partial\beta^{-1}(b)}{\partial b} - \psi_{m,n}(\beta^{-1}(b)).$$

Plugging $b = \beta(\theta)$ (as it is optimal for agent i to make the equilibrium bid) yields

$$\left[\frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p} - \beta(\theta_i)\right]\frac{\partial\psi(\theta_i)}{\partial\theta}\frac{1}{\beta'(\theta_i)} - \psi_{m,n}(\theta_i) = 0.$$

Rearranging for β' yields the following differential equation:

$$\beta'(\theta_i) = \frac{\psi'_{m,n}(\theta_i)}{\psi_{m,n}(\theta_i)}\left[\frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p} - \beta(\theta_i)\right]. \tag{24}$$

Since $\theta_i \mapsto \frac{\psi'_{m,n}(\theta_i)}{\psi_{m,n}(\theta_i)}$ is continuous and since the left-hand-side of (24) is uniformly Lipschitz in β, the ODE (24) admits a unique solution for every initial value. Set θ' to be the solution to the equation

$$\frac{h\left(\psi_{m,n}(\theta'),\theta'\right)}{\psi_{m,n}(\theta')} = \mathcal{R}.$$

If we set $\beta(\theta') = \mathcal{R}$ then the agent with type θ' is, by construction, indifferent between bidding $\mathcal{R}$ and bidding zero. For all higher types $\theta \in (\theta',\overline{\theta}]$ we extend the bidding function by solving the ODE (24). Since (24) is linear, we obtain the explicit solution

$$\beta(\theta_i) = \mathcal{R}\frac{\psi_{m,n}(\theta')}{\psi_{m,n}(\theta_i)} + \frac{1}{\psi_{m,n}(\theta_i)}\int_{\theta'}^{\theta_i}\psi'_{m,n}(z)\frac{\partial h\left(\psi_{m,n}(z),z\right)}{\partial p}dz$$

To verify, take the derivative to obtain

$$\begin{aligned}
\beta'(\theta_i) &= -\frac{\psi'_{m,n}(\theta_i)\psi_{m,n}(\theta')}{\left(\psi_{m,n}(\theta_i)\right)^2}\mathcal{R} - \frac{\psi'_{m,n}(\theta_i)}{\left(\psi_{m,n}(\theta_i)\right)^2}\int_{\theta'}^{\theta_i}\psi'_{m,n}(z)\frac{\partial h\left(\psi_{m,n}(z),z\right)}{\partial p}dz \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \frac{\psi'_{m,n}(\theta_i)}{\psi_{m,n}(\theta_i)}\frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p} \\
&= -\frac{\psi'_{m,n}(\theta_i)}{\psi_{m,n}(\theta_i)}\beta(\theta_i) + \frac{\psi'_{m,n}(\theta_i)}{\psi_{m,n}(\theta_i)}\frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p} \\
&= \frac{\psi'_{m,n}(\theta_i)}{\psi_{m,n}(\theta_i)}\left[\frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p} - \beta(\theta_i)\right].
\end{aligned}$$

We first need to show that the solution β of (24) is increasing for all $\theta_i \geq \theta'$. Observe that:

$$\beta' \geq 0 \Leftrightarrow \mathcal{R}\frac{\psi_{m,n}(\theta')}{\psi_{m,n}(\theta_i)} + \frac{1}{\psi_{m,n}(\theta_i)}\int_{\theta'}^{\theta_i}\psi'_{m,n}(z)\frac{\partial h\left(\psi_{m,n}(z),z\right)}{\partial p}dz \leq \frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p}$$

Plugging the expression for $\mathcal{R}$, this becomes:

$$\frac{h\left(\psi_{m,n}(\theta'),\theta'\right)}{\psi_{m,n}(\theta_i)} + \frac{1}{\psi_{m,n}(\theta_i)}\int_{\theta'}^{\theta_i}\psi'_{m,n}(z)\frac{\partial h\left(\psi_{m,n}(z),z\right)}{\partial p}dz \leq \frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p}$$

By super-modularity, we can bound the left hand side above:

$$\frac{h\left(\psi_{m,n}(\theta'),\theta'\right)}{\psi_{m,n}(\theta_i)} + \frac{1}{\psi_{m,n}(\theta_i)}\int_{\theta'}^{\theta_i}\psi'_{m,n}(z)\frac{\partial h\left(\psi_{m,n}(z),z\right)}{\partial p}dz$$

$$\leq \frac{h\left(\psi_{m,n}(\theta'),\theta'\right)}{\psi_{m,n}(\theta_i)} + \frac{\frac{\partial h(\psi_{m,n}(\theta_i),\theta_i)}{\partial p}}{\psi_{m,n}(\theta_i)}(\psi_{m,n}(\theta_i) - \psi_{m,n}(\theta'))$$

$$= \frac{h\left(\psi_{m,n}(\theta'),\theta'\right)}{\psi_{m,n}(\theta_i)} + \frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p} - \frac{\psi_{m,n}(\theta')}{\psi_{m,n}(\theta_i)}\frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p}$$

Thus, we need to show that:

$$\frac{h\left(\psi_{m,n}(\theta'),\theta'\right)}{\psi_{m,n}(\theta_i)} + \frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p} - \frac{\psi_{m,n}(\theta')}{\psi_{m,n}(\theta)}\frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p} \leq \frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p} \iff$$

$$\frac{1}{\psi_{m,n}(\theta_i)}[h\left(\psi_{m,n}(\theta'),\theta'\right) - \psi_{m,n}(\theta')\frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p}] \leq 0 \iff$$

$$\frac{h\left(\psi_{m,n}(\theta'),\theta'\right)}{\psi_{m,n}(\theta')} \leq \frac{\partial h\left(\psi_{m,n}(\theta_i),\theta_i\right)}{\partial p}$$

The last inequality holds by the convexity and super-modularity of h.

We need to verify that it is indeed optimal for the type θ_i to bid $\beta(\theta_i)$. We start by considering types $\theta_i \in [\theta', \bar{\theta}]$. By construction, the bid $\beta(\theta_i)$ satisfies the agent's first order condition. If the agent deviates by making the equilibrium bid that is optimal for type $\hat{\theta}$, her utility equals

$$h\left(\psi_{m,n}(\hat{\theta}),\theta_i\right) - \psi_{m,n}(\hat{\theta})\beta(\hat{\theta}). \tag{25}$$

The derivative with respect to $\hat{\theta}$ is

$$\left[\frac{\partial h\left(\psi_{m,n}(\hat{\theta}),\theta_i\right)}{\partial p} - \beta(\hat{\theta})\right]\psi'_{m,n}(\hat{\theta}) - \psi_{m,n}(\hat{\theta})\beta'(\hat{\theta})$$

$$= \underbrace{\left[\frac{\partial h\left(\psi_{m,n}(\hat{\theta}),\hat{\theta}\right)}{\partial p} - \beta(\hat{\theta})\right]\psi'_{m,n}(\hat{\theta})}_{\psi_{m,n}(\hat{\theta})\beta'(\hat{\theta})} - \psi_{m,n}(\hat{\theta})\beta'(\hat{\theta}) + \left[\frac{\partial h\left(\psi_{m,n}(\hat{\theta}),\theta_i\right)}{\partial p} - \frac{\partial h\left(\psi_{m,n}(\hat{\theta}),\hat{\theta}\right)}{\partial p}\right]$$

$$= \frac{\partial h\left(\psi_{m,n}(\hat{\theta}),\hat{\theta}\right)}{\partial p} - \frac{\partial h\left(\psi_{m,n}(\hat{\theta}),\hat{\theta}\right)}{\partial p}. \tag{26}$$

(at the last step we used the fact that β solves the ODE (24). As h is super-modular,

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expression (26) is increasing, and changes its sign from positive to negative at $\theta_i = \hat{\theta}$. Thus, the agent's objective (25) is concave, and is maximized at $\theta_i = \hat{\theta}$. An agent of type $\theta_i \in [\theta', \bar{\theta}]$ thus prefers to make the bid $\beta(\theta_i)$ over any other bid in $[\beta(\theta'), \beta((\bar{\theta})] = [R, \beta(\bar{\theta})]$. Clearly it can never be optimal for the agent to make a bid higher than $\beta(\bar{\theta})$ as a bid of $\beta(\bar{\theta})$ would already ensure that she wins and pays strictly less. It remains to verify that the agent does not want to make a bid of zero: observe that the agent could deviate to bid $\mathcal{R}$, which would yield a utility higher than the equilibrium utility of the type θ'. It thus suffices to verify that the equilibrium utility of the type θ' is non-negative. This type's equilibrium utility is given by

$$h\left(\psi_{m,n}(\theta'), \theta'\right) - \psi_{m,n}(\theta')\mathcal{R},$$

that, by the definition of the reserve price, equals zero. Finally, we verify that no type $\theta_i \in [\underline{\theta}, \theta']$ wants to deviate by making a non-zero bid. To see this, note that by deviating to any bid in $[\mathcal{R}, \beta(\bar{\theta})]$ such a type would get a utility which is lower than the utility the type θ' gets from making this bid. But, by construction, the optimal bid of type θ' equals $\mathcal{R}$ and yields her a utility of zero. This implies that the utility resulting from any bid greater zero must be less than zero for all lower types. $\qquad\square$

Proof of Proposition 8: With some abuse, we shall treat here m and n as continuous variables. Recall that the critical cutoff is given by

$$H\left(\psi_{m,n}\left(\theta^*\right), \theta^*\right) = 0.$$

Therefore,

$$\frac{\partial \theta^*}{\partial n} = -\frac{\frac{\partial H}{\partial p}\left(\psi_{m,n}\left(\theta^*\right), \theta^*\right) \frac{\partial \psi_{m,n}}{\partial n}\left(\theta^*\right)}{\frac{\partial H}{\partial \theta}\left(\psi_{m,n}\left(\theta^*\right), \theta^*\right) + \frac{\partial H}{\partial p}\left(\psi_{m,n}\left(\theta^*\right), \theta^*\right) \frac{\partial \psi_{m,n}}{\partial \theta}\left(\theta^*\right)}$$

Also, we know that

$$\mathcal{R}^* = \frac{h\left(\psi_{m,n}\left(\theta^*\right), \theta^*\right)}{\psi_{m,n}\left(\theta^*\right)}.$$

which implies

$$\frac{\partial \mathcal{R}^*}{\partial n} = \frac{\left[\frac{\partial h(\psi_{m,n}(\theta^*), \theta^*)}{\partial p}\left(\frac{\partial \psi_{m,n}(\theta^*)}{\partial n} + \frac{\partial \psi_{m,n}(\theta^*)}{\partial \theta}\frac{\partial \theta^*}{\partial n}\right) + \frac{\partial h(\psi_{m,n}(\theta^*), \theta^*)}{\partial \theta}\frac{\partial \theta^*}{\partial n}\right]\psi_{m,n}\left(\theta^*\right)}{\left(\psi_{m,n}\left(\theta^*\right)\right)^2}$$

$$- \frac{\left(\frac{\partial \psi_{m,n}(\theta^*)}{\partial n} + \frac{\partial \psi_{m,n}(\theta^*)}{\partial \theta}\frac{\partial \theta^*}{\partial n}\right)h\left(\psi_{m,n}\left(\theta^*\right), \theta^*\right)}{\left(\psi_{m,n}\left(\theta^*\right)\right)^2}.$$

The sign of the last derivative is just the sign of

$$
\left[\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial p} \left(\frac{\partial \psi_{m,n}\left(\theta^*\right)}{\partial n} + \frac{\partial \psi_{m,n}\left(\theta^*\right)}{\partial \theta}\frac{\partial \theta^*}{\partial n} \right) + \frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial \theta}\frac{\partial \theta^*}{\partial n} \right] \psi_{m,n}\left(\theta^*\right)
$$
$$
- \left(\frac{\partial \psi_{m,n}\left(\theta^*\right)}{\partial n} + \frac{\partial \psi_{m,n}\left(\theta^*\right)}{\partial \theta}\frac{\partial \theta^*}{\partial n} \right) h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)
$$

$$
\begin{aligned}
&\left[\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial p} \left(\frac{\partial \psi_{m,n}\left(\theta^*\right)}{\partial n} + \frac{\partial \psi_{m,n}\left(\theta^*\right)}{\partial \theta}\frac{\partial \theta^*}{\partial n} \right) + \frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial \theta}\frac{\partial \theta^*}{\partial n} \right] \psi_{m,n}\left(\theta^*\right) \\
&- \left(\frac{\partial \psi_{m,n}\left(\theta^*\right)}{\partial n} + \frac{\partial \psi_{m,n}\left(\theta^*\right)}{\partial \theta}\frac{\partial \theta^*}{\partial n} \right) h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \\
= \ & \left(\frac{\partial \psi_{m,n}\left(\theta^*\right)}{\partial n} + \frac{\partial \psi_{m,n}\left(\theta^*\right)}{\partial \theta}\frac{\partial \theta^*}{\partial n} \right) \left[\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial p}\psi_{m,n}\left(\theta^*\right) - h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \right] \\
&+ \psi_{m,n}\left(\theta^*\right) \frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial \theta}\frac{\partial \theta^*}{\partial n}
\end{aligned}
$$

Plugging the expression for $\frac{\partial \theta^*}{\partial n}$ into the last expression we get

$$\left(\frac{\partial \psi_{m,n}\left(\theta^*\right)}{\partial n} - \frac{\partial \psi_{m,n}\left(\theta^*\right)}{\partial \theta} \frac{\frac{\partial H}{\partial p}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \frac{\partial \psi_{m,n}}{\partial n}\left(\theta^*\right)}{\frac{\partial H}{\partial \theta}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) + \frac{\partial H}{\partial p}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \frac{\partial \psi_{m,n}}{\partial \theta}\left(\theta^*\right)} \right)$$

$$\times \left[\frac{\partial h}{\partial p}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \psi_{m,n}\left(\theta^*\right) - h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \right]$$

$$- \psi_{m,n}\left(\theta^*\right) \frac{\partial h}{\partial \theta}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \frac{\frac{\partial H}{\partial p}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \frac{\partial \psi_{m,n}}{\partial n}\left(\theta^*\right)}{\frac{\partial H}{\partial \theta}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) + \frac{\partial H}{\partial p}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \frac{\partial \psi_{m,n}}{\partial \theta}\left(\theta^*\right)}$$

$$= \frac{\partial \psi_{m,n}\left(\theta^*\right)}{\partial n} \left(1 - \frac{\frac{\partial H}{\partial p}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \frac{\partial \psi_{m,n}}{\partial n}\left(\theta^*\right)}{\frac{\partial H}{\partial \theta}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) + \frac{\partial H}{\partial p}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \frac{\partial \psi_{m,n}}{\partial \theta}\left(\theta^*\right)} \right)$$

$$\times \left[\frac{\partial h}{\partial p}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \psi_{m,n}\left(\theta^*\right) - h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \right]$$

$$- \psi_{m,n}\left(\theta^*\right) \frac{\partial h}{\partial \theta}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \frac{\frac{\partial H}{\partial p}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \frac{\partial \psi_{m,n}}{\partial n}\left(\theta^*\right)}{\frac{\partial H}{\partial \theta}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) + \frac{\partial H}{\partial p}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \frac{\partial \psi_{m,n}}{\partial \theta}\left(\theta^*\right)}$$

$$= \frac{\frac{\partial \psi_{m,n}}{\partial n}\left(\theta^*\right) \frac{\partial H}{\partial \theta}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\frac{\partial H}{\partial \theta}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) + \frac{\partial H}{\partial p}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \frac{\partial \psi_{m,n}}{\partial \theta}\left(\theta^*\right)}$$

$$\times \left[\frac{\partial h}{\partial p}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \psi_{m,n}\left(\theta^*\right) - h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \right]$$

$$- \psi_{m,n}\left(\theta^*\right) \frac{\partial h}{\partial \theta}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \frac{\frac{\partial H}{\partial p}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \frac{\partial \psi_{m,n}}{\partial n}\left(\theta^*\right)}{\frac{\partial H}{\partial \theta}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) + \frac{\partial H}{\partial p}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \frac{\partial \psi_{m,n}}{\partial \theta}\left(\theta^*\right)}$$

$$= \frac{\frac{\partial \psi_{m,n}}{\partial n}\left(\theta^*\right)}{\frac{\partial H}{\partial \theta}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) + \frac{\partial H}{\partial p}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \frac{\partial \psi_{m,n}}{\partial \theta}\left(\theta^*\right)}$$

$$\times \left[\begin{array}{c} \frac{\partial H}{\partial \theta}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \left(\frac{\partial h}{\partial p}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \psi_{m,n}\left(\theta^*\right) - h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \right) \\ - \psi_{m,n}\left(\theta^*\right) \frac{\partial h}{\partial \theta}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \frac{\partial H}{\partial p}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \end{array} \right]$$

Since $H(0,\theta) = 0$ for any θ, we have that $\frac{\partial H}{\partial \theta}(0,\theta) = 0$ for any θ. Therefore $\frac{\partial^2 H}{\partial p \partial \theta} \geq 0$ implies that $\frac{\partial H}{\partial \theta}(p,\theta) \geq 0$. Recall that $\psi_{m,n}(\theta)$ is strictly decreasing in n and increasing in θ. Since H is convex in p, and $H(0,\theta) = 0$ we can conclude that

$$\frac{\partial H}{\partial p}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right) \geq \frac{H\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\psi_{m,n}(\theta^*)} = 0$$

where the last equality follows from

$$H\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)=0.$$

Therefore

$$\frac{\frac{\partial\psi_{m,n}}{\partial n}\left(\theta^*\right)}{\frac{\partial H}{\partial\theta}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)+\frac{\partial H}{\partial p}\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)\frac{\partial\psi_{m,n}}{\partial\theta}\left(\theta^*\right)}<0.$$

Hence, we need to know the sign of

$$\frac{\partial H}{\partial\theta}\left(\frac{\partial h}{\partial p}\psi_{m,n}-h\right)-\psi_{m,n}\frac{\partial h}{\partial\theta}\frac{\partial H}{\partial p}.$$

Rearranging the above expression yields

$$\frac{\partial H}{\partial\theta}\left(\frac{\partial h}{\partial p}\psi_{m,n}-h\right)-\psi_{m,n}\frac{\partial h}{\partial\theta}\frac{\partial H}{\partial p}=\psi_{m,n}\left[\frac{\partial H}{\partial\theta}\frac{\partial h}{\partial p}-\frac{\partial h}{\partial\theta}\frac{\partial H}{\partial p}\right]-h\frac{\partial H}{\partial\theta}.$$

Recall that

$$H(p,\theta)=h(p,\theta)-\frac{\partial h(p,\theta)}{\partial\theta}\frac{1-F\left(\theta\right)}{f\left(\theta\right)}$$

Therefore,

$$
\begin{aligned}
\frac{\partial H(p,\theta)}{\partial\theta} &= \frac{\partial h(p,\theta)}{\partial\theta}-\frac{\partial^2 h(p,\theta)}{\partial\theta^2}\frac{1-F\left(\theta\right)}{f\left(\theta\right)}-\frac{\partial h(p,\theta)}{\partial\theta}\frac{\partial}{\partial\theta}\frac{1-F\left(\theta\right)}{f\left(\theta\right)}\\
\frac{\partial H(p,\theta)}{\partial p} &= \frac{\partial h(p,\theta)}{\partial p}-\frac{\partial^2 h(p,\theta)}{\partial\theta\partial p}\frac{1-F\left(\theta\right)}{f\left(\theta\right)}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\frac{\partial H\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial p}-\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}\frac{\partial H\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial p}\\
=\quad&\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial p}-\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial p}\frac{\partial^2 h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta^2}\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)}\\
&-\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial p}\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}\frac{\partial}{\partial\theta}\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)}\\
&-\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial p}\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}+\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}\frac{\partial^2 h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta\partial p}\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)}\\
=\quad&-\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial p}\frac{\partial^2 h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta^2}\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)}\\
&-\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial p}\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}\frac{\partial}{\partial\theta}\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)}\\
&+\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}\frac{\partial^2 h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta\partial p}\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)}.
\end{aligned}
$$

From the definition of the optimal reserve price we obtain that

$$
h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)=h_\theta\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)}
$$

(since θ^* solves $H\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)=0$). Therefore,

$$\psi_{m,n}\left(\theta^*\right)\left[\frac{\partial H\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial p}-\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}\frac{\partial H\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial p}\right]$$

$$-h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)\frac{\partial H\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}$$

$$=\;\psi_{m,n}\left(\theta^*\right)\left[\begin{array}{c}-\frac{\partial h(\psi_{m,n}(\theta^*),\theta^*)}{\partial p}\frac{\partial^2 h(\psi_{m,n}(\theta^*),\theta^*)}{\partial\theta^2}\frac{1-F(\theta^*)}{f(\theta^*)}-\frac{\partial h(\psi_{m,n}(\theta^*),\theta^*)}{\partial p}\frac{\partial h(\psi_{m,n}(\theta^*),\theta^*)}{\partial\theta}\frac{\partial}{\partial\theta}\frac{1-F(\theta^*)}{f(\theta^*)}\\+\frac{\partial h(\psi_{m,n}(\theta^*),\theta^*)}{\partial\theta}\frac{\partial^2 h(\psi_{m,n}(\theta^*),\theta^*)}{\partial\theta\partial p}\frac{1-F(\theta^*)}{f(\theta^*)}\end{array}\right]$$

$$-h_\theta\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)}\left[\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}-\frac{\partial^2 h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta^2}\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)}\right.$$

$$\left.-\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}\frac{\partial}{\partial\theta}\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)}\right]$$

$$=\;-\frac{\partial}{\partial\theta}\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)}\left[\psi_{m,n}\left(\theta^*\right)\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial p}\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}\right.$$

$$\left.+\left(\frac{\partial h\left(\psi_{m,n}\left(\theta^*\right),\theta^*\right)}{\partial\theta}\right)^2\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)}\right]$$

$$+\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)}\left[\begin{array}{c}-\psi_{m,n}\left(\theta^*\right)\frac{\partial h(\psi_{m,n}(\theta^*),\theta^*)}{\partial p}\frac{\partial^2 h(\psi_{m,n}(\theta^*),\theta^*)}{\partial\theta^2}+\psi_{m,n}\left(\theta^*\right)\frac{\partial h(\psi_{m,n}(\theta^*),\theta^*)}{\partial\theta}\frac{\partial^2 h(\psi_{m,n}(\theta^*),\theta^*)}{\partial\theta\partial p}\\-\left(\frac{\partial h(\psi_{m,n}(\theta^*),\theta^*)}{\partial\theta}\right)^2+\frac{\partial h(\psi_{m,n}(\theta^*),\theta^*)}{\partial\theta}\frac{\partial^2 h(\psi_{m,n}(\theta^*),\theta^*)}{\partial\theta^2}\frac{1-F(\theta^*)}{f(\theta^*)}\end{array}\right]$$

The assumptions $\frac{\partial^2 h}{\partial\theta^2}\leq 0$, and $\frac{\partial}{\partial\theta}(\frac{1-F}{f})\leq -1$ guarantee that the above expression is positive. Since

$$\frac{\partial}{\partial\theta}(\frac{1-F\left(\theta^*\right)}{f\left(\theta^*\right)})=\frac{-\left(f\left(\theta^*\right)\right)^2-f'\left(\theta^*\right)F\left(\theta^*\right)}{\left(f\left(\theta^*\right)\right)^2}=-1-\frac{f'\left(\theta^*\right)F\left(\theta^*\right)}{\left(f\left(\theta^*\right)\right)^2},$$

we obtain that $f'\left(\theta^*\right)\geq 0$ is sufficient for $\frac{\partial R^*}{\partial n}<0$. The proof that $\frac{\partial R^*}{\partial m}>0$ is similar, the only difference being that $\psi_{m,n}(\theta)$ is increasing in m. $\qquad\square$

Proof of Lemma 3: Recall that the expected revenue is given by

$$R=n\int_{\theta^*}^1 (p)^{\frac{l}{l-1}}\left(\frac{\theta}{b}\right)^{\frac{1}{l-1}}K(\theta)f\left(\theta\right)d\theta$$

where $\frac{l-1}{l}\theta^*-\frac{1-F(\theta^*)}{f(\theta^*)}=0$ and $p=F^{n-1}\left(\theta\right)$.

The derivative of the expected revenue with respect to n is

$$\int_{\theta^*}^1 F^{\frac{(n-1)l}{l-1}}(\theta)\left(\frac{\theta}{b}\right)^{\frac{1}{l-1}} K(\theta)f(\theta)\,d\theta + n\int_{\theta^*}^1 F^{\frac{(n-1)l}{l-1}}(\theta)\frac{l}{l-1}\ln F(\theta)\left(\frac{\theta}{b}\right)^{\frac{1}{l-1}} K(\theta)f(\theta)\,d\theta$$

$$= \int_{\theta^*}^1 K(\theta)\left(\frac{\theta}{b}\right)^{\frac{1}{l-1}} F^{\frac{(n-1)l}{l-1}}(\theta)\left[1+n\frac{l}{l-1}\ln F(\theta)\right]f(\theta)\,d\theta$$

This derivative can be written as:

$$\int_{\theta^*}^1 K(\theta)\frac{1}{F^{\frac{1}{l-1}}(\theta)}\left(\frac{\theta}{b}\right)^{\frac{1}{l-1}} F^{\frac{(n-1)l}{l-1}}(\theta)\left[1+n\frac{l}{l-1}\ln F(\theta)\right]F^{\frac{1}{l-1}}(\theta)\,f(\theta)\,d\theta$$

Because $1+n\frac{l}{l-1}\ln F(\theta)$ changes sign only once as a function of θ, from negative to positive, and by the concavity of F and monotonicity of K, the function $K(\theta)\left(\frac{\theta}{F(\theta)}\right)^{\frac{1}{l-1}}$ is positive and increasing. Thus, it is sufficient to show that[24]

$$\int_{\theta^*}^1 F^{\frac{(n-1)l}{l-1}}(\theta)\left[1+n\frac{l}{l-1}\ln F(\theta)\right]F^{\frac{1}{l-1}}(\theta)\,f(\theta)\,d\theta \geq 0.$$

Define a distribution $S(\theta)=F^{\frac{l}{l-1}}(\theta)$ and let $s(\theta)=\frac{l}{l-1}F^{\frac{1}{l-1}}(\theta)$ be its density. Using the new notation we have

$$\int_{\theta^*}^1 S^{n-1}(\theta)\left[1+n\ln S(\theta)\right]s(\theta)\,d\theta$$

$$= \int_{\theta^*}^1 S^{n-1}(\theta)s(\theta)\,d\theta + \int_{\theta^*}^1 S^{n-1}(\theta)n\ln S(\theta)s(\theta)\,d\theta =$$

$$= \int_{\theta^*}^1 S^{n-1}(\theta)s(\theta)\,d\theta + S^n(\theta)\ln S(\theta)\,|_{\theta=\theta^*}^1 - \int_{\theta^*}^1 S^n(\theta)\frac{s(\theta)}{S(\theta)}\,d\theta$$

$$= -S^n(\theta^*)\ln S(\theta^*) > 0. \quad \Box$$

Proof of Lemma 4. Assume, by contradiction, that the statement is not correct. Hence there exists an interval $[t',t'']$ such that the extreme point p is continuous and strictly in-

---

[24] Assume $k(\cdot)$ changes sign once at $x^* \in [a,b]$ from negative to positive and assume that $d(\cdot)$ is positive increasing on the interval $[a,b]$. Then $\int_a^b k(t)dt \geq 0$ implies

$$\int_a^b k(t)d(t)dt = \int_a^x k(t)d(t)dt + \int_x^b k(t)d(t)dt \geq d(x)\int_a^x k(t)dt + d(x)\int_x^b k(t)dt = d(x)\int_a^b k(t)dt \geq 0$$

creasing on this interval, but $p(t) \neq \phi_{m,n}(t)$.

We shall exhibit a function $u(t) \neq 0$ such that both functions $p + u$ and $p - u$ are non-decreasing and satisfy

$$p + u \prec \phi_{m,n} \quad \text{and} \quad p - u \prec \phi_{m,n}$$

contradicting the hypothesis that p is extreme.

By possibly choosing a subset of $[t', t'']$, we can assume without loss of generality that either $p(t) < \phi_{m,n}(t)$ for $t \in [t', t'']$ or $p(t) > \phi_{m,n}(t)$ for $t \in [t', t'']$, and we focus here on the first possibility (the latter is completely analogous).

Choose t_1, t_2, t_3 such that $t' < t_1 < t_2 < t_3 < t''$. Denote by $\delta_1 = \min_{t \in [t_1, t_2]} p'(t)$ (if the derivative does not exist we take the minimum between the left and right derivatives that always exist since p is monotonic), and similarly denote $\delta_2 = \min_{t \in [t_2, t_3]} p'(t)$. By assumptions, $\delta_1 > 0$ and $\delta_2 > 0$. We define u as follows:

$$
u(t) = \begin{cases}
\delta_1 (t - t_1) & \text{if } t \in [t_1, t_1 + \epsilon_1] \\
\delta_1 \epsilon_1 - \delta_1 (t - t_1 - \epsilon_1) & \text{if } t \in [t_1 + \epsilon_1, t_1 + 2\epsilon_1] \\
-\delta_2 (t - t_3 + 2\epsilon_2) & \text{if } t \in [t_3 - 2\epsilon_2, t_3 - \epsilon_2] \\
-\delta_2 \epsilon_2 + \delta_2 (t - t_3 + \epsilon_2) & \text{if } t \in [t_3 - \epsilon_2, t_3] \\
0 & \text{otherwise}
\end{cases}
$$

where ϵ_1 and ϵ_2 are chosen such that

$$
\begin{aligned}
(1)\ \epsilon_1 &\leq t_2 - t_1, \epsilon_2 \leq t_3 - t_2, \\
(2)\ \delta_1 \epsilon_1 - \delta_2 \epsilon_2 &= 0 \\
(3)\ p(t) - \delta_2 \epsilon_2 &< \phi_{m,n}(t)\ ,\ t \in [t_2, t_3].
\end{aligned}
$$

The second condition guarantees that $\int u(t)\, dt = \delta_1 \epsilon_1 - \delta_2 \epsilon_2 = 0$. Notice that, by construction, both $p + u$ and $p - u$ are monotone. Moreover by construction $p + u \prec \phi_{m,n}(t)$, while condition (3) guarantees that $p - u \prec \phi_{m,n}(t)$. $\qquad\square$
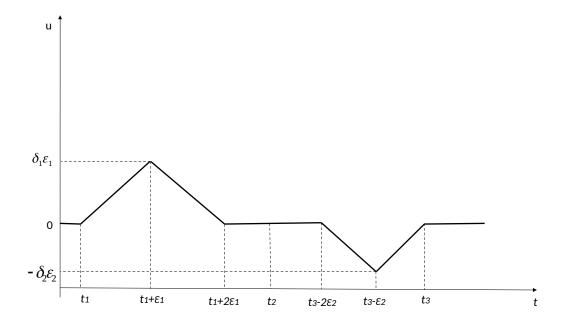
Figure 3: function u

10.1 Sufficient conditions for applying the FL Theorem

When does our "virtual utility" function $H((p, \theta)$

$$H(p, \theta) = h(p, \theta) - h_\theta(p, \theta) \times \frac{1 - F(\theta)}{f(\theta)}$$

satisfies the conditions in the Fan-Lorentz Theorem, i.e. when is the environment convex super-modular ? We have

$$\frac{\partial^2 H}{(\partial p)^2} = \frac{\partial^2 h}{(\partial p)^2} - \frac{\partial^3 h}{\partial \theta (\partial p)^2} \left(\frac{1 - F(\theta)}{f(\theta)} \right)$$

Since $\frac{\partial^2 h}{(\partial p)^2} \geq 0$ by assumption and because $\frac{1-F(\theta)}{f(\theta)} \geq 0$, a sufficient condition for $\frac{\partial^2 H}{(\partial p)^2} \geq 0$ is $\frac{\partial^3 h}{\partial \theta (\partial p)^2} \leq 0$.

We also have

$$\frac{\partial^2 H}{\partial \theta \partial p} = \frac{\partial^2 h}{\partial \theta \partial p} - \frac{\partial^2 h}{\partial \theta \partial p}\frac{d}{d\theta}(\frac{1-F(\theta)}{f(\theta)}) - \frac{\partial^3 h}{(\partial \theta)^2 \partial p}(\frac{1-F(\theta)}{f(\theta)}) =$$
$$\frac{\partial^2 h}{\partial \theta \partial p}(1 - \frac{d}{d\theta}(\frac{1-F(\theta)}{f(\theta)})) - \frac{\partial^3 h}{(\partial \theta)^2 \partial p}(\frac{1-F(\theta)}{f(\theta)})$$

A sufficient conditions for $\frac{\partial^2 H}{\partial \theta \partial p} \geq 0$ is an increasing hazard rate, i.e. $\frac{d}{d\theta}(\frac{f(\theta)}{1-F(\theta)}) \geq 0$ and $\frac{\partial^3 h}{(\partial \theta)^2 \partial p} \leq 0$. To conclude sufficient conditions are :

$$1. \quad \frac{\partial^3 h}{\partial \theta (\partial p)^2} \leq 0, \frac{\partial^3 h}{(\partial \theta)^2 \partial p} \leq 0$$
$$2. \ IFR$$

These conditions can be further decomposed into standard sufficient conditions on the functions v and c appearing in the micro-foundation.

The above derivations can be compared to the classical treatment, e.g., see Fudenberg and Tirole [1991] (Section 7.3.2). They only consider a single agent and hence need not consider the resource constraint that is our main concern here. Their sufficient conditions are:

$$FT1. \quad \frac{\partial^3 h}{\partial \theta (\partial p)^2} \geq 0, \frac{\partial^3 h}{(\partial \theta)^2 \partial p} \leq 0$$
$$FT2. \ IFR$$

The difference to our conditions is due to their assumption that the valuation is **concave** in the allocation. Thus, they initially assume $\frac{\partial^3 h}{(\partial p)^2} \leq 0$, and having $\frac{\partial^3 h}{\partial \theta (\partial p)^2} \geq 0$ ensures that their welfare function is concave so that the First Order Approach can be employed.

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