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Auctions vs. Negotiations: Optimal Selling Mechanism with Endogenous Bidder Values

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Abstract

This paper studies the design of the revenue maximizing selling mechanism in a scenario where bidders can make costly investments upfront to enhance their valuations. Unlike the case where bidders' values are exogenously fixed, here it may be profitable for the seller to discriminate among ex ante symmetric bidders. I first identify a sufficient and almost necessary condition under which symmetric auctions are optimal. When this condition fails, the optimal selling mechanism may be discriminatory. I further find that the optimal mechanism in general follows a structure which I call a threshold mechanism. Two extreme examples of the threshold mechanism are symmetric auctions and sequential negotiations. In general, any threshold mechanism can be implemented by a dynamic selling scheme which alternately utilizes auctions and negotiations.

Keywords: *Mechanism Design; R&D Investment; Endogenous Bidder Values; Favoritism*

JEL Codes: *D44, D82*

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1 Introduction

In classic studies on selling mechanisms, such as Myerson (1981) and Maskin and Riley (1984a), it is often assumed that all potential buyers' valuations for the object being sold are exogenously fixed. In this case, a revenue maximizing seller only needs to design a screening mechanism to elicit the buyers' private information. Under standard assumptions (e.g., symmetric bidders with private and independent values, risk neutrality, and regular distributions of buyer values), a symmetric auction will perform the best in extracting bidders' information rent and is thus optimal for the seller. In reality, however, there are many circumstances where bidders' values are at least partially determined by their investments. For instance, before competing in the market for a license for a wireless spectrum or a license to utilize a certain patent, firms need to invest in R&D to develop technologies that can make use of such licenses. Similarly, before bidding for procurement contracts, potential suppliers can invest in cost-reducing technologies to increase the profitability of such contracts. In all of these situations, the bidders' valuations are endogenously formed and thus may be affected by the allocation mechanism chosen by the seller.

In this paper, I study the design of the revenue maximizing selling mechanism for scenarios where bidders can make costly upfront investments to enhance their valuations. One interesting observation is that here, unlike the case where bidders' values are exogenously fixed, the seller (she) may find it profitable to discriminate among ex ante symmetric bidders (he). The intuition for this seemingly puzzling result is as follows: A non-discriminatory auction is the most successful mechanism for eliciting expected payments from bidders with given values and thus is ex post optimal for the seller. On the other hand, an auction is a very competitive selection process. Even bidders with relatively high values do not know for certain that they are going to win. It is not difficult to imagine that firms will be reluctant to invest too much in a technology they may never have a chance to utilize. What can the seller do to mitigate such concerns? She cannot promise to award the object to both bidders, as the supply is limited. Instead, she can arbitrarily choose one bidder and make him the *favored bidder*. By giving the favored bidder an unfair "head-start," or any other advantage, the seller can increase the favored bidder's investment incentives and shift the ex post distribution of his value upwards. If the investment technology is sufficiently efficient, it may be worthwhile for the seller to sacrifice some ex post information rent in exchange for a higher ex ante investment level. This observation naturally poses two questions, which will be addressed in this paper. First, when does the seller find it optimal to (to not) discriminate? Second, which selling format maximizes the seller's revenue when symmetric auctions fail to achieve such an outcome?

To answer the above questions, I consider a one-object two-bidder setting. The two bidders may have different initial valuations of the object, which are referred to as their *types*. Although each bidder's actual type is his private information, it is common knowledge that the types are drawn i.i.d. from the same commonly known distribution F.[1] In addition, both bidders can

[1] The methodology proposed in this paper can also be used to solve for the optimal selling mechanism with

make costly and irreversible investments to increase their valuations of the object before the selling mechanism is implemented.[2]

I demonstrate that the optimal selling mechanism can be characterized as a function of the bidders' type distribution. I first identify a sufficient and almost necessary condition, which I refer to as the *symmetry condition*, under which *symmetric auctions* are optimal. When this condition fails, the optimal mechanism may be discriminatory. I further find that, in general, the optimal mechanism follows a threshold structure and is thus referred to as a *threshold mechanism.* There are one or multiple *threshold point* (including two reserve types, one for each bidder), which I show how to compute in Section 4, that divide the type space into several intervals. If both bidders' types fell below their corresponding reserve types, the object is not assigned. If only one bidder's type is above the corresponding reserve type, this bidder is guaranteed to receive the object. If both bidders' types are above the corresponding reserve types and the two bidders' types fell into a same interval on which the symmetry condition fails, then a randomly selected but predetermined favored bidder always wins the object. Otherwise, the object goes to the bidder with the higher type. Symmetric auctions and *sequential negotiations* are two extreme examples of the threshold mechanism. By "sequential negotiations," I refer to the selling scheme in which the seller first approaches the favored bidder with a fixed price and, if she is turned down by the favored bidder, she turns to the other bidder with a possibly different fixed price. If the seller's offers are turned down by both bidders, she keeps the object. In general, there are several ways to implement a threshold mechanism. For example, any threshold mechanism can be implemented by a dynamic selling scheme which alternately utilizes auctions and negotiations. Moreover, any threshold mechanism can also be implemented by multi-round auctions with bidder-specific descending reserve prices. I will illustrate this point further in Section 4.

This paper contributes to the growing literature on mechanism design with investment options. A large portion of this literature (e.g., McAfee and McMillan, 1987; King, Welling, and McAfee, 2009; Hatfield, Kojima, and Kominers, 2015; Tomoeda, 2017) focuses on finding efficient mechanisms in various settings and thus are not closely related to the current article. There are also some papers considering the designer's revenue maximization problem. Most of these papers only consider symmetric mechanisms. For instance, Tan (1992) studies a procurement model where potential suppliers first invest in R&D and then compete for the procurement contract. He shows that the first-price and the second-price auctions are revenue equivalent when the investment technology exhibits non-increasing returns to scale. Gong, Li and McAfee (2012) also consider a procurement model with ex ante investment options. They show that, when the procurer is obligated to use a non-discriminatory generalized second-price auction to allocate the procurement contract, the procurer may find it profitable to split the contract among multiple suppliers to increase the suppliers' investment incentives. Gershkov, Moldovanu, and Strack (2018) investigate the seller-optimal allocation of multiple units of identical goods, when potential

asymmetric bidders. Details are available upon request.

[2]See Section 5.1 for a discussion of the timing of the investment decisions.

buyers can take costly actions to affect their values before entering the market. They find among all symmetric mechanisms, auctions are always optimal for the seller. In the current article, I focus on studying the use of discriminatory allocation rules.

The two papers most closely related to mine are Bag (1997) and Celik and Yilankaya (2009). Bag considers a setting where there are two bidders competing for a procurement contract. Each bidder can choose whether to invest in a stochastic R&D process to reduce its production cost. In his paper, Bag assumes that the two bidders are ex ante identical and that the investment decisions are binary. Furthermore, he assumes it is always optimal for the procurer to induce one and only one bidder to invest. Given these assumptions, he finds that a one-shot second-price auction with discriminatory entry fees is always revenue maximizing for the procurer. Selik and Yilankaya investigate an optimal auction design problem where potential buyers know their values but need to pay a fixed cost to participate.[3] Similar to Bag (1997), they also show that the revenue maximizing mechanism can always be implemented by a one-shot auction with bidder-specific entry fees. Moreover, they identify a condition under which the optimal entry fees are asymmetric for symmetric bidders. In this paper, I allow the bidders to have different initial valuations and to make continuous investment decisions. I also incorporate the possibility that the designer may want to induce both bidders to invest. As a result, I find there may exist some selling mechanism which strictly dominates any static auctions and provide a complete characterization of the optimal mechanism.

Relatedly, Obara (2008) examines the existence of a mechanism which extracts full surplus from the agents who can make investments upfront and shows that such a mechanism may not exist. Cisternas and Figueroa (2015) and Rosar and Muller (2015) consider a repeated procurement problem where the buyer can choose a different supplier in every period. However, only the incumbent supplier has the opportunity to invest in the cost reducing technology to reduce future production cost. Lu and Ye (2018) study a dynamic selling mechanism where buyer participations are costly and can occur over multiple periods. The settings considered in these papers are very different from the setting in the current paper. Moreover, with the exception of Lu and Ye (2018), none of the above papers provides a characterization of the optimal mechanism.

Finally, Seel and Wasser (2014) consider a two-player all-pay auction in which the designer maximizes a convex combination of the expected total and highest bid. Chawla, Hartline and Sivan (2015) study the design of optimal crowdsourcing contests. In crowdsourcing contests, entrants must exert costly effort upfront to enter. Each entrant's privately known ability and effort level jointly determine the quality of his/her submission. The designer's objective is to maximize his/her expected payoff, which equals the expected value of the best submission minus the expected payment. Perez-Castrillo and Wettstein (2016) investigate the design of an innovation contest, where the quality of the innovation achieved by an agent depends on his/her ability as well as the effort he/she devoted to the task. Although the problems of interest are

[3]As demonstrated by Gershkov, Moldovanu, and Strack (2018), costly participation can be considered as a special case of costly investment.

different, these papers also demonstrate that a discriminatory mechanism may outperform all non-discriminatory ones, even when all potential participants are symmetric. Therefore, they are related to my paper in a broad sense.

The rest of the paper proceeds as follows. Section 2 presents the model. The main results are summarized in Section 3 and Section 4. I discuss some of the assumptions and extensions of the model in section 5. Section 6 concludes.

2 The Model

I study the problem faced by a seller (she) who has a single non-divisible object to sell to one of two potential bidders (both he) - bidder 1 and bidder 2, respectively. The seller and both of the bidders are risk neutral. The seller bears an opportunity cost which is normalized to 0 and is known to all participants. Each bidder i, $i = 1, 2$, has a privately known type θ_i, which describes the bidder's initial valuation of the object, or equivalently, his R&D proficiency level. The bidders' types are drawn i.i.d. from a common distribution described by a strictly positive and continuous density function f and a continuous cumulative density function F, with support $\Theta = [\underline{\theta}, \bar{\theta}]$ $(0 < \underline{\theta} < \bar{\theta} < \infty)$ or $\Theta = \mathbb{R}_+$.[4] I assume the type distribution is *regular*, in the sense that the corresponding *virtual value function* $\mathcal{J}(\theta_i) \equiv \theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}$ is increasing. In addition, each bidder also selects an investment level $a_i \in [0, \infty)$ at cost $C(a_i) = \frac{a_i^2}{2K}$, where $K > 0$ scales the marginal cost of investment.[5] Note that a larger value of K represents a higher level of investment efficiency for both bidders. Bidder i's final valuation of the object, denoted v_i, is determined by the production function $v_i = \theta_i + a_i$. θ_i, a_i and v_i are all bidder i's private information and thus not contractible. To obtain neater results, I assume that there does not exist any non-degenerate interval over which $\mathcal{J} - KF$ is constant. As will become clear in Section 4, such a restriction serves only to simplify the results without causing any loss in generality. Finally, it is common knowledge that each bidder has an outside option of 0.

The game consists of three periods, $t = 0, 1, 2$, and there is no discounting. In period 0, Nature draws types θ_1 and θ_2 independently. The seller announces the mechanism she will use to allocate the object. In period 1, each bidder learns his private type, observes the mechanism and makes his irreversible investment decision. In period 2, each bidder sends a message to the mechanism. The mechanism announced in period 0 is implemented depending on the messages received.

To simplify analysis, I invoke the revelation principle and restrict attention to *direct mechanisms,* for which each bidder confidentially reports his type to the mechanism and subsequently, the mechanism chooses the outcomes based on the reported types. Formally, in a direct mechanism $\omega = (q, t, a)$, the seller first *recommends* an investment plan $a \equiv (a_1, a_2)$, where $a_i : \Theta \to \mathbb{R}_+$

[4]Throughout the paper, I write the proof for $\Theta = \mathbb{R}_+$, as the proof for $\Theta = [\underline{\theta}, \bar{\theta}]$ is essentially the same.

[5]I refer interested readers to supplementary materials of this paper for a discussion on more general cost functions.

maps each bidder's type to an investment level, for the two bidders to implement at $t = 1$. The bidders will take the seller's recommendation if and only if such plans are incentive compatible for them. Note that the recommendation can only depend on each bidder's own type, as investment decisions are made independently. The seller also announces an *allocation rule* $q \equiv (q_1, q_2)$, where $q : \Theta \times \Theta \to [0,1] \times [0,1]$ maps any pair of reported types $(\theta_1, \theta_2) \in \Theta \times \Theta$ to probabilities of winning the object; a *payment rule* $t \equiv (t_1, t_2)$, where $t : \Theta \times \Theta \to \mathbb{R} \times \mathbb{R}$ maps any pair of reported types to the transfers the two bidders need to make. For later use, I define $Q_i(\theta_i) \equiv \int_\Theta q_i(\theta_i, \theta_{-i}) dF(\theta_{-i})$ to be bidder i's expected winning probability, which is also known as the *reduced form allocation rule*; $T_i(\theta_i) \equiv \int_\Theta t_i(\theta_i, \theta_{-i}) dF(\theta_{-i})$ to be the *expected payment rule* for bidder i of type θ_i; and $\Omega = ((Q_1, Q_2), (T_1, T_2), (a_1, a_2))$ is called a *reduced form mechanism.*

3 Implementable Mechanisms

Given any mechanism $\omega = (q, t, a)$, bidder i of type θ_i chooses his reported type $\hat{\theta}_i$ and investment level $\hat{a}_i$ to maximize[6]

$$\begin{aligned} \pi_i(\theta_i, \hat{\theta}_i, \hat{a}_i) &= (\theta_i + \hat{a}_i) \int_\Theta q_i(\hat{\theta}_i, \theta_{-i}) dF(\theta_{-i}) - \int_\Theta t_i(\hat{\theta}_i, \theta_{-i}) dF(\theta_{-i}) - \frac{\hat{a}_i^2}{2K} \\ &= (\theta_i + \hat{a}_i) Q_i(\hat{\theta}_i) - T_i(\hat{\theta}_i) - \frac{\hat{a}_i^2}{2K}. \end{aligned}$$

The above formula shows that, fixing any mechanism $\omega = (q, t, a)$, the buyers are only interested in the reduced form version of ω, denoted by $\Omega = (Q, T, a)$. Similarly, the seller is only concerned with the expected payment T, since she is risk neutral. Hence, I can focus solely on reduced form mechanisms without losing any generality.

Given any prior distribution F, the seller's problem at $t = 0$ is to select an *implementable* reduced form mechanism Ω to maximize her expected profit. The set of implementable reduced form mechanisms is defined by the following conditions.

First, as a_i is non-contractible, any recommended investment plan must be incentive compatible for the bidder, assuming he plans to report truthfully at $t = 2$. That is, for $i = 1, 2$ and any $\theta_i \in \Theta$,

$$a_i(\theta_i) \in argmax_{\hat{a}_i \in \mathbb{R}_+} (\theta_i + \hat{a}_i) Q_i(\theta_i) - T_i(\theta_i) - \frac{\hat{a}_i^2}{2K}. \tag{1}$$

The first order condition directly gives the bidder's investment level as a function of θ_i and $Q_i(\cdot)$. That is,

$$a_i^*(\theta_i, Q_i) = K Q_i(\theta_i).$$

Moreover, the second order condition $-\frac{1}{K} < 0$ holds true for all $a_i \in \mathbb{R}_+$. Therefore, given any (Q, T), the only incentive compatible recommended effort level is $a_i(\theta_i) = a_i^*(\theta_i, Q_i) = K Q_i(\theta_i)$.

[6] If $i = 1$, then $-i = 2$; if $i = 2$, then $-i = 1$.

Similarly, one can verify that if bidder i of type θ_i plans to report to be type $\hat{\theta}_i$, he will choose investment level

$$a_i^*(\theta_i, \hat{\theta}_i, Q_i) = KQ_i(\hat{\theta}_i)$$

to maximize his expected utility.

Second, each bidder must be given the incentive to report his private information truthfully. That is, for $i = 1,2$ and for any $\theta_i \in \Theta$,

$$\theta_i \in argmax_{\hat{\theta}_i \in \Theta} (\theta_i + a_i(\theta_i, \hat{\theta}_i, Q_i))Q_i(\hat{\theta}_i) - T_i(\hat{\theta}_i) - \frac{a_i^2(\theta_i, \hat{\theta}_i, Q_i)}{2K} . \qquad (2)$$

Third, the reduced form allocation rule Q must be *feasible*, i.e., there exists an ex post allocation rule $q : \Theta \times \Theta \to [0,1] \times [0,1]$ such that

$$\sum_{i=1,2} q_i(\theta_1, \theta_2) \leq 1$$

and

$$Q_i(\theta_i) = \int_{\Theta} q_i(\theta_i, \theta_{-i}) dF(\theta_{-i})$$

for all $(\theta_1, \theta_2) \in \Theta \times \Theta$. I will discuss this constraint in more detail in the next section.

Finally, it is assumed that the seller cannot force either bidder to participate. Thus, to guarantee that the bidders remain in the mechanism whenever desired, the following individual rationality condition must be satisfied.

$$(\theta_i + a_i(\theta_i))Q_i(\theta_i) - T_i(\theta_i) - \frac{a_i^2(\theta_i)}{2K} \geq 0 \text{ for i=1,2 and for all } \theta_i \in \Theta. \qquad (3)$$

The following proposition identifies sufficient and necessary conditions for a mechanism to be implementable. Fix any (Q, T, a). Let $\theta_i^* = \inf\{\theta_i \mid Q_i(\theta_i) > 0\}$ be the corresponding *reserve type* for buyer i. Let

$$\mathcal{U}_i(\theta_i) = (\theta_i + a_i(\theta_i))Q_i(\theta_i) - T_i(\theta_i) - \frac{a_i^2(\theta_i)}{2K}$$

denote buyer i's utility in state θ_i, assuming $\Omega = (Q, T, a)$ is implemented.

Proposition 1. *A reduced form mechanism (Q, T, a) is implementable if and only if, for $i = 1,2$,*

(a) Q is feasible.

(b) $Q_i(\theta_i)$ is increasing in θ_i.

(c) $\mathcal{U}_i(\theta_i) = \mathcal{U}_i(\theta_i^*) + \int_{\theta_i^*}^{\theta_i} Q_i(\tilde{\theta}_i) d\tilde{\theta}_i$ for any $\theta_i \geq \theta_i^*$.

(d) $\mathcal{U}_i(\theta_i^*) \geq 0$.

(e) $a_i(\theta_i) = KQ_i(\theta_i)$

Somewhat surprisingly, except for the additional constraint on the investment plan (condition (e)), the implementability conditions are the same as those for the standard mechanism design problem (e.g. see Mas-Colell, Whinston, and Green, 1995, p. 888) and can be derived in a similar way. The reason is that, in the current setting, since the investment technology is deterministic, bidders do not earn any moral hazard rent. Thus, all additional revenue generated by the bidders' investment can be extracted by the seller and the bidders will only earn the same information rent as that in the no-investment model. Due to this set-up, I can focus on studying the trade-off between the seller's need to extract private information from the bidders and her need to encourage investment, which is of primary interest in this paper.

Since my aim is to select the optimal mechanism for the seller, henceforth I only consider mechanisms for which $\mathcal{U}_i(\theta_i^*) = 0$, $i = 1, 2$, as this is necessary for the mechanism to be optimal. Note that fixing any implementable mechanism $\Omega = (Q, T, a)$, a is uniquely determined by Q. Furthermore, the expected payment rule T is given by

$$\begin{aligned} T_i(\theta_i) &= (\theta_i + a_i(\theta_i))Q_i(\theta_i) - \frac{a_i^2(\theta_i)}{2K} - \mathcal{U}_i(\theta_i) \\ &= \left(\theta_i Q_i(\theta_i) - \int_{\theta_i^*}^{\theta_i} Q_i(\tilde{\theta}_i) d\tilde{\theta}_i\right) + \frac{K}{2} Q_i^2(\theta_i) \end{aligned}$$

and is also uniquely determined by Q. Thus, the seller's task is to choose an implementable mechanism (Q, T, a), or equivalently, to choose a feasible and increasing Q, to maximize

$$\begin{aligned} R(Q) &= \sum_{i=1,2} \int_\Theta T_i(\theta_i) dF(\theta_i) \\ &= \sum_{i=1,2} \int_\Theta \left[\left(\theta_i Q_i(\theta_i) - \int_{\theta_i^*}^{\theta_i} Q_i(\tilde{\theta}_i) d\tilde{\theta}_i\right) + \frac{K}{2} Q_i^2(\theta_i)\right] dF(\theta_i) \\ &= \sum_{i=1,2} \int_\Theta \theta_i Q_i(\theta_i) dF(\theta_i) + [1 - F(\theta_i)] \int_{\theta_i^*}^{\theta_i} Q_i(\tilde{\theta}_i) d\tilde{\theta}_i \,|_{\theta_i^*}^{\infty} \\ &\quad - \sum_{i=1,2} \int_\Theta Q_i(\theta_i) [1 - F(\theta_i)] d\theta_i + \frac{K}{2} \sum_{i=1,2} \int_\Theta Q_i^2(\theta_i) dF(\theta_i) \\ &= \sum_{i=1,2} \int_\Theta \mathcal{J}(\theta_i) Q_i(\theta_i) dF(\theta_i) + \frac{K}{2} \sum_{i=1,2} \int_\Theta Q_i^2(\theta_i) dF(\theta_i). \end{aligned}$$

In moving from the second to the third line, I used integration by parts; and in moving from the third to the last line, I used $1 - F(\theta) \to 0$ as $\theta \to \infty$ and $\int_{\theta_i^*}^{\theta_i^*} Q_i(\tilde{\theta}_i) d\tilde{\theta}_i = 0$. It is important to note that the first term of the above formula, $\sum_{i=1,2} \int_\Theta \mathcal{J}(\theta_i) Q_i(\theta_i) dF(\theta_i)$, measures how much the seller earns from screening (i.e. private information extraction), and that its maximum value can be obtained by utilizing a mechanism which always assigns the object to the bidder with the higher value, provided that such a bidder's value exceeds the reserve level (e.g. symmetric auctions with positive reserve prices). On the other hand, the second term, $\frac{K}{2} \sum_{i=1,2} \int_\Theta Q_i^2(\theta_i) dF(\theta_i)$, measures the seller's expected return from induced investment, and its maximum can be achieved using a mechanism for which the seller always assigns the object to the same bidder and charges a fixed price. Thus, the optimal choice of Q must balance the two concerns.

For illustrative convenience, I shall heretofore refer to feasible and increasing Q as *implementable*. Further, I define the term *generalized virtual value* as follows.

Definition 1. *Fixing any Q, the corresponding generalized virtual value $H_i : \Theta \times [0,1] \to \mathbb{R}$ is given by*

$$\mathcal{H}_i(\theta_i, Q_i(\theta_i)) = \mathcal{J}(\theta_i) + \frac{K}{2} Q_i(\theta_i).$$

Thus, the seller's revenue can be rewritten as

$$R(Q) \quad = \quad \sum_{i=1,2} \int_{\Theta} \mathcal{H}_i(\theta_i, Q_i(\theta_i)) Q_i(\theta_i) dF(\theta_i).$$

Henceforth, I will write $H_i(\theta_i, Q_i) = H_i(\theta_i, Q_i(\theta_i))$. Note that fixing any implementable Q, the corresponding generalized virtual value function must be strictly increasing in θ_i. Moreover, fixing any θ_i, $\mathcal{H}_i(\theta_i, Q_i)$ is strictly increasing in $Q_i(\theta_i)$ for any $\theta_i > \theta_i^*$.

4 Optimal Mechanisms

In the rest of this paper, I consider Q that are piecewise continuous and characterize optimal mechanisms up to sets of measure 0.

I first show that, similar to the findings of Myerson (1981), it is never optimal for the seller to assign the object to any bidder with a strictly negative generalized virtual value.

Lemma 1. *Fix any optimal Q. $\forall \theta_i \in \Theta$, if $\mathcal{H}_i(\theta_i, Q) < 0$, then $Q_i^*(\theta_i) = 0$.*

Proof. The proof is straightforward. Fix any implementable Q for which there exists some $i \in \{1,2\}$ and $\tilde{\theta}_i$ such that $\mathcal{H}_i(\tilde{\theta}_i, Q) < 0$ but $Q_i(\tilde{\theta}_i) > 0$. As is argued at the end of Section 3, $\mathcal{H}_i$ is increasing in θ_i, which implies that $\mathcal{H}_i(\theta_i, Q) < 0$ for any $\theta_i < \tilde{\theta}_i$. Thus, one can construct another allocation rule $\tilde{Q}$, for which $\tilde{Q}_i(\theta_i) = 0$ for all $\theta_i \leq \tilde{\theta}_i$ and $\tilde{Q}_i(\theta_i) = Q_i(\theta_i)$ otherwise; $\tilde{Q}_{-i}(\theta_{-i}) = Q_{-i}(\theta_{-i})$ for all $\theta_{-i} \in \Theta$. $\tilde{Q}$ is clearly implementable. By using $\tilde{Q}$ instead of Q, the change in the seller's expected revenue is

$$\triangle R \quad = \quad -\sum_{i=1,2} \int_0^{\tilde{\theta}} \mathcal{H}_i(\theta_i, Q_i) Q_i(\theta_i) dF(\theta_i) > 0.$$

Thus Q cannot be optimal. $\square$

On the other hand, the next lemma claims that, if the seller finds it optimal to assign the object to bidder i of type θ_i with a strictly positive probability, then the bidder should obtain the object at least when the other bidder's type fells below the corresponding reserve type.

Lemma 2. *Fix any optimal Q and let $\theta_i^* = \inf\{\theta_i \mid Q_i(\theta_i) > 0\}$ be the reserve type for bidder i, $i = 1,2$. Then $Q_i(\theta_i) \geq F(\theta_{-i}^*)$ for any $\theta_i > \theta_i^*$.*

Given the above lemma and that I characterize optimal mechanisms up to sets of measure 0, there is no loss of generality in only considering mechanisms for which $Q_i(\theta_i^*) = F(\theta_{-i}^*)$ for $i = 1,2$.[7]

Before proceeding to the next step, I would like to briefly illustrate the main technical difficulty in solving for the optimal mechanism. In the classical mechanism design problem (e.g., Myerson (1981) and Maskin and Riley (1984a)), the seller's objective function is often linear in the reduced form allocation rule (Q) and thus can be rewritten in terms of ex post winning probabilities (q). The new objective function can then be maximized pointwise for each type profile subject to the corresponding feasibility constraint, which is also characterized in terms of the ex post winning probability. The objective function in the current paper, however, is strictly convex in Q. Therefore, the classical approach fails and a characterization of the reduced form allocation rule is needed.

The *reduced form feasibility constraint*, first raised by Maskin and Riley (1984b) and Matthews (1983), requires that the probabilities that any subset of bidders will win the object are never higher than the probabilities that such bidders exist. Formally, a sufficient and necessary condition for any Q being feasible is that, for any $A_1, A_2 \subset \Theta$,

$$\int_{A_1} Q_1(\theta_1)dF(\theta_1) + \int_{A_2} Q_2(\theta_2)dF(\theta_2) \quad \leq \quad 1 - \int_{\theta_1 \notin A_1} dF(\theta_1) \int_{\theta_2 \notin A_2} dF(\theta_2) \tag{4}$$

(see Maskin and Riley (1984b), Matthews (1983), Border (2007) and Mierendorff (2011) for relevant proofs). The main difficulty in dealing with the reduced form feasibility constraint, relative to working with the constraints on q, is that such a condition needs to be satisfied for any subsets of the type space; furthermore, it is not clear a priori which constraints are binding.

In what follows, I first show that, to achieve the optimal outcome in the current setting, the feasibility constraint must always bind "at the top." This observation in turn allows me to narrowing the focus to a special class of Q which can be summarized by a cutoff function ϕ, as described in the following proposition.

Proposition 2. *Fix any optimal Q and let $\theta_i^* = \inf\{\theta_i \mid Q_i(\theta_i) > 0\}$ be the reserve type for bidder i. There exist an increasing function $\phi : [\theta_1^*, \infty) \to [\theta_2^*, \infty)$ and $\phi^{-1}(\theta_2) \equiv \inf\{t \mid \phi(t) = \theta_2\}$ such that*

$$Q_1(\theta_1) = \begin{cases} F(\phi(\theta_1)) & \theta_1 \geq \theta_1^* \\ 0 & \textit{otherwise} \end{cases}$$

[7] Fix any optimal Q. One can construct another allocation rule $\tilde{Q}$ such that $\tilde{Q}_i(\theta_i) = F(\theta_{-i}^*)$ for $\theta_i = \theta_i^*$ and $\tilde{Q}_i(\theta_i) = Q_i(\theta_i)$ otherwise. By Lemma 2, $\tilde{Q}$ is implementable. Thus $\tilde{Q}$ is clearly also optimal.

and

$$Q_2(\theta_2) = \begin{cases} F(\phi^{-1}(\theta_2)) & \theta_2 \geq \phi(\theta_1^*) \\ 0 & \textit{otherwise.} \end{cases}$$

The proof utilizes the reduced form feasibility constraint discussed above. Since any implementable Q is increasing and F is strictly increasing, there always exists a well-defined and increasing ϕ such that $Q_1(\theta_1) = F(\phi(\theta_1))$ for all $\theta_1 > \theta_1^*$. Further, by Lemma 2 and my previous assumption that $Q_i(\theta_i^*) = F(\theta_{-i}^*)$ for $i = 1, 2$, the corresponding ϕ^{-1} is also well-defined and increasing. The feasibility constraint requires that, for any $t > \theta_1^*$,

$$\int_t^\infty Q_1(\theta_1) dF(\theta_1) + \int_{\phi(t)}^\infty Q_2(\theta_2) dF(\theta_2) \quad \leq \quad 1 - F(t)F(\phi(t))$$

which is equivalent to

$$\int_{\phi(t)}^\infty Q_2(\theta_2) dF(\theta_2) \quad \leq \quad \int_{\phi(t)}^\infty F(\phi^{-1}(\theta_2)) dF(\theta_2).$$

To prove Proposition 2, it suffices to show that the above inequality is binding for any $t \geq \theta_1^*$. In other words, the feasibility constraint is always binding at the top. The detailed proof can be found in the appendix. Intuitively, fixing any Q_1, the seller always prefers to assign the object to the higher type bidder 2 as often as permissible by the feasibility constraint, because such a bidder has both a larger virtual value and a higher marginal return on investment.

I should note that any Q, as described in the above lemma, can be implemented by a deterministic mechanism with a cutoff function ϕ. Specifically, fixing any such Q, there exists a feasible ex post allocation rule q, for which

$$q_1(\theta_1, \theta_2) = \begin{cases} 1 & \text{if } \theta_1 \geq \theta_1^* \text{ and } \phi(\theta_1) \geq \theta_2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$q_2(\theta_1, \theta_2) = \begin{cases} 1 & \text{if } \theta_2 \geq \theta_2^* \text{ and } \phi(\theta_1) < \theta_2 \\ 0 & \text{otherwise} \end{cases}$$

to implement Q. Thus, one direct implication of Proposition 2 is that the seller never finds it optimal to employ lotteries, which is rather intuitive in the current setting. Consistent with Myerson (1981), utilizing lotteries would decrease the expected payment that the seller can extract from screening, taking both bidders' values as given. Moreover, the uncertainty associated with the use of lotteries would reduce the bidders' investment incentives. Another implication of Proposition 2 is that, as a complement to Lemma 1, the seller always assigns the object provided that at least one bidder's type is above the corresponding reserve type. To illustrate, according to

Lemma 1, $\mathcal{H}_i(\theta_i, Q_i) > 0$ for any $\theta_i > \theta_i^*$. Moreover, as is argued at the end of Section 3, $\mathcal{H}_i(\theta_i, Q_i)$ is strictly increasing in $Q_i(\theta_i)$ for any $\theta_i > \theta_i^*$. Thus, fixing any implementable mechanism for which the second implication does not hold, the seller can earn more expected profit by assigning the object more often to bidders whose types are above the corresponding reserve types. As a result, any such mechanism is not optimal.

Proposition 2 allows me to transform the task of finding the optimal pair of (Q_1, Q_2), subject to the constraints that Q must be increasing and feasible, to the task of searching for the optimal cutoff function ϕ which is increasing. The problem thus becomes tractable. However, in the rest of this paper, I will continue to state the results in terms of Q rather than the cutoff function ϕ for the sake of simplicity.

Next, I investigate the seller's optimal discriminatory strategy. I find that only one type of discriminatory treatment may be optimal for the seller. That is, the optimal mechanism may contain *favored bidder intervals*, the definition of which is given below.

Definition 2. *Fix any implementable Q with reserve types θ_1^* and θ_2^*. A non-degenerate interval $[x, y]$ is said to be a favored bidder interval and bidder i is said to be the favored bidder if and only if, for $i = 1, 2$ and any $\theta \in [x, y]$, $Q_i(\theta) = F(y)$ and $Q_{-i}(\theta) = F(x)$ or 0.*

By definition, if both bidders' types fell into a same favored bidder interval, the seller will always assign the object to a randomly selected but predetermined bidder, who is referred to as the *favored bidder*. Since the two bidders are symmetric, it is without loss of generality to always appoint bidder 1 as the favored bidder. Thus, I will do so henceforth to simplify elaboration.

I demonstrate that any optimal Q contains only regions where the object is not assigned $(Q_i(\theta_i) = 0)$, regions where the object is efficiently assigned $(Q_i(\theta_i) = F(\theta_i))$, and the favored bidder intervals.

Lemma 3. *Fix any optimal Q and let $\theta_i^* = \inf\{\theta_i \mid \mathcal{Q}_i(\theta_i) > 0\}$ be the reserve type for bidder i.*

(I) If there exists some non-degenerate interval $(x, z) \subset (\theta_i^*, \infty)$ on which $Q_i(\theta_i)$ is strictly increasing, then $Q_i(\theta_i) = F(\theta_i)$ $\forall \theta_i \in (x, z)$.

(II) If there exists some non-degenerate interval $[x', z'] \subset [\theta_i^*, \infty)$ on which $Q_i(\theta_i)$ is constant, then there exists a favored bidder interval d such that $[x', z'] \subset d$

The above lemma states that, if the seller wants to assign the object, then she will either assign the object efficiently, or she will choose to discriminate as much as the incentive compatibility constraint (i.e. Q must be increasing) permits. To understand the idea underlying the above results, recall that the seller's objective function consists of two parts. The first part, $\sum_{i=1,2} \int_\Theta \mathcal{J}(\theta_i) Q_i(\theta_i) dF(\theta_i)$, is linear in Q and measures the seller's gain from private information extraction (i.e. screening). The second part, $\frac{K}{2} \sum_{i=1,2} \int_\Theta Q_i^2(\theta_i) dF(\theta_i)$, is strictly convex in Q and measures the seller's gain from induced investment. As is shown by Myerson (1981), the seller can maximize the first part by utilizing a symmetric auction. Let Q^S represent the optimal symmetric auction and let θ^* denote the corresponding reserve type; further, let Q^D represents any allocation rule that favors bidder 1, with reserve types $\theta_1^* = \theta^*$ and θ_2^*. If the seller uses Q^D instead of Q^S, the change in her expected payoff equals the sum of

$$\triangle_1 \equiv \sum_{i=1,2} \int_\Theta \mathcal{J}(\theta_i) Q_i^D(\theta_i) dF(\theta_i) - \sum_{i=1,2} \int_\Theta \mathcal{J}(\theta_i) Q_i^S(\theta_i) dF(\theta_i)$$

and

$$\triangle_2 \equiv \frac{K}{2} \sum_{i=1,2} \int_\Theta [Q_i^D(\theta_i)]^2 dF(\theta_i) - \frac{K}{2} \sum_{i=1,2} \int_\Theta [Q_i^S(\theta_i)]^2 dF(\theta_i).$$

Roughly speaking, $\triangle_1$ measures the cost of discrimination and is always negative. By contrast, $\triangle_2$ measures the benefit of discrimination and is always positive. Moreover, the cost of discrimination is linear in the change of Q, while the benefit of discrimination is convex in the change of Q. Therefore, the seller either finds it optimal to not discriminate, or to discriminate as much as possible

Lastly, I identify the conditions under which the seller finds it optimal to (not) discriminate.

Theorem 1. *Fix any optimal Q. For $i = 1, 2$, let $\theta_i^* = \inf\{\theta_i \mid Q_i(\theta_i) > 0\}$ be the reserve type for bidder i.*
(I)If $\mathcal{J}(\theta) - KF(\theta)$ is strictly decreasing on $(\alpha, \beta) \subset (\theta_1^*, \infty)$, then there exists a favored bidder interval d such that $(\alpha, \beta) \subset d$;
(II) If $\mathcal{J}(\theta) - KF(\theta)$ is strictly increasing on $(\gamma, \delta) \subset (\theta_1^*, \infty)$, then:
(a) There does not exist any favored bidder interval $d' \subset (\gamma, \delta)$.
(b) If there exists some non-degenerate interval $(x, y) \subset (\gamma, \delta)$ on which $Q_i(\theta_i)$ is strictly increasing, then $Q_i(\theta_i) = F(\theta_i)$ $\forall \theta_i \in (x, y)$.

The above theorem has several important implications. First, it states that the seller always finds it optimal to discriminate in the region where $\mathcal{J}(\theta) - KF(\theta)$ is strictly decreasing. To see the intuition, suppose that both bidders' values fell into a same interval (α, β). The increasing rate of $\mathcal{J}(\theta)$ over (α, β) measures how much the seller can gain by always assigning the object to the bidder with the higher value relative to the case where she always gives the object to the same bidder regardless of the other bidder's value.[8] In other words, the increasing rate of the virtual value function represents the value of screening. On the other hand, note that $F(\theta)$ equals to a type θ bidder's winning probability in a symmetric auction, assuming that θ is above the reserve type. If $F(\theta)$ increases quickly, then the bidder's winning probability increases a lot by being favored in a favored bidder interval, relative to the case where he only gets the object if

[8]According to Lemma 3, any other form of allocation rules are never optimal for the seller and thus can be ignored.

he has the higher value. The parameter K measures the efficiency of the investment technology. Thus, the increasing rate of $KF(\theta)$ over (α,β) measures how much the seller would gain, in terms of increased investment, by utilizing a favored bidder interval rather than assigning the object efficiently to bidders with values in (α,β). In other words, the increasing rate of $KF(\theta)$ represents the value of discrimination. Therefore, if both bidders' values fell into the same interval on which $KF(\theta)$ increases faster than $\mathcal{J}(\theta)$, the seller will find that the expected value of discrimination exceeds the expected value of screening. Discriminatory treatment is thus optimal. I would like to note that, in the extreme case where $\mathcal{J}(\theta)-KF(\theta)$ is monotone decreasing, the optimal mechanism is fully discriminatory and can be implemented by *sequential negotiations*. That is, the seller first approaches the favored bidder with a fixed price, and turns to the other bidder with a possibly different fixed price if turned down by the first bidder. The seller keeps the object if her offers are turned down by both bidders. Formally, Q represents sequential negotiations if and only if, there exists $\theta_1^*,\theta_2^* \geq 0$ such that, for some $i \in \{1,2\}$, $Q_i(\theta_i) = 1$ for all $\theta_i \geq \theta_i^*$ and $Q_i(\theta_i) = 0$ otherwise; $Q_{-i}(\theta_{-i}) = F(\theta_i^*)$ for all $\theta_{-i} > \theta_{-i}^*$ and $Q_{-i}(\theta_{-i}) = 0$ otherwise.

Second, Theorem 1 also provides a sufficient and *almost* necessary condition under which the classic selling mechanism, symmetric auctions, are optimal. Note that Q represents a symmetric auction if and only if there exists $\theta^* \geq 0$ such that for $i = 1,2, Q_i(\theta_i) = F(\theta_i)$ for all $\theta_i \geq \theta^*$ and $Q_i(\theta_i) = 0$ otherwise. Let $\theta^*(F)$ denote the solution to $\mathcal{H}_i(\theta_i,F) = \mathcal{J}(\theta_i) + \frac{K}{2}F(\theta_i) = 0$.

Corollary 1. *Fix any optimal Q.*
(I) (Sufficiency) If $\mathcal{J}(\theta)-KF(\theta)$ is strictly increasing on Θ, then Q must represent a symmetric auction with reserve type $\theta^*(F)$.
(II)(Necessity) If Q represents a symmetric auction with reserve type $\theta^* \geq 0$, then $\mathcal{J}(\theta)-KF(\theta)$ is strictly increasing on (θ^*,∞) (or $(\theta^*,\bar{\theta}]$). [9]

The proof directly follows Theorem 1. For convenience, I refer to the condition that $\mathcal{J}(\theta) - KF(\theta)$ is strictly increasing on Θ as the *symmetry condition*. The symmetry condition is sufficient, but only almost necessary for auctions being optimal. The reason being that the seller is indifferent toward all allocation rules that only differ with regard to the region in which the object is not assigned. Clearly, fixing any distribution, the symmetry condition is more likely to fail when K is larger. Thus, discriminatory treatments are more likely to be profitable for the seller when the investment technology is more efficient.

Third, Theorem 1 does not cover instances in which $\mathcal{J}(\theta)-KF(\theta)$ is constant, since this is already ruled out by the assumption made in Section 2. In fact, if any two allocation rules Q and Q' only differ in states where $\mathcal{J}(\theta)-KF(\theta)$ is constant, then the two rules must generate the

[9] Note that the condition that $\mathcal{J}(\theta)-KF(\theta)$ is strictly increasing on $(\theta^*(F),\infty)$ is not sufficient for auctions with reserve type $\theta^*(F)$ being optimal. The reason is that, as the allocation rule varies, the corresponding optimal reserve types may also vary. Suppose, for example, there exists $\tilde{\theta}_i < \theta^*(F)$ such that $\mathcal{J}(\theta)-KF(\theta)$ is decreasing on $(\tilde{\theta}_i,\theta^*(F))$. Then there may exist a discriminatory allocation rule, with reserve types lower than $\theta^*(F)$, which generate more expected profit for the seller.

same expected revenue for the seller. Thus, narrowing the focus to cases in which the difference is never constant does not result in any loss of generality, but rather simplifies exposition.

Fourth, Theorem 1 does not provide a complete characterization of optimal mechanisms for any arbitrary distribution. However, it yields an algorithm to compute optimal Q for essentially any given regular distribution. The algorithm consists of the following two steps.

The first step is to determine the maximum possible number of favored bidder intervals in any optimal Q. For any F, let $\mathcal{I}(F)$ and $\mathcal{D}(F)$ denote the collection of pairwise disjoint intervals on which $\mathcal{J}(\theta) - KF(\theta)$ is strictly increasing and on which $\mathcal{J}(\theta) - KF(\theta)$ is strictly decreasing, respectively. Moreover, fix any Q, let $P(Q)$ denote the collection of favored bidder intervals. By Theorem 1, any element of $\mathcal{D}(F)$ must be contained in a favored bidder interval; and there is no favored bidder interval contained in any element of $\mathcal{I}(F)$. Thus, the number of favored bidder intervals, $|P(Q)|$, is clearly less than $|\mathcal{D}(F)|$. In some cases, we may be able to further reduce $\max|P(Q)|$ by analysis.

If $\max|P(Q)| = N < \infty$, $P(Q)$ can be written as $\{(x^j, y^j)\}_{j=1}^{N}$, where $x^j \leq y^j$ for $j = 1, 2, ...N$. The intervals are indexed by their positions in ascending order ($0 \leq x^1 < x^2 < \cdots < x^N$). Note that I allow for the possibility that $x^j = y^j$ for any j, which indicates that the resulting optimal Q may contain less than N favored bidder intervals. The seller's objective can be written as a function of these boundary points plus two reserve types. Then their values can be solved using standard optimization techniques. As can be seen from the above algorithm, it is possible to compute the optimal Q for essentially any regular F. The only exception is when $|\mathcal{D}(F)| = \infty$, but such cases appear to be too complicated to be of practical interest. At the end of this section, I provide an example (Example 2) to further illustrate the functioning of this algorithm.

Lastly, I discuss how to implement the optimal mechanism. In general, any Q as described in Theorem 1 can be implemented by q which follows a threshold structure, and thus is referred to as a *threshold mechanism*. The algorithm described above gives some threshold points (i.e. reserve types and the boundary points of all of the favored bidder intervals) which divide the type space into several intervals. If both bidders' types fell below the corresponding reserve types, the seller will keep the object. If only one bidder's type is above the corresponding reserve type, that bidder will certainly get the object. If both bidders' types are above the corresponding reserve types and fell into a same interval on which the symmetry condition fails (i.e. a favored bidder interval), then the seller will always assign the object to bidder 1. In all other cases, the seller will always assign the object to the bidder with the higher type.

There are multiple selling schemes to implement the optimal mechanism. For instance, any optimal Q can be implemented through a dynamic selling scheme that alternately utilizes auctions and negotiations. In addition, any threshold mechanism can also be implemented by multi-round auctions with bidder-specific descending reserve prices.[10] I will further illustrate this point at the

[10] Fix any optimal Q. If Q can be implemented through sequential negotiations with fixed prices (p_1, p_2), then it can also be implemented using the following two-round auction: At $t = 1$, the seller sets the reserve price for bidder 1 as p_1 and for bidder 2 as ∞. If at least one bidder bids, the one with the higher bid wins. If neither bidder bids,

end of this section using Example 2.[11]

To better illustrate my results, I present the following two examples.

Example 1- *Uniform Distribution.* Suppose that for $i = 1,2$, θ_i is uniformly distributed on $[0,1]$. The corresponding cumulative density function is $F(\theta) = \theta$ and the virtual value function is $\mathcal{J}(\theta) = 2\theta - 1$. Fixing any $K > 0$, $\mathcal{J}(\theta) - KF(\theta) = (2-K)\theta - 1$, which is strictly increasing when $K < 2$ and strictly decreasing when $K > 2$. It directly follows from Theorem 1 that, when $K < 2$, symmetric auctions are optimal. Conversely, when $K > 2$, sequential negotiations are optimal.

Example 2 - *Exponential Distribution.* Suppose that $\theta_i \sim \mathit{Exponential}(5)$ and $K = 0.8$. Calculation yields that $\mathcal{J}(\theta) - KF(\theta)$ decreases on $[0, 0.277]$ and increases on $[0.277, \infty)$. By Theorem 1, the optimal mechanism contains one favored bidder interval, (θ_1^*, y), which covers the region in which the symmetry condition fails (i.e. $[\theta_1^*, 0.277]$). Thus, the seller needs to solve the following optimization problem.

$$\max_{\theta_1^*, \theta_2^*, y} \int_{\theta_1^*}^{y} \mathcal{H}_1(\theta_1, F(y))dF(\theta_1) + \int_{\theta_2^*}^{y} \mathcal{H}_2(\theta_2, F(\theta_1^*))dF(\theta_2) + \sum_{i=1,2} \int_{y}^{\infty} \mathcal{H}_i(\theta_i, F(\theta_i))dF(\theta_i)$$

subject to the constraints that $0 \leq \theta_1^* \leq y$ and $0 \leq \theta_2^* \leq y$. By solving the above constrained optimization problem, I find the following assignment rule is optimal: $\theta_1^* = 0.548$, $\theta_2^* = 0$, and $y = 4.816$. That is, the optimal reserve type for bidder 1 is 0.548 and for bidder 2 is 0; there is an additional threshold point 4.8163. If $\theta_1 < \theta_1^*$, the seller always gives the object to bidder 2; if $\theta_1^* \leq \theta_1 < 0.548$ and $\theta_2 < 0.548$, the seller always gives the object to bidder 1. Otherwise, the seller assigns the object to the bidder with the higher type. The results are illustrated in the following graph.

the seller holds a second round of auction for which she sets the reserve prices for bidder 1 and bidder 2 as p_1 and p_2, respectively. Again, if at least one bidder submits a bid, then the object is assigned to the one with the higher bid. Otherwise the seller keeps the object.

[11] Furthermore, I refer interested readers to supplementary materials for a more complicated (and possibly more interesting) example.

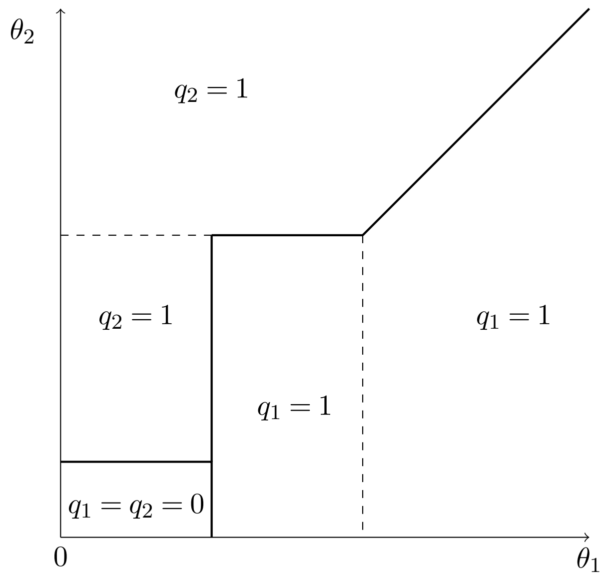

Figure 1: Exponential Distribution (1), K=2.5

To implement the optimal mechanism, the seller can first run a symmetric second-price auction with the reserve price of 5.22. If the seller fails to sell the object in the auction, she will then offer the object to bidder 1 with a fixed price of 0.948. If bidder 1 refuses the offer, the seller will offer the object to bidder 2 with a fixed price of 0.4, which bidder 2 will always accept. Alternatively, the optimal mechanism can also be implemented by a two-round second-price auction with asymmetric reserve prices. In the first round, the reserve prices are 0.948 for bidder 1 and 5.22 for bidder 2. In the second round, the seller keeps the reserve price for bidder 1 at 0.948 and lowers the reserve price for bidder 2 to 0.4. By implementing the optimal mechanism, the seller earns an expected revenue of 0.411.

For the sake of comparison, I have also determined that, if the seller must use a symmetric auction, the optimal reserve price is 0.075, and the seller earns an expected revenue of 0.373. If the seller is restricted to using sequential negotiations, she will always sell the object to bidder 1 at a fixed price of 0.4 and her expected profit is 0.4.

5 Discussions

5.1 Timing of Investment

In this paper, I consider an environment where potential buyers may invest to increase their valuations before competing in the market. As investment decisions are costly and irreversible, the loser's investment cost (if any) is wasted. One may argue, the efficiency loss can be avoided if investment decisions are delayed until after the object is allocated. Such a delay, however, may not always be practical in reality.

The main reason is that the R&D process, especially for large-scale projects, often requires a significant amount of time to finish. To develop new technology or to improve a corresponding technology to make use of a spectrum license, cell phone companies not only need to invest millions of dollars, but also spend several years on the project. Moreover, R&D is often a continuous process. If the amount of monetary input is fixed, the expected quality of the final product is likely to increase when researchers are given more time. As a result, despite the risk of not obtaining the license, the cell phone company may still prefer to start working on the project several years before the spectrum auction occurs. If the firm waits until it has already obtained the spectrum to invest, the spectrum may be used inefficiently or even remain idle for a long period of time, resulting in a larger efficiency loss than would have been created from a wasteful investment.

One alternative is to design a mechanism which allocates the "future use" of the object. For instance, the government can hold an auction (or any other selling scheme) in 2008 to sell the right to start using a certain spectrum in 2018. This alternative mechanism also eliminates inefficiency, but again may not be practical. First, it is more difficult to persuade stakeholders to agree to large stake bids based on the report of an R&D team without seeing the actual technique. Second, due to firms' liquidity constraints, it may not be feasible for the government to collect the payment in 2008. If the government collects the payment in 2018, however, the winner of the spectrum may want to re-negotiate with the government at that time. Being the only firm who has invested and obtained the proper technique to utilize the license, the winner faces essentially no competition and thus, holds a substantial amount of bargaining power. Unless the seller can fully commit to never renegotiating, which is often not possible in large stake auctions since not allocating the object creates significant social welfare loss, the previous winner may be able to obtain the object at a price much lower than the one previously agreed upon

5.2 Stochastic Investment Technology

In this subsection, I consider an alternative environment where the two bidders are ex ante identical and can each choose how much to invest in a stochastic investment technology. Specifically, the game still consists of three periods, $t = 0, 1, 2$, and the seller still announces the mechanism to use at $t = 0$. At $t = 1$, the bidders observe the mechanism and each chooses his investment level $a_i \in [0, \infty)$ at cost $C(a_i) = \frac{a_i^2}{2K}$, where $K > 0$. After the investments have been made, Nature draws a random variable ε_i for each bidder i. ε_1 and ε_2 are i.i.d. drawn from some common distribution described by a strictly positive and continuous density function g and a continuous cumulative density function G, with support $\Upsilon = [\underline{\varepsilon}, \bar{\varepsilon}]$ $(-\infty < \underline{\varepsilon} < \bar{\varepsilon} < \infty)$ or $\Upsilon = \mathbb{R}$. The corresponding virtual value function, $\mathcal{J}_G(\varepsilon_i) = \varepsilon_i - \frac{1-G(\varepsilon_i)}{g(\varepsilon_i)}$ is strictly increasing. Bidder i's valuation for the object, v_i, equals the sum of his investment level a_i and the stochastic factor ε_i. That is, for $i = 1, 2$, $v_i = a_i + \varepsilon_i$. At $t = 2$, each bidder privately observes v_i and the mechanism is implemented.

Again, I can invoke the revelation principle and restrict attention to direct mechanisms $\tilde{w} = (\tilde{a}_i, \tilde{q}_i, \tilde{t}_i)$: the seller first recommends an investment plan $\tilde{a} \equiv (\tilde{a}_1, \tilde{a}_2)$, where $\tilde{a}_1$ and $\tilde{a}_2$ are positive constants, for the two bidders to implement at $t = 1$. The seller also announces an allocation rule $\tilde{q} \equiv (\tilde{q}_1, \tilde{q}_2)$, where $q : \Upsilon \times \Upsilon \to [0,1] \times [0,1]$ maps any pair of reported types $(\varepsilon_1, \varepsilon_2) \in \Upsilon \times \Upsilon$ to probabilities of winning the object; a payment rule $\tilde{t} \equiv (\tilde{t}_1, \tilde{t}_2)$, where $\tilde{t} : \Upsilon \times \Upsilon \to \mathbb{R} \times \mathbb{R}$ maps any pair of reported types to the transfers which the two bidders need to make.

The methodology developed in this paper can also be used to characterize the optimal mechanism in this alternative setting. First, utilizing a similar analysis from section 3, one can verify that, in this alternative setting, the seller's objective is to choose a pair of feasible and increasing reduced form allocation rule $(\tilde{Q}_1, \tilde{Q}_2)$ to maximize[12]

$$\tilde{R}(\tilde{Q}) = \sum_{i=1,2} \int \mathcal{J}_G(\varepsilon_i) \tilde{Q}_i(\varepsilon_i) dG(\varepsilon_i) + \frac{K}{2} \sum_{i=1,2} E^2(\tilde{Q}_i(\varepsilon_i)).$$

Moreover, the optimal mechanism still follows a threshold structure. Specifically, the optimal mechanism only contains regions where the object is not assigned, the object is assigned to the bidder who reports the higher ε_i, and favored bidder intervals where the object is always assigned to the favored bidder (i.e. a modified version of Lemma 3 still holds). Note that here the favored bidder is always the one who invests more. In other words, the object is either assigned to the bidder who has obtained the luckier draw or the bidder who has invested more. Furthermore, when ε_i is uniformly distributed on $[0, 1]$, consistent with the findings of the benchmark model), the optimal mechanisms are either auctions or negotiations. Specifically, when $K < 1.35$, auctions are optimal; when $K \geq 1,35$ negotiations are optimal. Compared to the results of the benchmark model (negotiations are optimal if and only if $K \geq 2$), discriminatory treatment is more likely to be desirable for the seller, because in this alternative setting, a symmetric auction will always induce the two bidders to invest the same amount, resulting in a greater efficiency loss relative to the benchmark model where the lower type bidder invests less.

5.3 Auctions vs. Negotiations

As previously noted, auctions and negotiations are two extreme examples of the optimal mechanisms found in this paper. Both auctions and negotiations are widely used in reality, and numerous studies have compared the profitability of the two selling methods in various environments. For instance, Bulow and Klemperer (1996) consider a setting in which bidders' values are exogenously fixed. They reveal that under standard assumptions, an English auction with $N+1$ bidders always generates more expected profit for the seller than any negotiation with N bidders. This observation holds true even if the seller has all the bargaining power, as assumed in this paper. Conversely, in the current paper, negotiation with 1 bidder (i.e. sequential negotiations for which bidder 1 will always accept the offer) may outperform any symmetric auctions

[12] The proofs for all of the results in this subsection can be found in the supplementary materials of this paper.

with 2 bidders. The root cause of such opposite results is that, in my paper, bidders' values are endogenously determined. Symmetric auctions induce both bidders to invest and thus result in more expected efficiency loss than that of a 1 bidder negotiation. In addition, the assumption that the seller has all the bargaining power also plays an important role. Bulow and Klemperer (2009) study the relative profitability of auctions and negotiations when participation is costly. They find that if the buyers hold all the bargaining power in the negotiation, auctions will always perform better than negotiations under reasonable assumptions.[13]

6 Conclusion

This paper explores the design of the revenue maximizing selling mechanism for scenarios where bidders can make costly upfront investments to enhance their valuations. In such environments, committing to a very selective process (e.g. a symmetric auction) to determine which bidder will get the object results in low investment levels, because bidders are unwilling to invest too much without knowing whether they will win. Thus, unlike in the standard setting with exogenously given value distributions, the seller may prefer to adopt a discriminatory/less competitive selection process to increase the favored bidder's investment incentive.

There are two key findings in this paper. First, whether the seller finds it optimal to discriminate among ex ante symmetric bidders depends on the efficiency of the investment technology (K) and the relative increasing rates of the virtual value function ($\mathcal{J}$) and of the cumulative density function (F). Specifically, when the product of the investment efficiency parameter and the cumulative density function (KF) increases faster than the virtual value function ($\mathcal{J}$), the value of discrimination exceeds the value of screening. Thus, discriminatory treatment is optimal. Second, I show that the optimal mechanism follows a threshold structure and provide an algorithm to compute the threshold points. In an extreme case, i.e. the virtual value function always increases at a slower rate than that of the product of the investment efficiency parameter and the cumulative density function, sequential negotiations are optimal for the seller. In general, any optimal mechanism can be implemented by a dynamic selling scheme which alternately utilizes auctions and negotiations.

[13]I thank referee 1 for suggesting the comparison with Bulow and Klemperer (1996) and Paul Klemperer for suggesting Bulow and Klemperer (2009). Bulow and Klemperer (2009) consider a dynamic environment where entry can occur over multiple periods. If, instead, entry decisions need to be made simultaneously as in this paper, then negotiations become even less profitable and are less likely to outperform auctions.

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Appendix: Proofs for Main Results

Proof of Proposition 1

Condition (a) repeats the feasibility constraint and condition (d) simply restates the individual rationality constraint. Further, as is already shown in the text, a necessary and sufficient condition for a being incentive compatible is that $a_i(\theta) = a_i^*(\theta_i, \theta_i, Q_i) = KQ_i(\theta_i)$. So condition (e) also holds. I only need to prove that any (Q, T, a) induces truthtelling if and only if conditions (b) and (c) hold.

(1) Necessity

First note that, fix any (Q, T, a), if bidder i is of actual type θ_i and plans to report type $\hat{\theta}_i$, he will chooses $\hat{a}_i$ to maximize

$$(\theta_i + \hat{a}_i)Q_i(\hat{\theta}_i) - T_i(\hat{\theta}_i) - \frac{\hat{a}_i^2}{2K}.$$

The first order condition directly gives

$$a_i^*(\theta_i, \hat{\theta}_i, Q_i) = KQ_i(\hat{\theta}_i).$$

and the second order condition $-\frac{1}{K} < 0$ always hold. So $a_i^*(\theta_i, \hat{\theta}_i, Q_i)$ is indeed optimal for bidder i of actual type θ_i and reported type $\hat{\theta}_i$.

The truthtelling constraint requires that for any $\theta_i > \theta_i' > \theta_i^*$,

$$\begin{aligned}
\mathcal{U}_i(\theta_i) &\geq (\theta_i + a_i^*(\theta_i, \theta_i', Q_i))Q_i(\theta_i') - T_i(\theta_i') - \frac{a_i^{*2}(\theta_i, \theta_i', Q_i)}{2K} \\
&= \theta_i Q_i(\theta_i') + \frac{K}{2}Q_i^2(\theta_i') - T_i(\theta_i') \\
&= \mathcal{U}_i(\theta_i') + (\theta_i - \theta_i')Q_i(\theta_i')
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{U}_i(\theta_i') &\geq (\theta_i' + a_i^*(\theta_i', \theta_i, Q_i))Q_i(\theta_i) - T_i(\theta_i) - \frac{a_i^{*2}(\theta_i', \theta_i, Q_i)}{2K} \\
&= \mathcal{U}_i(\theta_i) + (\theta_i' - \theta_i)Q_i(\theta_i)
\end{aligned}$$

Thus,

$$Q_i(\theta_i) \geq \frac{\mathcal{U}_i(\theta_i) - \mathcal{U}_i(\theta_i')}{\theta_i - \theta_i'} \geq Q_i(\theta_i')$$

The above expression immediately implies that Q_i must be non-decreasing. In addition, letting $\theta_i' \to \theta_i$ implies that for any $\theta_i > \theta_i^*$ I have

$$\mathcal{U}_i'(\theta_i) = Q_i(\theta_i)$$

and thus for any $\theta_i \geq \theta_i^*$,

$$\mathcal{U}(\theta_i) = \mathcal{U}_i(\theta_i^*) + \int_{\theta_i^*}^{\theta_i} Q_i(\tilde{\theta}_i) d\tilde{\theta}_i$$

(2) Sufficiency

Consider any $\theta_i > \theta_i' > \theta_i^*$. (b) and (c) implies

$$\begin{aligned}
\mathcal{U}_i(\theta_i) - \mathcal{U}_i(\theta_i') &= \int_{\theta_i'}^{\theta_i} Q_i(\tilde{\theta}_i) d\tilde{\theta}_i \\
&\geq Q_i(\theta_i')(\theta_i - \theta_i')
\end{aligned}$$

Hence, I have

$$\begin{aligned}
\mathcal{U}_i(\theta_i) &\geq \mathcal{U}_i(\theta_i') + (\theta_i - \theta_i')Q_i(\theta_i') \\
&= Q_i(\theta_i')\left(\theta_i + KQ_i(\theta_i')\right) - T_i(\theta_i') - \frac{K^2 Q_i^2(\theta_i')}{2K}
\end{aligned}$$

So bidder i of type θ_i has no incentive to misreport to be θ_i'. Similarly, I can derive that

$$\mathcal{U}_i(\theta_i') \geq Q_i(\theta_i)\left(\theta_i' + KQ_i(\theta_i)\right) - T_i(\theta_i) - \frac{K^2 Q_i^2(\theta_i)}{2K}$$

So bidder i of type θ_i' has no incentive to misreport to be θ_i. Since the choice of θ_i and θ_i' are arbitrary, the truthtelling constraint holds for all $\tilde{\theta}_i > \theta_i^*$ and $i = 1, 2$.

Proof of Lemma 2

Suppose not. Then there exists an optimal allocation rule $Q' = (Q_1', Q_2')$ for which there exists some $i \in \{1, 2\}$ and $\tilde{\theta}_i > \theta_i^*$ such that $Q_i'(\theta_i) < F(\theta_{-i}^*)$ for any $\theta_i^* < \theta_i \leq \tilde{\theta}_i$. Letting $\theta_i^{\max} = \sup\{\theta_i' \mid Q_i'(\theta_i) < F(\theta_{-i}^*)\}$. One can construct another allocation rule Q'', for which $Q_i''(\theta_i) = F(\theta_{-i}^*)$ for all $\theta_i^* < \theta_i \leq \theta_i^{\max}$ and $\tilde{Q}_i(\theta_i) = Q_i(\theta_i)$ otherwise; $\tilde{Q}_{-i}(\theta_{-i}) = Q_{-i}(\theta_{-i})$ for all $\theta_{-i} \in \Theta$. It is easy to verify that if Q' is implementable, then Q'' is also implementable.

By using Q'' instead of Q, the change in the seller's expected revenue is

$$\triangle R = \sum_{i=1,2} \int_{\theta_i^*}^{\theta_i^{\max}} \left[\mathcal{H}_i(\theta_i, F(\theta_{-i}^*))F(\theta_{-i}^*) - \mathcal{H}_i(\theta_i, Q'))Q_i'(\theta_{-i}^*)\right] dF(\theta_i).$$

$\triangle R$ is strictly positive as $\mathcal{H}_i$ is strictly increasing in Q_i and, by Lemma 1, $\mathcal{H}_i(\theta_i, Q')) > 0$ for all $\theta_i > \theta_i^{\max}$ (otherwise Q' is not optimal anyway). Thus Q' cannot be optimal.

Proof of Proposition 2

Fix any optimal $Q = (Q_1, Q_2)$. As Q_1 is increasing and F is strictly increasing, there always exists a well-defined and increasing ϕ such that $Q_1(\theta_1) = F(\phi(\theta_1))$ for all $\theta_1 > \theta_1^*$. Also note that $\phi(\theta_1^*) = \theta_2^*$ by my previous assumption. Thus the corresponding ϕ^{-1} is also well-defined and increasing. The statement that for any optimal Q, $Q_1(\theta_1) = 0$ for all $\theta_1 < \theta_1^*$ and $Q_2(\theta_2) = 0$ for all $\theta_2 < \theta_2^*$ directly follows the definitions of θ_1^* and θ_2^*. I need only show that $Q_2(\theta_2) = F(\phi^{-1}(\theta_2))$ for any $\theta_2 > \phi(\theta_1^*)$.

Recall that any Q is implementable iff for any $A_1 \subset \Theta$ and $A_2 \subset \Theta$,

$$\int_{A_1} Q_1(\theta_1)dF(\theta_1) + \int_{A_2} Q_2(\theta_2)dF(\theta_2) \quad \leq \quad 1 - \int_{\theta_1 \notin A_1} dF(\theta_1) \int_{\theta_2 \notin A_2} dF(\theta_2)$$

Fix any $t > \theta_1^*$, let $A_1 = (t, \infty)$ and $A_2 = (\phi(t), \infty)$, the above inequality becomes

$$\int_t^\infty Q_1(\theta_1)dF(\theta_1) + \int_{\phi(t)}^\infty Q_2(\theta_2)dF(\theta_2) \quad \leq \quad 1 - F(t)F(\phi(t))$$

Note that

$$\begin{aligned}
\int_t^\infty F(\phi(\theta_1))dF(\theta_1) + \int_{\phi(t)}^\infty F(\phi^{-1}(\theta_2))dF(\theta_2) \quad &= \quad \int_t^\infty F(\phi(\theta_1))dF(\theta_1) + \int_t^\infty F(\phi^{-1}(\phi(\theta_1)))dF(\phi(\theta_1)) \\
&= \quad \int_t^\infty F(\phi(\theta))dF(\theta) + \int_t^\infty F(\theta)dF(\phi(\theta)) \\
&= \quad F(\phi(\theta))F(\theta) \mid_t^\infty \\
&= \quad 1 - F(t)F(\phi(t))
\end{aligned}$$

where in obtaining the first line I used the transformation of variables $\theta_2 = \phi(\theta_1)$ and in going from the second to the third line I used integration by parts. Thus the above inequality can be rewritten as

$$\int_t^\infty Q_1(\theta_1)dF(\theta_1) + \int_{\phi(t)}^\infty Q_2(\theta_2)dF(\theta_2) \quad \leq \quad \int_t^\infty F(\phi(\theta_1))dF(\theta_1) + \int_{\phi(t)}^\infty F(\phi^{-1}(\theta_2))dF(\theta_2)$$

which implies

$$\int_{\phi(t)}^\infty Q_2(\theta_2)dF(\theta_2) \quad \leq \quad \int_{\phi(t)}^\infty F(\phi^{-1}(\theta_2))dF(\theta_2) \tag{5}$$

for all $t > \theta_1^*$. To show that (Q_1, Q_2^*), where $Q_2^*(\theta_2) = F(\phi^{-1}(\theta_2))$ for $\theta_2 > \phi(\theta_1^*) = \theta_2^*$ is optimal among all implementable Q, it suffices to show that (Q_1, Q_2^*) is optimal among all (Q_1, Q_2) that satisfies Inequality 5.

Suppose not. Then there exists an optimal pair of (Q_1, Q_2) for which one of the following statement holds.

(1)

$$\int_{\phi(\theta_1^*)}^\infty Q_2(\theta_2)dF(\theta_2) \quad < \quad \int_{\phi(\theta_1^*)}^\infty F(\phi^{-1}(\theta_2))dF(\theta_2)$$

(2)

$$\int_{\phi(\theta_1^*)}^\infty Q_2(\theta_2)dF(\theta_2) \quad = \quad \int_{\phi(\theta_1^*)}^\infty F(\phi^{-1}(\theta_2))dF(\theta_2)$$

and

$$\int_{\phi(t)}^{\infty} Q_2(\theta_2) dF(\theta_2) \quad \leq \quad \int_{\phi(t)}^{\infty} F(\phi^{-1}(\theta_2)) dF(\theta_2)$$

hold for all $t > \theta_1^*$ and strictly so for some t.

(1) Fix any Q as described in (1). There clearly exists another $\tilde{Q} = (\tilde{Q}_1, \tilde{Q}_2)$ satisfying Inequality 5 for which (a) $\tilde{Q}_1(\theta_1) = Q_1(\theta_1)$ for all θ_1 and (b) $\tilde{Q}_2(\theta_2) = Q_2(\theta_2) = 0$ for all $\theta_2 < \phi(\theta_1^*)$ and $\tilde{Q}_2(\theta_2) \geq Q_2(\theta_2)$ for all $\theta_2 > \phi(\theta_1^*)$ with strict inequality holds on some sets of positive measure. Thus if the seller uses $\tilde{Q}$ instead of Q, the change of her expected profit is

$$\triangle = \int_{\Theta} \mathcal{J}(\theta_2) \left[\tilde{Q}_2(\theta_2) - Q_2(\theta_2) \right] dF(\theta_2) + \frac{K}{2} \int_{\Theta} \left[\tilde{Q}_2^2(\theta_2)) - Q_2^2(\theta_2) \right] dF(\theta_2) > 0.$$

So any such Q is not optimal.

(2) Fix any Q as described in (2). I construct two distributions

$$H(\theta_2) = \frac{Q_2(\theta_2) f(\theta_2)}{\int_{\phi(\theta_1^*)}^{\infty} Q_2(s) dF(s)}$$

and

$$H^*(\theta_2) \equiv \frac{F(\phi^{-1}(\theta_2)) f(\theta_2)}{\int_{\phi(\theta_1^*)}^{\infty} F(\phi^{-1}(s)) dF(s)}$$

for any $\theta_2 \geq \theta_2^*$. By construction, $H^*(\theta_2)$ first order stochastic dominates $H(\theta_2)$. Both Q_2 and $F(\phi^{-1})$ are increasing on Θ and $\mathcal{J}$ is strictly increasing on Θ. The expectation of a strictly increasing function increases with a first order stochastic dominance shift, thus I have

$$\begin{aligned} \int_{\phi(\theta_1^*)}^{\infty} Q_2(\theta_2) Q_2(\theta_2) dF(\theta_2) \quad &< \quad \int_{\phi(\theta_1^*)}^{\infty} Q_2(\theta_2) F(\phi^{-1}(\theta_2)) dF(\theta_2) \\ &< \quad \int_{\phi(\theta_1^*)}^{\infty} F(\phi^{-1}(\theta_2)) F(\phi^{-1}(\theta_2)) dF(\theta_2) \end{aligned}$$

and

$$\int_{\phi(\theta_1^*)}^{\infty} \mathcal{J}(\theta_2) Q_2(\theta_2) dF(\theta_2) < \int_{\phi(\theta_1^*)}^{\infty} \mathcal{J}(\theta_2) F(\phi^{-1}(\theta_2)) dF(\theta_2).$$

That is,

$$\begin{aligned} R((Q_1, Q_2^*)) - R((Q_1, Q_2)) \quad &= \quad \left[\int_{\phi(\theta_1^*)}^{\infty} \mathcal{J}(\theta_2) F(\phi^{-1}(\theta_2)) dF(\theta_2) - \int_{\phi(\theta_1^*)}^{\infty} \mathcal{J}(\theta_2) Q_2(\theta_2) dF(\theta_2) \right] \\ &\quad + \frac{K}{2} \left[\int_{\phi(\theta_1^*)}^{\infty} F^2(\phi^{-1}(\theta_2)) dF(\theta_2) - \int_{\phi(\theta_1^*)}^{\infty} Q_2^2(\theta_2)) dF(\theta_2) \right] \\ &> \quad 0 \end{aligned}$$

Thus any such Q is not optimal.

Proof of Lemma 3

(I) Suppose not. Then there exists an optimal Q for which at least one of the following statements is true.

(a) There exists a non-degenerate interval $(x,z) \subset (\theta_i^*, \infty)$ such that $Q_i(\theta_i)$ is strictly increasing on (x,z) and $Q_i(\theta_i) > F(\theta_i)$ for any $\theta_i \in (x,z)$;

(b) There exists a non-degenerate interval $(\tilde{x}, \tilde{z}) \subset (\theta_i^*, \infty)$ such that $Q_1(\theta_1)$ is strictly increasing on $(\tilde{x}, \tilde{z})$ and $Q_i(\theta_i) < F(\theta_i)$ for any $\theta_i \in (x', z')$.

I write the proof for case (a). The proof for (b) is essentially the same and thus I omit the details. Further, since the two bidders are symmetric, I can let $i = 1$ without causing any loss of generality. Recall that, by Proposition 2, I can restrict attention to the set of Q which can be described by a "cutoff function" ϕ.

Suppose (a) is true. Fix any $y \in (x,z)$ at which $\phi(\cdot)$ is continuous and differentiable. Such a point (y) always exists as I only consider piecewise continuous Q and ϕ is monotone. Fix any $t \in (x,y)$, I construct a new rule Q^t such that $Q_1^t(\theta_1) = Q_1(y)$ for all $\theta \in (t,y)$ and $Q_1^t(\theta_1) = Q_1(\theta_1)$ otherwise; $Q_2^t(\theta_2) = F(t)$ for $\theta \in (\phi(t), \phi(y))$ and $Q_2^t(\theta_2) = Q_2(\theta_2)$ otherwise. It is easy to check that any such Q^t is implementable as long as Q is implementable.

If the seller uses Q^t instead of Q, the change in her expected utility is

$$\triangle(t) \quad = \quad \triangle_1(t) + \triangle_2(t)$$

where

$$\begin{aligned} \triangle_1(t) &= \int_t^y [F(\phi(y)) - F(\phi(\theta_1))]\mathcal{J}(\theta_1) dF(\theta_1) \\ &\quad + \int_{\phi(t)}^{\phi(y)} [F(t) - F(\phi^{-1}(\theta_2))]\mathcal{J}(\theta_2) dF(\theta_2) \end{aligned}$$

and

$$\begin{aligned} \triangle_2(t) &= \frac{K}{2} \int_t^y [F^2(\phi(y)) - F^2(\phi(\theta_1))] dF(\theta_1) \\ &\quad + \frac{K}{2} \int_{\phi(t)}^{\phi(y)} [F^2(t) - F^2(\phi^{-1}(\theta_2))] dF(\theta_2). \end{aligned}$$

Roughly speaking, in the above representations, $\triangle_1(t)$ measures the change in the seller's expected rent from screening and $\triangle_2(t)$ measures the change in bidders' investment incentives.

Taking derivative of $\triangle_1(t)$ gives

$$\begin{aligned} \triangle_1'(t) &= -[F(\phi(y)) - F(\phi(t))]\mathcal{J}(t) f(t) + f(t) \int_{\phi(t)}^{\phi(y)} \mathcal{J}(\theta_2) dF(\theta_2) \\ &= f(t) \int_{\phi(t)}^{\phi(y)} [\mathcal{J}(\theta_2) - \mathcal{J}(t)] dF(\theta_2) \\ &= f(t) \int_t^y [\mathcal{J}(\phi(\theta)) - \mathcal{J}(t)] dF(\phi(\theta)) \end{aligned}$$

where in going from the second to the third line I used the transformation of variable $\theta_2 = \phi(\theta)$. Similarly, the first derivative of $\triangle_2(t)$ is

$$
\begin{aligned}
\triangle_2'(t) &= -\frac{K}{2} f(t)\left[F^2(\phi(y)) - F^2(\phi(t))\right] + \frac{K}{2}\int_{\phi(t)}^{\phi(y)} 2F(t)f(t)dF(\theta_2) \\
&= Kf(t)\int_t^y \left[F(t) - F(\phi(\theta))\right] dF(\phi(\theta))
\end{aligned}
$$

Thus I have

$$
\begin{aligned}
\triangle'(t)\frac{1}{f(t)} &= \int_t^y \left[\mathcal{J}(\phi(\theta)) - \mathcal{J}(t)\right] dF(\phi(\theta)) \\
&\quad -K\int_t^y \left[F(\phi(\theta)) - F(t)\right] dF(\phi(\theta)) \\
&= \int_t^y \left[\mathcal{J}(\phi(\theta)) - KF(\phi(\theta))\right] dF(\phi(\theta)) \\
&\quad -\int_t^y \left[\mathcal{J}(t) - KF(t)\right] dF(\phi(\theta))
\end{aligned}
$$

Similarly, fix any $s \in (y, z)$, I construct a new rule $\hat{Q}^s$ such that $\hat{Q}_1^s(\theta_1) = Q_1(y)$ for all $\theta_1 \in (y, s)$ and $\hat{Q}_1^s(\theta_1) = Q_1(\theta_1)$ otherwise; $\hat{Q}(\theta_2) = F(s)$ for $\theta \in (\phi(y), \phi(s))$ and $\hat{Q}_2^2(\theta_2) = Q_2(\theta_2)$ otherwise. It is easy to check that any such $\hat{Q}^s$ is implementable as long as Q is implementable. If the seller uses $\hat{Q}^s$ instead of Q, the change in his expected utility is

$$
\begin{aligned}
\hat{\triangle}(s) = &\int_y^s [F(\phi(y)) - F(\phi(\theta_1))]\mathcal{J}(\theta_1)dF(\theta_1) + \int_{\phi(y)}^{\phi(s)} [F(s) - F(\phi^-(\theta_2))]\mathcal{J}(\theta_2)dF(\theta_2) \\
&+ \frac{K}{2}\int_y^s [F^2(\phi(y)) - F^2(\phi(\theta_1))]dF(\theta_1) + \frac{K}{2}\int_{\phi(y)}^{\phi(s)} [F^2(s) - F^2(\phi^-(\theta_2))]dF(\theta_2)
\end{aligned}
$$

Taking derivative of $\hat{\triangle}(s)$ gives

$$
\begin{aligned}
\hat{\triangle}'(s)\frac{1}{f(s)} &= \int_y^s \left[\mathcal{J}(\phi(\theta)) - KF(\phi(\theta))\right] dF(\phi(\theta)) \\
&\quad -\int_y^s \left[\mathcal{J}(s) - KF(s)\right] dF(\phi(\theta))
\end{aligned}
$$

It is trivial that $\triangle(y) = \hat{\triangle}(y) = 0$ and $\triangle'(y) = \hat{\triangle}'(y) = 0$.

I further take derivatives of $\triangle'(t)\frac{1}{f(t)}$ and $\hat{\triangle}'(s)\frac{1}{f(s)}$ respectively, which yield

$$
\begin{aligned}
\frac{d\frac{\triangle'(t)}{f(t)}}{dt} &= -f(\phi(t))\phi'(t)\left\{\left[\mathcal{J}(\phi(t)) - \mathcal{J}(t)\right] - K\left[F(\phi(t)) - F(t)\right]\right\}) \\
&\quad +\int_t^y \left[Kf(t) - \mathcal{J}'(t)\right] dF(\phi(\theta))
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d\frac{\hat{\triangle}'(s)}{f(s)}}{ds} &= f(\phi(s))\phi'(s)\left\{[J(\phi(s))-\mathcal{J}(s)]-K[F(\phi(s))-F(s)]\right\} \\
&\quad + \int_y^s [Kf(s)-\mathcal{J}'(s)]\, dF(\phi(\theta))
\end{aligned}
$$

It directly follows that

$$
\begin{aligned}
-\frac{d\frac{\triangle'(t)}{f(t)}}{dt}\Big|_{t=y} &= \frac{d\frac{\hat{\triangle}'(s)}{f(s)}}{ds}\Big|_{s=y} \\
&= f(\phi(y))\phi'(y)\left\{[\mathcal{J}(\phi(y))-KF(\phi(y))]-[\mathcal{J}(y)-KF(y)]\right\}
\end{aligned}
$$

There are 3 possible cases.

(1) If $\mathcal{J}(\phi(y))-KF(\phi(y))>\mathcal{J}(y)-KF(y)$, then $\hat{\triangle}''(y)>0$. By continuity of $\phi(\theta)$ at $\theta=y$, there exists $\varepsilon>0$ such that $\hat{\triangle}''(s)>0$ for all $s\in(y,y+\varepsilon)$ which further implies that $\hat{\triangle}(y+\varepsilon)>0$.

(2) If $\mathcal{J}(\phi(y))-KF(\phi(y))<\mathcal{J}(y)-KF(y)$, then $\triangle''(y)<0$. By continuity of $\phi(\theta)$ at $\theta=y$, there exists $\varepsilon'>0$ such that$\triangle''(t)<0$ for all $t\in(y-\varepsilon',y)$ which further implies that $\triangle(y-\varepsilon')>0$.

(3) If $\mathcal{J}(\phi(y))-KF(\phi(y))=\mathcal{J}(y)-KF(y)$ for any y at which ϕ is continuous and differentiable, then $\mathcal{J}(\phi(y))-KF(\phi(y))=\mathcal{J}(y)-KF(y)$ almost everywhere on (x,z) as ϕ is monotone increasing. Note that the differentiability of f implies the differentiability of $\mathcal{J}-KF$ and the continuity of f implies the continuity of $\mathcal{J}'-Kf$.

If $\mathcal{J}-KF$ is strictly decreasing (i.e. $\mathcal{J}'-Kf<0$) at y, then there exists $\tilde{\varepsilon}>0$ such that $\mathcal{J}'-Kf<0$ for all $s\in(y,y+\tilde{\varepsilon})$. Thus I have, for all $s\in(y,y+\tilde{\varepsilon})$,

$$
\frac{d\frac{\hat{\triangle}'(s)}{f(s)}}{ds} = \int_y^s [Kf(s)-\mathcal{J}'(s)]\, dF(\theta)>0
$$

That is, $\hat{\triangle}''(s)>0$ for all $s\in(y,y+\tilde{\varepsilon})$, which further implies $\hat{\triangle}(y+\tilde{\varepsilon})>0$. Similarly , if $\mathcal{J}-KF$ is strictly increasing at y, then there exists $\tilde{\varepsilon}'>0$ such that $\triangle(y-\tilde{\varepsilon}')>0$.

The foregoing analysis shows that there either exists $s\in(y,z)$ such that $\hat{\triangle}(s)>0$ or there exists $t\in(x,y)$ such that $\triangle(t)>0$. So I conclude that, as desired, any Q as described in case (a) is not optimal.

(II)To show (II), it suffices to show the following claim is true.

If there exists some non-degenerate interval $[x',z']\subset[\theta_i^*,\infty)$ *such that* $\forall\theta_i,\theta_i'\in[x',z']$ *and* $\forall\theta_i''\notin[x',z']$, $Q_i(\theta_i)=Q_i(\theta_i')\neq Q_i(\theta_i'')$, *then for any* $\theta_i\in[x',z']$, $Q_i(\theta_i)=\inf\{Q_i(\theta_i')\mid\theta_i'>z'\}$ *or* $Q_i(\theta_i)=\sup\{Q_i(\theta_i')\mid\theta_i'<x'\}$.

Suppose there exists some non-degenerate interval $[x',z']\subset[\theta^*,\infty)$ such that $\forall\theta_i,\theta_i'\in[x',z']$,

$Q_i(\theta_i) = Q_i(\theta_i')$, but

$$\sup\{Q_i(\theta_i') \mid \theta_i' < x\} < Q_i(\theta_i) < \inf\{Q_i(\theta_i') \mid \theta_i' > z\},$$

For notational simplicity, let $F(a) \equiv Q_1(\theta_1)$ for $\theta_1 \in [x,z]$. Fix any $y \in (x,z)$. For any t such that $F(t) \in (\sup\{Q_i(\theta_i') \mid \theta_i' < x\}, F(a))$, I construct another rule $Q^{y,t}$ such that $Q^{y,t}(\theta) = F(t)$ for $\theta_1 \in (x,y)$ and $Q_1^{y,t}(\theta_1) = Q_1(\theta_1)$ otherwise; $Q_2^{y,t}(\theta_2) = F(y)$ for $\theta_2 \in (t,a)$ and $Q_2^{y,t}(\theta_2) = Q_2(\theta_2)$ otherwise. Similarly, for any s such that $F(s) \in (F(a), \inf\{Q_i(\theta') \mid \theta' > z\})$, I construct another rule $\hat{Q}_1^{y,s}$ such that $\hat{Q}_1^{y,s}(\theta) = F(s)$ for $\theta_1 \in (y,z)$ and $\hat{Q}_1^{y,s}(\theta_1) = Q_1(\theta_1)$ otherwise; $\hat{Q}_2^{y,s}(\theta_2) = F(y)$ for $\theta_2 \in (a,s)$ and $\hat{Q}_2^{y,s}(\theta_2) = Q_2(\theta_2)$ otherwise. By following a similar procedure as in (I), one can show that there either exists a pair of y and t such that $\triangle(y,t) = R(Q^{y,t}) - R(Q) > 0$ or there exists a pair of y and s such that $\hat{\triangle}(y,s) = R(\hat{Q}^{y,s}) - R(Q) > 0$. In either case, Q is not optimal.

Proof of Theorem 1

Part (I) Suppose there exists an optimal $Q = (Q_1, Q_2)$ for which Part (I) does not hold. Then by Lemma 3, there exists a non-degenerate $(x,y) \subset (\alpha,\beta)$ such that $Q_i(\theta_i) = F(\theta_i)$ on (x,y) for $i = 1,2$. I construct another $\tilde{Q}$ for which $\tilde{Q}_1(\theta_1) = F(y)$ for $\theta_1 \in (x,y)$ and $\tilde{Q}_1(\theta_1) = Q_1(\theta_1)$ otherwise; $\tilde{Q}_2(\theta_2) = F(x)$ for $\theta_2 \in (x,y)$ and $\tilde{Q}_2(\theta_2) = Q_2(\theta_2)$ otherwise. $\tilde{Q}$ is clearly implementable as long as Q is implementable. If the seller uses $\tilde{Q}$ instead of Q, the change in her expected revenue is

$$\begin{aligned}\tilde{\triangle} \quad = \quad & \int_x^y F(x)\mathcal{J}(\theta)dF(\theta) + \int_x^y F(y)\mathcal{J}(\theta)dF(\theta) \\ & + \frac{K}{2}\int_x^y F^2(x)dF(\theta) + \frac{K}{2}\int_x^y F^2(y)dF(\theta) \\ & -2\int_x^y F(\theta)\mathcal{J}(\theta)dF(\theta) - K\int_x^y F^2(\theta)dF(\theta)\end{aligned}$$

For any $s \in (x,y)$, let

$$\begin{aligned}\tilde{\triangle}(s) \quad = \quad & \int_x^s F(x)\mathcal{J}(\theta)dF(\theta) + \int_x^s F(s)\mathcal{J}(\theta)dF(\theta) \\ & + \frac{K}{2}\int_x^s F^2(x)dF(\theta) + \frac{K}{2}\int_x^s F^2(s)dF(\theta) \\ & -2\int_x^s F(\theta)\mathcal{J}(\theta)dF(\theta) - K\int_x^s F^2(\theta)dF(\theta)\end{aligned}$$

Note that $\tilde{\triangle}(x) = 0$, $\tilde{\triangle}(y) = \tilde{\triangle}$, and for any $s \in (x,y)$,

$$\begin{aligned}
\tilde{\triangle}'(s) &= F(x)\mathcal{J}(s)f(s) + F(s)\mathcal{J}(s)f(s) + f(s)\int_x^s \mathcal{J}(\theta)dF(\theta) \\
&\quad -2F(s)\mathcal{J}(s)f(s) + \frac{K}{2}F^2(x)f(s) + \frac{K}{2}F^2(s)f(s) \\
&\quad +KF(s)f(s)\int_x^s dF(\theta) - KF^2(s)f(s) \\
&= f(s)\int_x^s \{[\mathcal{J}(\theta) - KF(\theta)] - [\mathcal{J}(s) - KF(s)]\}\, dF(\theta)
\end{aligned}$$

is strictly positive as $\mathcal{J} - KF$ is strictly decreasing on (α,β). Thus $\tilde{\triangle} > 0$ and Q is not optimal.

Part (II) (a) Suppose not. Then there exists $[x',y'] \subset (\delta,\gamma)$ such that $Q_1(\theta_1) = F(y')\ \forall \theta_1 \in [x',y']$ and $Q_2(\theta_2) = F(x')\ \forall \theta_2 \in [x',y']$. I construct another allocation rule $\hat{Q}$ for which $\hat{Q}_1(\theta_1) = F(\theta_1)$ for $\theta_1 \in [x',y']$ and $\hat{Q}_1(\theta_1) = Q_1(\theta_1)$ otherwise; $\hat{Q}_2(\theta_2) = F(\theta_2)$ for $\theta_2 \in [x',y']$ and $\hat{Q}_2(\theta_2) = Q_2(\theta_2)$ otherwise. $\hat{Q}$ is clearly implementable as long as Q is implementable. If the seller uses $\hat{Q}$ instead of Q, the change in her expected revenue is

$$\begin{aligned}
\hat{\triangle} &= 2\int_x^y F(\theta)\mathcal{J}(\theta)dF(\theta) + K\int_x^y F^2(\theta)dF(\theta) \\
&\quad - \int_x^y F(x)\mathcal{J}(\theta)dF(\theta) - \int_x^y F(y)\mathcal{J}(\theta)dF(\theta) \\
&\quad - \frac{K}{2}\int_x^y F^2(x)dF(\theta) - \frac{K}{2}\int_x^y F^2(y)dF(\theta)
\end{aligned}$$

One can show that $\hat{\triangle} > 0$ by using similar arguments as in Part (I) and that $\mathcal{J} - KF$ is strictly increasing on (γ,δ).

Part (II) (b) directly follows from Lemma 3 (II).