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Steady States in Search-and-Matching Models

Stephan Lauermann*
Georg Nöldeke**
Thomas Tröger***

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*Department of Economics, University of Bonn, Germany, s.lauermann@uni-bonn.de
**Faculty of Business and Economics, University of Basel, Switzerland, georg.noeldeke@unibas.ch
***Department of Economics, University of Mannheim, Germany, troeger@uni-mannheim.de

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Stephan Lauermann† Georg Nöldeke‡ Thomas Tröger§

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Abstract: Most of the literature that studies frictional search-and-matching models with heterogeneous agents and random search investigates steady states. Steady state requires that the flows of agents into and out of the population of unmatched agents balance. Here, we investigate the structure of this steady-state condition. We build on the "fundamental matching lemma" for quadratic search technologies in Shimer and Smith (2000) and establish the existence, uniqueness, and comparative-static properties of the solution to the steady-state condition for *any* search technology that satisfies minimal regularity conditions.

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†Department of Economics, University of Bonn, Germany, s.lauermann@uni-bonn.de

‡Faculty of Business and Economics, University of Basel, Switzerland, georg.noeldeke@unibas.ch

§Department of Economics, University of Mannheim, Germany, troeger@uni-mannheim.de

1 Introduction

Search-and-matching models with heterogeneous agents are the cornerstone of the literature investigating the economic causes and consequences of sorting and mismatch in labor markets (Marimon and Zilibotti, 1999; Teulings and Gautier, 2004; Eeckhout and Kircher, 2011; Hagedorn, Law, and Manovskii, 2017). More generally, such models, surveyed by Burdett and Coles (1999), Smith (2011) and Chade, Eeckhout, and Smith (2017, Section 4), have provided key insights into the functioning of decentralized markets with trading frictions. Examples include the conditions for assortative matching with transferable utility (Shimer and Smith, 2000), block segregation with nontransferable utility (Burdett and Coles, 1997; Smith, 2006), the emergence of the law of one price (Gale, 1987; Lauermann, 2013), and the convergence of equilibria to stable matches as frictions vanish in marriage markets with nontransferable utility (Lauermann and Nöldeke, 2014). With few exceptions (e.g., Smith, 2009), the literature considers steady-state equilibria.

As emphasized in Burdett and Coles (1999), identifying a steady-state equilibrium in a search-and-matching model involves solving two distinct problems. First, taking a steady-state population of unmatched agents as given, determine stationary equilibrium strategies in the induced strategic interaction between the agents. Such a "partial equilibrium" (Burdett and Coles, 1997) specifies, in particular, who matches with whom. More precisely, it specifies what we refer to as the matching affinities, i.e., the probability that a meeting between agents of given types leads to a match. Second, taking the matching affinities as given, determine a steady-state population of unmatched agents, where steady state requires that the outflow from the population of unmatched agents (due to the formation of matches or exit) is balanced by a corresponding inflow (due to match dissolution or entry) for every type of agent. A steady-state equilibrium obtains if both, the partial-equilibrium and the steady-state condition hold.[1]

Here, we investigate the structure of the steady-state condition for a search-and-matching model with random search, a continuum of types, and an exogenous separation of matches as in Shimer and Smith (2000) and Smith (2006).[2] Our motivation for doing so is that, despite its importance for the determination of a steady-state equilibrium, much less is known about the structure of the steady-state condition than about the structure of the partial-equilibrium condition. Indeed, almost no progress has been made in elucidating the structure of the steady-state condition since the seminal contribution in Shimer and Smith (2000), who consider the steady-state condition under the assumption that the search technology that generates the meetings between unmatched agents is quadratic. Shimer and

[1]Some papers (e.g. McNamara and Collins, 1990; Morgan, 1998; Burdett and Wright, 1998; Bloch and Ryder, 2000; Chade, 2001; Adachi, 2003; Chen, 2005) assume that agents who leave the pool of unmatched agents are immediately replaced by identical clones. In such cloning models, the steady-state condition is trivially satisfied for *any* population of unmatched agents. Thus, the conditions for a steady-state equilibrium reduce to the partial-equilibrium condition, and none of the issues we discuss in this paper arise.

[2]Our results carry over to the common alternative specification in which there is an exogenous inflow of unmatched agents and unmatched agents abandon their attempt to find a partner at an exogenous rate; see Section 5.

Smith (2000) establish the existence of a unique unmatched population that satisfies the steady-state condition and the continuity of this solution in the matching affinities. This result has later been labeled the "fundamental matching lemma" in Smith (2006), and it was extended to linear search technologies in Nöldeke and Tröger (2009).

Our main results (Propositions 1 and 2) show that the fundamental matching lemma does not require the assumption that the search technology is quadratic or linear. Rather, the fundamental matching lemma holds whenever the search technology induces an aggregate meeting rate that is continuous and non-decreasing in the mass of unmatched agents and satisfies the boundary condition that the aggregate meeting rate is zero if there are no unmatched agents. These regularity conditions on the behavior of the aggregate meeting rate are not only natural; we also show that they are minimal in the following sense: if any one of these three conditions (continuity, monotonicity, boundary behavior) fails, examples featuring either no steady state or multiple steady states can be constructed.

The proofs of our main results build on two observations. The first observation, which is immediate from the arguments proving the fundamental matching lemma for quadratic search technologies in Shimer and Smith (2000), is that the steady states for quadratic search technologies are a continuous function of a parameter that scales the velocity of the search technology. The second observation is that every steady state for any general search technology (satisfying our regularity conditions) corresponds to a steady state for a quadratic search technology with a suitably chosen velocity parameter. Taken together, these observations imply that for given matching affinities the steady states of general search technologies can be identified with the roots of a continuous real function. Proving the fundamental matching lemma for general search technologies thus boils down to showing that this function has a unique root which changes continuously in the matching affinities. This is easy to do *if* an increase in the velocity of a quadratic search technology does (i) not increase the associated steady-state mass of unmatched agents and (ii) increases the aggregate steady-state meeting rate. Establishing these intuitive comparative-static properties lies at the heart of our argument and provides an economic understanding of the forces driving our generalization of the fundamental matching lemma.

The most immediate implication of our analysis for steady-state *equilibrium*—with matching affinities satisfying the additional partial-equilibrium condition—is that the results showing existence of steady-state equilibrium in Shimer and Smith (2000) and Smith (2006) hold for all search technologies satisfying our regularity conditions and not only for quadratic search technologies. This is so because the only implication of the steady-state condition exploited in the existence proofs in Shimer and Smith (2000) and Smith (2006) is the fundamental matching lemma.[3] Conditions under which our results are liable to imply uniqueness of steady-state equilibrium are discussed at the end of the paper in Section 5, where we also consider extensions and related literature on the uniqueness of steady states.

[3]For search-and-matching models with a finite number of types, Lauermann and Nöldeke (2015) and Manea (2017) present alternative proofs of the existence of steady-state equilibria based on Kakutani's fixed point theorem.

2 The Steady-State Condition

We consider a search-and-matching process akin to the one considered in Shimer and Smith (2000) and Smith (2006). Time is continuous and there is a continuum of infinitely lived agents. Agents are characterized by their types $x \in [0,1]$. For any (measurable) $X \subset [0,1]$, the mass of agents with types in this set is given by $\int_X \ell(x)dx$, where the (exogenous) population density $\ell : [0,1] \to (0,\infty)$ is measurable, bounded, and bounded away from zero. Agents are either unmatched or matched with a single partner. For any $X \subset [0,1]$, the mass of unmatched agents with types in this set is given by $\int_X u(x)dx$, where the (endogenous) unmatched density $u : [0,1] \to (0,\infty)$ is measurable and satisfies $u \leq \ell$. The mass of matched agents with types in X is $\int_X (\ell(x) - u(x))dx$.

Unmatched agents search for partners and meet other unmatched agents, drawn at random from the population of all unmatched agents.[4] The mass of unmatched agents involved in meetings per unit of time—the aggregate meeting rate—is determined by the search technology as a function of the mass of unmatched agents $\bar{u} = \int_0^1 u(x)dx$ and a velocity parameter $\sigma \geq 0$. Let $\mathcal{D}$ denote the set of real-valued functions on $[0,1]$ that are positive, measurable, and bounded.[5]

Assumption 1 (General Search Technology)**.** *For any unmatched density $u \in \mathcal{D}$, the aggregate meeting rate is given by $\sigma \cdot m(\bar{u})$, where $\sigma \geq 0$ is the velocity parameter of the search technology and the contact function $m : [0,\infty) \to [0,\infty)$ is continuous, non-decreasing, and satisfies $m(0) = 0$.*

As meetings are random, every unmatched agent meets other unmatched agents with types in $X \subset [0,1]$ at the rate $\sigma \cdot r(\bar{u}) \int_X u(x)dx$, where the rate function $r : (0,\infty) \to [0,\infty)$ is given by

$$r(\bar{u}) = \frac{m(\bar{u})}{\bar{u}^2}. \tag{1}$$

We refer to $\sigma \cdot r(\bar{u}) \cdot \bar{u}$, i.e., the rate at which an unmatched agent meets some other unmatched agent, as the individual meeting rate.

Matches form whenever two unmatched agents meet and agree to match with each other. The proportion of meetings between agents with types x and y that lead to a match is described by a symmetric measurable function $\alpha : [0,1]^2 \to [0,1]$. Let $\mathcal{A}$ denote the set of all such functions. We treat this matching affinity $\alpha \in \mathcal{A}$ as exogenous, thereby taking the behavior of agents as given.

All matches from the pool of matched agents dissolve at an exogenous rate $\delta > 0$. At the moment a match is dissolved, both agents return to the pool of unmatched agents.

In a steady state, the flow creation and flow destruction of matches for every type of agent must balance. The flow of matches that are created and involve type x is the product of the

[4] We consider the (non-trivial) extension of our results to the case in which there are two distinct groups of agents (workers and firms, or women and men) and agents in one group only meet agents from the other group in Section 5.

[5] We follow the convention of using terms such as positive, negative, increasing, and decreasing in the strict sense, with a prefix "non-" indicating the opposite weak sense.

unmatched density $u(x)$ of type x and the individual matching rate $\sigma \cdot r(\bar{u}) \int_0^1 \alpha(x,y)u(y)dy$ for agents of type x. The flow of matches that are destroyed and involve type x is the product of the dissolution rate δ and the matched density $\ell(x) - u(x)$ of type x. Therefore, steady state requires

$$\delta \left[\ell(x) - u(x)\right] = u(x) \cdot \sigma \cdot r(\bar{u}) \int_0^1 \alpha(x,y)u(y)dy \quad \forall x \in [0,1]. \tag{2}$$

From equation (2), it is obvious that there is no loss of generality in normalizing the dissolution rate δ to 1. We do so throughout the following and rewrite (2) as

$$\ell(x) = u(x) \left[1 + \sigma \cdot r(\bar{u}) \int_0^1 \alpha(x,y)u(y)dy\right] \quad \forall x \in [0,1]. \tag{3}$$

We refer to (3) as the general balance condition.

Remark 1 (A Finite Number of Types)**.** When providing intuition and constructing examples, we find it convenient to consider models with a finite number of types, say, $x = 1, \ldots, n$. The counterpart to (3) for such a model is

$$\ell(x) = u(x) \left[1 + \sigma \cdot r(\bar{u}) \sum_{y=1}^{n} \alpha(x,y)u(y)\right] \quad \forall x = 1, \ldots, n, \tag{4}$$

where $\ell(x) > 0$ is now the mass of agents of type x in the population and $u(x)$ is the corresponding unmatched mass. It is easily verified that all of our subsequent analysis and results carry over to (4), cf. footnote 16 in Appendix A.4. ■

3 Quadratic Search Technologies

A broad class of search technologies is compatible with Assumption 1, including the quadratic search technology considered in Shimer and Smith (2000) and Smith (2006).

Assumption 2. *[Quadratic Search Technologies] For any unmatched density $u \in \mathcal{D}$, the aggregate meeting rate is given by $\sigma \cdot m(\bar{u})$, where $\sigma = 1$ and*

$$m(\bar{u}) = \rho \cdot \bar{u}^2 \tag{5}$$

with $\rho \geq 0$.

In Assumption 2 we have embedded a velocity parameter $\rho \geq 0$ in the description of the contact function m, while setting the velocity parameter premultiplying the contact function m in Assumption 1 to $\sigma = 1$. This proves convenient in Section 4 where we relate the steady states for a general search technology with velocity σ to the steady states for a quadratic search technology with a suitably chosen velocity ρ that is, in general, different from σ.

From equations (1) and (5), the rate function associated with a quadratic search technology is simply given by its velocity (i.e., $r(\bar{u}) = \rho$ holds for all $\bar{u} > 0$). Thus, under Assumption 2 the general balance condition (3) simplifies to

$$\ell(x) = u(x)\left[1 + \rho \int_0^1 \alpha(x,y)u(y)dy\right] \quad \forall x \in [0,1]. \tag{6}$$

We refer to (6) as the quadratic balance condition.

Shimer and Smith (2000, Lemma 4) have shown the following "fundamental matching lemma" (Smith, 2006) for quadratic search technologies:

Lemma 1 (Fundamental Matching Lemma for Quadratic Search Technologies)**.** *There exists a unique unmatched density $u \in \mathcal{D}$ solving the quadratic balance condition* (6) *and this solution is a jointly continuous function of ρ and α.*

The continuity claim in the statement of Lemma 1 means that $\lim_{n\to\infty} \int |\alpha_n(x,y) - \alpha(x,y)|dxdy + |\rho_n - \rho| = 0$ implies $\lim_{n\to\infty} \int |u_{\rho_n,\alpha_n}(x) - u_{\rho,\alpha}(x)|dx = 0$, where $(\rho,\alpha) \to u_{\rho,\alpha}$ is the map from $[0,\infty) \times \mathcal{A}$ to the unmatched density implied by (6).[6]

In the remainder of this section, we investigate properties of the solution to the quadratic balance condition as a function of the velocity parameter ρ. To simplify notation, we thus write u_ρ for the unique solution of (6) given α.

By Lemma 1 the map $\rho \to u_\rho$ is continuous. The following lemma asserts that the unmatched steady-state density is non-increasing in the velocity of the search technology.[7] The proof is in Appendix A.1.

Lemma 2. *The map $\rho \to u_\rho(x)$ is non-increasing for all $x \in [0,1]$.*

From the quadratic balance condition (6), proving Lemma 2 is equivalent to showing that the individual steady-state matching rates $\rho \int_0^1 \alpha(x,y)u_\rho(y)dy$ are non-decreasing in ρ no matter what type x is considered. It is clear that it cannot be the case that all the individual matching rates are decreasing in ρ: if this were so, then the quadratic balance conditions would imply that $u_\rho(x)$ is increasing in ρ for all x, which in turn would imply that none of the individual matching rates decreases, yielding a contradiction. Hence, what

[6] For fixed ρ, this is equivalent to the continuity claim in Shimer and Smith (2000, Lemma 4), who assert that the map $\alpha \to u_{\rho,\alpha}$ is continuous with respect to the pseudometrics on $\mathcal{A}$ and $\mathcal{D}$ induced by the L^1-norms. While Shimer and Smith (2000) do not consider the (joint) continuity in ρ, this is immediate upon noting that (i) their arguments do not require α to be bounded above by 1 and, therefore, show that the solution to (6) is continuous in $\hat{\alpha} = \rho \cdot \alpha$, and that (ii) the map $(\rho,\alpha) \to \hat{\alpha}$ is continuous, too.

[7] Lemma 2 is stronger than what we require for the generalization of the fundamental matching lemma to the solution of the general balance condition (3); as advertised in the Introduction (and shown in the proofs of Propositions 1 and 2) for that purpose the essential implication of Lemma 2 is that $\bar{u}_\rho$ is non-increasing in ρ. Of course, Lemma 2 also allows us to conclude that the mass of any subset of types must be decreasing in ρ – a property that we require when extending the fundamental matching lemma to a model with two groups of agents (cf. Section 5).

is required to obtain Lemma 2 is that a change in the velocity of the search technology cannot cause some matching rates to decrease and others to increase or, equivalently, cause the unmatched density of some types to increase and the unmatched density of some other types to decrease. Establishing this is straightforward if the matching behavior of the agents is homogenous in the sense that the matching affinity α is constant: in this case the quadratic balance conditions implies that all individual matching rates are proportional to $\rho \cdot \bar{u}_\rho$ and therefore must all change in the same direction when ρ changes. The general argument relies on a more subtle implication of the quadratic balance condition: for any type x and for any change in the velocity, the absolute value of the percentage change in the unmatched density of type x must be smaller than the percentage change in the matching rate of type x.

To see how the relationship between the percentage changes just mentioned delivers the comparative-static result in Lemma 2, it is instructive to consider an example with a finite number of types (cf. Remark 1). Suppose, for instance, that the velocity ρ increases by 5 percent. As there is a finite number of types, there must be some type x' for whom the associated percentage change in $u(x')$ is maximal. Now suppose that, contrary to what is asserted in Lemma 2, the increase in velocity leads to an increase in $u(x')$ by, say, 1 percent. To maintain a steady-state, such an increase in the unmatched mass of type x' by 1 percent requires that its matching rate decreases by more than 1 percent. Given that the velocity has increased by 5 percent, this, in turn, requires the unmatched mass $u(y)$ of some other type y to decrease by more than 6 percent. Similar reasoning to the one just given shows that this is only possible if there is some further type x'' whose unmatched mass $u(x'')$ has increased by more than 1 percent. But the existence of such a type x'' contradicts the hypothesis that the percentage change in the unmatched mass is maximal for type x'. Thus, $u(x)$ must be non-increasing in ρ for all types x.[8]

As we have noted above, Lemma 2 is equivalent to the assertion that all individual steady-state matching rates $\rho \int_0^1 \alpha(x,y) u_\rho(y) dy$ are non-decreasing in the velocity of the quadratic search technology. It is natural to conjecture that this comparative statics goes hand-in-hand with an increase in the individual steady-state meeting rates $\rho \cdot \bar{u}_\rho$. Indeed, as suggested by the fact that Lemma 2 holds for any specification of $\alpha \in \mathcal{A}$, a stronger result holds: It not only becomes easier to meet *some* other unmatched agent when the velocity goes up. Rather, for any set $Y \subset [0,1]$ of types, it becomes easier to meet agents with types in this set. The proof of the following lemma is in Appendix A.2. It is straightforward from the arguments proving Lemma 2.

Lemma 3. *The map $\rho \to \rho \cdot u_\rho(x)$ is increasing for all $x \in [0,1]$.*

For a given unmatched density u, it is trivial that an increase in ρ causes an increase in $\rho \cdot u(x)$. The point of Lemma 3 is that the countervailing effect of an increase in velocity ρ

[8] Analogous reasoning can be used to infer the uniqueness claim in the fundamental matching lemma for the quadratic search technology. See the remark at the end of the proof of Lemma 2 in Appendix A.1.

on the steady-state unmatched density u_ρ identified in Lemma 2 cannot overturn this direct effect.

Next, we inquire whether not only the individual steady-state meeting rate $\rho \cdot \bar{u}_\rho$ but also the aggregate steady-state meeting rate $\rho \cdot \bar{u}_\rho^2$ is increasing in ρ, i.e., whether an increase in the velocity of the search technology implies that there are more meetings even when taking the resulting steady-state adjustment in the mass of unmatched agents into account. The following result provides an affirmative answer:

Lemma 4. *The map $\rho \to \rho \cdot \bar{u}_\rho^2$ is increasing in ρ.*

The proof of Lemma 4 is in Appendix A.3. It proceeds in two steps. The first step uses a transformation of the unmatched density to replace u_ρ by the unique solution $w_\gamma \in \mathcal{D}$ of the condition

$$\ell(x) = w(x)\left[\gamma + \int_0^1 \alpha(x,y)w(y)dy\right] \quad \forall x \in [0,1] \tag{7}$$

and shows that the claim in Lemma 4 is equivalent to the claim that the mass $\bar{w}_\gamma$ is decreasing in $\gamma > 0$. The second step adapts an argument that Decker, Lieb, McCann, and Stephens (2013) have developed in a related context.[9] It shows that the equalities in (7) correspond to the first-order conditions of a convex minimization problem to infer that $\bar{w}_\gamma$ is indeed decreasing in γ.

In light of the fact that Lemmas 2 and 3 provide monotone comparative statics results that hold pointwise (i.e., for all types), it is tempting to think that there should be a corresponding pointwise counterpart to Lemma 4, asserting that $\rho \cdot u_\rho(x) \cdot u_\rho(y)$ is increasing in ρ for all $(x,y) \in [0,1]^2$. Appendix A.4 presents an example to demonstrate that such a result need not hold.[10] Indeed, an increase in ρ can decrease the expression

$$\rho \cdot \alpha(x,y) \cdot u_\rho(x) \cdot u_\rho(y) \tag{8}$$

for some pairs (x,y). We find this last observation both surprising and interesting in its own right because the expression in (8) is nothing but the steady-state density of matched pairs featuring agents of type x and y.[11] Consequently, while the monotonicity result for the unmatched density established in Lemma 2 directly implies that the matched density $\ell(x) - u(x)$ must be non-decreasing in the velocity of the search technology for all x, the same property need not hold for the share of agents of type x who find themselves matched with partners in some set Y.

[9] Decker, Lieb, McCann, and Stephens (2013) establish the existence of a unique equilibrium in the model of a marriage market from Choo and Siow (2006) and derive comparative-statics properties of this equilibrium. These results are generalized in Galichon and Salanié (2015).

[10] Of course, integrating $\rho \cdot u_\rho(x) \cdot u_\rho(y)$ over x and y yields $\rho \cdot \bar{u}_\rho^2$, so that Lemma 4 implies that it is impossible for $\rho \cdot u_\rho(x) \cdot u_\rho(y)$ to decrease for all pairs (x,y).

[11] To see this, observe that in a steady state pairs with agents of type x and y form whenever agents of types x and y meet and agree to match, which happens at the rate given in (8). Recalling that we have normalized the dissolution rate δ to 1, the expression in (8) thus coincides with the steady-state density of matched pairs of agents with types x and y.

Remark 2 (Additional Comparative Statics)**.** To simplify the exposition, we have used the normalization $\delta = 1$ to obtain the quadratic balance condition (6) instead of an analogous condition in which the scaled velocity ρ/δ appears in the place of the velocity ρ. It is clear that Lemmas 1 to 4 remain applicable to describe the comparative statics with respect to a change in scaled velocity and, therefore (with the obvious sign reversals), also with respect to a change in the dissolution rate δ. Similarly, multiplying all $\alpha(x, y)$ by the same factor $0 < \xi < 1$ is tantamount to replacing ρ by $\xi \cdot \rho$ in (6).

Finally, we observe that the arguments yielding Lemma 4 imply that a proportionate increase in the population density, i.e., replacing ℓ by $\tau \cdot \ell$ for $\tau > 1$, must increase the mass of unmatched agents in the associated steady state. Appendix A.5 provides the details of the argument. ■

4 General Search Technologies

In this section, we show that the results for quadratic search technologies stated in the preceding section carry over to general search technologies. What makes this possible is an intimate relationship between the solutions of the general balance condition (3) and the quadratic balance condition (6). Indeed, comparing (3) and (6) it is immediate that an unmatched density $v \in \mathcal{D}$ solves the general balance condition if and only if solves the quadratic balance condition for a suitable choice of the velocity parameter ρ, namely

$$\rho = \sigma \cdot r(\bar{v}). \tag{9}$$

Note that in writing (9) we have used v rather than u to denote an unmatched density solving the general balance condition. We do so throughout this section as it helps to distinguish the solutions to the general balance condition from solutions to the quadratic balance condition.

Applying (1) to rewrite (9) we have the following result:

Lemma 5. *An unmatched density $v \in \mathcal{D}$ solves the general balance condition* (3) *if and only if v solves the quadratic balance condition* (6) *for $\rho \geq 0$ satisfying*

$$\rho \cdot \bar{v}^2 - \sigma \cdot m(\bar{v}) = 0. \tag{10}$$

While Lemma 5 holds without any assumptions on the contact function m, our subsequent results require the properties of the contact function stated in Assumption 1.

Equipped with the results from the previous section and Lemma 5 it is straightforward to establish the following existence and uniqueness result:

Proposition 1. *Let Assumption 1 hold. Then there exists a unique unmatched density $v \in \mathcal{D}$ solving the general balance condition* (3)*.*

Proof. Let u_ρ denote the unique solution of the quadratic balance condition (6) for given values of the other parameters, as in the previous section. Define $F : [0,\infty) \to (-\infty,\infty)$ by

$$F(\rho) = \rho \cdot \bar{u}_\rho^2 - \sigma \cdot m(\bar{u}_\rho). \tag{11}$$

From Lemma 5, it suffices to show that there exists a unique $\rho^* \geq 0$ satisfying $F(\rho^*) = 0$, with $v = u_{\rho^*}$ then being the unique solution to (3).

The function F is increasing in ρ: $\rho \cdot \bar{u}_\rho^2$ is increasing (Lemma 4), $\bar{u}_\rho$ is non-increasing (Lemma 2), m is non-decreasing (Assumption 1), and $\sigma \geq 0$ holds. Hence, the equation $F(\rho) = 0$ has at most one solution.

We have $F(0) \leq 0$ because $m(\overline{u}_0)$ is non-negative. In addition, the function F is continuous in ρ because $\bar{u}_\rho$ is continuous in ρ (Lemma 1) and m is continuous (Assumption 1). Hence, by the intermediate value theorem, a solution to the equation $F(\rho) = 0$ exists if the increasing function F grows without bounds as $\rho \to \infty$ or satisfies $F^* > 0$, where $F^* = \lim_{\rho\to\infty} F(\rho)$.

As $\rho \cdot \bar{u}_\rho^2$ is increasing and non-negative, it either grows without bounds as $\rho \to \infty$ or satisfies $a^* > 0$, where $a^* = \lim_{\rho\to\infty} \rho \cdot \bar{u}_\rho^2$. In the first of these cases, we have that F also grows without bounds as $\rho \to \infty$ (because, by Assumption 1 and Lemma 2, $m(\bar{u}_\rho)$ is non-increasing in ρ and bounded below by zero). In the second of these cases, we must have $\lim_{\rho\to\infty} \bar{u}_\rho = 0$, which implies (by Assumption 1) $\lim_{\rho\to\infty} m(\bar{u}_\rho) = 0$. Therefore, we have $F^* = a^* > 0$ in this case. □

The regularity conditions in Assumption 1 are natural. In addition, they are the minimal conditions under which Proposition 1 can be obtained. Specifically, the condition that the aggregate meeting rate is non-decreasing is necessary to exclude the possibility of multiple steady-states and, given that the aggregate meeting rate is non-decreasing, the other two conditions (continuity and boundary behavior) are necessary to exclude the possibility that a steady state does not exist. Appendix A.6 exhibits simple examples that validate these claims.

Having established existence and uniqueness, we next generalize the continuity claim in Lemma 1 to general search technologies, thereby showing that the fundamental matching lemma from Shimer and Smith (2000) holds for all search technologies satisfying Assumption 1.[12]

Proposition 2. *Let Assumption 1 hold. The unique solution $v \in \mathcal{D}$ to the general balance condition* (3) *is a jointly continuous function of σ and α.*

Proof. To allow us to distinguish between the solution of the quadratic balance condition (6) as a function of ρ and α and the solution to the general balance condition (3) as a function of σ and α, we denote the former (as in Section 3) by $u_{\rho,\alpha}$ and the latter by $v_{\sigma,\alpha}$.

[12] As in Lemma 1, the continuity claim in Proposition 2 is with respect to the pseudometrics induced by the L^1 norms on $\mathcal{A}$ and $\mathcal{D}$. Thus (cf. footnote 6), for given σ, the continuity claim in Proposition 2 generalizes the continuity result in the fundamental matching lemma as stated in Shimer and Smith (2000, Lemma 4) from quadratic search technologies to all search technologies satisfying Assumption 1.

Let $(\sigma_n, \alpha_n)_{n\in\mathbb{N}}$ be a sequence in $[0,\infty) \times \mathcal{A}$ converging to $(\sigma_0, \alpha_0) \in [0,\infty) \times \mathcal{A}$. Let $v_n = v_{\sigma_n,\alpha_n}$ for all $n \in \mathbb{N}_0$. We have to show that the sequence $(v_n)_{n\in\mathbb{N}}$ converges to v_0.

From Lemma 5, there exists a sequence $(\rho_n)_{n\in\mathbb{N}}$ in $[0,\infty)$ such that $v_n = u_{\rho_n,\alpha_n}$ holds for all n. Similarly, there exists $\rho_0 \geq 0$ such that $v_0 = u_{\rho_0,\alpha_0}$. Lemma 1 has established that $u_{\rho,\alpha}$ is continuous in (ρ, α). Hence, to complete the proof, it suffices to show that $(\rho_n)_{n\in\mathbb{N}}$ converges to ρ_0.

Lemma 5 implies

$$\rho_n \cdot \bar{v}_n^2 - \sigma_n \cdot m(\bar{v}_n) = 0, \quad \forall n \in \mathbb{N}_0. \tag{12}$$

Suppose that $(\rho_n)_{n\in\mathbb{N}}$ has a converging subsequence with finite limit ρ^*. By continuity of $u_{\rho,\alpha}$, the corresponding subsequence of $(v_n)_{n\in\mathbb{N}}$ then converges to $v^* = u_{\rho^*,\alpha_0}$. As m is continuous (Assumption 1) and (12) holds along the subsequence, we obtain

$$\rho^* \cdot \overline{u}^2_{\rho^*,\alpha_0} - \sigma_0 \cdot m(\overline{u}_{\rho^*,\alpha_0}) = 0,$$

which implies $\rho^* = \rho_0$ (because, for given α and σ, equation (10) has a unique solution; cf. the proof of Proposition 1). As this holds for all converging subsequences of $(\rho_n)_{n\in\mathbb{N}}$, it remains to exclude the possibility that $(\rho_n)_{n\in\mathbb{N}}$ has a subsequence diverging to infinity. We do so in Appendix A.7. $\square$

Finally, we consider the counterparts of the comparative static results in Lemmas 2 to 4 for general search technologies. Using v_σ to denote the solution to (3) as a function of the parameter σ, the counterpart to Lemma 2 is the claim that the steady-state density $v_\sigma(x)$ is non-decreasing in σ, the counterpart to Lemma 3 is that $\sigma \cdot r(\bar{v}_\sigma) \cdot v_\sigma(x)$ is increasing in σ, and the counterpart to Lemma 4 is that the aggregate meeting rate $\hat{m}(\bar{v}_\sigma, \sigma) = \sigma \cdot m(\bar{v}_\sigma)$ is increasing in σ. All of these statements are true if the search technology not only satisfies Assumption 1 but the mild additional condition that the contact function m generates a positive mass of meetings when there is a positive mass of unmatched agents:

Proposition 3. *Let Assumption 1 hold, let m be positive on $(0,\infty)$, and let $v_\sigma \in \mathcal{D}$ denote the unique solution to the general balance condition (3) for given values of the other parameters. Then*

1. *The map $\sigma \to v_\sigma(x)$ is non-increasing for all $x \in [0,1]$.*
2. *The map $\sigma \to \sigma \cdot r(\bar{v}_\sigma) \cdot v_\sigma(x)$ is increasing for all $x \in [0,1]$.*
3. *The map $\sigma \to \sigma \cdot m(\bar{v}_\sigma)$ is increasing.*

Proof. Let u_ρ denote the unique solution of the quadratic balance condition (6) for given values of the other parameters. Let $\sigma_2 > \sigma_1 \geq 0$. From Lemma 5 there exist ρ_2 and ρ_1 satisfying $v_{\sigma_i} = u_{\rho_i}$ and

$$\rho_i \cdot \bar{u}^2_{\rho_i} - \sigma_i \cdot m(\bar{u}_{\rho_i}) = 0 \tag{13}$$

for $i = 1, 2$. As $u_{\rho_i} \in \mathcal{D}$ implies $\bar{u}_{\rho_1} > 0$, we have $m(\bar{u}_{\rho_1}) > 0$ from the assumption that m is positive on $(0, \infty)$. Therefore, (13) implies

$$\rho_1 \cdot \bar{u}_{\rho_1}^2 - \sigma_2 \cdot m(\bar{u}_{\rho_1}) < 0.$$

As $\rho \cdot \bar{u}_{\rho}^2 - \sigma_2 \cdot m(\bar{u}_{\rho})$ is increasing in ρ (cf. the proof of Proposition 1), it follows from the above inequality and (13) that $\rho_2 > \rho_1$. The first claim in the statement of the proposition is then immediate from Lemma 2 and $v_{\sigma_i} = u_{\rho_i}$. The second claim follows upon observing that equation (9) implies $\rho_i = \sigma_i \cdot r(\bar{v}_{\sigma_i})$, so that the inequality $\sigma_2 \cdot r(\bar{v}_{\sigma_2}) \cdot v_{\sigma_2}(x) > \sigma_1 \cdot r(\bar{v}_{\sigma_1}) \cdot v_{\sigma_1}(x)$ follows from Lemma 3 and $v_{\sigma_i} = u_{\rho_i}$. Finally, Lemma 4 implies $\rho_2 \cdot \bar{u}_{\rho_2}^2 > \rho_1 \cdot \bar{u}_{\rho_1}^2$, so that $v_{\sigma_i} = u_{\rho_i}$ and (13) imply $\sigma_2 \cdot m(\bar{v}_{\sigma_2}) > \sigma_1 \cdot m(\bar{v}_{\sigma_1})$. $\square$

5 Discussion

We conclude by discussing extensions of our analysis, conditions for the uniqueness of steady-state equilibrium, and related work identifying conditions for the uniqueness of steady states in other fields.

5.1 Extensions

We have followed Shimer and Smith (2000) and Smith (2006) in supposing that there is an exogenous population of infinitely lived agents who exit the unmatched pool only if they have found a partner and return to this pool when their partnership dissolves due to exogenous separation. Alternatively, we could have followed Burdett and Coles (1997) and others in supposing that (i) there is a constant exogenous inflow of "newborn" agents into the pool of unmatched agents given by a density $\hat{\ell}$, (ii) matches are permanent, so that matched partners never return to the pool of unmatched agents, and (iii) all unmatched agents abandon the search for a partner with an exogenous exit rate $\delta > 0$. Balancing in- and outflows for such a model yields the steady-state condition

$$\hat{\ell}(x) = u(x) \left[\delta + \sigma \cdot r(\bar{u}) \int_0^1 \alpha(x, y) u(y) dy \right], \quad \forall x \in [0, 1]. \tag{14}$$

Upon defining $\ell(x) = \hat{\ell}(x)/\delta$, it is immediate that (14) is equivalent to (2). Thus, all of our results carry over to this alternative specification of the search-and-matching process.

We have also followed Shimer and Smith (2000) and Smith (2006) in assuming that the search technology generates random meetings between *all* agents. This is in contrast to much of the theoretical and applied literature on search-and-matching that considers scenarios in which there are two distinct groups of agents (workers and firms, or men and women); the only meetings that are accounted for in these model are those that feature a pair of agents from distinct groups. All of our results can be extended to this case. We explain how this can be done in Appendix A.8. In particular, the fundamental matching lemma holds for all continuous aggregate meeting rates that are non-decreasing in the unmatched masses of both groups and satisfy the boundary condition that there are no meetings if either of the two groups has no unmatched agents. With the singular exception of the two-group version of the linear search technology (discussed in detail and dismissed as a reasonable model in Stevens, 2007), these conditions are satisfied for all specifications of the aggregate meeting rate considered in the extensive literature studying random search models in labor markets (Petrongolo and Pissarides, 2001; Rogerson, Shimer, and Wright, 2005). Combining the validity of the fundamental matching lemma for general search technologies with the arguments for the existence of steady-state equilibrium in Smith (2006) yields the existence of steady-state equilibrium in the two-group models in Burdett and Coles (1997) and Eeckhout (1999).[13]

It is an open question whether our results carry over to other variations of the search-and-matching-process, involving, for instance, different search intensities for different types or different dissolution rates for different matches. We are confident, though, that the techniques we have developed here are well-suited to study such extensions and to determine the conditions under which uniqueness of steady-states obtains for search technologies more general than the quadratic one.

Looking further afield, we believe that the logic underlying our comparative-statics results in Section 3 is of broader interest. We have already observed (cf. footnote 9) that arguments reminiscent of the ones proving Lemma 4 have been used in other contexts. Further, it is straightforward to verify that the proof of Lemma 2 goes through when the expression for the individual matching rate $\rho \int_0^1 \alpha(x, y)u(y)dy$ in (6) is replaced by any non-decreasing linear homogenous function of u. Given the prevalence of linear homogenous relationships in economic theory, this suggests that the techniques used in the proof of Lemma 2 may be applicable in other economic problems.[14]

5.2 Uniqueness of Steady-State Equilibrium

Recall from the Introduction that a steady-state equilibrium requires two conditions: the unmatched density must satisfy the steady-state condition and the matching affinities must form a partial equilibrium given the population of agents available for meetings (Burdett and Coles, 1999). The uniqueness of the solution to the steady-state condition eliminates one of the potential sources of multiplicity of steady-state equilibria. The uniqueness of the solution to the partial-equilibrium condition (which obtains, for instance, in the models considered in Burdett and Coles (1997), Eeckhout (1999), Shimer and Smith (2000), and Smith (2006)) eliminates another one. This leaves the feedback effects between the steady-state condition

[13] Eeckhout (1999) considers only the partial-equilibrium condition. Burdett and Coles (1997) establish the existence of steady-state equilibrium under restrictive assumptions, including the requirement that both groups have identical masses and that the type distributions are log-concave.

[14] As an illustrative example, we note that the proof of the "uniqueness of proportions" in the balanced growth model with constant returns to scale from Solow and Samuelson (1953, Section 6) follows a logic quite similar to the one we have explained after the statement of Lemma 2.

and the partial-equilibrium condition as the only possible source of a multiplicity of steady-state equilibria. Burdett and Coles (1997, Theorem 2) demonstrate that such feedback effects can be sufficiently strong to generate multiple steady-state equilibria when search frictions (which are captured by the velocity of the search technology in our setting) are intermediate. On the other hand, there is little scope for the distribution of unmatched types to influence the partial-equilibrium matching affinities when search frictions are either very high or very low (because, in the first case, the agents are inclined to accept all matches that provide them with strictly positive surplus, and, in the second case, matching affinities resemble those that arise in the frictionless limit). Thus, a natural conjecture is that our results can be used to obtain uniqueness results for steady-state equilibria in such circumstances. Support for this conjecture is provided by the fact that with the general version of the fundamental matching lemma in place, the uniqueness result for steady-state equilibria in Lauermann and Nöldeke (2014, Corollary 3) extends immediately from quadratic search technologies to all search technologies satisfying our regularity conditions.

5.3 Related Work in other Fields

Steady-state conditions similar to the one considered here (albeit with a finite number of types) have been investigated in disciplines other than economics, namely, in the demographic literature concerned with modeling the contact structure that leads to marriages (and the subsequent production of offspring) and in the literature on chemical reaction networks.[15] An example of the former is Heesterbeek and Metz (1993, Section 2), who prove the existence and uniqueness of steady state for quadratic search technologies in a model with match-dependent dissolution rates.

To see why there is a relationship between the kind of search-and-matching process we study and chemical reaction networks, observe that the formation and dissolution of partnerships in our model is akin to the formation and dissolution of a chemical bond between two molecules. A recent survey of the conditions yielding uniqueness of steady states in chemical reactions networks is contained in Shinar and Feinberg (2012). These conditions are different from our Assumption 1 (as they do not allow for the kind of "crowding externality" captured in the dependence of the aggregate meeting rate on $\bar{u}$), with the only overlap occurring for the case of a quadratic search technology (known as mass action kinetics in the natural sciences), for which existence and uniqueness results were first obtained in the late 1970s. For a recent application of a result from the literature on chemical reaction networks to an economic problem see Häfner and Nöldeke (2017, Lemma 1), who obtain the existence of a unique steady state in a model of iterated incumbency contests by adapting a result from Banaji and Baigent (2008).

[15] There is also an extensive literature in behavioral ecology using search models to study the evolution of sex roles, but most of this literature does not consider the possibility that agents of a given sex may differ in their types. Notable exceptions are Johnstone, Reynolds, and Deutsch (1996), who study a model that is, in essence, a version of the model considered in Burdett and Coles (1997) with a quadratic search technology, and Kokko and Johnstone (2002).

Appendix

A.1 Proof of Lemma 2

We proceed in three steps. The first step establishes that an unmatched density solving the quadratic balance condition (6) is not only bounded above but also bounded away from zero. This ensures that all expressions considered in the following two steps are finite. The second step shows that for any two velocities ρ_1 and ρ_2 the corresponding solutions to (6), denoted by u_1 and u_2 for simplicity, are ordered, that is, either $u_1(x) \geq u_2(x)$ holds for all $x \in [0,1]$ or the reverse inequality holds for all $x \in [0,1]$. The third step excludes the possibility that a higher meeting rate leads to a larger unmatched steady-state density, thereby finishing the proof that $u_\rho(x)$ is non-increasing in ρ.

Throughout the proof we eschew making use of the uniqueness result from Lemma 1, thereby clarifying that this property plays no role in our argument. Rather, as we explain in a remark at the end of proof, uniqueness of the solution to the quadratic balance condition (6) can be inferred from our argument.

Step 1: Consider any $(\rho, \alpha) \in [0, \infty) \times \mathcal{A}$ and let $u \in \mathcal{D}$ be an unmatched density solving (6). As the population density ℓ is bounded above and bounded away from zero, there exist $\underline{\ell}$ and $\overline{\ell}$ such that $0 < \underline{\ell} \leq \ell(x) \leq \overline{\ell} < \infty$ holds for all $x \in [0,1]$. As u satisfies $0 < u(x) \leq \ell(x)$ for all $x \in [0,1]$, it is immediate that $u(x) \leq \overline{\ell}$ holds for all $x \in [0,1]$. Using this upper bound and that $0 \leq \alpha(x,y) \leq 1$ holds for all (x,y), we also have

$$\int_0^1 \alpha(x,y)u(y)dy \leq \overline{\ell} \tag{15}$$

for all x. Hence, (6) implies $\ell(x) \leq u(x)\left[1 + \rho\overline{\ell}\right]$ for all x. Thus,

$$0 < \frac{\underline{\ell}}{1 + \rho\overline{\ell}} \leq u(x) \leq \overline{\ell} < \infty \quad \forall x \in [0,1]. \tag{16}$$

Step 2: Let $u_1 \in \mathcal{D}$ and $u_2 \in \mathcal{D}$ be solutions to the quadratic balance conditions (6) for velocities $\rho_1 \geq 0$ and $\rho_2 \geq 0$ respectively:

$$\ell(x) = u_1(x)\left[1 + \rho_1 \int_0^1 \alpha(x,y)u_1(y)dy\right] \quad \forall x \in [0,1], \tag{17}$$

$$\ell(x) = u_2(x)\left[1 + \rho_2 \int_0^1 \alpha(x,y)u_2(y)dy\right] \quad \forall x \in [0,1]. \tag{18}$$

From (16) in Step 1 we have that

$$\lambda_1 = \sup_{x\in[0,1]} \frac{u_1(x)}{u_2(x)} \tag{19}$$

$$\lambda_2 = \sup_{x\in[0,1]} \frac{u_2(x)}{u_1(x)} \tag{20}$$

are both finite and positive. We now argue that at most one of these two numbers can be strictly greater than 1, which implies that u_1 and u_2 are ordered, i.e., either $u_1 \leq u_2$ or $u_2 \leq u_1$ holds.

Suppose $\lambda_2 > 1$ holds. The following shows that this implies $\lambda_1\rho_1 > \lambda_2\rho_2$. Suppose not, so that we have $\lambda_2\rho_2 \geq \lambda_1\rho_1$. Then

$$\begin{aligned} u_2(x)\left[1+\rho_2\int_0^1 \alpha(x,y)u_2(y)dy\right] &= u_1(x)\left[1+\rho_1\int_0^1 \alpha(x,y)u_1(y)dy\right] \\ &\leq u_1(x)\left[1+\lambda_1\rho_1\int_0^1 \alpha(x,y)u_2(y)dy\right] \\ &\leq u_1(x)\left[1+\lambda_2\rho_2\int_0^1 \alpha(x,y)u_2(y)dy\right], \end{aligned}$$

for all $x \in [0,1]$, where the equality in the first line is from (17) and (18), the first inequality holds because (19) implies $\lambda_1 u_2(y) \geq u_1(y)$, and the second inequality is from the hypothesis $\lambda_2\rho_2 \geq \lambda_1\rho_1$. Consequently, we obtain

$$\frac{u_2(x)}{u_1(x)} \leq \frac{1+\lambda_2\rho_2\int_0^1 \alpha(x,y)u_2(y)dy}{1+\rho_2\int_0^1 \alpha(x,y)u_2(y)dy}$$

for all $x \in [0,1]$ and thus

$$\lambda_2 \leq \sup_{x\in[0,1]} \left[\frac{1+\lambda_2\rho_2\int_0^1 \alpha(x,y)u_2(y)dy}{1+\rho_2\int_0^1 \alpha(x,y)u_2(y)dy}\right]. \tag{21}$$

On the other hand, using the hypothesis $\lambda_2 > 1$ and (15), we have

$$\frac{1+\lambda_2\rho_2\int_0^1 \alpha(x,y)u_2(y)dy}{1+\rho_2\int_0^1 \alpha(x,y)u_2(y)dy} \leq \frac{1+\lambda_2\rho_2\bar{\ell}}{1+\rho_2\bar{\ell}} < \lambda_2 \quad \forall x \in [0,1]. \tag{22}$$

From (22) we obtain

$$\lambda_2 > \sup_{x\in[0,1]} \left[\frac{1+\lambda_2\rho_2\int_0^1 \alpha(x,y)u_2(y)dy}{1+\rho_2\int_0^1 \alpha(x,y)u_2(y)dy}\right].$$

and thereby a contradiction to (21). Consequently, $\lambda_2 > 1$ implies $\lambda_1\rho_1 > \lambda_2\rho_2$.

Exchanging the roles of λ_1 and λ_2 in the above argument yields that $\lambda_1 > 1$ implies $\lambda_2\rho_2 > \lambda_1\rho_1$. As at most one of the inequalities $\lambda_1\rho_1 > \lambda_2\rho_2$ and $\lambda_2\rho_2 > \lambda_1\rho_1$ can hold, it follows that $\lambda_1 \leq 1$ or $\lambda_2 \leq 1$ (or both) must hold.

STEP 3: Suppose $\rho_2 \geq \rho_1$. If $\lambda_1 > 1$ holds, then $\lambda_2 \leq 1$ is immediate from the conclusion of Step 2. By the definition of λ_2 in (20) this implies $u_2(x) \leq u_1(x)$ for all $x \in [0,1]$, which is the desired result.

It remains to consider the case $\lambda_1 \leq 1$. From the definition of λ_1 in (19) this implies $u_2(y) \geq u_1(y)$ for all $y \in [0,1]$. Together with the inequality $\rho_2 \geq \rho_1$ this yields

$$\rho_2 \int_0^1 \alpha(x,y)u_2(y)dy \geq \rho_1 \int_0^1 \alpha(x,y)u_1(y)dy \quad \forall x \in [0,1].$$

It is then immediate from (17) and (18) that $u_2(x) \leq u_1(x)$ holds for all $x \in [0,1]$, finishing the proof.

REMARK: Applying the argument from Step 3 to the case $\rho_1 \geq \rho_2$ yields $u_1(x) \leq u_2(x)$ for all $x \in [0,1]$. Consequently, for $\rho_1 = \rho_2$ we have $u_1(x) = u_2(x)$ for all $x \in [0,1]$, showing that (6) cannot have more than one solution for given ρ (and α).

A.2 Proof of Lemma 3

We use the same notation as in the proof of Lemma 2.

Suppose $\rho \cdot u_\rho(x)$ is not increasing in ρ for all $x \in [0,1]$. There then exists $\rho_2 > \rho_1 > 0$ and $x' \in [0,1]$ such that $\rho_2 \cdot u_2(x') \leq \rho_1 \cdot u_1(x')$ holds. We then have $u_1(x') > u_2(x')$ and therefore $\lambda_1 > 1$. From Step 2 in the proof of Lemma 2 this implies $\lambda_2 \leq 1$ and $\lambda_2\rho_2 > \lambda_1\rho_1$. In particular, we have $\rho_2 > \lambda_1\rho_1$. From the definition of λ_1 in (19) this yields $\rho_2 \cdot u_2(x) > \rho_1 \cdot u_1(x)$ for all $x \in [0,1]$, contradicting the hypothesis that the inequality $\rho_2 \cdot u_2(x') \leq \rho_1 \cdot u_1(x')$ holds for some $x' \in [0,1]$.

A.3 Proof of Lemma 4

As $\rho \cdot \bar{u}_\rho^2$ is equal to zero for $\rho = 0$ and positive otherwise, it suffices to consider $\rho > 0$. We proceed in two steps.

STEP 1: For any $\rho > 0$ define $s_\rho \in \mathcal{D}$ by

$$s_\rho(x) = \sqrt{\rho} \cdot u_\rho(x), \quad \forall x \in [0,1]. \tag{23}$$

Using that u_ρ is the unique positive solution to (6), it is immediate from (23) that s_ρ is the unique positive solution to (7) for $\gamma = 1/\sqrt{\rho} > 0$. From (23) we also have

$$\bar{s}_\rho = \sqrt{\rho \cdot \bar{u}_\rho^2}.$$

Therefore, using w_γ to denote the unique positive solution to (7) as a function of γ, it suffices to show that the map $\gamma \to \bar{w}_\gamma$ from $(0,\infty)$ to $(0,\infty)$ is decreasing in γ to establish the lemma.

STEP 2: Let $\mathcal{V}$ be the subspace of essentially bounded functions in $L^2[0,1]$. For all $v \in \mathcal{V}$ and $\gamma > 0$ define

$$H(v,\gamma) = \gamma \int_0^1 e^{v(x)}dx + \frac{1}{2}\int_0^1\int_0^1 \alpha(x,y)e^{v(x)+v(y)}dxdy - \int v(x)\ell(x)dx. \tag{24}$$

The function H is convex in v. It is also continuous in v. Its derivative (with respect to v) is the bounded linear operator H_v on $L^2[0,1]$ defined by

$$\begin{aligned} H_v(v,\gamma)(h) &= \lim_{t\to 0}\frac{H(v+th,\gamma)-H(v,\gamma)}{t} \\ &= \gamma\int_0^1 e^{v(x)}h(x)dx + \frac{1}{2}\int_0^1\int_0^1 \alpha(x,y)e^{v(x)+v(y)}\left[h(x)+h(y)\right]dxdy \\ &- \int_0^1 \ell(x)h(x)dx. \end{aligned}$$

Using the symmetry condition $\alpha(x,y) = \alpha(y,x)$, we obtain

$$H_v(v,\gamma)(h) = \gamma\int_0^1 e^{v(x)}h(x)dx + \int_0^1\int_0^1 \alpha(x,y)e^{v(x)+v(y)}h(x)dxdy - \int_0^1 \ell(x)h(x)dx. \tag{25}$$

Because H is convex, it follows from (25) that v minimizes $H(v,\gamma)$ over $\mathcal{V}$ if and only if the first order condition

$$\gamma e^{v(x)} + e^{v(x)}\int_0^1 \alpha(x,y)e^{v(y)}dy - \ell(x) = 0 \tag{26}$$

holds for almost all $x \in [0,1]$.

Applying the transformation $w = e^v$ to (24) and (26) and comparing the resulting first order condition with (7) we obtain that w_γ minimizes

$$G(w,\gamma) = \gamma\int_0^1 w(x)dx + \frac{1}{2}\int_0^1\int_0^1 \alpha(x,y)w(x)w(y)dxdy - \int_0^1 \ell(x)\ln(w(x))dx \tag{27}$$

over the set of all positive, essentially bounded functions in $L^2[0,1]$. Further, as w_γ is uniquely determined, any other such minimizer of $G(w,\gamma)$ must agree with w_γ for almost all $x \in [0,1]$.

Consider now $\gamma_2 > \gamma_1 > 0$. Because w_{γ_1} minimizes $G(w,\gamma_1)$ and w_{γ_2} minimizes $G(w,\gamma_2)$, these minimizers are essentially unique, and (7) precludes the possibility that $w_{\gamma_1} = w_{\gamma_2}$ holds for almost all x, we have

$$[G(w_{\gamma_1}, \gamma_1) - G(w_{\gamma_2}, \gamma_1)] + [G(w_{\gamma_2}, \gamma_2) - G(w_{\gamma_1}, \gamma_2)] < 0. \tag{28}$$

Substituting from (27) into (28) yields

$$[\gamma_2 - \gamma_1]\,[\bar{w}_{\gamma_2} - \bar{w}_{\gamma_1}] < 0.$$

Hence, as was to be shown, $\bar{w}_\gamma$ is decreasing in γ.

A.4 Proof that $\rho \cdot u_\rho(x) \cdot u_\rho(y)$ may fail to to be increasing in ρ for all (x, y)

To establish our claim, we consider an example with a finite number of types (cf. Remark 1). As we can always reinterpret such a model as one with a continuum of types, this suffices to establish our claim.[16]

Let $n = 2$, so that there are only two types, let the matching affinities be given by $\alpha(1,1) = \alpha(2,2) = 0$ and $\alpha(1,2) = \alpha(2,1) = 1$, and impose the normalization $\delta = 1$. Then, with a quadratic search technology, the steady-state conditions (4) can be written as

$$u_\rho(1) + \rho \cdot u_\rho(1) \cdot u_\rho(2) - \ell(1) = 0, \tag{29}$$

$$u_\rho(2) + \rho \cdot u_\rho(1) \cdot u_\rho(2) - \ell(2) = 0. \tag{30}$$

Applying the implicit function theorem to (29) – (30) we obtain the derivatives

$$\frac{du_\rho(1)}{d\rho} = \frac{du_\rho(2)}{d\rho} = -\frac{u_\rho(1) \cdot u_\rho(2)}{\Delta_\rho} \tag{31}$$

where

$$\Delta_\rho = 1 + \rho \cdot u_\rho(1) + \rho \cdot u_\rho(2) > 0. \tag{32}$$

To determine the effect of an increase in ρ on, say, $\rho \cdot u_\rho(1) \cdot u_\rho(1) = \rho \cdot u_\rho(1)^2$, we apply the product rule, (31), and (32) to obtain

$$\begin{aligned}\frac{d(\rho \cdot u_\rho(1)^2)}{d\rho} &= u_\rho(1)^2 - \frac{2\rho \cdot u_\rho(1)^2 \cdot u_\rho(2)}{1 + \rho \cdot u_\rho(1) + \rho \cdot u_\rho(2)} \\ &= \frac{u_\rho(1)^2}{1 + \rho \cdot u_\rho(1) + \rho \cdot u_\rho(2)} \left(1 + \rho \cdot (u_\rho(1) - u_\rho(2))\right). \end{aligned} \tag{33}$$

[16] Formally, the model with a finite number of types can be embedded into our setting by supposing that the type space $[0,1]$ can be partitioned into n measurable sets with the property that both $\ell(x)$ and $\alpha(x, \cdot)$ are constant on each of the sets in this partition. The uniqueness result (Lemma 1) for the quadratic search technology under consideration here implies that in steady-state $u(x)$ is constant on each element of the partition, so that there are only n steady-state conditions, corresponding to (4), to consider.

The sign of (33) is identical to the sign of $1 + \rho \cdot (u_\rho(1) - u_\rho(2))$ and thus is strictly negative whenever $u_\rho(2) - u_\rho(1) > 1/\rho$ holds. As it is immediate from (29) and (30) that $u_\rho(2) - u_\rho(1) = \ell(2) - \ell(1)$ holds, it follows that $\rho \cdot u_\rho(1)^2$ is unimodal (rather than increasing) in ρ whenever $\ell(2) - \ell(1) > 0$ holds (with the unique maximum of $\rho \cdot u_\rho(1)^2$ occurring at $\hat{\rho} = 1/(\ell(2) - \ell(1))$.

The above establishes that with $x = y = 1$ the expression $\rho \cdot u_\rho(x) \cdot u_\rho(y)$ is decreasing in ρ for sufficiently large ρ. To show that, as claimed in the main body of the paper, the expression $\rho \cdot \alpha(x, y) \cdot u_\rho(x) \cdot u_\rho(y)$ may also be decreasing in ρ, it suffices to observe that by the joint continuity of the steady-state density in ρ and α (Lemma 1), it is still the case that $\rho \cdot u_\rho(x) \cdot u_\rho(y)$ is decreasing in ρ for sufficiently large ρ if the assumption $\alpha(1, 1) = 0$ in the above example is replaced by $\alpha(1, 1) = \epsilon > 0$ for sufficiently small ϵ.

A.5 Comparative Statics for the Population Density in Remark 2.

For $\tau > 0$, consider the counterpart to the quadratic balance condition (6)

$$\tau \cdot \ell(x) = u(x) \left[1 + \rho \int_0^1 \alpha(x, y) u(y) dy\right] \quad \forall x \in [0, 1] \tag{34}$$

and let u_τ denote its unique solution as a function of τ.

Letting $\gamma = 1/\sqrt{\tau}$ and $v = \gamma \cdot u$, (34) can be rewritten as

$$\ell(x) = v(x) \left[\gamma + \rho \int_0^1 \alpha(x, y) v(y) dy\right] \quad \forall x \in [0, 1],$$

which is analogous to (7) and therefore has a unique solution v_γ satisfying that $\bar{v}_\gamma$ is decreasing in γ (cf. Step 2 in the proof of Lemma 4). Consequently, $\bar{u}_\tau/\sqrt{\tau}$ is increasing in τ (as an increase in τ corresponds to a decrease in γ), so that $\bar{u}_\tau$ is increasing in τ.

A.6 Minimality of the Regularity Conditions in Assumption 1

Suppose that the population density is constant and thus equal to $l > 0$ for all $x \in [0, 1]$. Suppose, in addition that all meetings lead to matches, so that the matching affinities are given by $\alpha(x, y) = 1$ for all (x, y). The general balance condition (3) then simplifies to

$$l = u(x) \left[1 + \sigma \cdot r(\bar{u})\bar{u}\right], \quad \forall x \in [0, 1].$$

This condition is satisfied if and only if $u(x) = \bar{u}$ holds for all $x \in [0, 1]$ and, using (1), the mass of unmatched agents solves

$$l = \bar{u} + \sigma \cdot m(\bar{u}). \tag{35}$$

In particular, a solution to (3) exists (is unique) if and only if a solution to (35) exists (is unique). Therefore, to validate the claims in the main body of the paper, it suffices to show that (i) for suitable choices of the parameters $\delta > 0$ and $l > 0$, condition (35) has multiple solutions if the aggregate meeting rate fails to be non-decreasing, and, if the aggregate meeting rate is non-decreasing, the parameters δ and l can be chosen such that no solution to (35) exists if (ii) m fails the boundary condition or (iii) m fails to be continuous.

i) Suppose that m fails to be non-decreasing. Then, there exist $\hat{u}$ and $u^\dagger$ satisfying $\hat{u} > u^\dagger > 0$ and $m(u^\dagger) > m(\hat{u})$. Setting $\sigma = [\hat{u} - u^\dagger]/[m(u^\dagger) - m(\hat{u})] > 0$ and $l = \hat{u} + \sigma \cdot m(\hat{u})$, it is immediate that $\hat{u}$ satisfies (35) and easily verified that $u^\dagger$ does so, too.

ii) Suppose that m is non-decreasing and $m(0) > 0$ holds. Then, as the right side of (35) is increasing in $\bar{u}$, equation (35) has no solution for $\sigma > 0$ and $l > 0$ satisfying $0 < l/\sigma < m(0)$.

iii) Suppose that m is non-decreasing but not continuous. There then exists at least one $\hat{u} \geq 0$ such that $\tilde{m}$ jumps up at $\hat{u}$. If $\hat{u} = 0$, then an argument analogous to the one in the preceding paragraph shows that (35) has no solution if $m(0) < l/\sigma < \lim_{\bar{u}\downarrow 0} m(\bar{u})$ holds. If $\hat{u} > 0$, then there exists a number $M > 0$ such that $M \neq m(\hat{u})$, $\bar{u} < \hat{u}$ implies $m(\bar{u}) < M$ and $\bar{u} > \hat{u}$ implies $m(\bar{u}) > M$. Fix such M and let $\sigma > 0$ and $l > 0$ be such that $l = \hat{u} + \sigma M$. By construction, we then have that $\hat{u}$ does not solve (35). Further, as the right side of (35) is increasing in $\bar{u}$ and $\bar{u} > \hat{u}$ implies $m(\bar{u}) > M$, also no $\bar{u} > \hat{u}$ solves (35). An analogous argument shows that (35) also has no solution with $\bar{u} < \hat{u}$.

A.7 Completion of the Proof of Proposition 2

Suppose, without loss of generality, that $(\rho_n)_{n\in\mathbb{N}}$ itself diverges to infinity. Observe that the converging sequence $(\sigma_n)_{n\in\mathbb{N}}$ is bounded and that $(m(\bar{v}_n))_{n\in\mathbb{N}}$ is bounded, too, because $v_n \leq \ell$ holds for all $n \in \mathbb{N}$ and so Assumption 1 implies $m(\bar{v}_n) \leq m(\int_0^1 \ell(x)dx))$. Thus, $(\sigma_n \cdot m(\bar{v}_n))_{n\in\mathbb{N}}$ is bounded, and it follows from (12) that $(\bar{v}_n)_{n\in\mathbb{N}}$ converges to zero. From Assumption 1, this implies that $(\sigma_n \cdot m(\bar{v}_n))_{n\in\mathbb{N}}$ converges to zero. Thus,

$$\lim_{n\to\infty} \bar{v}_n + \sigma_n \cdot m(\bar{v}_n) = 0.$$

But this is impossible: the population density ℓ is bounded away from zero and we have

$$\begin{aligned}\int_0^1 \ell(x) &= \bar{v}_n + \sigma_n \cdot r(\bar{v}_n) \int_0^1 \int_0^1 \alpha_n(x,y) v_n(y) v_n(x) dy dx \\ &\leq \bar{v}_n + \sigma_n \cdot r(\bar{v}_n) \cdot \bar{v}_n^2 \\ &= \bar{v}_n + \sigma_n \cdot m(\bar{v}_n)\end{aligned}$$

for all $n \in \mathbb{N}$, where the equality in the first line is from integrating (3) with respect to x, the inequality in the second line uses $0 \leq \alpha_n(x,y) \leq 1$, and the equality in the third line is from (1).

A.8 Random Meetings between Two Groups

We modify the search-and-matching process from Section 2 to incorporate two groups as follows: First, we suppose that the interval $[0,1]$ is partitioned into two measurable sets A and B such that $\int_A \ell(x)dx > 0$ and $\int_B \ell(x)dx > 0$ holds. Agents with types in A are the members of group A and agents with types in B are the members of group B. Second, the aggregate meeting rate only accounts for meetings featuring unmatched agents from distinct groups and depends on the unmatched masses $\bar{u}^A = \int_A u(x)dx$ and $\bar{u}^B = \int_B u(x)dx$ rather than just on $\bar{u}$. The counterpart to Assumption 1 is

Assumption 3 (General Search Technology with Two Groups)**.** *For any unmatched density $u \in \mathcal{D}$, the aggregate meeting rate is given by $\sigma \cdot m(\bar{u}^A, \bar{u}^B)$, where $\sigma \geq 0$ and $m : [0,\infty)^2 \to [0,\infty)$ is continuous, non-decreasing in both arguments, and satisfies $m(\bar{u}^A, 0) = m(0, \bar{u}^B) = 0$.*

Let $r : (0,\infty)^2 \to [0,\infty)$ be given by

$$r(\bar{u}^A, \bar{u}^B) = \frac{m(\bar{u}^A, \bar{u}^B)}{\bar{u}^A \cdot \bar{u}^B}. \tag{36}$$

The assumption of random meetings between the unmatched members of the two groups implies that each unmatched agent from group A meets some unmatched agent from group B at the individual meeting rate $\sigma \cdot r(\bar{u}_A, \bar{u}_B) \cdot \bar{u}_B$, whereas each unmatched agent from group B meets some unmatched agent from group A at the individual meeting rate $\sigma \cdot r(\bar{u}_A, \bar{u}_B) \cdot \bar{u}_A$.

With these modifications, the counterpart to the general balance condition (3) is

$$\ell(x) = u(x) \left[1 + \sigma \cdot r(\bar{u}^A, \bar{u}^B) \int_B \alpha(x,y) u(y) dy \right] \quad \forall x \in A, \tag{37a}$$

$$\ell(x) = u(x) \left[1 + \sigma \cdot r(\bar{u}^A, \bar{u}^B) \int_A \alpha(x,y) u(y) dy \right] \quad \forall x \in B, \tag{37b}$$

where $0 \leq \alpha(x,y) = \alpha(y,x) \leq 1$ is probability that a meeting between a pair of agents with types (x,y) leads to a match, for $(x,y) \in (A \times B) \cup (B \times A)$. Extending the definition of the matching affinities α to $[0,1]^2$ by setting $\alpha(x,y) = \alpha(y,x) = 0$ for all $(x,y) \in (A \times A) \cup (B \times B)$, the steady-state condition (37) can also be written as

$$\ell(x) = u(x) \left[1 + \sigma \cdot r(\bar{u}^A, \bar{u}^B) \int_0^1 \alpha(x,y) u(y) dy \right] \quad \forall x \in [0,1], \tag{38}$$

which provides us with a more natural counterpart to (3).

With a quadratic search technology, the aggregate meeting rate is equal to the contact rate (i.e., we again set $\sigma = 1$ for quadratic search technologies) and given by

$$m(\bar{u}^A, \bar{u}^B) = \rho \cdot \bar{u}^A \cdot \bar{u}^B. \tag{39}$$

Using (36) and $\sigma = 1$, this implies that $\sigma \cdot r(\bar{u}^A, \bar{u}^B)$ is constant and equal to ρ for such technologies. Hence, for quadratic search technologies (38) is identical to (6). Consequently, all results from Section 3 hold for the model with two groups.

Using (38) in lieu of (3), the counterpart to (9) for the two-group model is

$$\rho = \sigma \cdot r(\bar{v}^A, \bar{v}^B). \tag{40}$$

Using (36), equation (40) in turn implies that condition (10) in Lemma 5 becomes

$$\rho \cdot \bar{v}^A \cdot \bar{v}^B - \sigma \cdot m(\bar{v}^A, \bar{v}^B) = 0. \tag{41}$$

If the map $\rho \to \rho \cdot \bar{u}^A_\rho \cdot \bar{u}^B_\rho$ is increasing and Assumption 3 holds, then the proofs of Propositions 1 – 3 go through with minor modifications to show that these results hold when the general balance condition is given by (38) rather than (3).[17] In addition, it is immediate that the third statement in Proposition 3 can be modified to obtain that the aggregate steady-state meeting rate $\sigma \cdot m(\bar{v}^A_\sigma, \bar{v}^B_\sigma)$ is increasing in the velocity σ of a general search technology.

To finish the argument it remains to show that the map $\rho \to \rho \cdot \bar{u}^A_\rho \cdot \bar{u}^B_\rho$ is increasing. To do so, we make substantial use of the assumption that matches only occur between agents of distinct groups, so that (cf. (37)) the quadratic balance condition can be written as

$$\ell(x) = u(x)\left[1 + \rho \int_B \alpha(x,y)u(y)dy\right] \quad \forall x \in A, \tag{42a}$$

$$\ell(x) = u(x)\left[1 + \rho \int_A \alpha(x,y)u(y)dy\right] \quad \forall x \in B. \tag{42b}$$

For $\rho > 0$ let

$$\gamma^A(\rho) = \frac{\sqrt{\bar{u}^A_\rho}}{\sqrt{\rho \cdot \bar{u}^B_\rho}} > 0 \text{ and } \gamma^B(\rho) = \frac{\sqrt{\bar{u}^B_\rho}}{\sqrt{\rho \cdot \bar{u}^A_\rho}} > 0, \tag{43}$$

and define $s_\rho \in \mathcal{D}$ by

$$s_\rho(x) = \begin{cases} u_\rho(x)/\gamma^A(\rho) & \text{if } x \in A, \\ u_\rho(x)/\gamma^B(\rho) & \text{if } x \in B. \end{cases} \tag{44}$$

Arguments analogous to those in Step 1 of the proof of Lemma 4 in Appendix A.3 then

[17] The only somewhat substantial modification required is in the part of the proof of Proposition 2 that we have relegated to Appendix A.7. An argument analogous to the one in the first paragraph in Appendix A.7 allows us to conclude that $\lim_{n\to\infty} \bar{v}^A_n + \sigma_n m(\bar{v}^A_n, \bar{v}^B_n) = 0$ or $\lim_{n\to\infty} \bar{v}^B_n + \sigma_n m(\bar{v}^A_n, \bar{v}^B_n) = 0$ must hold when $(\rho_n)_{n\in\mathcal{N}}$ diverges to infinity. On the other hand, integrating (37a) over the types in A and (37b) over the types in B yields $\int_A \ell(x) \le \bar{v}^A_n + \sigma_n m(\bar{v}^A_n, \bar{v}^B_n)$ and $\int_B \ell(x) \le \bar{v}^B_n + \sigma_n m(\bar{v}^A_n, \bar{v}^B_n)$ respectively. Because we have assumed $\int_A \ell(x) > 0$ and $\int_B \ell(x) > 0$, this yields a contradiction.

show that s_ρ is the unique solution to

$$\ell(x) = w(x)\left[\gamma^A(\rho) + \int_B \alpha(x,y)w(y)dy\right] \quad \forall x \in A, \tag{45a}$$

$$\ell(x) = w(x)\left[\gamma^B(\rho) + \int_A \alpha(x,y)w(y)dy\right] \quad \forall x \in B. \tag{45b}$$

In addition, considering the function

$$H(v,\gamma^A,\gamma^B) = \gamma^A \int_A e^{v(x)}dx + \gamma^B \int_B e^{v(y))}dy + \int_0^1 \int_0^1 \alpha(x,y)e^{v(x)+v(y)}dxdy - \int_0^1 v(x)\ell(x)dx$$

as a starting point, arguments analogous to the ones in Step 2 of the proof of Lemma 4 in Appendix A.3 show that, for $\rho_2 > \rho_1 > 0$, we have

$$\left[\gamma^A(\rho_2) - \gamma^A(\rho_1)\right]\left[\bar{s}^A_{\rho_2} - \bar{s}^A_{\rho_1}\right] + \left[\gamma^B(\rho_2) - \gamma^B(\rho_1)\right]\left[\bar{s}^B_{\rho_2} - \bar{s}^B_{\rho_1}\right] < 0. \tag{46}$$

Observing that Lemmas 2 and 3 imply that the expressions defining $\gamma^A(\rho)$ and $\gamma^B(\rho)$ in (43) are decreasing in ρ and that (44) implies

$$\bar{s}^A_\rho(x) = \bar{s}^B_\rho(x) = \sqrt{\rho \cdot \bar{u}^A_\rho \cdot \bar{u}^B_\rho}, \tag{47}$$

the inequality in (46) yields that $\rho \cdot \bar{u}^A_\rho \cdot \bar{u}^B_\rho$ is increasing in ρ.

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