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A Common-Value Auction with State-Dependent Participation*

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This paper analyzes a common-value, first-price auction with state-dependent participation. The number of bidders, which is unobservable to them, depends on the true value. For exogenously given participation patterns that involve many bidders in each state, the bidding equilibrium may be of a "pooling" type—with high probability, the winning bid is the same across states and is below the ex-ante expected value—or of a "partially revealing" type—with no significant atoms in the winning bid distribution and an expected winning bid increasing in the true value. Which of these forms will arise is determined by the likelihood ratio at the top of the signal distribution and the participation across states. When the state-dependent participation is endogenized as the strategic solicitation by an informed seller who bears a small cost for each solicited bidder, an equilibrium of the partially revealing type always exists and is unique of this type; for certain signal distributions there also exist equilibria of the pooling type.

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1 Introduction

In various auctions and similar trading scenarios, participation is state dependent—its extent may be correlated with information that is relevant for the bidding. This might be the case when the decisions on the costly recruitment of participants are made by an informed seller, or when the participants are induced to participate by the value of correlated outside options. Strategic participants take this into account, and it may affect behavior and the resulting prices. The main objective of this paper is to shed light on these considerations that are obviously present in many different scenarios, be it a sale of an asset of uncertain value or the shopping around of a venture by an entrepreneur to potential lenders.

Price formation with state-dependent participation can take different forms. This paper explores it by studying auctions in which the number of bidders varies across states and bidders can learn about the state from their own participation. We view the auction model as a convenient abstraction of a free-form price-formation process that takes place in a decentralized market environment rather than in a formal mechanism. The specific auction format and some of the other features are selected to facilitate the clear exposition of the insights concerning the strategic effects of state-dependent participation rather than tailored to fit a specific application.

Specifically, we analyze a first-price auction for a single good with two value-states, ℓ and h, such that the common value of the good, v_ω, $\omega = \ell, h$, satisfies $v_h > v_\ell$. In state ω, there are n_ω bidders. They do not observe ω or n_ω but get private, conditionally independent signals that are drawn from a distribution G_ω with support $[\underline{x}, \bar{x}]$ and density g_ω. The likelihood ratio $\frac{g_h(x)}{g_\ell(x)}$ is increasing, so higher signals are relatively more likely in state h. This is the same basic model as in our companion paper, Lauermann and Wolinsky (2017).[1] In this world, bidders obtain information about the total participation (and hence the state) through their own presence at the auction. This augments their private signal information, and the compound posterior likelihood ratio of the states depends both on the signal likelihood ratio $\frac{g_h(x)}{g_\ell(x)}$ (as it would in a standard auction environment) and on the participation ratio $\frac{n_h}{n_\ell}$. The objective of this paper is to explore the implications of this feature.

[1] We discuss this relationship later. For now, we note that there is essentially no overlap in results.

The first part of the paper, consisting of Sections 2—4, is a self-contained exploration of the bidding equilibria with exogenously given state-dependent participation, (n_ℓ, n_h). Our main characterization result (Theorem 1) concerns the forms of the bidding equilibria when n_ℓ and n_h are large. Specifically, the key magnitude is the "compound" posterior likelihood ratio, $\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\frac{n_h}{n_\ell}$ and the form of the equilibrium varies dramatically according to whether this ratio is below or above 1. If this ratio is below 1, then any bidding equilibrium is necessarily of a pooling type: there is some bid b below the ex-ante expected value such that, with probability close to 1, the winning bid is equal to b in both states. In fact, in this case, any b in some interval just below the ex-ante expected value is such pooling outcome of some such equilibrium. If this ratio is above 1, then any bidding equilibrium is of a partially revealing type: there are no significant atoms in the winning bid distribution, and the expected winning bid is higher in state h than in ℓ. In being partially revealing, the equilibrium in this case resembles the equilibrium of an ordinary common-value auction. The special insight beyond what we know from the analysis of ordinary auctions concerns the manner in which the state-dependent participation determines the degree of "revelation" through the ratio $\frac{n_h}{n_\ell}$.

These results are explained by the form of the "winner's inference," $\frac{\Pr(\text{all other bids} \leq b|h)}{\Pr(\text{all other bids} \leq b|\ell)}$, given a common bidding strategy β. When there are many bidders, for a strictly increasing bidding strategy β to be an equilibrium, the expected value conditional on winning must be increasing in the bid. But this is the case only if this ratio is increasing in b. The analysis will show that, given a common, strictly increasing bidding strategy β, and large n_ℓ and n_h, the relationship between the ratio $\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\frac{n_h}{n_\ell}$ and 1 determines whether the winner's inference of the relevant bidders (those with x near $\bar{x}$) is increasing or decreasing in x.

In the second part of the paper, the state-dependent participation is endogenized in a specific way. An informed seller knows ω and invites n_ω bidders at a constant cost per bidder.[2] Costly strategic solicitation is interesting in its own right, in that stimulating participation is an important element of the seller's activities in bidding scenarios. Here, however, it primarily serves to demonstrate that the different patterns of state-dependent participation considered in the first part may arise in a

[2]State-dependent participation may arise for a variety of reasons. In Murto and Välimäki (2019), it is the result of partially informed bidders' costly entry decisions; in Atakan and Ekmekci (2016), bidders' entry decisions differ across states due to differences in the value of outside options. A range of behavioral considerations might have a similar effect as well.

natural setting. We focus on the case of a small solicitation cost, which naturally results in large participation. Theorem 2 establishes the existence[3] of a sequence of equilibria whose limit (as the solicitation costs vanish) outcome is of the partially revealing type. The optimal solicitation pins down the ratio $\frac{n_h}{n_\ell}$ uniquely, and hence there is a unique limit outcome of this type. The theorem also claims that, for some signal distributions, there also exist sequences of equilibria whose limit outcomes are of the pooling type placing mass 1 on some bid below the ex-ante expected value.

The extent of information aggregation by the price can be thought of casually as reflected by the closeness of the price to the true value and, more formally, as how informative is the price as a signal of the true state. It depends on the form of the equilibrium and on the ratio $\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\frac{n_h}{n_\ell}$. The price aggregates no information in the pooling equilibrium and aggregates some information in the partially revealing equilibria (the distribution of the winning bid in state h stochastically dominates that of state ℓ). The extent of information aggregation in the partially revealing equilibria increases in $\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\frac{n_h}{n_\ell}$. That is, the expected price is closer to the true value and, more generally, the price is more informative signal of the state.[4]

In an ordinary large common-value auction without state-dependent participation, the price aggregates only the bidders' information, and the extent of information aggregation depends on the informativeness of the private signals as captured by $\frac{g_h(\bar{x})}{g_\ell(\bar{x})}$ (Wilson (1977) and Milgrom (1979)). With state-dependent participation, the seller's information is also aggregated into the price via $\frac{n_h}{n_\ell}$, either dampening or enhancing the effect of the bidders' information. For a given value of $\frac{g_h(\bar{x})}{g_\ell(\bar{x})}$, the larger $\frac{n_h}{n_\ell}$ is, the more information is incorporated into the price. In particular, the price aggregates information better than it does in a large ordinary auction with the same signal structure whenever $\frac{n_h}{n_\ell}$ is larger than 1, and it worse at aggregating the information when $\frac{n_h}{n_\ell}$ is smaller than 1.

We analyze large, first-price auctions in a binary-state world and, strictly speaking, the results apply to that environment. However, this model is just a means to illustrate the main insights concerning the effects of state-dependent participation that are likely to be relevant for a broader set of environments. In Sections 8.4 and 5.3, after having presented our model and analysis, we will discuss in some more

[3] All statements regarding existence pertain to the limit with respect to the step size of a finite grid of the set of feasible bids.

[4] More precisely, the limit price distribution is equivalent to a distribution over posteriors that is Blackwell more informative about the state as $\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\frac{n_h}{n_\ell}$ increases.

detail this speculative claim concerning the wider scope.

1.1 Literature Connections

Lauermann and Wolinsky (2017) uses the same model as this paper and shares with it some auxiliary observations. Our unapologetic use of the same model reflects our view of it as a fundamental model of an important situation that is not exhausted after one use. Except for the model, there is no overlap between the two papers: the results of the current paper have no counterpart in Lauermann and Wolinsky (2017). The main result from Lauermann and Wolinsky (2017) is that there exists an equilibrium with an atom at the top of the bid distribution when the bidders' signals are binary. In the current paper, the inevitability of atoms for certain participation patterns is introduced for the first time, as is everything about the partially revealing equilibria, including their general form, their existence and uniqueness under endogenous solicitation, and the corresponding insights regarding information aggregation.

From the perspective of auction theory, the closest papers are Murto and Valimaki (2019) and Atakan and Ekmekci (2016). They also have a common-value auction with state-dependent participation,[5] but they explore other mechanisms that generate it.

Our discussion of information aggregation continues the discussion of this question by Milgrom (1979) and Wilson (1977) in the context of an ordinary common-value auction. Translated to the two-state model considered here, Milgrom's (1979) result is that the winning bid in an ordinary common-value auction approaches the true value as the number of bidders grows if and only if the likelihood ratio of the two states is unbounded over the support of the signal distribution. Our analysis recognizes the additional information due to the state-dependent strategic solicitation and points out that this solicitation may dampen or enhance information revelation.

Broecker (1990) and Riordan (1993) model competition among incompletely informed banks over the business of potential borrowers as an ordinary auction—the borrowers contact all the banks for quotes. This and our companion paper recognize that such competition may be significantly affected when borrowers choose how many banks to contact based on their private information. The inevitability of

[5] Remark to the referees: These papers became available after earlier versions of our paper.

atoms established in the present paper implies that certain state dependent contact patterns of a broad class will *necessarily* result in banks pooling on a unique quote, which resembles a collusive outcome.

In markets of the sort we are interested in, the contacts made by agents do not always follow a rigid protocol—sometimes they are indeed simultaneous, as in the present model, sometimes sequential, and sometimes a combination of the two. We explored the sequential scenario in Lauermann and Wolinsky (2016). A central qualitative difference is due to the absence of direct price competition in the sequential-search-with-bargaining model. There, uninformed agents with promising signals cannot actively overbid, and, therefore, the extent of information aggregation is determined by the interaction of search and the signal technology. In contrast, the auction setting assigns a prominent role to price competition. The uninformed may try to evade adverse selection by bidding more aggressively, and in the process inject their information into the price. This explains why, with sufficiently informative signals, the partially revealing equilibrium with bidder solicitation is nearly competitive and also aggregates information well, unlike the corresponding unique equilibrium of the search model.

2 The Bidding Game and Preliminary Characterization

This section and the following one discuss the bidding behavior for an exogenously given pattern of state-dependent participation. The solicitation by an informed seller is one possible such scenario. But, as mentioned above, state-dependent participation may arise for other reasons as well. Therefore, the understanding of this situation is both of interest in its own right and used as a building block for the subsequent analysis of endogenous solicitation.

2.1 The Bidding Game and its Equilibrium

Basics.—This is a single-good, common-value, first-price auction environment with two underlying states, h and ℓ. There are N potential bidders (buyers). The common values of the good for all potential bidders in the two states are v_ℓ and v_h, respectively, with $0 \leq v_\ell < v_h$.

Nature draws a state $\omega \in \{\ell, h\}$ with prior probabilities $\rho_\ell > 0$ and $\rho_h > 0$,

$\rho_\ell + \rho_h = 1$, and, in state ω, randomly draws n_ω bidders from the pool, $1 \le n_\omega \le N$. A bold $\mathbf{n}$ denotes the vector (n_ℓ, n_h).[6]

Each of the n_ω bidders observes a private signal $x \in [\underline{x}, \bar{x}]$. Conditional on the state $\omega \in \{\ell, h\}$, signals are independently and identically distributed according to a cumulative distribution function (c.d.f.) G_ω. A bidder does not observe ω or n_ω, but she believes that her probability of being invited to the auction in state ω is $\frac{n_\omega}{N}$.

The set of feasible bids, P_Δ, is a grid with step size $\Delta \ge 0$

$$P_\Delta \triangleq \begin{cases} [0, v_\ell] \cup \{v_\ell + \Delta, v_\ell + 2\Delta, \cdots, v_h - \Delta, v_h\} & \text{for} \quad \Delta > 0 \\ [0, v_h] & \text{for} \quad \Delta = 0 \end{cases},$$

Notice that even for the case of $\Delta > 0$, contains the continuum of prices on $[0, v_\ell]$.[7] The grid is introduced to deal with later existence issues.

The n_ω bidders simultaneously submit bids $b \in P_\Delta$. The highest bid wins, and ties are broken randomly with equal probabilities. If the winning bid is p in state $\omega \in \{h, \ell\}$, then the payoffs are $v_\omega - p$ for the winning bidder and 0 for all others.

We call this the "bidding game" and denote it by $\Gamma_0(\mathbf{n}, N, \Delta)$. The ordinary common-value auction is a special case with $n_\ell = n_h$.

The Signal.—The signal distributions G_ω, $\omega \in \{\ell, h\}$, have no atoms and strictly positive densities g_ω on an identical support, $[\underline{x}, \bar{x}]$. The likelihood ratio $\frac{g_h(x)}{g_\ell(x)}$ is nondecreasing and right-continuous, with $\frac{g_h(\bar{x})}{g_\ell(\bar{x})} = \lim_{x \to \bar{x}} \frac{g_h(x)}{g_\ell(x)}$. This is the (weak) monotone likelihood ratio property (MLRP): larger values of x indicate a (weakly) higher likelihood of the higher state. The signals are nontrivial and boundedly informative, i.e.,

$$0 < \frac{g_h(\underline{x})}{g_\ell(\underline{x})} < 1 < \frac{g_h(\bar{x})}{g_\ell(\bar{x})} < \infty.$$

A bidder's posterior probability of ω, conditional on being solicited and receiving signal x, is

$$\Pr[\omega | x, \text{sol}; \mathbf{n}] \triangleq \frac{\rho_\omega g_\omega(x) \frac{n_\omega}{N}}{\rho_\ell g_\ell(x) \frac{n_\ell}{N} + \rho_h g_h(x) \frac{n_h}{N}},$$

where ρ_ω, $g_\omega(x)$, and $\frac{n_\omega}{N}$, respectively, reflect the information contained in the prior belief, in the signal x, and in the bidder being invited. We use "sol" to denote the

[6] The solicitation is modeled here as a move of nature to focus on the bidding, but, as mentioned above, it can be endogenized in several ways, and we will do this later.

[7] This avoids some irrelevant distinctions between the case in which the bottom equilibrium bid is v_ℓ and the case in which it is $v_\ell - \Delta$.

event that the bidder was solicited. Notice that N cancels out, and, hence, it does not play any role in the analysis.

Bidding.—A bidding strategy β prescribes a bid as a function of the signal realization,

$$\beta : [\underline{x}, \bar{x}] \rightarrow P_{\Delta}.$$

We study strategies that are symmetric and pure.

Given a bidding strategy β employed by n other bidders, the probability of winning with a bid b in state ω is $\pi_{\omega}(b; \beta, n)$. From here on, $(\beta, \mathbf{n})$ and (β, n) will typically be suppressed from the arguments, and we write expressions such as $\Pr[\omega|x,\text{sol}]$ and $\pi_{\omega}(b)$ with the understanding that they depend on a specific profile $(\beta, \mathbf{n})$.

Expected Payoff.— Given the bidding strategy β and the participation $\mathbf{n} = (n_{\ell}, n_h)$, the interim expected payoff to a bidder who bids b, conditional on participating and observing the signal x, is

$$U(b|x, \text{sol}) = \Pr[\text{win at } b|x, \text{sol}] \left(\mathbb{E}[v|x, \text{sol,win at } b] - b \right), \tag{1}$$

where

$$\Pr[\text{win at } b|x\text{,sol}] = \frac{\rho_{\ell} g_{\ell}(x) n_{\ell} \pi_{\ell}(b) + \rho_h g_h(x) n_h \pi_h(b)}{\rho_{\ell} g_{\ell}(x) n_{\ell} + \rho_h g_h(x) n_h}, \tag{2}$$

and

$$\mathbb{E}[v|x\text{,sol,win at } b] = \frac{\rho_{\ell} g_{\ell}(x) n_{\ell} \pi_{\ell}(b) v_{\ell} + \rho_h g_h(x) n_h \pi_h(b) v_h}{\rho_{\ell} g_{\ell}(x) n_{\ell} \pi_{\ell}(b) + \rho_h g_h(x) n_h \pi_h(b)}, \tag{3}$$

where $(\beta, \mathbf{n})$ is suppressed from the arguments of $\mathbb{E}[v|...]$ and $\Pr[\text{win at } b|...]$, according to the convention adopted above.

Bidding Equilibrium.—A *bidding equilibrium* of $\Gamma_0(\mathbf{n}, N, \Delta)$ is a bidding strategy β such that $b = \beta(x)$ maximizes $U(\cdot|x\text{,sol})$ for all x.

3 Equilibrium Monotonicity

With state-dependent participation, monotonicity is not immediate since the signals inform bidders also about the number of competitors rather than just about the value. If fewer bidders are solicited when $\omega = h$, a higher signal implies both, a higher value and less competition. The following example illustrates this consideration.

Example of a Non-monotone Bidding Equilibrium: Let $[\underline{x}, \bar{x}] = [0, 1]$, with $g_h(x) = 2x$ and $g_\ell(x) = 2 - 2x$. Thus, the signals $x = 1$ and $x = 0$ reveal the state.[8] Suppose that $v_\ell > 0$, $n_h = 1$, and $n_\ell = 100$. It follows that $\pi_h(b; \beta, 1) = 1$ for all $b \geq 0$. Hence, $\beta(1) = 0$ in every bidding equilibrium. So, if β were weakly increasing, then $\beta(x) = 0$ for all x. However, this strategy cannot be an equilibrium. At $x = 0$, the expected payoff from bidding $b = 0$ is $\frac{1}{100} v_\ell$, whereas the expected payoff from bidding $b' = \varepsilon$ is $v_\ell - \varepsilon$. Because $v_\ell > 0$, a deviation to b' is profitable for small ε. Thus, in this example, there is no weakly increasing bidding equilibrium.

However, when either at least two bidders participate in the auction in both states or $v_\ell = 0$ (unlike in the example), a bidding equilibrium strategy β is monotonic in the sense that, for any bidding equilibrium there is an equivalent monotone bidding equilibrium. A bidding equilibrium $\widetilde{\beta}$ is said to be *equivalent* to a bidding equilibrium β if the implied joint distributions over bids and states are identical.

Proposition 1 **(Monotonicity of Bidding Equilibrium)** *Suppose that either $v_\ell = 0$ or $n_\omega \geq 2$, $\omega = \ell, h$ and β is a bidding equilibrium.*

1. *If $x' > x$, then $U(\beta(x') | x', sol) \geq U(\beta(x) | x, sol)$. The inequality is strict if and only if $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$.*

2. *There exists an equivalent bidding equilibrium $\widetilde{\beta}$, such that $\widetilde{\beta}$ is nondecreasing on $[\underline{x}, \bar{x}]$ and coincides with β over intervals over which $\frac{g_h}{g_\ell}$ is strictly increasing.*

Thus, if the likelihood ratio $\frac{g_h}{g_\ell}$ is strictly increasing everywhere, then a bidding equilibrium β is necessarily monotonic; if $\frac{g_h}{g_\ell}$ is constant over some interval, then β need not be monotonic over it, since all those signals contain the same information. However, in this case, there is an equivalent monotone bidding equilibrium that is obtained by reordering the bids over such intervals.[9]

This proposition is not proved separately since it is a special case of a more general version, called Proposition 7, which will be stated and proved in Appendix B.2.

The main observation in the proof is that, for $b \geq v_\ell$, $U(b | x, \text{sol}; \beta, \mathbf{n})$ satisfies single crossing with respect to b and x, for any β (monotone or not). Thus, above v_ℓ, best responses are monotone and so are equilibrium bids.

[8] The example violates the bounded likelihood-ratio assumption. This simplifies the argument but is not essential.

[9] Although strict MLRP evidently simplifies the argument, we chose to require only weak MLRP, since this admits discrete signals as a special case that is useful for some examples and results.

The two conditions in the proposition ensure that equilibrium bids are necessarily above v_ℓ. First, if $v_\ell = 0$, then this simply follows from the restriction of bids to be positive. Second, if there are at least two bidders, then a "Bertrand" argument implies that bids must be at least v_ℓ. For an intuition, note that it is common knowledge that the value is at least v_ℓ. As in the standard Bertrand game with complete information, already 2 bidders are sufficient. The assumption that $[0, v_\ell] \subset P_\Delta$ is used in this part of the proof.

The single-crossing property implies that the proof does not have to distinguish between the cases of $\Delta > 0$ and $\Delta = 0$ above v_l. Moreover, the single-crossing property implies that our restriction to pure strategies is without loss of generality.

In light of Proposition 1, from now on, whenever $n_\omega \geq 2$, $\omega = \ell, h$, attention is confined to nondecreasing bidding equilibria.

4 Bidding Equilibria with Many Bidders

This section characterizes bidding equilibria when there are many bidders in each state. From a substantive point of view, the many bidders case is the relevant case for the questions of competitiveness and information aggregation in markets. From an analytical point of view, this case makes it easier to get clean characterization results and identify the underlying economic mechanism.

4.1 Preliminaries

We look at a sequence of bidding games $\Gamma_0\left(\mathbf{n}^k, N^k, \Delta^k\right)$ such that $\Delta^k \geq 0$, $\lim \Delta^k = 0$,

$$\lim_{k\to\infty} n_\omega^k = \infty \text{ for } \omega = \ell, h,$$

and

$$\lim_{k\to\infty} \frac{n_h^k}{n_\ell^k} = r \in [0, \infty],$$

and at a corresponding sequence of bidding equilibria β^k. We are interested in the limits of equilibrium magnitudes as $k \to \infty$.[10]

With many bidders, only bids associated with signals that are sufficiently close to $\bar{x}$ have a significant probability of winning. Therefore, the object of interest is

[10] By assumption, $N^k \geq n_\omega^k$ for $\omega = \ell, h$, and so $N^k \to \infty$.

the equilibrium distribution of the *winning* bid in state ω, namely,

$$F_\omega(p|\beta,n) \triangleq (G_\omega(\{x : \beta(x) \leq p\}))^n,$$

and its pointwise limit, rather than the distribution of all the bids.

The notation's density is reduced as follows. First, when we discuss a fixed sequence $\left\{(\beta^k, \mathbf{n}^k)\right\}_{k=1}^\infty$, then magnitudes induced by $(\beta^k, \mathbf{n}^k)$ are typically written as $U^k(b|x,\text{sol})$, $F_\omega^k(p)$, etc. (rather than as $U(b|x,\text{sol};\beta^k,\mathbf{n}^k)$, $F_\omega\left(p|\beta^k, n_\omega^k\right)$, etc.). Second, since nearly all limits are with respect to k, we generally omit the delimiter $k \to \infty$. Finally, we sometimes use the abbreviations

$$\bar{g} \triangleq \frac{g_h(\bar{x})}{g_\ell(\bar{x})} \text{ and } \rho \triangleq \frac{\rho_h}{\rho_\ell}.$$

4.2 Winning Bid Distribution: Pooling vs. Partially Revealing

Our main characterization result shows that, for large k, the form of $F_\omega^k(p)$ is determined by $\frac{g_h(\bar{x})}{g_\ell(\bar{x})} \lim \frac{n_h^k}{n_\ell^k} = \bar{g}r$. It exhibits a large atom at the top if $\bar{g}r < 1$, and it is essentially free of atoms if the reverse inequality holds.

Let $\mathbb{E}[v]$ denote the expected ex-ante value of the good, $\mathbb{E}[v] = \rho_\ell v_\ell + \rho_h v_h$, and let

$$\overline{\mathbb{E}}[v|\bar{x},\text{sol}] \triangleq \lim \mathbb{E}^k[v|\bar{x},\text{sol}] \equiv \frac{v_\ell + \rho\bar{g}r v_h}{1+\rho\bar{g}r}, \tag{4}$$

be the limit posterior conditional on the highest signal $\bar{x}$ and being solicited. Note that $\overline{\mathbb{E}}[v|\bar{x},\text{sol}] > \mathbb{E}[v]$ if $\bar{g}r > 1$ and $\overline{\mathbb{E}}[v|\bar{x},\text{sol}] < \mathbb{E}[v]$ if $\bar{g}r < 1$. Thus, if $\bar{g}r < 1$, then just being included in the auction already involves a "participation curse" that depresses the value estimate held by any bidder below the prior.

Intuitively, $\bar{g}r \gtrless 1$ determines whether the expected number of relevant bidders (those with signals close to $\bar{x}$) is higher in state h or ℓ. This is because the expected number of bidders having signals in an ε-neighborhood of $\bar{x}$ is $n_\omega^k(1 - G_\omega(\bar{x} - \varepsilon)) \approx n_\omega^k g_\omega(\bar{x})\varepsilon$.

Theorem 1 *For every sequence of bidding games $\Gamma_0(\mathbf{n}^k, N^k, \Delta^k)$ with $\Delta^k > 0$ for all k, $\Delta^k \to 0$, $\min\{n_\ell^k, n_h^k\} \to \infty$, and $\lim \frac{n_h^k}{n_\ell^k} = r$, there exists a sequence of bidding equilibria β^k.*

1. *If $\bar{g}r > 1$, then for any such sequence,*

$$\lim F_\omega^k(p) = \Phi_\omega(p|r),$$

where $\Phi_\omega(\cdot|r)$ is an atomless distribution that is uniquely determined by r with support $\left[v_\ell, \overline{\mathbb{E}}[v|\bar{x}, sol]\right]$.

2. *If $\bar{g}r < 1$, then*

   (a) *for any such sequence, there is a sequence of bids $\hat{b}^k$ such that*

$$\lim \left[F_\omega^k(\hat{b}^k + \Delta^k) - F_\omega^k(\hat{b}^k - \Delta^k)\right] = 1,$$

   with $\overline{\mathbb{E}}[v|\bar{x}, sol] \leq \liminf \hat{b}^k$ and $\limsup \hat{b}^k \leq \mathbb{E}[v]$.

   (b) *for any $\hat{b}$ with $\overline{\mathbb{E}}[v|\bar{x}, sol] < \hat{b} < \mathbb{E}[v]$, there is a sequence of equilibria β^k such that*

$$\lim \left[F_\omega^k(\hat{b}) - F_\omega^k(\hat{b} - \Delta^k)\right] = 1.$$

3. *If $\bar{g}r = 1$, then for any such sequence, $\lim F_\omega^k(p)$ has mass 1 on $\mathbb{E}[v]$.*

Note that the theorem speaks about the (limit of the) distribution of the *winning* bid rather than the distribution of the submitted bids. Thus, Part 2 does not mean that, for large k, most equilibrium bids are $\hat{b}^k$ or $\hat{b}^k + \Delta^k$ but rather that the winning bid is very likely to be either $\hat{b}^k$ or $\hat{b}^k + \Delta^k$.

The proof shows that the distributions Φ_ω mentioned in Part 1 are

$$\Phi_\ell(p|r) \triangleq \begin{cases} 1 & \text{if } \overline{\mathbb{E}}[v|\bar{x}, \text{sol}] \leq p \\ \left(\frac{1}{\rho\bar{g}r}\frac{p - v_\ell}{v_h - p}\right)^{\frac{1}{\bar{g}r-1}} & \text{if } v_\ell < p \leq \overline{\mathbb{E}}[v|\bar{x}, \text{sol}] \\ 0 & \text{if } p \leq v_\ell \end{cases} \tag{5}$$

and

$$\Phi_h(\cdot|r) \triangleq (\Phi_\ell(\cdot|r))^{\bar{g}r}, \tag{6}$$

and thus are uniquely determined by r, as claimed.

For the special case of $n_\ell = n_h = n$ (i.e., $r = 1$), the explicit characterization of the winning bid distribution is essentially implied by the analysis of Murto and Valimaki (2015).

Part 3 of the theorem implies that the limit equilibrium outcome is continuous in $\bar{g}r$ at $\bar{g}r = 1$. As $\bar{g}r \to 1$ from above, the distributions Φ_ω converge to a mass point at $\mathbb{E}[v]$, and, as $\bar{g}r \to 1$ from below, $\overline{\mathbb{E}}[v|\bar{x},\text{sol}]$ converges to $\mathbb{E}[v]$, and so the interval of outcomes in Part 2 collapses.

The assumption that $\Delta^k > 0$ along the sequence is only used to show the existence of equilibrium.[11] The characterization results concerning the forms of the equilibria hold with $\Delta = 0$ as well.

Proposition 2 *Consider any sequence of bidding games $\Gamma_0(\boldsymbol{\eta}^k, N^k, \Delta)$ with $\Delta = 0$, such that $\min\{n_\ell^k, n_h^k\} \to \infty$, and $\lim \frac{n_h^k}{n_\ell^k} = r$. Then, the characterization results of Theorem 1 (i.e., items 1, 2a, and 3) hold for any corresponding sequence of bidding equilibria β^k.*

The proof in Appendix A.1 shows Proposition 2 first, before allowing for a grid and proving Theorem 1.

4.3 Key Ideas from the Proof of Theorem 1

The following two observations highlight the key intuition of the theorem. First, if bidders with signals close to $\bar{x}$ are tied at a common bid, it must be that $\bar{g}r < 1$. Second, if bidders with signals close to $\bar{x}$ use a strictly increasing bidding strategy, it must be that $\bar{g}r > 1$.

Pooling on a Common Bid. Suppose the equilibrium bidding strategies β^k are such that bidders with signals close to the top are tied at a common bid, that is, for all large k and some $\hat{b}$ and x^k,

$$\beta^k(x) = \hat{b} \text{ for all } x \in \left[x^k, \bar{x}\right], \tag{7}$$

and suppose that the winning bid is equal to $\hat{b}$ with probability 1 in the limit,

$$\lim \left[G_h\left(x^k\right)\right]^{n_h^k - 1} = \lim \left[G_\ell\left(x^k\right)\right]^{n_\ell^k - 1} = 0,$$

that is, x^k is not too close to $\bar{x}$.

Since the auction ends with a winning bid of $\hat{b}$ in both states when k is large, the bidders' ex-ante rationality requires

$$\mathbb{E}[v] \geq \hat{b}.$$

[11] Existence without a grid is discussed in Section 5.1.

When bidding $\hat{b}$, the winning probability and, hence, the expected payoffs vanish to zero. A bidder who ε-overbids $\hat{b}$, however, wins with probability 1 in both states, and the expected value conditional on winning is $\mathbb{E}[v| \ \bar{x}, \text{sol}]$ in the limit (since winning contains no further information). Thus, for a bidder with a signal $\bar{x}$ not to overbid $\hat{b}$, it must be that

$$\hat{b} \geq \mathbb{E}[v|\bar{x}, sol].$$

For both of the above inequalities to hold simultaneously, it must be that $\mathbb{E}[v|\bar{x}, sol] \leq \mathbb{E}[v]$, which holds if and only if $\bar{g}r \leq 1$; see (4). Thus, $\bar{g}r \leq 1$ is necessary for an atom of the form (7).

Strictly Increasing Bids. Suppose the bidding strategy β^k is strictly increasing near the top (so, there are no atoms). In particular, suppose that, for all k large enough,

$$\beta^k(\bar{x}) > \beta^k\left(x^k\right),$$

for any bidder with signal x^k sufficiently close to $\bar{x}$ for which

$$\left[G_\ell\left(x^k\right)\right]^{n_\ell^k-1} \in (0,1),$$

for all k, that is, a bidder submitting $\beta^k\left(x^k\right)$ has a fixed and constant probability of winning. Of course, $x^k \to \bar{x}$ as $k \to \infty$.

Since there are an increasingly large number of bidders, the bidders' expected equilibrium profits are zero in the limit. However, if β^k is strictly increasing, then bidders with signals $\bar{x}$ and x^k expect to win with a strictly positive, nonvanishing probability. For their profits to go to zero, it must therefore be that

$$\beta^k(\bar{x}) \approx \mathbb{E}^k[v|\bar{x}, \text{sol,win at } \beta^k(\bar{x})],$$

and

$$\beta^k\left(x^k\right) \approx \mathbb{E}^k[v|x^k, \text{sol,win at } \beta^k\left(x^k\right)]. \tag{8}$$

Given $\beta^k(\bar{x}) > \beta^k\left(x^k\right)$, it must be that

$$\lim \mathbb{E}^k[v|\bar{x}, \text{sol,win at } \beta^k(\bar{x})] \geq \lim \mathbb{E}^k[v|x^k, \text{sol,win at } \beta^k\left(x^k\right)]. \tag{9}$$

Since $x^k \to \bar{x}$, whether inequality (9) holds depends on the "winner's inference" from winning at $\beta^k(\bar{x})$ versus $\beta^k\left(x^k\right)$. In the following, we show that (9) requires $\bar{g}r \geq 1$.

Obviously, the probability of winning is 1 in both states at $\beta^k(\bar{x})$, and so the winner's inference is

$$\frac{\pi^k(\beta^k(\bar{x})\,|h)}{\pi^k(\beta^k(\bar{x})\,|\ell)} = 1,$$

for all k. At $\beta^k\left(x^k\right)$, we have

$$\frac{\pi^k(\beta^k\left(x^k\right)|h)}{\pi^k(\beta^k\left(x^k\right)|\ell)} = \frac{\left[G_h(x^k)\right]^{n_h^k-1}}{\left[G_\ell(x^k)\right]^{n_\ell^k-1}}.$$

Simple algebra shows that, when $\lim\left[G_\ell(x^k)\right]^{n_\ell^k-1} = q \in (0,1)$, then $\lim\left[G_h(x^k)\right]^{n_h^k-1} = q^{\bar{g}r}$.[12] Therefore,

$$\lim\frac{\pi^k(\beta^k\left(x^k\right)|h)}{\pi^k(\beta^k\left(x^k\right)|\ell)} = q^{\bar{g}r-1}.$$

The expected value conditional on winning is

$$\lim\mathbb{E}^k[v|x^k,\text{sol,win at }\beta^k\left(x^k\right)] = \frac{v_\ell + \rho\lim\frac{g_h\left(x^k\right)}{g_\ell\left(x^k\right)}\frac{\pi^k(\beta^k\left(x^k\right)|h)}{\pi^k(\beta^k\left(x^k\right)|\ell)}v_h}{1+\rho\lim\frac{g_h\left(x^k\right)}{g_\ell\left(x^k\right)}\frac{\pi^k(\beta^k\left(x^k\right)|h)}{\pi^k(\beta^k\left(x^k\right)|\ell)}},$$

and using the above we have

$$\lim\mathbb{E}^k[v|x^k,\text{sol,win at }\beta^k\left(x^k\right)] = \frac{v_\ell+\rho\bar{g}rq^{\bar{g}r-1}v_h}{1+\rho\bar{g}rq^{\bar{g}r-1}}. \tag{10}$$

So, for (9) to hold it must be that (10) is increasing in q, which is the case if and only if $\bar{g}r \geq 1$; thus, $\bar{g}r \geq 1$ is necessary for β^k to be strictly increasing at the top.[13]

4.4 Revenue and Information Aggregation in Large Auctions

Theorem 1 has implications for how the parameters $\bar{g}(\equiv \frac{g_h(\bar{x})}{g_\ell(\bar{x})})$ and $r(\equiv \lim\frac{n_h^k}{n_\ell^k})$ affect the expected equilibrium revenue and the extent of information aggregation in the limit and for large k.

[12] With $Q_\ell \equiv -\lim n_\ell^k\left(1-G_\ell(x^k)\right)$, we have $\lim\left[G_\ell(x^k)\right]^{n_\ell^k-1} = e^{Q_\ell} = q$. By l'Hospital's rule, $Q_h \equiv -\lim n_h^k\left(1-G_h(x^k)\right) = \bar{g}rQ_\ell$, and so $\lim\left[G_h(x^k)\right]^{n_h^k-1} = e^{Q_h} = e^{Q_\ell\bar{g}r} = q^{\bar{g}r}$. Intuitively, the number of bidders with signals $\geq x^k$ is approximately Poisson distributed with means $n_\ell^k\left(1-G_\ell(x^k)\right)$ and $n_h^k\left(1-G_h(x^k)\right)$, respectively.

[13] The explicit solution in (10) and $\beta^k\left(x^k\right) \approx \mathbb{E}[v|x^k,\text{sol,win at }\beta^k\left(x^k\right)]$ are used in the proof to derive the closed form of Φ_ω (the winning bid distribution in the limit).

The interim expected revenue is $\mathbb{E}^k[p|\omega] \triangleq \mathbb{E}\left[p|\omega;\beta^k,n_\omega^k\right]$. In the partially revealing case of $\bar{g}r>1$, the revenue converges to a unique limit, $\overline{\mathbb{E}}[p|\omega]=\lim\mathbb{E}^k[p|\omega]$, the ex-ante revenue $\rho_\ell\overline{\mathbb{E}}[p|\ell]+\rho_h\overline{\mathbb{E}}[p|h]$ is equal to the ex-ante value $\mathbb{E}[v]$, and $\overline{\mathbb{E}}[p|h]>\mathbb{E}[v]>\overline{\mathbb{E}}[p|\ell]$.

In the pooling case of $\bar{g}r<1$, the revenue is approximately equal to the atom for large k, $\mathbb{E}^k[p|\omega]\approx\hat{b}^k$, and so it is independent of the state (i.e., $\lim\left[\mathbb{E}^k[p|h]-\mathbb{E}^k[p|\ell]\right]=0$). The atom, and, hence, the revenue may vary along the sequence but is bounded, $\limsup\mathbb{E}^k[p|\omega]\le\mathbb{E}[v]$, with a strict inequality for some sequences of equilibria.

Corollary 1 *Consider a sequence of bidding games $\Gamma_0\left(\mathbf{n}^k,N^k,\Delta^k\right)$ such that $\Delta^k\geq 0$, $\Delta^k\to 0$, $\min\{n_\ell^k,n_h^k\}\to\infty$ and $\lim\frac{n_h^k}{n_\ell^k}=r$, and a corresponding sequence of bidding equilibria β^k, with $\overline{\mathbb{E}}[p|\omega]=\lim\mathbb{E}^k[p|\omega]$ (when it exists).*

1. *If $\bar{g}r>1$, then*
$$\overline{\mathbb{E}}[p|\ell]<\mathbb{E}[v]<\overline{\mathbb{E}}[p|h], \tag{11}$$
and
$$\rho_\ell\overline{\mathbb{E}}[p|\ell]+\rho_h\overline{\mathbb{E}}[p|h]=\mathbb{E}[v]. \tag{12}$$

2. *If $\bar{g}r<1$, then*
$$\lim\left[\mathbb{E}^k[p|h]-\mathbb{E}^k[p|\ell]\right]=0,$$
and $\limsup\mathbb{E}^k[p|\omega]\le\mathbb{E}[v]$.

Proof: The equality in Part 2 of the result is immediately implied by Part 2 of Theorem 1.

For Part 1, (12) follows from direct calculation using the explicit form of the winning bid distribution Φ_ω given by (5).[14] Then, (11) follows from (12) and the fact that Φ_h first-order stochastically dominates Φ_ℓ. ∎

Recall from Theorem 1 that Part 2 of the corollary applies not only to $\mathbb{E}^k[p|\omega]$ but also to the realized price.

Information Aggregation by the Price. We use the term information aggregation to describe the information conveyed by the price about the state. We will examine how it depends on the parameters $\bar{g}$ and r first informally and then more formally. When $\bar{g}r<1$, the price fails to aggregate the information since exactly the same price prevails in both states with high probability.

[14] The calculation is simplified by changing the integration variable to $y=\frac{1}{\rho\bar{g}r}\frac{p-v_\ell}{v_h-p}$ in the integral $\rho_\ell\lim\mathbb{E}^k[p|\ell]+\rho_h\lim\mathbb{E}^k[p|h]=\rho_\ell\int pd\Phi_\ell(p)+\rho_h\int pd\Phi_\ell(p)$. Alternatively, it follows from (8) and the law of iterated expectations.

In the partially revealing case of $\bar{g}r > 1$, the extent of aggregation can be evaluated by comparing the limit distributions of the winning bid Φ_ω in the two states. Inspection of (5)—(6) reveals the following facts. First, when $\bar{g}r$ is near 1, then Φ_h and Φ_ℓ are nearly identical. Second, when $\bar{g}r$ is large, then Φ_ω is concentrated near v_ω in both states and actually approaches a mass point on v_ω as $\bar{g}r \to \infty$. Thus, a price observation is not a very informative signal of the state if $\bar{g}r$ is near 1, but it is so if $\bar{g}r$ is very large.

More formally, we claim that $\bar{g}r$ determines the informativeness of the price about the state in the sense of Blackwell's criterion. Recall an information structure is a set of signals S and a conditional distribution $H(s|\omega)$ over S, for every state $\omega \in \{\ell, h\}$. In the auction environment at hand, $S = [\underline{x}, \bar{x}]$, $s \in S$ is the first order statistic of the individual signals of the participating bidders and hence $H(s|\omega) = (G_\omega(s))^{n_\omega}$. For any prior likelihood ratio ρ, this information structure induces a distribution $\Psi_\omega(\rho)$ over a set of posteriors in each state ω. The functions Ψ_ω are an equivalent representation of the underlying information structure. In the case of a monotone bidding equilibrium, the distributions of the winning bid, viewed as functions of ρ, are equivalent to the Ψ_ω's, since there is a one-to-one relationship between the bid and the posterior. In the limit of a sequence of equilibria such that $\lim \frac{n_h^k}{n_\ell^k} = r$, these Ψ_ω's are equivalent to the limits of the winning bid distributions $\Phi_\omega(\cdot|\rho, \bar{g}, r)$. As we just said, although the elements in the support of $\Phi_\omega(\cdot|\rho, \bar{g}, r)$ are expected values, they are in one-to-one relationship with the posteriors. Thus, when we say below that $\Phi_\omega(\cdot|\rho, \bar{g}, r)$, $\omega = \ell, h$, is more informative than $\Phi_\omega(\cdot|\rho, \bar{g}', r')$, $\omega = \ell, h$, the statement is about the underlying information structure in which the decision maker's signal is the winning bid. Now, we can inquire formally how the parameters $\bar{g}$ and r affect the informativeness of the equilibrium price.

Corollary 2 *1. If $\bar{g}r > \bar{g}'r' > 1$, then $\Phi_\omega(\cdot|\rho, \bar{g}, r)$, $\omega = \ell, h$, is more informative than $\Phi_\omega(\cdot|\rho, \bar{g}', r')$ according Blackwell's criterion.*

2. $\overline{\mathbb{E}}[p|\ell]$ is decreasing in $\bar{g}r$ and $\overline{\mathbb{E}}[p|h]$ is increasing.

3. $\overline{\mathbb{E}}[p|\omega] \to v_\omega$ as $\bar{g}r \to \infty$ $\qquad \omega = \ell, h$.

The proof is in Appendix A.2. Notice the asymmetry between the cases of $\bar{g}r > 1$ and $\bar{g}r < 1$. While in the former region the informativeness of the price varies monotonically with $\bar{g}r$, in the latter region it is the same for all values of $\bar{g}r$.

Finally, the case of $n_\ell = n_h$ (i.e., $r = 1$) is just the ordinary common value auction. Milgrom (1979) shows that information aggregation in such large auction is nearly perfect—in the sense of the winning bid approaching the true value—iff $\bar{g} = \infty$. Adapting Milgrom's analysis to the case of finite $\bar{g}$, it is intuitive that the winning bid gets closer to the true value as $\bar{g}$ grows. The corollary verifies this and also shows that the price becomes more informative in the more general sense of Blackwell's criterion[15].

4.5 Failure of Affiliation of Beliefs

Another way to describe the role of $\bar{g}r$ in determining the equilibrium outcome is in terms of the affiliation between the value and the highest signal. Let $y_{[\mathbf{n}]}$ denote the highest signal realization given participation $\mathbf{n} = (n_\ell, n_h)$. The c.d.f. of $y_{[\mathbf{n}]}$ conditional on ω is $(G_\omega(x))^{n_\omega - 1}$. Therefore, the likelihood ratio of the states at $y_{[\mathbf{n}]} = x$ is

$$\frac{n_h}{n_\ell} \frac{g_h(x)}{g_\ell(x)} \frac{(G_h(x))^{n_h - 1}}{(G_\ell(x))^{n_\ell - 1}}. \tag{13}$$

In ordinary auctions with $n_h = n_\ell = n$, this likelihood ratio is increasing in x, which means that $y_{[\mathbf{n}]}$ is affiliated with the value. In contrast, with state-dependent participation, the likelihood ratio (13) need not be increasing—in fact, it is *decreasing* for x sufficiently close to $\bar{x}$ if $\frac{n_h}{n_\ell} \frac{g_h(\bar{x})}{g_\ell(\bar{x})} < 1$. Therefore, $y_{[\mathbf{n}]}$ might not be positively affiliated with the value.

5 Discussion

5.1 Existence without Grid

The difficulty in establishing existence directly in the model with a continuum of bids owes to the possible presence of atoms in the bid distribution. Therefore, the bidders' *equilibrium* payoffs might not be continuous in their bids, and this precludes the application of "off-the-shelf" existence results. This is why we look instead at the limit of a sequence of equilibria for a vanishingly small grid (whose existence is guaranteed by established results, e.g., Athey (2001)).

One issue with this approach is that such limit is not necessarily an equilibrium of the continuum case, since the limit might exhibit atoms that are absent in the

[15] This seems to be a somewhat novel observation for ordinary common value auctions as well.

sequence. To see this, consider a sequence of games with grid P_{Δ^k}, with $\beta^k(x) = b$ for $x < \hat{x}$ and $\beta^k(x) = b + \Delta^k$ for $x \geq \hat{x}$. In the limit as $\Delta^k \to 0$, $\lim \beta^k(x) = b$ for all x. Therefore, the winning probability in the limit is strictly higher than the limit of winning probabilities for bidders with $x < \hat{x}$ and it is strictly lower for bidders with $x \geq \hat{x}$. Such merging of atoms may imply that the limit strategy does not need to be an equilibrium of the game with a continuum of bids, even if the elements of the sequence are.

This issue may be resolved by simply defining equilibrium to be the limit outcome as the grid's step goes to zero, or by using the related approach of Jackson, Simon, Swinkels, and Zame (2002). Roughly speaking, bidders submit two numbers, their actual bid and their "eagerness to trade"; the winning bidder is selected from among those who are tied for the "most eager" designation within the group of those who are tied at the highest bid. In the example of the previous paragraph, the limit strategy will have all bidders bid b, but those with $x \geq \hat{x}$ (who bid $b + \Delta^k$ along the sequence) express eagerness $\overline{e}$, while those with $x < \hat{x}$ (who bid b along the sequence) express eagerness $\underline{e} < \overline{e}$. In case of a tie at b, the winner is chosen randomly from among those with $\overline{e}$ if such exists and otherwise from those with $\underline{e}$ (b bidders who announce anything else have even lower priority). With this approach, the winning probabilities and payoffs with a strategy that is the limit of a convergent sequence of bidding strategies are the limit of the winning probabilities and payoffs along the sequence. Therefore, the limit of a convergent sequence of equilibrium bidding strategies, for a vanishingly small grid, is an equilibrium of the continuum limit (of the modified game) itself.[16]

5.2 Random State-Dependent Participation

In the model considered so far, participation $\mathbf{n} = (n_\ell, n_h)$ is deterministic. In many cases of interest, however, participation is random. Let $\boldsymbol{\eta} = (\eta_\ell, \eta_h)$ denote participation distributions, where $\eta_\omega(n)$ is the probability with which $n = 1, ..., N$ bidders are invited in state ω. The expected payoff $U(b|x; \beta, \boldsymbol{\eta})$ and the probability of winning $\pi_\omega(b|\beta, \boldsymbol{\eta})$ are now functions of $\boldsymbol{\eta}$. The bidding game given $\boldsymbol{\eta} = (\eta_\ell, \eta_h)$ is $\Gamma_0(\boldsymbol{\eta}, N, \Delta)$ and a bidding equilibrium is defined as before.

[16] Lauermann and Speit (2019) show that equilibrium will not exist in a related auction model with a state-independent, Poisson distributed number of bidders. They also verify the applicability of Jackson et al (2002) with a suitable message space.

Appendix B.1 presents the explicit expressions of U and π_ω for this case. It also presents the proof that any bidding equilibrium is monotone using the single crossing property of the buyers' preferences, see Proposition 7. In addition, for certain forms of random state-dependent participation, the characterization of the bidding equilibria of large auctions in Theorem 1 holds. In one such form, the support of η_ω^k is contained in $\{n_\omega^k,..., n_\omega^k+m\}$ for some fixed integer $m>0$. We need this special case for other purposes below.

Proposition 3 *Consider any sequence of bidding games $\Gamma_0(\boldsymbol{\eta}^k, N^k, \Delta^k)$ such that, for every k, the support of η_ω^k is contained in $\{n_\omega^k, ..., n_\omega^k+m\}$ for some fixed integer $m>0$ and $\Delta^k \to 0$, $\min\{n_\ell^k, n_h^k\} \to \infty$, and $\lim \frac{n_h^k}{n_\ell^k} = r$. Then, the conclusions of Theorem 1 hold.*

This observation is not surprising, since in this case the randomness becomes relatively negligible as $\min\left\{n_\ell^k, n_h^k\right\} \to \infty$. and $n_\omega^k \to \infty$. The proof is in Appendix B.3.

5.3 Broader Class of Environments

We analyze large, first-price auctions in a binary state world and, strictly speaking, the results pertain to that environment. However, our main insights carry over to a broader class of environments. The previous subsection already presents another scenario to which this analysis applies (random state-dependent participation).

Other auction formats. Although we have not performed the full analysis, it seems that the qualitative results continue to hold for a second-price auction as well. In this case, the functional forms of the limit price distribution (5) will be different, but the main insights would not change.

Large auctions. The focus on large auctions is natural for discussing information aggregation. But the strategic effects of state-dependent participation are just as relevant for trading scenarios with few participants. Still, we focus on large auctions because the analysis is simpler. For example, in the partially revealing case, large numbers guarantee that bids are near the expected values and thus simplify the argument. But such proximity may already hold for fairly low numbers and perhaps other arguments utilizing more directly the structure of the equilibrium might be used.

Two states. The qualitative insights of the strategic inference from the state-dependent participation do not seem to depend on the two-state assumption. We use this assumption to establish the monotonicity of the equilibrium bidding strategy. If monotonicity can be established for the multiple state case, perhaps by resorting to stronger assumptions, then the extension to a world of multiple states would probably be quite straightforward.

6 Strategic Solicitation—Model and Preliminaries

Up to this point, the state-dependent participation $\mathbf{n} = (n_\ell, n_h)$ was exogenously given and the underlying reasons for it have not been modeled. The second part of the paper, from here on, endogenizes the participation in a specific way. An informed seller solicits bidders optimally at a cost s per invited bidder. So, if n bidders are solicited and the winning bid is p, then the seller's payoff is $p - ns$. Given the bidding behavior derived above, we will inquire about the participation patterns (n_ℓ, n_h) that may emerge in equilibrium. This part both "confirms" that the different forms of state-dependent participation may arise in a "closed" model, and is of interest in its own right as a self-contained piece that takes as given the bidding behavior derived in the first part and analyzes the solicitation equilibrium.

6.1 Strategic Solicitation and Equilibrium

Let $\Gamma(s, \Delta)$ be the game that includes both the strategic bidder solicitation by the seller and the strategic bidding by the buyers. A bidding strategy β is as before. A solicitation strategy $\mathbf{n} = (n_\ell, n_h)$ with $1 \leq n_\omega \leq N_s$, prescribes the number of bidders solicited by the seller in each state. The potential number of bidders N_s is such that $N_s \geq \frac{v_h}{s}$, which guarantees that it is never profitable for the seller to solicit all potential bidders. The restriction $n_\omega \geq 1$ is imposed to avoid dealing with trivial equilibria without participation. The expected winning bid in state ω when there are n bidders who use β is $\mathbb{E}[p|\omega; \beta, n]$.

A *pure equilibrium* of $\Gamma(s, \Delta)$ consists of a strategy β and a solicitation strategy $\mathbf{n} = (n_\ell, n_h)$, such that (i) β is a bidding equilibrium of $\Gamma_0(\mathbf{n}, N_s, \Delta)$ and (ii) the solicitation strategy is optimal for the seller, i.e.,

$$n_\omega \in \underset{n\in\{1,2,\ldots,N_s\}}{\arg\max} \left(\mathbb{E}\left[p|\omega;\beta,n\right]-ns\right). \tag{14}$$

Since a pure equilibrium might not exist, we allow for mixed solicitation strategies. Let $\boldsymbol{\eta}=(\eta_\ell,\eta_h)$ be a mixed solicitation strategy (where $\eta_\omega(n)$ is the probability of $n=1,\ldots,N_s$ bidders being invited in state ω), and let $\Gamma_0\left(\boldsymbol{\eta},N_s,\Delta\right)$ be the corresponding bidding game as introduced in Section 5.2. The expected payoff $U(b|x;\beta,\boldsymbol{\eta})$ and the probability of winning $\pi_\omega\left(b|\beta,\boldsymbol{\eta}\right)$ are now functions of the mixed strategy $\boldsymbol{\eta}$.[17] The equilibrium definitions are completely analogous with $\boldsymbol{\eta}$ replacing $\mathbf{n}$ and the optimality of η_ω stated in (14) is required to hold for each n in its support.

6.2 Optimal Solicitation Strategies are Essentially Pure

If η_ω is optimal, its support is either a single integer n or two adjacent integers $\{n,n+1\}$. This is because the seller's payoff, $\mathbb{E}\left[p|\omega;\beta,n\right]-ns$, is strictly concave in n, unless β is constant, in which case $n=1$ is optimal.

Lemma 1 **(Optimal Solicitation)** *Given any bidding strategy β, there is an integer n^*_ω such that*

$$\{n^*_\omega,n^*_\omega+1\}\supseteq \underset{n\in\{1,2,\cdots,N\}}{\arg\max}\ \mathbb{E}\left[p|\omega;\beta,n\right]-ns.$$

This result is familiar from other contexts, and it is an immediate consequence of the concavity of the expectation of the first-order statistic in n.[18]

Given the lemma, we restrict attention to mixed strategies η_ω whose support contains at most two adjacent integers. Any such mixed strategy η_ω can be described by two numbers, namely, $n_\omega\in\{1,\ldots,N\}$ and $\gamma_\omega\in(0,1]$, where $\gamma_\omega=\eta_\omega\left(n_\omega\right)>0$ and $1-\gamma_\omega=\eta_\omega\left(n_\omega+1\right)\geq 0$. A solicitation strategy is pure if $\gamma_\omega=1$. Thus, from here on, when we talk about n_ω in the context of a strategy η_ω, we mean the bottom of the support of η_ω.

As noted in Section 5.2, the introduction of mixed strategies does not alter the main qualitative features of the bidding equilibria.

[17] Some explicit expressions of these magnitudes that are needed for the proofs are in Appendix B.

[18] A self-contained proof for the current setting can be found in Lauermann and Wolinsky (2017).

7 Equilibria with Small Sampling Costs

The many-bidders case of Section 4 was captured by a sequence of bidding games, $\Gamma_0\left(\mathbf{n}^k, N^k, \Delta^k\right)$ in which $n_\omega^k \to \infty$, $\omega = \ell, h$ and $\Delta^k \to 0$. Since the number of bidders is now endogenous, we look at the sequences

$$\left(s^k\right)_{k=1}^{\infty}, \text{ with } s^k > 0 \text{ and } s^k \to 0. \tag{15}$$

and $\Delta^k \geq 0$ s.t. $\Delta^k \to 0$, which induce a sequence of games $\Gamma\left(s^k, \Delta^k\right)$. We consider a corresponding sequence of equilibrium strategies $(\beta^k, \boldsymbol{\eta}^k)$ and look at the limits of equilibrium magnitudes as $k \to \infty$. As it will turn out, with $s^k \to 0$, optimal solicitation usually results in $n_\omega^k \to \infty$. Therefore, here too, $k \to \infty$ is associated with ever larger numbers of solicited bidders. We continue with the simplifications adopted above of omitting $(\beta^k, \boldsymbol{\eta}^k)$ from the arguments of equilibrium magnitudes and omitting the delimiter $k \to \infty$ from the expression lim.

7.1 Existence and Characterization of Equilibrium with Endogenous Solicitation

Recall the shorthand $F_\omega^k(p)$ for the c.d.f. of the winning bid. We are interested in the shape of $F_\omega^k(p)$ as $s^k \to 0$, and denote its limit as $\bar{F}_\omega(p) = \lim_{k\to\infty} F_\omega^k(p)$ (when it exists).

Theorem 2 (**Equilibrium Characterization**) *Consider a sequence of games $\Gamma\left(s^k, \Delta^k\right)$ where $s^k \to 0$, $\Delta^k \geq 0$ and $\Delta^k \to 0$.*

1. *If $\Delta^k > 0$ for all k, there exists a sequence of equilibria $\left(\beta^k, \boldsymbol{\eta}^k\right)$ such that $\bar{F}_\omega = \Phi_\omega(\cdot|r^*)$, for $\omega = \ell, h$, where Φ_ω is described by (5)-(6) and r^* is a fixed number determined by ρ and $\overline{g}$. This sequence is such that* $\min\left\{n_\ell^k, n_h^k\right\} \to \infty$, *and* $\lim \frac{n_h^k}{n_\ell^k} = r^*$;
2. *If $\Delta^k > 0$ for all k, for some signal distributions G_ω, there exists a sequence of equilibria $\left(\beta^k, \boldsymbol{\eta}^k\right)$ for which* $\min\left\{n_\ell^k, n_h^k\right\} \to \infty$ *and $\bar{F}_\omega$ has mass 1 on some $C \leq \mathbb{E}[v]$, for $\omega = \ell, h$;*
3. *For any sequence of equilibria $\left(\beta^k, \boldsymbol{\eta}^k\right)$ for which* $\min\left\{n_\ell^k, n_h^k\right\} \nrightarrow \infty$ *and $\bar{F}_\omega$ exists, it has mass 1 on some $C \leq \mathbb{E}[v]$, for $\omega = \ell, h$;*
4. *For any sequence of equilibria $\left(\beta^k, \boldsymbol{\eta}^k\right)$ for which $\bar{F}_\omega$ exists, it must be of one of the forms described above: $\Phi_\omega(\cdot|r^*)$ or a mass 1 atom on some $C \leq \mathbb{E}[v]$.*

Theorem 2 brings both existence results (Parts 1 and 2) and characterization results (Parts 3 and 4). The existence results provide one possible foundation to the results of Theorem 1 by showing that the two different forms of $\bar{F}_\omega$—the continuous Φ_ω and the mass 1 atom below $\mathbb{E}[v]$—indeed arise with endogenous solicitation.

The characterization results present all the $\bar{F}_\omega$ forms that might arise in equilibrium[19,20] and make two new points relative to what we already know from Theorem 1. First, the case of $\min\left\{n_\ell^k, n_h^k\right\} \nrightarrow \infty$ is covered as well to account for the possibility that optimal sampling results in a bounded number even as $s \to 0$. This point is fairly obvious, though it requires some work to establish.[21] Second, optimal sampling pins down the ratio $\lim_{k\to\infty} \frac{n_h^k}{n_\ell^k}$ for the partially revealing equilibrium at r^*. As we show below, the ratio r^* is the unique $r > \frac{1}{\bar{g}}$ that solves

$$\int_0^1 (x - \frac{1}{\bar{g}}) x^{\frac{1}{\bar{g}r-1}} \frac{\ln x}{(1 + x\rho\bar{g}r)^2} dx = 0. \tag{16}$$

Hence, r^* depends only on $\bar{g}$ and ρ, and is independent of the v_ω's and of the other parameters of the G_ω's. This second point is perhaps the more distinct contribution of the theorem.

As was the case of Theorem 1, the characterization results do not require a finite grid; so, the no-grid ($\Delta^k = 0$) case is covered as well. The finite grid is only used to claim existence.

That a pooling equilibrium exists under certain conditions (Part 2) was already established in Lauermann and Wolinsky (2017) and therefore will not be proven in this in this paper.[22]

[19] One may worry about the condition that F_ω^k converges. However, by Helly's selection theorem, every sequence of c.d.f.s has a pointwise everywhere convergent subsequence, which can be adjusted to be a c.d.f..

[20] The assumption $n_\omega \geq 1$ excludes the equilibrium with no participation, which is otherwise present as well.

[21] This includes, in particular, the equilibrium with $\mathbf{n} = (1, 1)$ and $\beta(x) \equiv 0$ (subject to the constraint $n_\omega \geq 1$).

[22] Lauermann and Wolinsky (2017) show that a pooling equilibrium exists when signals are binary. Lauermann and Wolinsky (2018b) generalizes this result and shows that a pooling equilibrium exists for a class of discrete signals. This does not mean that such equilibria exist only under those circumstances, but rather that we know how to construct an equilibrium in these cases.

7.2 Proof of Theorem 2

We prove first the characterization results (Part 3 and 4 of the theorem). Since, as mentioned above, the finite grid is not needed for these results, we will omit it in this part of the proof (e.g., write $\Gamma\left(s^k\right)$ rather than $\Gamma\left(s^k, \Delta^k\right)$). Then, we turn to existence (Parts 1 and 2) and resurrect the finite grid.

7.2.1 Proving the Characterization Results (Part 3 and 4)

We use a result of Lauermann and Wolinsky (2018a) that establishes a relationship between the total solicitation cost and the distribution of the winning bid in first-price auctions with bidder solicitation.[23] Specifically, for the common-value environment of this paper, it implies the following lemma.

Lemma 2 (**Total Solicitation Costs**) *Consider a sequence $s^k \to 0$ and a sequence of bidding strategies β^k. Suppose that η^k_ω is an optimal solicitation strategy given β^k in state ω, and the implied winning bid distribution F^k_ω converges pointwise to $\bar{F}_\omega$. Then,*

$$\lim n^k_\omega s^k = -\int_0^{v_h} \left(\bar{F}_\omega\left(p\right)\right) \ln\left(\bar{F}_\omega\left(p\right)\right) dp.$$

That is, the total cost of the optimal solicitation is proportional to a certain dispersion measure of the winning bid. Obviously, at the optimum, a non-degenerate winning bid distribution is incompatible with zero total solicitation cost, since in such a case doubling the number of bidders would strictly increase the expected winning bid without increasing the cost. Moreover, a more dispersed distribution implies larger a reward to such doubling of the number, and hence must be counterbalanced by a larger total solicitation cost.

Proposition 4 addresses the main new insight of Theorem 2: the optimal sampling pins down $\lim_{k\to\infty} \frac{n^k_h}{n^k_\ell}$.

Proposition 4 *Consider a sequence of games $\Gamma\left(s^k\right)$ such that $s^k \to 0$, and a corresponding sequence of equilibria $\left(\beta^k, \boldsymbol{\eta}^k\right)$. If $\min\left\{n^k_\ell, n^k_h\right\} \to \infty$ and $\bar{g}\lim_{k\to\infty} \frac{n^k_h}{n^k_\ell} > 1$, then $\lim_{k\to\infty} \frac{n^k_h}{n^k_\ell} = r^*$, which is the unique $r > \frac{1}{\bar{g}}$ solution of (16).*

Proof of Proposition 4: We show that, for every subsequence along which $\frac{n^k_h}{n^k_\ell}$ converges (in the extended reals), its limit is r^*, proving that the sequence itself converges.

[23] The note is posted at SSRN at http://dx.doi.org/10.2139/ssrn.3294964.

In the following, we utilize the characterization of bidding equilibria from Theorem 1 that is shown to hold also for a mixed solicitation strategy in Proposition 3.

Let $r = \lim \frac{n_h^k}{n_\ell^k}$, and suppose that $r < \infty$ and $\overline{g}r > 1$. By Theorem 1, $\lim_{k\to\infty} F_\omega\left(p|\beta^k, \eta_\omega^k\right) = \Phi_\omega\left(\cdot|r\right)$. This and Lemma 2 together imply that

$$\lim_{k\to\infty} n_\omega^k s^k = -\int_{v_\ell}^{\frac{v_\ell+\rho\overline{g}rv_h}{1+\rho\overline{g}r}} \left(\Phi_\omega\left(p|r\right)\right)\ln\left(\Phi_\omega\left(p|r\right)\right)dp.$$

Since $\lim\left(n_h^k s^k\right) = r\lim\left(n_\ell^k s^k\right)$, it follows that

$$\frac{1}{r}\int_{v_\ell}^{\frac{v_\ell+\rho\overline{g}rv_h}{1+\rho\overline{g}r}} \left(\Phi_h\left(p|r\right)\right)\ln\left(\Phi_h\left(p|r\right)\right)dp = \int_{v_\ell}^{\frac{v_\ell+\rho\overline{g}rv_h}{1+\rho\overline{g}r}} \left(\Phi_\ell\left(p|r\right)\right)\ln\left(\Phi_\ell\left(p|r\right)\right)dp. \quad (17)$$

In Appendix **C** we rewrite (17) using the explicit characterization of Φ_ω to prove the following lemma.

Lemma 3 *For any $\rho > 0$ and $\overline{g} > 1$, there is a unique number $r^* \in \left(\frac{1}{\overline{g}}, \infty\right)$ such that equation (17) holds. It is the unique $r > \frac{1}{\overline{g}}$ that solves (16).*

It follows from (17) and Lemma 3 that, for any sequence of equilibria with $\overline{g}\lim \frac{n_h^k}{n_\ell^k} > 1$ and $\lim \frac{n_h^k}{n_\ell^k} < \infty$, it must be the case that $\lim \frac{n_h^k}{n_\ell^k} = r^*$.

Thus, Proposition 4 holds if $\lim \frac{n_h^k}{n_\ell^k} < \infty$ for any such sequence. Suppose to the contrary that $\lim \frac{n_h^k}{n_\ell^k} = \infty$. Then, Theorem 1 implies that $\lim F_\omega\left(\cdot|\beta^k, \eta_\omega^k\right)$ is a degenerate distribution with support v_ω. Lemma 2 implies that $\lim n_\omega^k s^k = 0$, so that seller type ω's equilibrium payoff converges to v_ω.

By analogous reasoning to Lemma 5, if $\lim F_h\left(p|\beta^k, \eta_h^k\right) = 0$, then $\lim F_\ell\left(p|\beta^k, \eta_h^k\right) = 0$. Therefore, if seller type ℓ solicits n_h^k bidders, $\lim \mathbb{E}\left[p|\ell;\beta^k, n_h^k\right] \geq v_h$. Since $\lim n_h^k s^k = 0$, for large k, seller type ℓ's payoff with this strategy is near v_h, which is larger than her equilibrium payoff near v_ℓ. Thus, $\lim \frac{n_h^k}{n_\ell^k} = \infty$ cannot hold. ∎

The following Proposition 5 establishes part 3 of the theorem.

Proposition 5 *Consider a sequence of games $\Gamma\left(s^k\right)$ such that $s^k \to 0$. Suppose that $\left(\beta^k, \boldsymbol{\eta}^k\right)$ is a corresponding sequence of equilibria such that $\min\left\{n_\ell^k, n_h^k\right\} \nrightarrow \infty$. Then, $\bar{F}_\omega$ has probability mass 1 on some $C \leq \mathbb{E}\left[v\right]$, for both $\omega = \ell$ and $\omega = h$.*

The proof of Proposition 5 in Appendix **C** has to deal with the complication that β^k might not be monotone since $n_\omega^k \geq 2$ may not be assumed in this case.

Part 4 of Theorem 2 can now be proved using Propositions 4 and 5 and Theorem 1. First, suppose that $\bar{F}_\omega$ is not a mass 1 atom. Proposition 5 then implies that $\min\left\{n_\ell^k, n_h^k\right\} \to \infty$. Hence, Theorem 1 implies that $\bar{g}\lim_{k\to\infty}\frac{n_h^k}{n_\ell^k} > 1$ and then, Proposition 4 implies that $\lim_{k\to\infty}\frac{n_h^k}{n_\ell^k} = r^*(\rho, \bar{g})$.

Second, suppose $\bar{F}_h$ has an atom of mass 1 at C. If $\min\left\{n_\ell^k, n_h^k\right\} \to \infty$, then Proposition 4 implies that $\bar{g}\lim_{k\to\infty}\frac{n_h^k}{n_\ell^k} \leq 1$.[24] Hence, by Theorem 1, $C \leq \mathbb{E}(v)$. Otherwise, if $\min\left\{n_\ell^k, n_h^k\right\} \nrightarrow \infty$, then Proposition 5 implies $C \leq \mathbb{E}(v)$.

This establishes Part 4 of Theorem 2 and hence concludes the proof of the characterization results of this theorem. ∎

7.2.2 Proving the Existence Results (Parts 1 and 2)

Existence alone is not an issue, since there is always a trivial equilibrium with $n_\ell = n_h = 1$ and $\beta \equiv 0$ (subject to the constraint $n_\omega \geq 1$). At issue is the existence of the nontrivial equilibria described in Parts 1 and 2. Since some version of Part 2 was already established in Lauermann and Wolinsky (2017), we prove in this paper only Part 1 as restated by the following proposition.

Proposition 6 *For any sequence of games $\Gamma\left(s^k, \Delta^k\right)$ for which $\Delta^k > 0$ and $\lim\left(s^k, \Delta^k\right) = (0,0)$, there exists a sequence of equilibria $\left(\beta^k, \boldsymbol{\eta}^k\right)$ such that $\bar{F}_\omega = \Phi_\omega\left(\cdot|r^*\right)$.*

The proof introduces "constrained equilibria" $\left(\beta^k, \boldsymbol{\eta}^k\right)$ such that β^k is a bidding equilibrium given $\boldsymbol{\eta}^k$ (as before) but $\boldsymbol{\eta}^k$ are optimal subject to the following constraints: (i) n_ω^k is forced to increase without a bound, i.e., $n_\omega^k \geq \underline{n}_\omega^k$ for some exogenous $\underline{n}_\omega^k \to \infty$ and (ii) $\frac{n_h^k}{n_\ell^k} \geq \bar{r}$ for some $\bar{r} \in (\frac{1}{\bar{g}}, r^*)$. By Theorem 1, such a constrained equilibrium $\left(\beta^k, \boldsymbol{\eta}^k\right)$ must induce a non-degenerate winning bid distribution. The proof then uses two Lemmas. Lemma 16 establishes that, if a sequence of constrained equilibria $\left(\beta^k, \boldsymbol{\eta}^k\right)_{k=1}^\infty$ exists, then for large enough k the constraints do not bind whenever the bound $\underline{n}_\omega^k$ grows sufficiently slowly; therefore, $\left(\beta^k, \boldsymbol{\eta}^k\right)$ is an unconstrained equilibrium. This and the characterization established above then imply that the limiting winning bid distribution is $\Phi_\omega\left(\cdot|r^*\right)$. Lemma 15 uses essentially standard existence arguments (e.g., Athey (2001)) to establish that, for any sequence $\left(s^k, \Delta^k\right) \to (0,0)$ for which $\Delta^k > 0$ for all k, a constrained equilibrium exists. Together these lemmas establish the proposition. ∎

The proposition completes the proof of Theorem 2. ∎

[24] This conclusion relies on Proposition 4 ruling out that $C = v_h$ and $\lim_{k\to\infty}\frac{n_h^k}{n_\ell^k} = \infty$.

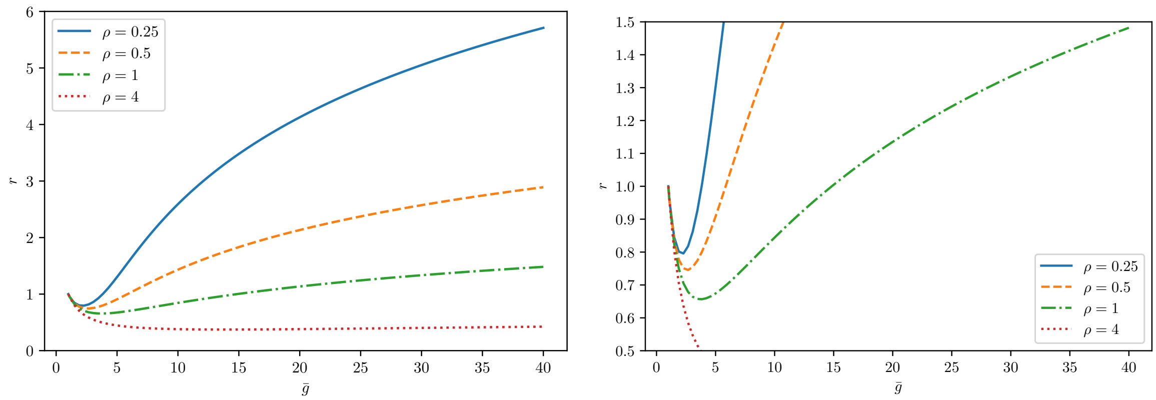


Figure 1: The ratio $r^*(\bar{g}, \rho)$ as a function of $\bar{g}$ for 4 different levels of $\rho(\equiv \frac{\rho_h}{\rho_\ell})$.

The key step in the last (existence) part of the proof is Lemma 16, which shows that, for k large enough, the constraint $\frac{n_h^k}{n_\ell^k} \geq \bar{r}$ does not bind if $\bar{r}\bar{g} > 1$. In particular, it cannot be the case that $\lim \frac{n_h^k}{n_\ell^k} = \bar{r}$ over a sequence of constrained equilibria. This is easier to see when $\bar{r}$ is close to $\frac{1}{\bar{g}}$. In this case, $\lim \frac{n_h^k}{n_\ell^k} = \bar{r}$ implies that $\bar{g} \lim \frac{n_h^k}{n_\ell^k}$ is close to 1, and hence $\Phi_\ell(\cdot|\bar{r})$ is close to $\Phi_h(\cdot|\bar{r})$. Therefore, by Lemma 2, the optimal n_ℓ^k and n_h^k would be similar as well, so that $\lim \frac{n_h^k}{n_\ell^k} \approx 1 > \frac{1}{\bar{g}} \approx \bar{r}$. Thus, $\lim \frac{n_h^k}{n_\ell^k} = \bar{r}$ cannot be a fixed point of the limit of constrained equilibria.

8 Discussion for Strategic Solicitation

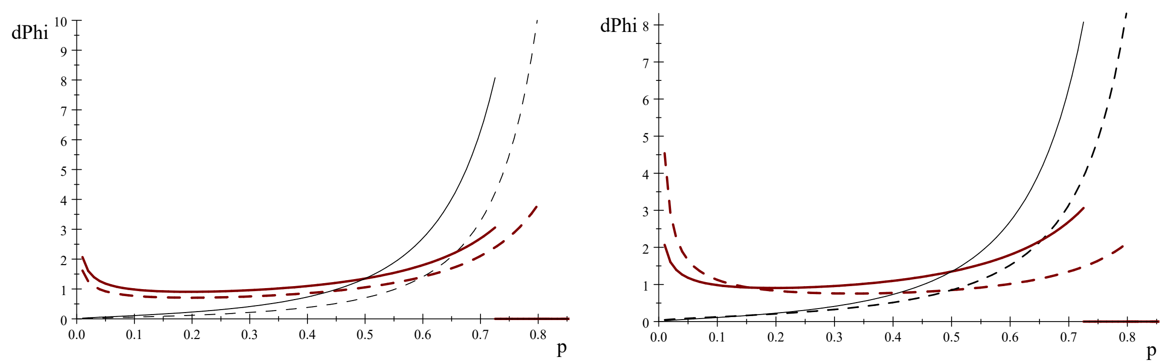
8.1 Information Aggregation and its Relation to Parameters

By Theorem 2, strategic state-dependent solicitation pins down r at $r^* = r^*(\bar{g}, \rho)$. Let us examine how $r^*(\bar{g}, \rho)$ depends on the parameters. Figure 1 illustrates the shape of the ratio $r^*(\bar{g}, \rho)$ as a function of $\bar{g}$ for 4 different levels of $\rho(\equiv \frac{\rho_h}{\rho_\ell})$. The right-hand panel of Figure 1 just offers a closer look at the dip that occurs at low values of $\bar{g}$ and is also visible in the left-hand panel. The negative relationship between $r^*(\bar{g}, \rho)$ and ρ depicted by the diagram holds for all $\bar{g}$ and ρ as can be confirmed by implicit differentiation of (16). We conjecture[25] that the behavior of $r^*(\bar{g}, \rho)$ as a function of $\bar{g}$ at other levels of ρ is well represented by the curves

[25] The analysis on which this conjecture is based is too complicated to present or confirm. Since the exact shape is not too important for our discussion, we leave it as a conjecture rather than a fact.

displayed in the graph. That is, for any ρ, there is a cutoff $\hat{g}(\rho)$ such that $r^* \gtrless 1$ if $\bar{g} \gtrless \hat{g}(\rho)$ and for $\bar{g} > \hat{g}(\rho)$, we have $r^*(\bar{g}, \rho) \approx a(\rho) + b(\rho) \ln \bar{g}$. This means that r^* is increasing very slowly at large $\bar{g}$'s.

Partial intuition. Since r^* is a fixed point, the intuitive explanation for its dependence on the parameters is not entirely straightforward. A partial intuition can be obtained by inspecting the densities of the winning bid distributions, $d\Phi_\omega$ for $\omega = \ell, h$. The more dispersed the distribution is, the larger the incentive for sampling in state ω. Figure 2 shows how the dispersion of Φ_ω, $\omega = \ell, h$, changes for a fixed $r^* = r^*(4, 1)$ with ρ and $\bar{g}$ respectively. The left-hand panel shows that, as ρ goes from $\rho = 1$ to $\rho = 1.5$, Φ_h becomes relatively more concentrated than Φ_ℓ, which explains stronger incentive to sample in state ℓ, and hence lower r^*. The right panel shows that when $\bar{g}$ changes from $\bar{g} = 4$ to $\bar{g} = 6$, both Φ_ω's become more concentrated (Φ_h becomes more concentrated at the top, and Φ_ℓ more concentrated both at the bottom and at the top). So, inspection of the diagram does not suggest a clear conclusion on how this change of $\bar{g}$ affects the relative incentives to sample in the two states. Indeed, as we know from Figure 1, r^* is not monotone in $\bar{g}$ and is not affected much when $\bar{g}$ is large.



(a) $d\Phi_\omega$ for $\rho = 1$ (straight) and $\rho = 1.5$ (dashed). (b) $d\Phi_\omega$ for $\bar{g} = 4$ (straight) and $\bar{g} = 6$ (dashed).

Figure 2: The figure shows the densities $d\Phi_h$ (in black/thin) and $d\Phi_\ell$ (in red/thick) for two different levels of $\bar{g}$ and ρ at r^* given $\bar{g} = 4$ and $\rho = 1$.

Effects on information aggregation. Recall from Corollary 2 that, in the partially revealing case ($\bar{g}r > 1$) of auctions with state-dependent participation, the extent of information aggregation increases with $\bar{g}r$ and becomes nearly perfect as $\bar{g}r \to \infty$. One immediate question is whether a higher $\bar{g}$ that corresponds to more informative basic signals results in a higher $\bar{g}r^*$, and hence improved information aggregation, or whether its effect is mitigated or even offset by a lower r^*. As the Figure 1 shows, when $\bar{g}$ is small, $r^* < 1$ and when $\bar{g}$ is large, $r^* > 1$ (recall that in any case $\bar{g}r^* > 1$). Thus, the endogenous participation ratio r^* dampens information aggregation further when $\bar{g}$ is small and enhances it further when $\bar{g}$ is large. Furthermore, since over the latter range, $r^*(\bar{g})$ is increasing, a larger $\bar{g}$ is associated with a more significant reinforcement of the information aggregation by the endogenous participation (though, for large $\bar{g}$, r^* is increasing at a slow rate).

Since r^* is decreasing in ρ, a higher ρ reduces the extent of information aggregation.

The large ordinary common value auction ($n_\ell = n_h$) is of course the special case of $r = 1$. Claim 2 immediately implies that, in a large ordinary common value auctions the extent of information aggregation increases with $\bar{g}$, since $\bar{g}r$ is obviously increasing with $\bar{g}$ given the fixed $r = 1$. The above observations on r^* imply that, with large $\bar{g}$, more information is aggregated into the price in the auction with state-dependent solicitation than in the ordinary large auction, and the opposite is true when $\bar{g}$ is small.

With endogenous solicitation, the case of $n_\ell = n_h$ would arise when the seller is uninformed about the state. Thus, the above is also a comparison of the extent of information aggregation in the alternative scenarios of an informed and an uniformed seller. When $\bar{g}$ is large, the informed seller's actions inject information into the price and the extent of information aggregation is larger than it is with an uniformed seller. But when $\bar{g}$ is small, the informed seller's actions actually reduce the extent of information aggregation relative to the uniformed seller's case.

The comparison between these two regimes depends on ρ as well. When ρ is large, the advantage of the ordinary auction in information aggregation at low $\bar{g}$'s is more pronounced, but at large $\bar{g}$'s there is no big difference between the two regimes. This is reversed for small ρ: less pronounced difference at lower $\bar{g}$'s, but significant differences at higher $\bar{g}$'s.

These conclusions together with Claim 2 have implications for the revenue comparison between the informed and uninformed seller regimes. The revenue of the informed seller is higher in state h and lower in state ℓ when $\bar{g}$ is larger or ρ is smaller, while the opposite relations hold when $\bar{g}$ is smaller or ρ is larger. The ex-ante expected revenue is, of course, $\mathbb{E}[v]$ in all cases, but its distribution across states varies.

8.2 Seller's Commitment and Ex-Ante Optimal Participation

If the seller can commit ex-ante to a solicitation strategy (while otherwise the game remains unchanged), she can extract nearly the entire surplus when s is small. For example, commitment to $n_\ell = n_h = 1/\sqrt{s}$—which is large when s is small—implies via Corollary 1 that the ex-ante expected revenue is approximately equal to the ex-ante expected value $\mathbb{E}[v]$. Since the total solicitation cost is just $\sqrt{s}$, the seller's ex-ante expected payoff is approximately $\mathbb{E}[v] - \sqrt{s}$, which for small s is strictly higher than the seller's payoff in the partially revealing equilibrium without commitment where the total solicitation cost remains bounded away from zero even as $s \to 0$. Since in this specific example of commitment $n_\ell = n_h$, this conclusion is also valid for the case of an uninformed seller who does not know the true state.

8.3 Unboundedly Informative Signals

It has been assumed throughout that the signals are boundedly informative, $\frac{g_h(\bar{x})}{g_\ell(\bar{x})} < \infty$. While this assumption was used in the analysis, some of the results extend to a setting with an unboundedly informative signal; see the working paper version of this article, Lauermann and Wolinsky (2018b). Specifically, with an unbounded likelihood ratio, all equilibria of the full game are still either of the pooling form or of the partially revealing form (which, in this case, is perfectly revealing in the limit when the numbers of bidders go to ∞). Moreover, for any unboundedly informative signal, a perfectly revealing equilibrium exists. Finally, a pooling equilibrium can actually exist for some signal distributions exhibiting an unbounded likelihood ratio.

8.4 Simultaneous Search

Although our model is framed using the terminology of auctions, it can also be read as a simultaneous search model along the lines of Burdett and Judd (1983), with adverse selection as the added element. In that model, a buyer obtains a sample of prices from sellers of a homogeneous product. The seller in our model is the counterpart of the buyer in their model.[26] The important difference is that our

[26] The roles of the seller and the buyers in our model can be reversed to make the models exactly parallel.

variation on their model endows this buyer with private information that might affect the seller's cost. This could be relevant for markets of certain services, such as repair or the credit markets mentioned in the introduction. The private information implies both additional substantive insights and some additional analytical challenges. In particular, in Burdett and Judd's (1983) model, the more convincing equilibrium becomes competitive when the sampling cost becomes negligible, while this is not necessarily the case in our model.

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A Bidding Equilibrium with State-Dependent Participation

Auxiliary Result: Winning Probability at Atoms. The following lemma is restated from Lauermann and Wolinsky (2017). It derives an expression for the winning probability in the case of a tie. Define

$$x_-(b) \triangleq \inf \left\{x \in [\underline{x}, \bar{x}] \,|\beta(x) \geq \bar{b}\right\}$$

and

$$x_+ \triangleq \sup \left\{x \in [\underline{x}, \bar{x}] \,|\beta(x) \leq \bar{b}\right\}.$$

Lemma 4 [Lauermann and Wolinsky, 2017.] *Suppose β is nondecreasing and, for some $\bar{b}$, $x_- = x_-\left(\bar{b}\right) < x_+\left(\bar{b}\right) = x_+$. Then,*

$$\pi_\omega\left(\bar{b}\right) = \frac{G_\omega\left(x_+\right)^n - G_\omega\left(x_-\right)^n}{n\left(G_\omega\left(x_+\right) - G_\omega\left(x_-\right)\right)} = \int_{x_-}^{x_+} \frac{\left(G_\omega\left(x\right)\right)^{n-1} g_\omega\left(x\right) dx}{G_\omega\left(x_+\right) - G_\omega\left(x_-\right)}. \tag{18}$$

Observe that the last expression is the expected probability of a randomly drawn signal from $[x_+, x_-]$ to be the highest. Thus, $\pi_\omega\left(\bar{b}\right)$ "averages" what would be the winning probabilities of the types in $[x_+, x_-]$ if β were strictly increasing.

A.1 Proof of Proposition 2 and Theorem 1 (Large Bidding Equilibria)

Here, and in the rest of the appendix, we often use the abbreviation

$$\lambda \triangleq \overline{g} \lim \frac{n_h^k}{n_\ell^k}.$$

A.1.1 Preliminary Comments

The finite grid ($\Delta^k > 0$) is needed only for the existence claims but not for the characterization results. We therefore proceed as follows. First, we show the characterization results for the no-grid case of $\Delta^k = 0$ because this case is less cluttered, proving Proposition 2. Second, we resurrect the finite grid with $\Delta^k > 0$ to explain the adaptations of the proof that it requires, proving the characterization parts of Theorem 1. Finally, we establish the existence of equilibria, especially those described in Part 2 of Theorem 1.

We prepare the proof with a number of auxiliary lemmas that hold for $\Delta^k \geq 0$.

A.1.2 Auxiliary Lemmas

The next lemma formalizes the idea that the number of bidders with signals close to $\bar{x}$ is Poisson distributed.

Lemma 5 (Poisson-Approximation.) *Consider some sequence $\left(x^k, \mathbf{n}^k\right)$ with $\min\{n_\ell^k, n_h^k\} \to \infty$ and $\lim \frac{n_h^k}{n_\ell^k} = r < \infty$. If*

$$\lim \left(G_\ell(x^k)\right)^{n_\ell^k} = q$$

for some $q \in [0,1]$, *then*

$$\lim \left(G_h(x^k)\right)^{n_h^k} = q^{\bar{g}r}.$$

Proof of Lemma 5. Let $Q_\omega \triangleq \lim \left(1 - G_\omega\left(x^k\right)\right) n_\omega^k \in [0,\infty) \cup \infty$. Observe that

$$\lim \left(G_\omega(x^k)\right)^{n_\omega^k} = \lim(1 - \frac{1 - G_\omega\left(x^k\right)}{n_\omega^k} n_\omega^k)^{n_\omega^k} = e^{-Q_\omega}.$$

The lemma clearly holds with $q = 0$ if $\lim x^k < \bar{x}$. So, suppose $\lim x^k = \bar{x}$. Then, $\lim \frac{1-G_h\left(x^k\right)}{1-G_\ell\left(x^k\right)} = \bar{g}$, and so we have $Q_h = Q_\ell \bar{g} \lim(n_h^k/n_\ell^k)$. Therefore, $q = e^{-Q_\ell}$ implies

$$\lim \left(G_h(x^k)\right)^{n_h^k} = e^{-Q_h} = e^{Q_\ell \bar{g} \lim(n_h^k/n_\ell^k)} = q^{\bar{g}r}. \qquad \blacksquare$$

Recall that

$$U(b|x, \text{sol}; \beta, \mathbf{n}) = \frac{\rho_\ell g_\ell\left(x\right) n_\ell \pi_\ell\left(b; \beta, n_\ell\right)\left(v_\ell - b\right) + \rho_h g_h\left(x\right) n_h \pi_h\left(b; \beta, n_h\right)\left(v_h - b\right)}{\rho_\ell g_\ell\left(x\right) n_\ell + \rho_h g_h\left(x\right) n_h}. \tag{19}$$

Lemma 6 ("Zero Profit") *For any* $\varepsilon > 0$*, there is an* $M(\varepsilon)$ *such that, if* $n_\omega > M(\varepsilon)$*,* $\omega = \ell, h$*, then* $U(\beta(x)|x, \text{sol}; \beta, \mathbf{n}) < \varepsilon$ *for all* x *in every bidding equilibrium* β*.*

Remark: We do not suppress here $\beta, \mathbf{n}$ from the arguments of U since the claim concerns a range of $\mathbf{n}$ and all corresponding equilibria β.

Proof of Lemma 6. By (19) and the right-continuity of $\frac{g_h}{g_\ell}$, $(U(b|\cdot, \text{sol}; \beta, \mathbf{n}))_{b,\beta,\mathbf{n}}$ is a family of functions that is uniformly (right-)equi-continuous: For every $\varepsilon > 0$ and x, there is some $z_\varepsilon > 0$ such that

$$|U(b|x', \text{sol}; \beta, \mathbf{n}) - U(b|x, \text{sol}; \beta, \mathbf{n})| \leq \frac{\varepsilon}{2},$$

for all b, all $(\beta, \mathbf{n})$ and all x' such that $0 \leq x' - x \leq z_\varepsilon$; similarly at $\bar{x}$ for all x' s.t. $\bar{x} - x' \leq z_\varepsilon$.[27]

Suppose $U(\beta(x)|x, \text{sol}; \beta, \mathbf{n}) = \varepsilon > 0$ for some $x < \bar{x}$ (the case $x = \bar{x}$ is analogous and omitted). From β being a bidding equilibrium, for all $x' > x$ s.t. $x' - x \leq z_\varepsilon$,

$$|U(\beta(x)|x, \text{sol}; \beta, \mathbf{n}) - U(\beta(x')|x', \text{sol}; \beta, \mathbf{n})| \leq \frac{\varepsilon}{2}. \tag{20}$$

[27] The monotonicity of $U(\beta(x)|x, \text{sol}; \beta, \mathbf{n})$ in x, which is established in Lemma 11, implies that it would be sufficient to argue the result for $\bar{x}$.

Therefore,

$$U(\beta(x)|x,\text{ sol};\beta,\mathbf{n})-\frac{\varepsilon}{2}\leq\inf_{x'\in[x,x+z_\varepsilon]}U(\beta(x')|x',\text{ sol};\beta,\mathbf{n})$$
$$\leq\sum_{\omega=\ell,h}\rho_\omega\frac{\int_x^{x+z_\varepsilon}[v_\omega-\beta(x')]\pi_\omega(\beta(x');\beta,n_\omega)dG_\omega(x')}{G_\omega(x+z_\varepsilon)-G_\omega(x)}\leq\sum_{\omega=\ell,h}\rho_\omega\frac{v_\omega\int_{\underline{x}}^{\bar{x}}\pi_\omega(\beta(x');\beta,n_\omega)dG_\omega(x')}{G_\omega(x+z_\varepsilon)-G_\omega(x)}$$
$$=\sum_{\omega=\ell,h}\frac{\rho_\omega v_\omega}{n_\omega\left(G_\omega(x+z_\varepsilon)-G_\omega(x)\right)}\leq\frac{\mathbb{E}[v]}{\min_{\omega\in\{\ell,h\}}\left(n_\omega\left(G_\omega(x+z_\varepsilon)-G_\omega(x)\right)\right)},$$

where the first inequality follows from (20), the second follows from the definition of U, the third owes to increasing the term in the numerator, and the fourth from the fact that the expected probability of winning over all signals is $1/n_\omega$. Now, let $M(\varepsilon)$ be large enough so that, for $n_\omega\geq M(\varepsilon)$, the RHS is smaller than $\frac{\varepsilon}{2}$. Therefore, for any $\mathbf{n}$ such that $n_\omega\geq M(\varepsilon)$, $U(\beta(x)|x,\text{ sol};\beta,\mathbf{n})<\varepsilon$. ∎

Corollary 3 *Let $(\mathbf{n}^k)_{k=1}^\infty$ be such that $\min\{n_\ell^k,n_h^k\}\to\infty$ and $(\beta^k)_{k=1}^\infty$ be a corresponding sequence of bidding equilibria.*
(i)

$$\lim\sup_{x\in[\underline{x},\bar{x}]}U^k(\beta^k(x)|x,\ sol)=0. \tag{21}$$

(ii) If, for some sequence $(b^k)_{k=1}^\infty$ of bids and some ω, $\lim\pi_\omega^k(b^k)>0$, then for any sequence $(x^k)_{k=1}^\infty$,

$$\lim\mathbb{E}^k[v|x^k,\textit{sol,win at }b^k]\leq\lim b^k. \tag{22}$$

(iii) If $\lim\pi_\omega^k\left(\beta^k(x^k)\right)>0$ for some ω and sequence $(x^k)_{k=1}^\infty$, then

$$\lim\beta^k(x^k)=\lim\mathbb{E}^k[v|x^k,\textit{sol,win at }\beta^k(x^k)]. \tag{23}$$

Proof of Corollary 3: From Lemma 5, $\lim\pi_h^k\left(\beta^k(x^k)\right)>0\Leftrightarrow\lim\pi_\ell^k\left(\beta^k(x^k)\right)>0$. Therefore, $\lim\pi_\omega^k(b^k)>0$ for some ω is sufficient for $\lim\pi_\omega^k(b^k)>0$ for all ω.

Part (i) and (ii) follow immediately from Lemma 6 that would be contradicted if (21) or (22) did not hold. Part (iii) is immediate from (22) and the individual rationality condition,

$$\beta^k(x^k)\leq\mathbb{E}^k[v|x^k,\text{sol,win at }\beta^k(x^k)].$$

∎

Recall that $\overline{g}\triangleq\frac{g_h(\bar{x})}{g_\ell(\bar{x})}$.

Lemma 7 *Let $\mathbf{n}^k$ be such that $\min\{n_\ell^k, n_h^k\} \to \infty$ and $\overline{g} \lim \frac{n_h^k}{n_\ell^k} < 1$, and let $(\beta^k)_{k=1}^\infty$ be a corresponding sequence of (nondecreasing) bidding strategies. If $(b^k)_{k=1}^\infty$ is a sequence of bids such that $b^k < \beta^k(\bar{x})$ for all k and $\lim \pi_\ell^k(b^k) \in (0,1)$, then,*

$$\lim \mathbb{E}^k[v|\bar{x}, \textit{sol,win at } b^k] > \lim \mathbb{E}^k[v|\bar{x}, \textit{sol,win at } \beta^k(\bar{x})].$$

Proof of Lemma 7. Divide through the numerator and denominator of (3) by $\rho_\ell g_\ell(x) n_\ell \pi_\ell(b)$ to express it in terms of the compound likelihood ratio $\frac{\rho_h}{\rho_\ell}\frac{g_h(x)}{g_\ell(x)}\frac{n_h}{n_\ell}\frac{\pi_h(b)}{\pi_\ell(b)}$ as

$$\mathbb{E}[v|x,\text{sol,win at } b] = \frac{v_\ell + \frac{\rho_h g_h(x) n_h \pi_h(b)}{\rho_\ell g_\ell(x) n_\ell \pi_\ell(b)} v_h}{1 + \frac{\rho_h g_h(x) n_h \pi_h(b)}{\rho_\ell g_\ell(x) n_\ell \pi_\ell(b)}}. \tag{24}$$

Hence, we have to show that

$$\lim \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} > \lim \frac{\pi_h^k(\beta^k(\bar{x}))}{\pi_\ell^k(\beta^k(\bar{x}))}. \tag{25}$$

Let

$$\begin{aligned} \hat{q} &\triangleq \lim \left(G_\ell(x_+^k(b^k))\right)^{n_\ell^k - 1}, \\ \hat{q}_- &\triangleq \lim \left(G_\ell(x_-^k(b^k))\right)^{n_\ell^k - 1}, \end{aligned}$$

with $1 \geq \hat{q} \geq \hat{q}_- > 0$ by $\lim \pi_\ell^k(b^k) \in (0,1)$. Recall $\lambda \triangleq \overline{g} \lim \frac{n_h^k}{n_\ell^k}$. We first show the following:

$$\lim \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} = \hat{q}^{\lambda - 1} > 1 \text{ if } \hat{q}_- = \hat{q} \tag{26}$$

and

$$\lim \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} = \frac{(\hat{q})^\lambda - (\hat{q}_-)^\lambda}{\lambda(\hat{q} - \hat{q}_-)} > \hat{q}^{\lambda - 1} \geq 1 \text{ if } \hat{q}_- < \hat{q}. \tag{27}$$

To derive (26), note that[28]

$$\left(G_\omega(x_-^k)\right)^{n_\omega^k - 1} \leq \pi_\omega(b^k|\beta^k, n_\omega^k) \leq \left(G_\omega(x_+^k)\right)^{n_\omega^k - 1}.$$

[28] This can be verified using Lemma 4. For example, expanding the formula for π_ω gives

$$\pi_\omega\left(b^k|\beta^k, n_\omega^k\right) = \frac{1}{n_\omega^k}[G_\omega(x_+^k)^{n_\omega^k - 1} + G_\omega(x_+^k)^{n_\omega^k - 2} G_\omega(x_-^k) + \cdots + G_\omega(x_-^k)^{n_\omega^k - 1}] \geq \frac{n_\omega^k G_\omega(x_+^k)^{n_\omega^k - 1}}{n_\omega^k}.$$

Hence, whenever $\lim \left(G_\ell\left(x_-^k\right)\right)^{n_\ell^k-1} = q_- = \hat{q} = \lim \left(G_\ell\left(x_+^k\right)\right)^{n_\ell^k-1}$, Lemma 5 implies

$$\lim \frac{\pi_h^k\left(b^k\right)}{\pi_\ell^k\left(b^k\right)} = \frac{\hat{q}^\lambda}{\hat{q}} = \hat{q}^{\lambda-1}.$$

To derive (27), recall from Lemma 4 that

$$\pi_\omega^k\left(b^k\right) = \frac{\left(G_\omega\left(x_+^k\right)\right)^{n_\omega^k} - \left(G_\omega\left(x_-^k\right)\right)^{n_\omega^k}}{n_\omega^k[G_\omega\left(x_+^k\right) - G_\omega\left(x_-^k\right)]}, \tag{28}$$

and hence using Lemma 5,

$$\lim \frac{\pi_h^k\left(b^k\right)}{\pi_\ell^k\left(b^k\right)} = \lim \frac{n_\ell^k}{n_h^k} \frac{G_\ell\left(x_+^k\right) - G_\ell\left(x_-^k\right)}{G_h\left(x_+^k\right) - G_h\left(x_-^k\right)} \frac{G_h\left(x_+^k\right)^{n_h^k} - G_h\left(x_-^k\right)^{n_h^k}}{G_\ell\left(x_+^k\right)^{n_\ell^k} - G_\ell\left(x_-^k\right)^{n_\ell^k}} = \frac{(\hat{q})^\lambda - (\hat{q}_-)^\lambda}{\lambda\left(\hat{q} - \hat{q}_-\right)}.$$

To show the inequality $\frac{(\hat{q})^\lambda - (\hat{q}_-)^\lambda}{\lambda(\hat{q}-\hat{q}_-)} > \hat{q}^{\lambda-1}$, let $Q \triangleq \frac{\hat{q}_-}{\hat{q}} < 1$. Then, the inequality is equivalent to $Q^\lambda - \lambda Q + \lambda < 1$. Since $\lambda < 1$, the LHS is increasing in Q over $[0,1)$ and is equal to 1 at $Q = 1$, so the inequality holds.

Let

$$\bar{x}_-^k \triangleq x_-^k\left(\beta^k\left(\bar{x}\right)\right) \text{ and } q \triangleq \lim \left(G_\ell\left(\bar{x}_-^k\right)\right)^{n_\ell^k}.$$

Since, by the hypothesis, $b^k < \beta^k\left(\bar{x}\right)$ for all k, we have $q \geq \hat{q}$.

Case 1. Suppose that $q = 1$. Since

$$\pi_\omega^k\left(\beta^k(\bar{x})\right) \geq \left(G_\omega\left(\bar{x}_-^k\right)\right)^{n_\omega^k-1},$$

we have $\lim \pi_\ell^k\left(\beta^k(\bar{x})\right) = q(=1)$. By Lemma 5, $\lim \left(G_h\left(\bar{x}_-^k\right)\right)^{n_h^k} = q^\lambda = 1$ as well. So, $\lim \frac{\pi_h^k\left(\beta^k(\bar{x})\right)}{\pi_\ell^k\left(\beta^k(\bar{x})\right)} = 1$. This, (26), and (27) imply (25).

Case 2. Suppose that $q < 1$. So, there is an atom at $\beta^k(\bar{x})$. First, consider $\lambda \in (0,1)$. As before, using Lemmas 5 and 4, we have

$$\lim \frac{\pi_h^k(\beta^k\left(\bar{x}\right))}{\pi_\ell^k(\beta^k\left(\bar{x}\right))} = \frac{1-q^\lambda}{\lambda\left(1-q\right)} < q^{\lambda-1}, \tag{29}$$

where the last inequality follows from $\lambda \in (0,1)$, $q \in (0,1)$, and straightforward algebraic manipulation.[29]

[29] With $Q = \frac{1}{q}$, the inequality is equivalent to $(Q)^\lambda - \lambda Q + \lambda < 1$. The right-hand side equals 1 if $Q = 1$ and is increasing in Q on $(0,1)$ by $\lambda \in (0,1)$; hence, the inequality holds.

Since $q \geq \hat{q} > 0$ and $\lambda < 1$, we have $\hat{q}^{\lambda-1} \geq q^{\lambda-1}$. Now, this together with (26), (27), and (29) imply (25).

If $\lambda = 0$, by Lemma 5, $\lim G_h(x_-^k(b^k))^{n_h^k} = 1$, and hence $\lim \pi_h^k\left(\beta^k(\bar{x})\right) = 1$. Thus, (25) follows from $\lim \pi_\ell^k\left(b^k\right) < \lim \pi_\ell^k\left(\beta^k(\bar{x})\right)$. ∎

A.1.3 Proof of Proposition 2 (Characterization for the Case of no Grid)

We use the following lemma in the proof.

Lemma 8 *Suppose $\Delta^k = 0$ for all k. Let $\mathbf{n}^k$ be such that $\min\{n_\ell^k, n_h^k\} \to \infty$ and $\overline{g} \lim \frac{n_h^k}{n_\ell^k} > 1$. Let $(\beta^k)_{k=1}^\infty$ be a corresponding sequence of equilibrium bidding strategies. If $(\beta^k)_{k=1}^\infty$ contains a sequence of nonvanishing atoms $(b^k)_{k=1}^\infty$, i.e., $\lim\left(G_\ell\left(x_+^k\left(b^k\right)\right)\right)^{n_\ell^k} > \lim\left(G_\ell\left(x_-^k\left(b^k\right)\right)\right)^{n_\ell^k}$, then*

$$\lim_{k\to\infty} b^k < \lim_{k\to\infty}\lim_{\varepsilon\to 0} \mathbb{E}^k[v|x_+^k, \textit{sol,win at } b^k + \varepsilon].$$

Proof of Lemma 8. By bidders' individual rationality, $\mathbb{E}^k[v|x_-^k,\text{sol,win at } b^k] \geq b^k$. Therefore, the claim will follow from $\lim\lim_{\varepsilon\to 0}\mathbb{E}^k[v|x_+^k,\text{sol,win at } b^k+\varepsilon] > \lim \mathbb{E}^k[v|x_-^k,\text{sol,win at } b^k]$, which in turn will follow from

$$\lim \frac{g_h\left(x_-^k\right)}{g_\ell\left(x_-^k\right)}\frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} < \lim \frac{g_h\left(x_+^k\right)}{g_\ell\left(x_+^k\right)}\frac{\left(G_h\left(x_+^k\left(b^k\right)\right)\right)^{n_h^k}}{\left(G_\ell\left(x_+^k\left(b^k\right)\right)\right)^{n_\ell^k}}. \tag{30}$$

Let $q_- = \lim G_\ell\left(x_-^k\right)^{n_\ell^k}$ and $q_+ = \lim G_\ell\left(x_+^k\right)^{n_\ell^k}$. Note that $G_\omega\left(x_+^k\right)^{n_\omega^k} \approx G_\omega\left(x_+^k\right)^{n_\omega^k-1}$ for large k. By the hypothesis of the lemma, $q_+ > 0$. By Lemma 5, $\lim G_h\left(x_-^k\right)^{n_h^k} = (q_-)^\lambda$ and $\lim G_h\left(x_+^k\right)^{n_h^k} = (q_+)^\lambda$. Recall from Lemma 4 that

$$\lim \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} = \lim \frac{n_\ell^k}{n_h^k}\frac{G_\ell\left(x_+^k\right)-G_\ell\left(x_-^k\right)}{G_h\left(x_+^k\right)-G_h\left(x_-^k\right)}\frac{G_h\left(x_+^k\right)^{n_h^k}-G_h\left(x_-^k\right)^{n_h^k}}{G_\ell\left(x_+^k\right)^{n_\ell^k}-G_\ell\left(x_-^k\right)^{n_\ell^k}}. \tag{31}$$

Using this and the above observations,

$$\lim \frac{g_h\left(x_-^k\right)}{g_\ell\left(x_-^k\right)}\frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} = \lim\left(\frac{g_h\left(x_-^k\right)}{g_\ell\left(x_-^k\right)}\frac{G_\ell\left(x_+^k\right)-G_\ell\left(x_-^k\right)}{G_h\left(x_+^k\right)-G_h\left(x_-^k\right)}\right)\frac{g_h\left(\bar{x}\right)}{g_\ell\left(\bar{x}\right)}\frac{(q_+)^\lambda-(q_-)^\lambda}{\lambda\left(q_+-q_-\right)}. \tag{32}$$

Now, $\lim \frac{g_h\left(x_+^k\right)}{g_\ell\left(x_+^k\right)} = \frac{g_h(\bar{x})}{g_\ell(\bar{x})}$ and by MLRP

$$\frac{g_h\left(x_-^k\right)}{g_\ell\left(x_-^k\right)} \frac{G_\ell\left(x_+^k\right) - G_\ell\left(x_-^k\right)}{G_h\left(x_+^k\right) - G_h\left(x_-^k\right)} \leq 1.$$

Therefore, we may establish (30) by showing that,

$$\frac{(q_+)^\lambda - (q_-)^\lambda}{\lambda\left(q_+ - q_-\right)} < (q_+)^{\lambda-1}. \tag{33}$$

Letting $Q = \frac{q_-}{q_+} < 1$, (33) is equivalent to $Q^\lambda - \lambda Q + \lambda > 1$. Since $\lambda > 1$, the LHS is decreasing in Q over $[0,1)$ and is equal to 1 at $Q = 1$. Therefore, (33) holds and so does (30). ∎

We now prove proposition 2. By Proposition 1, we may assume that each bidding strategy β^k is monotone.

Case 1: Suppose $\bar{g}r < 1$. Given any $\varepsilon \in (0,1)$, let (x^k) be such that $\lim \left(G_h\left(x^k\right)\right)^{n_h^k} = \varepsilon$ for all k. We show that

$$\lim \left(G_h\left(x_+^k\left(\beta^k\left(x^k\right)\right)\right)\right)^{n_h^k} = 1,$$

with $x_+^k(b) = \sup\{x|\beta^k(x) = b\}$. This implies

$$\lim \left(G_h\left(x_+^k\left(b^k\right)\right)\right)^{n_h^k} - \left(G_h\left(x_-^k\left(b^k\right)\right)\right)^{n_h^k} \geq 1 - \varepsilon.$$

Then by Lemma 5 and $\bar{g}r < 1$, this inequality holds for $\omega = \ell$ as well. Since we can choose ε arbitrarily small, this establishes the claim.

Let $y_+^k \equiv x_+^k(b)$, and suppose to the contrary that

$$\lim \left(G_h\left(y_+^k\right)\right)^{n_h^k} < 1. \tag{34}$$

Since $\beta^k\left(x^k\right) < \beta^k\left(\bar{x}\right)$, (34) implies that there exists b^k with $\beta^k\left(x^k\right) < b^k < \beta^k\left(\bar{x}\right)$ and

$$\lim \pi_\ell^k(b^k) \in (0,1). \tag{35}$$

Hence, the zero-profit condition (22) from Corollary 3 requires that

$$\lim b^k \geq \lim \mathbb{E}^k[v|\bar{x}, \text{sol,win at } b^k]. \tag{36}$$

Given (35) and (36), Lemma 7 implies that

$$\lim \mathbb{E}^k[v|\bar{x}, \text{sol,win at } b^k] > \lim \mathbb{E}^k[v|\bar{x}, \text{sol,win at } \beta^k(\bar{x})]. \tag{37}$$

Individual rationality requires that

$$\lim \mathbb{E}^k[v|\bar{x}, \text{sol,win at } \beta^k(\bar{x})] \geq \lim \beta^k(\bar{x}). \tag{38}$$

Hence, (36)-(38) together imply a contradiction to $b^k < \beta^k(\bar{x})$. Thus, (34) cannot hold, which proves the claim.

Case 2a: Suppose $\overline{g}r > 1$ and $r \neq \infty$. Let us establish first that there are no atoms in the limit. Suppose to the contrary that $\beta^k(x) = b^k$ for all $x \in (x_-^k, x_+^k)$ and $\lim \left(G_\ell\left(x_+^k\right)\right)^{n_\ell^k} > \lim \left(G_\ell\left(x_-^k\right)\right)^{n_\ell^k} \geq 0$. Thus,

$$\lim_{k\to\infty} \lim_{\varepsilon\to 0} \pi_\ell^k\left(b^k + \varepsilon\right) = \lim_{k\to\infty} \left(G_\ell\left(x_+^k\right)\right)^{n_\ell^k} > 0. \tag{39}$$

This and Lemma 8 implies that

$$\lim_{k\to\infty} \lim_{\varepsilon\to 0} U^k(b^k + \varepsilon | x_+^k, \text{ sol}) > 0. \tag{40}$$

contradicting the zero-profit condition (21). Thus, there can be no atom.

Next, let us derive the functional form. Take any $\alpha \in (0,1)$. Let $(x^k)_{k=1}^\infty$ be such that $\left(G_\ell\left(x^k\right)\right)^{n_\ell^k - 1} = \alpha$ for all k. By the absence of atoms (just established above), $\lim \pi_\omega^k(\beta^k\left(x^k\right)) = \lim \left(G_\omega\left(x^k\right)\right)^{n_\omega^k - 1} = \lim F_\omega^k(\beta^k\left(x^k\right))$. By Corollary (3)-(iii) ,

$$\lim \beta^k\left(x^k\right) = \lim \mathbb{E}^k[v|x^k, \text{sol, win at } \beta^k\left(x^k\right)]$$

Therefore, expressing $\mathbb{E}^k[v|x^k,\text{sol,win at } \beta^k\left(x^k\right)]$ in terms of the compound likelihood ratio as in (24) and using $\lim \pi_\omega^k(\beta^k\left(x^k\right)) = \lim \left(G_\omega\left(x^k\right)\right)^{n_\omega^k - 1}$,

$$\lim \beta^k\left(x^k\right) = \lim \frac{v_\ell + \frac{\rho_h}{\rho_\ell} \frac{g_h\left(x^k\right)}{g_\ell\left(x^k\right)} \frac{n_h^k}{n_\ell^k} \frac{\pi_h^k(\beta^k\left(x^k\right))}{\pi_\ell^k(\beta^k\left(x^k\right))} v_h}{1 + \frac{\rho_h}{\rho_\ell} \frac{g_h\left(x^k\right)}{g_\ell\left(x^k\right)} \frac{n_h^k}{n_\ell^k} \frac{\pi_h^k(\beta^k\left(x^k\right))}{\pi_\ell^k(\beta^k\left(x^k\right))}} = \lim \frac{v_\ell + \frac{\rho_h}{\rho_\ell} \frac{g_h\left(x^k\right)}{g_\ell\left(x^k\right)} \frac{n_h^k}{n_\ell^k} \frac{\left(G_h\left(x^k\right)\right)^{n_h^k - 1}}{\left(G_\ell\left(x^k\right)\right)^{n_\ell^k - 1}} v_h}{1 + \frac{\rho_h}{\rho_\ell} \frac{g_h\left(x^k\right)}{g_\ell\left(x^k\right)} \frac{n_h^k}{n_\ell^k} \frac{\left(G_h\left(x^k\right)\right)^{n_h^k - 1}}{\left(G_\ell\left(x^k\right)\right)^{n_\ell^k - 1}}}. \tag{41}$$

From $\lim \left(G_\omega\left(x^k\right)\right)^{n_\omega^k-1} > 0$, we have $x^k \to \bar{x}$. This, Lemma 5, and $\lim \left(G_\omega\left(x^k\right)\right)^{n_\omega^k-1} = \lim F_\omega^k(\beta^k\left(x^k\right))$ imply

$$\lim \frac{\left(G_h\left(x^k\right)\right)^{n_h^k-1}}{\left(G_\ell\left(x^k\right)\right)^{n_\ell^k-1}} = \lim \frac{\left(G_h\left(x^k\right)\right)^{n_h^k}}{\left(G_\ell\left(x^k\right)\right)^{n_\ell^k}} = [\lim \left(G_\ell\left(x^k\right)\right)^{n_\ell^k}]^{\lambda-1} = \alpha^{\lambda-1},$$

where, as before, $\lambda = \overline{g} \lim \frac{n_h^k}{n_\ell^k}$. Using this observation and letting $\lim \beta^k\left(x^k\right) = p$, we can rewrite (41) as

$$p = \frac{v_\ell + \rho\lambda\alpha^{\lambda-1}v_h}{1+\rho\lambda\alpha^{\lambda-1}}. \tag{42}$$

Thus, for every $\alpha \in (0,1)$, we can find the unique p such that $\lim F_\ell^k(p) = \alpha$. This gives a function $\hat{p}\left(\alpha\right)$ that is continuous and strictly increasing on $(0,1)$. The limit distribution $\Phi_\ell\left(p\right)$ is simply the inverse of $\hat{p}$, meaning, the α solution of (42) for given p. Finally, from Lemma 5, $\lim F_h^k(p) = \Phi_h\left(p\right)$.

Case 2b: Suppose $r = \infty$. In this case, $\Phi_\omega\left(\cdot|r\right)$ is degenerate with probability mass 1 on v_ω. Given bidders' individual rationality constraint, it is sufficient to show that $\Phi_h\left(\cdot|r\right)$ is degenerate with probability mass 1 on v_h. But this follows directly from the zero profit condition and the observation that, given $r = \infty$, if $\lim F_h^k(p) > 0$ for some $p < v_h$, then $\lim \mathbb{E}^k[v|x^k\text{,sol, win at } p] = v_h$.

Case 3: Suppose $\overline{g}r = 1$. From bidders' individual rationality,

$$\rho_\ell \lim \mathbb{E}^k[p|\ell] + \rho_h \lim \mathbb{E}^k[p|h] \leq \mathbb{E}\left[v\right]. \tag{43}$$

We show that, for any $p < \mathbb{E}\left[v\right]$, $\lim F_\omega^k\left(p\right) = 0$. This together with (43) implies the proposition, since if $\lim F_\omega^k\left(p\right) < 1$ for some $p > \mathbb{E}\left[v\right]$, (43) would be violated.

Suppose to the contrary that, for some $p < \mathbb{E}\left[v\right]$, $\lim F_\omega^k\left(p\right) > 0$. Therefore, given that bids are from the continuum, there is $p' < \mathbb{E}\left[v\right]$, such that $q \triangleq \lim \pi_\ell^k(p') > 0$. Then, there is a sequence $\left(b^k\right)_{k=1}^\infty$ such that β^k has no atom at b^k for any k, $b^k \geq p'$, and $\lim b^k = p'$. Letting $\hat{q} \triangleq \lim \pi_\ell^k(b^k)$, Lemma 5 and $\lambda = \overline{g} \lim \frac{n_h^k}{n_\ell^k} = 1$ imply

$$\lim \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} = \frac{\hat{q}^\lambda}{\hat{q}} = 1.$$

Thus, from (3), $\lim \mathbb{E}^k[v|\bar{x}, \text{sol,win at } b^k] = \mathbb{E}[v] > \lim b^k$. Since also $\lim \pi_\omega^k(b^k) > 0$ from $b^k > p'$ and $\lim \pi_\ell^k(p') > 0$, we have

$$\lim U^k(b^k|\bar{x},\ \text{sol}) > 0,$$

contradicting the zero-profit condition (21). Thus, such $\left(b^k\right)_{k=1}^{\infty}$ cannot exist. Therefore, $\lim \pi_\omega^k(p) = 0$ for all $p < \mathbb{E}[v]$, as needed.

This shows the characterization results for the no-grid case, proving Proposition 2.

A.1.4 Proving the Characterization Results of Theorem 1

We now consider the finite grid ($\Delta^k > 0$). Most of the above proof goes through with no change. We will therefore only present the arguments that have to be adjusted, rather than reproduce the entire proof. These are in the instances where a "slight undercutting" argument is used, and the adjusted arguments ensure that, for a sufficiently fine grid, the above proof goes through.

Case 1: $\bar{g}r < 1$**.** Given any $\varepsilon \in (0,1)$, let x^k be such that $\left(G_h\left(x^k\right)\right)^{n_h^k} = \varepsilon$ for all k. Let $b^k = \beta^k\left(x^k\right)$. As before, the result holds if

$$\lim \left(G_h\left(x_+^k\left(b^k + \Delta^k\right)\right)\right)^{n_h^k} = 1. \tag{44}$$

Suppose to the contrary that (44) fails and $\lim \left(G_h\left(x_+^k\left(b^k + \Delta^k\right)\right)\right)^{n_h^k} < 1$. Then,

$$b^k + \Delta^k < \beta^k\left(\bar{x}\right).$$

Moreover,

$$\lim \pi_h^k(b^k + \Delta^k) \leq \lim \left(G_h\left(x_+^k\left(b^k + \Delta^k\right)\right)\right)^{n_h^k} < 1$$

and

$$\lim \pi_h^k(b^k + \Delta^k) \geq \lim \left(G_h\left(x_-^k\left(b^k + \Delta^k\right)\right)\right)^{n_h^k} \geq \lim \left(G_h\left(x^k\right)\right)^{n_h^k} = \varepsilon > 0.$$

Hence, the zero-profit condition (21) requires that

$$\lim \left(b^k + \Delta^k\right) \geq \lim \mathbb{E}^k[v|\bar{x}, \text{sol,win at } b^k + \Delta^k].$$

Now, $\lim \pi_h^k(b^k + \Delta^k) \in (0,1)$ and $b^k + \Delta^k < \beta^k\left(\bar{x}\right)$ for all k implies via Lemma 7 (β^ks that have support only on the grid are a special case considered in that lemma) that

$$\lim \mathbb{E}^k[v|\bar{x}, \text{sol,win at } b^k + \Delta^k] > \lim \mathbb{E}^k[v|\bar{x}, \text{sol,win at } \beta^k(\bar{x})].$$

The bidders' individual rationality requires that

$$\lim \mathbb{E}^k[v|\bar{x}, \text{sol,win at } \beta^k(\bar{x})] \geq \lim \beta^k(\bar{x}).$$

Together, the last three displayed inequalities contradict $b^k + \Delta^k < \beta^k(\bar{x})$. ∎

Case 2: $\bar{g}r > 1$. The critical lemma for this case was Lemma 8, which should be adapted as follows.

Lemma 9 *Suppose $\Delta^k > 0$ for all k. Let n^k be such that $\min\{n_\ell^k, n_h^k\} \to \infty$ and $\lim \frac{n_h^k}{n_\ell^k} \frac{g_h(\bar{x})}{g_\ell(\bar{x})} > 1$. Let $(\beta^k)_{k=1}^\infty$ be a corresponding sequence of bidding equilibria. If $(\beta^k)_{k=1}^\infty$ exhibits a sequence of non-vanishing atoms $(b^k)_{k=1}^\infty$, i.e., $\lim \left(G_\ell\left(x_+^k\left(b^k\right)\right)\right)^{n_\ell^k} > \lim \left(G_\ell\left(x_-^k\left(b^k\right)\right)\right)^{n_\ell^k}$, then*

$$\lim b^k < \lim \mathbb{E}^k[v|x_+^k, \textit{sol,win at } b^k + \Delta^k].$$

Proof of Lemma 9. If, in the limit, there is no atom at $b^k+\Delta^k$, i.e., if $\lim \left(G_\ell\left(x_+^k\left(b^k+\Delta^k\right)\right)\right)^{n_\ell^k} = \lim \left(G_\ell\left(x_-^k\left(b^k+\Delta^k\right)\right)\right)^{n_\ell^k}$, then the original proof of the lemma works directly. If in the limit there is an atom at $b^k + \Delta^k$, then instead of (30) we have to establish

$$\lim \frac{g_h\left(x_-^k\right)}{g_\ell\left(x_-^k\right)} \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} < \lim \frac{g_h\left(x_+^k\right)}{g_\ell\left(x_+^k\right)} \frac{\pi_h^k(b^k+\Delta^k)}{\pi_\ell^k(b^k+\Delta^k)}. \tag{45}$$

Let $x_{++}^k = x_+^k\left(b^k+\Delta^k\right)$ and note that $x_-^k\left(b^k+\Delta^k\right) = x_+^k\left(b^k\right) = x_+^k$. Also, recall $q_+ = \lim G_\ell\left(x_+^k\right)^{n_\ell^k}$ and let $q_{++} = \lim G_\ell\left(x_{++}^k\right)^{n_\ell^k}$. We already know from (30) that

$$\lim \frac{g_h\left(x_-^k\right)}{g_\ell\left(x_-^k\right)} \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} < \lim \frac{g_h\left(x_+^k\right)}{g_\ell\left(x_+^k\right)} \frac{\left(G_h\left(x_+^k\left(b^k\right)\right)\right)^{n_h^k}}{\left(G_\ell\left(x_+^k\left(b^k\right)\right)\right)^{n_\ell^k}} = \lim \frac{g_h(\bar{x})}{g_\ell(\bar{x})} \left(q_+\right)^{\lambda-1}$$

Analogous calculation to that of (31)-(32) in the proof yields

$$\begin{aligned}
\lim \frac{g_h\left(x_+^k\right)}{g_\ell\left(x_+^k\right)} \frac{\pi_h^k(b^k+\Delta^k)}{\pi_\ell^k(b^k+\Delta^k)} &= \lim \frac{n_\ell^k}{n_h^k} \frac{g_h\left(x_+^k\right)}{g_\ell\left(x_+^k\right)} \frac{G_\ell\left(x_{++}^k\right) - G_\ell\left(x_+^k\right)}{G_h\left(x_{++}^k\right) - G_h\left(x_+^k\right)} \frac{G_h\left(x_{++}^k\right)^{n_h^k} - G_h\left(x_+^k\right)^{n_h^k}}{G_\ell\left(x_{++}^k\right)^{n_\ell^k} - G_\ell\left(x_+^k\right)^{n_\ell^k}} \\
&= \frac{g_h(\bar{x})}{g_\ell(\bar{x})} \frac{\left(q_{++}\right)^\lambda - \left(q_+\right)^\lambda}{\lambda\left(q_{++} - q_+\right)}.
\end{aligned}$$

Finally,

$$(q_+)^{\lambda-1} < \frac{(q_{++})^\lambda - (q_+)^\lambda}{\lambda\left(q_{++} - q_+\right)}, \tag{46}$$

since letting $Q = \frac{q_{++}}{q_+} > 1$, (46) is equivalent to $Q^\lambda - \lambda Q + \lambda > 1$. Since $\lambda > 1$, the LHS is increasing in Q over $[1, \infty)$ and is equal to 1 at $Q = 1$. Therefore, (46) and so does (45). This completes the adaptation of Lemma 8 for the case of finite price grid. ∎

We can now adapt the proof from Proposition 2. The proof uses a slight overbidding argument. The paragraph containing equations (39) and (40) should be modified as follows

$$\lim_{k\to\infty} \lim_{\varepsilon^k \to 0} \pi_\ell^k(b^k + \Delta^k) \geq \lim_{k\to\infty} \left(G_\ell\left(x_+^k\right)\right)^{n_\ell^k} > 0, \tag{47}$$

where the first inequality is strict if in the limit there is an atom at $b^k + \Delta^k$, i.e., if $\lim \left(G_\ell\left(x_+^k\left(b^k + \Delta^k\right)\right)\right)^{n_\ell^k} < \lim\left(G_\ell\left(x_-^k\left(b^k + \Delta^k\right)\right)\right)^{n_\ell^k}$. This and Lemma 9 implies that

$$\lim U^k(b^k + \Delta^k | x_+^k, \text{sol}) > 0. \tag{48}$$

Beyond that point, the proof from Proposition 2 continues unchanged. ∎

Case 3: $\bar{g}r = 1$**.** The only necessary change required in the original proof of Proposition 2 is with respect to the choice of the sequence b^k. Note that given $\Delta^k \to 0$, under the stated hypothesis, there must still be a sequence b^k such that $b^k \geq p'$, $\lim b^k = p'$, and the probability of a tie at b^k is vanishing,

$$\lim \frac{\pi_\omega^k(b^k)}{\left(G_\omega\left(x_+^k\left(b^k\right)\right)\right)^{n_\omega^k}} = 1.$$

The remainder of the proof from Proposition 2 applies as before. ∎

A.1.5 Proving the Existence Claims of Theorem 1

Recall that $P_\Delta = [0, v_\ell) \cup \{v_\ell, v_\ell + \Delta, v_\ell + 2\Delta, ..., v_h - \Delta, v_h\}$. Let $m = \| \{v_\ell, v_\ell + \Delta, ..., v_h - \Delta, v_h\} \|$. Using the idea of Athey (2001), Σ_Δ is a set of vectors of dimension $m+1$ whose coordinates belong to $[\underline{x}, \bar{x}]$

$$\Sigma_\Delta = \{\sigma = (\sigma_0, \sigma_1, ..., \sigma_m) \in [\underline{x}, \bar{x}]^{m+1} \mid \underline{x} \triangleq \sigma_0 \leq \sigma_1 \leq ... \leq \sigma_m \triangleq \bar{x}\},$$

where σ determines a monotone bidding strategy β_σ by $\beta_\sigma(x) = v_\ell + i\Delta$ if $x \in [\sigma_i, \sigma_{i+1})$, $i = 0, ..., m-1$. Given $\varepsilon > 0$ and some $\hat{b} \in P_\Delta$, let $m(\hat{b}) = \| \left\{ v_\ell, v_\ell + \Delta, ..., \hat{b} - \Delta, \hat{b} \right\} \|$ and

$$\Sigma_\Delta(\hat{b}, \varepsilon) = \left\{ \sigma \in \Sigma_\Delta | \sigma_{m(\hat{b})-1} = \bar{x} - \varepsilon, \, \sigma_{m(b)} = \bar{x} \right\};$$

that is, for $\sigma \in \Sigma_\Delta \left(\hat{b}, \varepsilon \right)$, the strategy $\beta_\sigma(x) = \hat{b}$ for all $x \in [\bar{x} - \varepsilon, \bar{x}]$.

Define the correspondence Ψ from $\Sigma_\Delta(\hat{b}, \varepsilon)$ into itself: For any $\sigma' \in \Sigma_\Delta(\hat{b}, \varepsilon)$, let

$$\Psi(\sigma') = \left\{ \sigma \in \Sigma_\Delta(\hat{b}, \varepsilon) | \beta_\sigma(x) \in \arg\max_{b \leq \hat{b}} U(b | x, \text{sol}; \beta_\sigma, \mathbf{n}) \text{ for all } x \leq \bar{x} - \varepsilon \right\},$$

that is, $\Psi(\sigma')$ is the best-response correspondence for $x \leq \bar{x} - \varepsilon$ when bidders are restricted to bid at most $\hat{b}$. The correspondence Ψ is non-empty, convex valued, and upper hemi-continuous. That Ψ is non-empty and convex valued follows immediately from the single-crossing property identified in Lemma 10, shown directly below, just as in Athey (2001). The upper hemi-continuity follows from the theorem of the maximum. Thus, by Kakutani's fixed-point Theorem, there exists some $\sigma^*(\hat{b}, \varepsilon)$ such that $\sigma^* = \Psi(\sigma^*)$.

General Existence Claim: If we choose $\varepsilon = 0$ and $\hat{b} = v_h$, then $\sigma^* = \Psi(\sigma^*)$ implies that σ^* is an equilibrium of the original game $\Gamma_0(\mathbf{n}, N, \Delta)$, proving the general existence claim at the start of Theorem 1.

Now, fix some sequence of bidding games $\Gamma_0(\mathbf{n}^k, N^k, \Delta^k)$ such that $\Delta^k > 0$, $\Delta^k \to 0$, $\min\{n_\ell^k, n_h^k\} \to \infty$, and $\lim \frac{n_h^k}{n_\ell^k} = r$, with $\bar{g}r < 1$.

Now, take any $q \in (0,1)$ and let ε^k be such that $\left(G_\ell \left(\bar{x} - \varepsilon^k \right) \right)^{n_\ell^k} = q$. Given some $\hat{b}$, let Ψ^k be the correspondence given $\hat{b}$, ε^k, Δ^k, $\mathbf{n}^k$ and let σ^k be one of its fixed points.

Claim 1 *For every $\hat{b}$ with $\overline{\mathbb{E}}[v|\bar{x}, sol] < \hat{b} < \mathbb{E}[v]$ and q small enough, the strategy $\beta^k = \beta_{\sigma^k}$ is a bidding equilibrium of $\Gamma_0(\mathbf{n}^k, N^k, \Delta^k)$ for k large enough.*

The claim implies the last remaining item from Theorem 1. To prove the claim, it is sufficient to show that $\bar{x}$ does not have an incentive to bid higher than $\hat{b}$ and $\bar{x} - \varepsilon$ has a strict incentive to bid $\bar{b}$, shown in Steps 2 and 3 below. This implies that β_{σ^k} is an optimal bid for all signals given the single-crossing property from Lemma 10 because the constraints are slack.

Step 1. For q small enough and every $\sigma^k \in \Sigma_k(\hat{b}, \varepsilon^k)$,

$$\lim \mathbb{E}^k[v|\bar{x} - \varepsilon^k, \text{sol,win at } \hat{b}] > \hat{b}.$$

By definition, $x_-^k(\hat{b}) = \sigma^k_{m^k(\hat{b})-1}$. Let $q_- = \lim \left(G_\ell\left(\sigma^k_{m^k(b)-1}\right)\right)^{n_\ell^k}$, and note that $q_- \leq q$. From before, with $\lambda = \bar{g}r$, $\lim \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} = \frac{1-q_-^\lambda}{\lambda(1-q_-)}$ and so

$$\lim \frac{n_h}{n_\ell} \frac{g_h\left(\bar{x} - \varepsilon^k\right)}{g_\ell\left(\bar{x} - \varepsilon^k\right)} \frac{\pi_h^k\left(\hat{b}\right)}{\pi_\ell^k\left(\hat{b}\right)} = \frac{1 - q_-^\lambda}{1 - q_-},$$

which is arbitrarily close to 1 for q_- small enough. It follows that, for ever δ, there is some q small enough such that

$$\lim \mathbb{E}^k[v|\bar{x} - \varepsilon^k, \text{sol,win at } \hat{b}] \geq \mathbb{E}[v] - \delta.$$

Since $\hat{b} < \mathbb{E}[v]$, the claim follows.

Step 2. For q small enough and k large enough,

$$U\left(\hat{b}|\bar{x}, \text{ sol}; \beta_{\sigma^k}, \mathbf{n}^k\right) > U\left(b'|\bar{x}, \text{ sol}; \beta_{\sigma^k}, \mathbf{n}^k\right) \text{ for all } b' > \hat{b}.$$

From $U\left(b'|\bar{x}, \text{ sol}; \beta_{\sigma^k}, \mathbf{n}^k\right) = \mathbb{E}^k[v|\bar{x}, \text{sol}] - b'$ and $\overline{\mathbb{E}}[v|\bar{x}, \text{sol}] < \hat{b} < b'$, we have $U\left(b'|\bar{x}, \text{ sol}; \beta_{\sigma^k}, \mathbf{n}^k\right) < 0$.

From Step 1, $U\left(\hat{b}|\bar{x}, \text{ sol}; \beta_{\sigma^k}, \mathbf{n}^k\right) > 0$ for q small enough and k large enough.

Step 3. For q small enough and k large enough,

$$U\left(\hat{b}|\bar{x} - \varepsilon^k, \text{ sol}; \beta_{\sigma^k}, \mathbf{n}^k\right) > U\left(b'|\bar{x} - \varepsilon^k, \text{ sol}; \beta_{\sigma^k}, \mathbf{n}^k\right) \text{ for all } b' < \hat{b}.$$

Note that

$$\frac{U\left(\hat{b}|\bar{x} - \varepsilon^k, \text{ sol}; \beta_{\sigma^k}, \mathbf{n}^k\right)}{U\left(b'|\bar{x} - \varepsilon^k, \text{ sol}; \beta_{\sigma^k}, \mathbf{n}^k\right)} = \frac{\Pr^k\left(\text{win at } \hat{b}|\bar{x} - \varepsilon^k\right)}{\Pr^k\left(\text{win at } b'|\bar{x} - \varepsilon^k\right)} \frac{\left(\mathbb{E}^k[v|\bar{x} - \varepsilon^k, \text{sol,win at } \hat{b}] - \hat{b}\right)}{\left(\mathbb{E}^k[v|\bar{x} - \varepsilon^k, \text{sol,win at } b'] - b'\right)}.$$

Also,

$$\mathbb{E}^k[v|\bar{x} - \varepsilon^k, \text{sol,win at } b'] - b' \geq v_h,$$

and

$$\lim \mathbb{E}^k[v|\bar{x} - \varepsilon^k, \text{sol,win at } \hat{b}] - \hat{b} > 0.$$

Therefore, it is sufficient to show that, for every $R > 1$ there is some q small enough such that

$$\lim \frac{\Pr^k\left(\text{win at } \hat{b}|\bar{x} - \varepsilon^k\right)}{\Pr^k\left(\text{win at } b'|\bar{x} - \varepsilon^k\right)} > R.$$

For this, in turn, it is sufficient to show that, for $\omega \in \{\ell, h\}$,

$$\lim \frac{\pi^k_\omega\left(\hat{b}\right)}{\pi^k_\omega\left(b'\right)} > R.$$

With $x^k_- = \sigma^k_{m^k(\hat{b})-1}$, we have $\beta^k(x) < \hat{b}$ iff $x \leq x^k_-$. Therefore, $\pi^k_\omega(b') \leq \left(G_\omega\left(x^k_-\right)\right)^{n^k_\omega}$, and we have

$$\frac{\pi^k_\omega\left(\hat{b}\right)}{\pi^k_\omega\left(b'\right)} \geq \frac{1}{\left(G_\omega\left(x^k_-\right)\right)^{n^k_\omega}} \frac{1 - \left(G_\omega\left(x^k_-\right)\right)^{n^k_\omega}}{n^k_\omega[1 - G_\omega\left(x^k_-\right)]}.$$

If $\left(G_\omega\left(x^k_-\right)\right)^{n^k_\omega} \to q_- \in (0, 1)$, then

$$\lim \frac{1}{\left(G_\omega\left(x^k_-\right)\right)^{n^k_\omega}} \frac{1 - \left(G_\omega\left(x^k_-\right)\right)^{n^k_\omega}}{n^k_\omega[1 - G_\omega\left(x^k_-\right)]} = \frac{1 - q_-}{-q_- \ln q_-}.$$

Now, the claim follows since $q_- \leq q$ and we can choose q small enough such that $\frac{1-q}{-q \ln q} < R$ (recall that $-q_- \ln q_- \to 0$ for $q_- \to 0$).

If $n^k_\omega[1 - G_\omega\left(x^k_-\right)] \to \infty$, then the claim follows because $\left(G_\omega\left(x^k_-\right)\right)^{n^k_\omega} n^k_\omega[1 - G_\omega\left(x^k_-\right)]$ is increasing in $G_\omega\left(x^k_-\right)$ for $n^k_\omega[1 - G_\omega\left(x^k_-\right)] \geq 1$, and, hence, $\left(G_\omega\left(x^k_-\right)\right)^{n^k_\omega} n^k_\omega[1 - G_\omega\left(x^k_-\right)] \leq -q \ln q$ for q small enough. (To see it is increasing, write the expression as $\xi^n n[1 - \xi]$ and note that $\frac{d}{d\xi}\left(\xi^n n[1 - \xi]\right) = n\xi^{n-1} n[1 - \xi] - \xi^n n = n\xi^{n-1}\left[n\left(1 - \xi\right) - \xi\right] > 0$ for $n\left(1 - \xi\right) > 1 > \xi$.)

This finishes the proof of Step 3. As noted before, Step 2 and Step 3 imply the Claim. ∎

A.2 Proof of Corollary 2

Proof: 1. Observe first that $\Phi_\omega(\cdot|\rho, \bar{g}, r)$, $\omega = \ell, h$, is more informative in the sense of Lehman (1988)'s criterion. To see this, consider a probability q of a Type I error (rejecting the hypothesis that $\omega = h$ when it is true), and let $p_q$and p'_q be the thresholds that achieve it, $q = \Phi_h(p_q|\rho, \bar{g}, r) = \Phi_h(p'_q|\rho, \bar{g}', r')$. The corresponding Type II errors satisfy $1 - \Phi_\ell(p_q|\rho, \bar{g}, r) = 1 - q^{1/\bar{g}r} < 1 - q^{1/\bar{g}'r'} = 1 - \Phi_\ell(p'_q|\rho, \bar{g}', r')$, which implies Lehman's ranking. Since in this two-state environment Lehman's ranking is equivalent to Blackwell's ranking (Jewitt, 2007), $\Phi_\omega(\cdot|\rho, \bar{g}, r)$, $\omega = \ell, h$, is more informative by that criterion as well.

2. Consider the following decision problem. A decision maker DM observes the winning bid p and has to select a value estimate $\widehat{v} \in [v_\ell, v_h]$. Its utility function is $u(\widehat{v}, \omega) = -(\widehat{v} - v_\omega)^2$. DM's posterior after observing p is $\Pr[\omega|$ winning bid $= p]$. The optimal $\widehat{v}$ maximizes

$$U(\widehat{v}) = -\Pr[\ell|\text{winning bid} = p]\mathbb{E}(\widehat{v} - v_\ell)^2 - \Pr[h|\text{winning bid} = p]\mathbb{E}(\widehat{v} - v_h)^2$$

and hence it is $\Pr[\ell|\text{winning bid} = p]v_\ell + \Pr[h|\text{winning bid} = p]v_h = \mathbb{E}[v|$ win at $p]$. Since, as we observed before, at the (limit) equilibrium $p = \mathbb{E}[v|$ win at $p]$, the optimal $\widehat{v}$ is p itself.

Since $\Phi_\omega(\cdot|\rho, \bar{g}, r)$ is Blackwell more informative than $\Phi_\omega(\cdot|\rho, \bar{g}', r')$, it has to yield higher optimal expected payoff for any payoff function and any prior. In particular, for any ρ,

$$\sum_\omega \rho_\omega \mathbb{E}\left[U(p)|\omega\right] \geq \sum_\omega \rho_\omega \mathbb{E}'\left[U(p)|\omega\right] \tag{49}$$

where $\mathbb{E}$ and $\mathbb{E}'$ are the expectations with respect to $\Phi_\omega(\cdot|\rho, \bar{g}, r)$ and $\Phi_\omega(\cdot|\rho, \bar{g}', r')$, respectively. Now,

$$\sum_\omega \rho_\omega \mathbb{E}\left[U(p)|\omega\right] = -\sum_\omega \rho_\omega \mathbb{E}\left[(p - v_\omega)^2|\omega\right]$$

$$= -\sum_\omega \rho_\omega \mathbb{E}\left(p^2|\omega\right) + 2\left(v_h - v_\ell\right)\rho_h \mathbb{E}(p|h) + 2v_\ell \rho_h \mathbb{E}(p|h) + 2v_\ell \rho_\ell \mathbb{E}(p|\ell) + C$$

$$= -\sum_\omega \rho_\omega \mathbb{E}\left(p^2|\omega\right) + 2\left(v_h - v_\ell\right)\rho_h \mathbb{E}(p|h) + 2v_\ell \mathbb{E}(v) + C,$$

where $C = -\rho_l v_\ell^2 - \rho_h v_h^2$and we used $\mathbb{E}(v) = \rho_h \mathbb{E}(p|h) + \rho_\ell \mathbb{E}(p|\ell)$ from Claim 1. Analogously,

$$\sum_\omega \rho_\omega \mathbb{E}'\left[U(p)|\omega\right] =$$

$$-\sum_\omega \rho_\omega \mathbb{E}'\left(p^2|\omega\right) + 2\left(v_h - v_\ell\right)\rho_h \mathbb{E}'(p|h) + 2v_\ell \mathbb{E}(v) + C.$$

Therefore, (49) is equivalent to

$$-\sum_{\omega}\rho_{\omega}\mathbb{E}\left(p^{2}|\omega\right)+2\left(v_{h}-v_{\ell}\right)\rho_{h}\mathbb{E}(p|h)\geq-\sum_{\omega}\rho_{\omega}\mathbb{E}'\left(p^{2}|\omega\right)+2\left(v_{h}-v_{\ell}\right)\rho_{h}\mathbb{E}'(p|h). \tag{50}$$

Now, since $\Phi_{\omega}(\cdot|\rho,\bar{g},r)$ is Blackwell more informative than $\Phi_{\omega}(\cdot|\rho,\bar{g}',r')$ and so the posteriors are a mean-preserving spread of the latter, and since p^2 is a convex function of the posterior,

$$\sum_{\omega}\rho_{\omega}\mathbb{E}\left[p^{2}|\omega\right]\geq\sum_{\omega}\rho_{\omega}\mathbb{E}'\left[p^{2}|\omega\right].$$

Therefore, for (50) to hold we must have

$$\mathbb{E}\left[(p|h)\geq\mathbb{E}'\left[p|h\right].\right.$$

Since $\mathbb{E}\left[v\right]=\rho_{h}\mathbb{E}(p|h)+\rho_{\ell}\mathbb{E}(p|\ell)$, the reverse inequality holds for $\mathbb{E}(p|\ell)$

3. Since $\Phi_{\omega}(\cdot|\rho,\bar{g},r)$ converges to a mass point on v_{ω} when $\bar{g}r\to\infty$, the result follows[30]. ∎

B Bidding Equilibrium with a Mixed Solicitation Strategy

B.1 Notation for Mixed Strategies

Given a mixed solicitation strategy $\boldsymbol{\eta}=(\eta_{\ell},\eta_{h})$, let

$$\overline{n}_{\omega}(\eta_{\omega})\triangleq\sum_{n=1}^{N}n\eta_{\omega}\left(n\right),\text{ and }\overline{\pi}_{\omega}(b;\beta,\eta_{\omega})\triangleq\sum_{n=1}^{N}\eta_{\omega}\left(n\right)n\pi_{\omega}\left(b;\beta,n\right)/\overline{n}_{\omega}. \tag{51}$$

These are the expected number of bidders and the weighted average probability of winning in state ω. To make the expressions less dense, we omit here and later the argument of $\overline{n}_{\omega}(\eta_{\omega})$ and write just $\overline{n}_{\omega}$ instead. Also, as before, when there is no danger of confusion, we will continue to omit the argument β and $\boldsymbol{\eta}$ from U, π_{ω}, E, etc. The counterpart of (19)—the expected payoff to a bidder who bids b given a

[30]Recalling that $p\in[v_{\ell},v_{h}]$ and hence bounded.

mixed solicitation strategy $\boldsymbol{\eta} = (\eta_\ell, \eta_h)$—is

$$U(b|x,\ \mathrm{sol}) = \frac{\rho_\ell g_\ell(x)\,\overline{n}_\ell \overline{\pi}_\ell(b)\,(v_\ell - b) + \rho_h g_h(x)\,\overline{n}_h \overline{\pi}_h(b)\,(v_h - b)}{\rho_\ell g_\ell(x)\,\overline{n}_\ell + \rho_h g_h(x)\,\overline{n}_h}. \tag{52}$$

Expressions (1)—(2) can also be adapted to mixed strategies, with $\overline{n}_\omega$ and $\overline{\pi}_\omega$ just taking everywhere the place of n_ω and π_ω.

B.2 Proof of Monotonicity with Random Participation

Proposition 7 *Suppose either $v_\ell = 0$ or η is such that $\eta_\ell(1) = \eta_h(1) = 0$, and β is a bidding equilibrium.*

1. *If $x' > x$, then $U(\beta(x')\,|x',sol) \geq U(\beta(x)\,|x,sol)$. The inequality is strict if and only if $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$.*

2. *There exists an equivalent bidding equilibrium $\widetilde{\beta}$, such that $\widetilde{\beta}$ is nondecreasing on $[\underline{x}, \bar{x}]$ and coincides with β over intervals over which $\frac{g_h}{g_\ell}$ is strictly increasing.*

The proof of Proposition 7 relies on two lemmas.

Lemma 10 **(Single-Crossing)** *Given any bidding strategy β, any solicitation strategy $\boldsymbol{\eta}$ and any bids $b' > b \geq v_\ell$.*

1. *If $\overline{\pi}_\omega(b') > 0$ for some $\omega \in \{\ell, h\}$, then, for all $x' > x$,*

   $$U(b'|x,\ sol) \geq U(b|x,\ sol) \Rightarrow U(b'|x',\ sol) \geq U(b|x',\ sol);$$

   where the second inequality is strict if $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$.

2. *If $\overline{\pi}_\omega(b') = 0$ for some $\omega \in \{\ell, h\}$, then $\overline{\pi}_\omega(b) = 0$ for both ω, and $U(b'|x,$ $sol) = U(b|x,\ sol) = 0$ for all x.*

Remark. The proof of Lemma 10 relies on the assumption that there are only two states. If bids are necessarily above v_ℓ (as is indeed implied by the next lemma), conditional on state ℓ, a higher bid is necessarily worse than a lower one. So, if two bids are optimal for some belief, the higher bid must be better if the state is h–implying that a higher belief must make the higher bid more attractive. This is the key role of that assumption.

The following lemma collects a number of additional properties of a bidding equilibrium β. One of them is a straightforward Bertrand property: when the seller solicits two or more bids in both states, then $\beta(x) \geq v_\ell$, for all x.

Lemma 11 **(Bertrand and Other Properties)** *Suppose either* $v_\ell = 0$ *or* $\eta_\ell(1) = \eta_h(1) = 0$ *and* β *is a bidding equilibrium.*

1. $\overline{\pi}_\omega(\beta(x)) > 0$ *if* $\frac{g_h(x)}{g_\ell(x)} > \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$.
2. $\beta(x) \in [v_\ell, v_h)$ *for almost all* x.
3. $U(\beta(x')|x', \, sol) \geq U(\beta(x)|x, \, sol)$ *if* $x' > x$. *The inequality is strict if and only if* $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$.

The proof of the lemma utilizes that the set of feasible bids is dense below v_ℓ. If the price grid is finite below v_ℓ as well, equilibrium may involve bids just below v_ℓ—just like in the usual Bertrand pricing game with price grid—but such equilibria would not add anything important.

Proof of Lemma 10: $b' > b \geq v_\ell$ implies $(v_\ell - b') < (v_\ell - b)$ and $\overline{\pi}_\ell(b') \geq \overline{\pi}_\ell(b)$. These together with the hypothesis $\overline{\pi}_\ell(b') > 0$ and $b' > b \geq v_\ell$ imply

$$\overline{\pi}_\ell(b')\,(v_\ell - b') < \overline{\pi}_\ell(b)\,(v_\ell - b)\,. \tag{53}$$

Hence, $U(b'|x, \text{sol}) \geq U(b|x, \text{sol})$ requires

$$\overline{\pi}_h(b')\,(v_h - b') > \overline{\pi}_h(b)\,(v_h - b)\,. \tag{54}$$

Rewriting $U(b'|x, \text{sol})$ yields

$$\frac{\rho_\ell g_\ell(x)\,\bar{n}_\ell}{\rho_\ell g_\ell(x)\,\bar{n}_\ell + \rho_h g_h(x)\,\bar{n}_h}\left[\bar{\pi}_\ell(b)(v_\ell - b) + \frac{\rho_h g_h(x)\,\bar{n}_h}{\rho_\ell g_\ell(x)\,\bar{n}_\ell}\bar{\pi}_h(b)\,(v_h - b)\right]. \tag{55}$$

It follows from $U(b'|x, \text{sol}) \geq U(b|x, \text{sol})$ and (53) that

$$\begin{aligned}&\frac{\rho_h g_h(x)\,\overline{n}_h}{\rho_\ell g_\ell(x)\,\overline{n}_\ell}\left[\overline{\pi}_h(b')\,(v_h - b') - \overline{\pi}_h(b)\,(v_h - b)\right]\\ \geq\ & \overline{\pi}_\ell(b)(v_\ell - b) - \overline{\pi}_\ell(b')(v_\ell - b') > 0.\end{aligned}$$

Since $x' > x$ and $\frac{g_h(x)}{g_\ell(x)}$ is nondecreasing,

$$\begin{aligned}&\frac{\rho_h g_h(x')\,\overline{n}_h}{\rho_\ell g_\ell(x')\,\overline{n}_\ell}\left[\overline{\pi}_h(b')\,(v_h - b') - \overline{\pi}_h(b)\,(v_h - b)\right]\\ \geq\ & \overline{\pi}_\ell(b)(v_\ell - b) - \overline{\pi}_\ell(b')(v_\ell - b') > 0.\end{aligned} \tag{56}$$

which implies

$$
\begin{aligned}
& U(b'|x',\ \text{sol}) \\
= \quad & \frac{\rho_\ell g_\ell\left(x'\right)\overline{n}_\ell}{\rho_\ell g_\ell\left(x'\right)\overline{n}_\ell+\rho_h g_h\left(x'\right)\overline{n}_h}\left[\overline{\pi}_\ell(b')(v_\ell-b')+\frac{\rho_h g_h\left(x'\right)\overline{n}_h}{\rho_\ell g_\ell\left(x'\right)\overline{n}_\ell}\overline{\pi}_h(b')\left(v_h-b'\right)\right] \\
\geq \quad & \frac{\rho_\ell g_\ell\left(x'\right)\overline{n}_\ell}{\rho_\ell g_\ell\left(x'\right)\overline{n}_\ell+\rho_h g_h\left(x'\right)\overline{n}_h}\left[\overline{\pi}_\ell(b)(v_\ell-b)+\frac{\rho_h g_h\left(x'\right)\overline{n}_h}{\rho_\ell g_\ell\left(x'\right)\overline{n}_\ell}\overline{\pi}_h(b)\left(v_h-b\right)\right] \qquad (57) \\
= \quad & U(b|x',\ \text{sol}).
\end{aligned}
$$

If $\frac{g_h(x')}{g_\ell(x')}>\frac{g_h(x)}{g_\ell(x)}$, then (56) and (57) hold with strict inequalities.

The last part of the lemma is immediate because G_h and G_ℓ are mutually absolutely continuous, so that $G_h\left(\{x|\beta(x)\leq b\}\right)=0\Leftrightarrow G_\ell\left(\{x|\beta(x)\leq b\}\right)=0$. ∎

Proof of Lemma 11:

Step 0: If $\pi_\omega\left(b\right)>0$ for some $n\geq 2$ and $\omega=\ell$ or h, then $\overline{\pi}_\omega\left(b\right)>0$ for both ω and any η_ω.

Proof of Step 0: $\pi_\omega\left(b;\beta,n\right)>0$ for some n and ω implies that $G_\omega\left(\{x|\beta(x)\leq b\}\right)>0$. Since G_h and G_ℓ are mutually absolutely continuous, it follows that $G_{\omega'}\left(\{x|\beta(x)\leq b\}\right)>0$ also for $\omega'\neq\omega$. Therefore, $\overline{\pi}_\omega\left(b\right)>0$ for both ω and any η_ω. □

Step 1. $\beta\left(x\right)\geq v_\ell$ for almost all x.

Proof of Step 1: This is immediate if $v_\ell=0$. So, suppose $\eta_\ell(1)=\eta_h(1)=0$.

Let $\underline{b}\equiv\inf\{b|\pi_\omega\left(b\right)>0$ for some n and $\omega\}$. Suppose $\underline{b}<v_\ell$. It may not be that β has an atom at $\underline{b}$ (i.e., $\int_{\{x:\beta(x)=\underline{b}\}}g_\omega(x)dx>0$) since by a standard Bertrand argument $U(\underline{b}+\varepsilon|x,\text{sol})>U(\underline{b}|x,\text{sol})$ for sufficiently small $\varepsilon\in(0,v_\ell-\underline{b})$. Therefore, there exists a sequence of x^k such that $\beta\left(x^k\right)\to\underline{b}$ and $\overline{\pi}_\omega\left(\beta\left(x^k\right)\right)\to 0$ (owing to $\eta_\omega(1)=0$). Hence, equilibrium payoffs $U(\beta\left(x^k\right)|x^k,\text{sol})\to 0$. However, by the definition of $\underline{b}$ and monotonicity of $\overline{\pi}_\omega$, $\overline{\pi}_\omega(b)$ is strictly positive for all $b\in(\underline{b},v_\ell)$. Thus, for all $b\in(\underline{b},v_\ell)$, the payoff $U(b|x,\text{sol})>0$. This contradicts the optimality of $\beta\left(x^k\right)$ for sufficiently large k, a standard Bertrand argument. Thus, $\underline{b}\geq v_\ell$. Finally, $\pi_\omega\left(b\right)=0$ for all $b<v_\ell$ implies that $G_\omega\left(\{x|\beta\left(x\right)\geq v_\ell\}\right)=1$, proving the step. □

Step 2. $\beta\left(x\right)<v_h$ for all x. □

Proof of Step 2: It clearly cannot be that $G_\omega\left(\{x|\beta(x)>v_h\}\right)=1$ for any ω, since this would imply that bidders have strictly negative payoffs in expectations. Suppose that $\beta(x')\geq v_h$ for some x'. From $G_\ell\left(\{x|\beta(x)>v_h\}\right)<1$, $\beta(x')\geq v_h$ implies $\overline{\pi}(\beta(x'))>0$ and $U(\beta(x')|x',\text{sol})<0$, a contradiction to the optimality of

$\beta(x')$. $\square$

Step 3. $\overline{\pi}_\omega\left(\beta(x)\right) > 0$ for almost all x for $\omega \in \{\ell, h\}$.

Proof of Step 3: Fix $\omega \in \{\ell, h\}$. Let $X = \{x | \overline{\pi}_\omega\left(\beta(x)\right) = 0\}$. The probability that in state ω all bidders are from that set is $\Sigma_n \eta_\omega(n)[G_\omega(X)]^n$. Since in that event some bidder has to win, we have $\Sigma_n \eta_\omega(n)[G_\omega(X)]^n \leq \Pr[\{\text{Winning bidder has signal } x \in X\} | \omega] \leq \overline{n}_\omega \int_{x \in X} \overline{\pi}_\omega\left(\beta(x)\right) g\left(x\right) dx = 0$. Hence, $G_\omega(X) = 0$. $\square$

Step 4. For any $x' > x$, $U(\beta\left(x'\right)|x', \text{sol}) \geq U(\beta\left(x\right)|x, \text{sol})$. The inequality is strict if and only if $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$. Thus, $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$ implies that $U(\beta\left(x'\right)|x', \text{sol})$ is strictly positive.

Proof of Step 4: From (52) it follows (after dividing the numerator and denominator by $g_\ell\left(x\right)$) that

$$U(b|x,\ \text{sol}) = \frac{\rho_\ell \overline{n}_\ell \overline{\pi}_\ell(b)(v_\ell - b) + \rho_h \frac{g_h(x)}{g_\ell(x)} \overline{n}_h \overline{\pi}_h(b)\left(v_h - b\right)}{\rho_\ell \overline{n}_\ell + \rho_h \frac{g_h(x)}{g_\ell(x)} \overline{n}_h}. \tag{58}$$

Therefore, for any $x' > x$,

$$U(\beta(x')|x',\ \text{sol}) \geq U(\beta(x)|x',\ \text{sol}) \geq U(\beta(x)|x,\ \text{sol}) \geq 0, \tag{59}$$

where the first and last inequalities are equilibrium conditions; the second inequality owes to $\frac{g_h(x')}{g_\ell(x')} \geq \frac{g_h(x)}{g_\ell(x)}$ and $\overline{\pi}_h(\beta(x))\left(v_h - \beta(x)\right) \geq 0 \geq \overline{\pi}_\ell(\beta(x))(v_\ell - \beta(x))$, which follows from Steps 1 and 2.

Suppose $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$. Now, either $\overline{\pi}_\omega\left(\beta(x)\right) > 0$, in which case $\overline{\pi}_h(\beta(x))\left(v_h - \beta(x)\right) > 0$, and it follows from (58) and $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$ that the second inequality in (59) is strict, or $\overline{\pi}_\omega\left(\beta(x)\right) = 0$ and hence $U(\beta(x)|x, \text{sol}) = 0$. In the latter case, by Step 3, there is some $y \in (\underline{x}, x')$ such that $\overline{\pi}_\omega\left(\beta(y)\right) > 0$. We can choose y such that $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(y)}{g_\ell(y)}$ (recall that $\frac{g_h(\underline{x})}{g_\ell(\underline{x})} = \lim_{x \to \underline{x}} \frac{g_h(x)}{g_\ell(x)}$). By Step 2, $\overline{\pi}_h(\beta(y))\left(v_h - \beta(y)\right) > 0$. Since $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(y)}{g_\ell(y)}$, it follows from (58) and the fact that β is a bidding equilibrium that

$$U(\beta(x')|x',\ \text{sol}) \geq U(\beta(y)|x',\ \text{sol}) > U(\beta(y)|y,\ \text{sol}) \geq 0 = U(\beta(x)|x,\ \text{sol}).$$

Conversely, $\frac{g_h(x')}{g_\ell(x')} = \frac{g_h(x)}{g_\ell(x)}$ implies

$$U(\beta(x')|x',\ \text{sol}) = U(\beta(x')|x,\ \text{sol}) \leq U(\beta(x)|x,\ \text{sol}) = U(\beta(x)|x',\ \text{sol}) \leq U(\beta(x')|x',\ \text{sol}),$$

where the inequalities are equilibrium conditions while the equalities owe to the fact that x and x' contain the same information. Therefore, $U(\beta(x')|x', \text{sol}) = U(\beta(x)|x, \text{sol})$. $\square$

Step 5. The strict positivity of $U(\beta(x)|x, \text{sol})$ implies immediately that $\overline{\pi}_\omega(\beta(x)) > 0$ for any x for which $\frac{g_h(x)}{g_\ell(x)} > \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$. (Step 3 established this only for almost all x). This proves Part 1 of the Lemma.

This completes the proof of the lemma: Part 1 of the Lemma is established in Step 5. Part 2 is established in Step 1 and 2. Part 3 is established in Step 4. ∎

Proof of Proposition 7:

Part 1: Proved by Lemma 11.

Part 2**:** Suppose that $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$ for some $x, x' \in (\underline{x}, \bar{x}]$, but $\beta(x') < \beta(x)$. Since β is a bidding equilibrium, $U(\beta(x)|x, \text{sol}) \geq U(\beta(x')|x, \text{sol})$. By Lemma 11, $\overline{\pi}_\omega(\beta(x')) > 0$ and $\beta(x') \geq v_\ell$. Therefore, by Lemma 10, $U(\beta(x)|x', \text{sol}) > U(\beta(x')|x', \text{sol})$, contradicting the optimality of $\beta(x')$ for x'. Thus, the supposition $\beta(x') < \beta(x)$ is false. Hence, $\beta(x') \geq \beta(x)$ whenever $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$.

Next, suppose that $\frac{g_h(x')}{g_\ell(x')} = \frac{g_h(x)}{g_\ell(x)}$ for some $x, x' \in (\underline{x}, \bar{x}]$, but $\beta(x') < \beta(x)$. Then there is some interval containing x and x' over which $\frac{g_h(x)}{g_\ell(x)}$ is constant, say, C. Let $[x_-, x_+]$ be the closure of this interval. By the above argument, $\beta(x'') \leq \beta(x)$ whenever $x'' < x_- < x$ and $\beta(x) \leq \beta(x''')$ whenever $x < x_+ < x'''$. Define $\widetilde{\beta}_1(x)$ by

$$\widetilde{\beta}_1(x) = \ \inf\{b : G_h(x) \leq G_h(\{t|\beta(t) \leq b\})\} \quad \text{if} \quad x \in [x_-, x_+]$$

Thus, on $[x_-, x_+]$ the signals are essentially "reordered" to make $\widetilde{\beta}_1(x)$ monotone. Outside $[x_-, x_+]$, $\widetilde{\beta}_1(x)$ coincides with $\beta(x)$. Note that $\tilde{\beta}(x') \leq \tilde{\beta}(x) \leq \tilde{\beta}(x'')$ for all $x' < x_-$ and $x_+ < x''$. With this definition,

$$G_h(\{x|\widetilde{\beta}_1(x) \leq b\}) = G_h(\{x|\beta(x) \leq b\}),$$

for all b. That is, the distribution of bids induced by $\widetilde{\beta}_1$ is equal to the distribution of bids induced by β in state h. It is also the same in state ℓ because $\widetilde{\beta}_1 = \beta$ outside $[x_-, x_+]$ and because the distributions G_ℓ and G_h conditional on $x \in (x_-, x_+)$ are identical (owing to the constant $\frac{g_h(x)}{g_\ell(x)}$).

The equality of the distributions of bids under $\widetilde{\beta}_1$ and β implies that, for any $x \notin \{x_-, x_+\}$, $\tilde{\beta}_1(x)$ is optimal: for $x \notin [x_-, x_+]$ this follows immediately from $\widetilde{\beta}_1(x) = \beta(x)$; for $x \in (x_-, x_+)$ this follows from $\tilde{\beta}_1(x) = \beta(y)$ where y is some value of the signal such that $\frac{g_h(y)}{g_\ell(y)} = \frac{g_h(x)}{g_\ell(x)}$. For $x \in \{x_-, x_+\}$, note that we can represent the distribution of signals by an equivalent pair of densities that is equal to the original densities almost everywhere, so that the resulting equilibrium still corresponds to the same distributional strategy. Here, $\tilde{\beta}_1$ can be rationalized at $\{x_-, x_+\}$ by changing the densities at the points $x \in \{x_-, x_+\}$. At x_-, if $\tilde{\beta}_1(x_-) = \tilde{\beta}_1(x_- + \varepsilon)$ for some ε (an atom), $\tilde{\beta}_1(x_-)$ is rationalized by setting $g_\omega(x_-) = \lim_{\varepsilon \to 0} g_\omega(x_- + \varepsilon)$. Otherwise, $\tilde{\beta}_1(x_-)$ is rationalized by setting $g_\omega(x_-) = \lim_{\varepsilon \to 0} g_\omega(x_- - \varepsilon)$. Similarly for x_+. It follows that $\widetilde{\beta}_1$ is monotone on $[x_-, x_+]$ and that it is equivalent to β.

Repeating this construction for all intervals over which $\frac{g_h(x)}{g_\ell(x)}$ is constant, we get a sequence of bidding strategies (constructing the sequence by starting with the longest interval of signals on which $\frac{g_h(x)}{g_\ell(x)}$ is constant). Let $\tilde{\beta}$ be the pointwise limit of this sequence on $(\underline{x}, \bar{x}]$ and let $\tilde{\beta}(\underline{x}) = \lim_{\varepsilon \to 0} \beta(\underline{x} + \varepsilon)$. Then, $\tilde{\beta}$ is an equivalent bidding equilibrium that is monotone on $[\underline{x}, \bar{x}]$, as claimed. ∎

B.3 Proof of Proposition 3 for Random Participation

The following lemma shows that, for the purposes of this proof, $\boldsymbol{\eta}^k$ may be replaced by $\mathbf{n}^k$ without loss of generality. Once this is established, the proof of Theorem 1 applies and need not be repeated. Recall $\overline{n}_\omega(\eta_\omega)$ and $\overline{\pi}_\omega(b; \beta, \eta_\omega)$, $\omega = \ell, h$, from (51). Since we deal here explicitly with $\boldsymbol{\eta}$ and $\mathbf{n}$, we do not suppress them in the arguments of π and $\mathbb{E}[v|...]$.

Lemma 12 *Consider a sequence of bidding games $\Gamma_0\left(N^k, \boldsymbol{\eta}^k, \Delta^k\right)$ such that the support of η_ω^k is contained in $\{n_\omega^k, ..., n_\omega^k + m\}$ for some fixed integer $m > 0$ and $\Delta^k \to 0$, $\min\{n_\ell^k, n_h^k\} \to \infty$ and $\lim_{k\to\infty} \frac{n_h^k}{n_\ell^k} = r$, and a corresponding sequence of bidding equilibria β^k.*
(i)

$$\lim \frac{\overline{n}_h^k}{\overline{n}_\ell^k} = \lim \frac{n_h^k}{n_\ell^k}, \tag{60}$$

(ii) For any (b^k) with $\lim \left(G_\omega\left(x_+\left(b^k\right)\right)\right)^{n_\omega^k} > 0$,

$$\lim \frac{\overline{\pi}_h(b^k; \beta^k, \eta_h^k)}{\overline{\pi}_\ell(b^k; \beta^k, \eta_\ell^k)} = \lim \frac{\pi_h(b^k; \beta^k, n_h^k)}{\pi_\ell(b^k; \beta^k, n_\ell^k)}.$$

(iii) For any (b^k) with $\lim \left(G_\omega\left(x_+\left(b^k\right)\right)\right)^{n_\omega^k} > 0$,

$$\lim \mathbb{E}[v|x^k, \textit{sol,win at } b^k; \beta^k, \boldsymbol{\eta}^k] = \lim \mathbb{E}[v|x^k, \textit{sol,win at } b^k; \beta^k, \mathbf{n}^k].$$

Remark: The condition $\lim \left(G_\omega\left(x_+\left(b^k\right)\right)\right)^{n_\omega^k} > 0$ is needed for part (ii). For any fixed $x < \bar{x}$, if β^k is strictly increasing, it follows from $\pi_\omega(\beta^k(x);\beta^k,n_\omega^k) = (G_\omega(x))^{n_\omega^k-1}$ that

$$\frac{\pi_h(\beta^k(x);\beta^k,n_h^k+1)}{\pi_\ell(\beta^k(x);\beta^k,n_\ell^k)} = G_h(x)\,\frac{\pi_h(\beta^k(x);\beta^k,n_h^k)}{\pi_\ell(\beta^k(x);\beta^k,n_\ell^k)} < \frac{\pi_h(\beta^k(x);\beta^k,n_h^k)}{\pi_\ell(\beta^k(x);\beta^k,n_\ell^k)}.$$

Therefore, since $G_h(x) < 1$, the difference between these ratios is not vanishing as would be required for the result of the lemma to hold for this x. However, when $\lim \left(G_\omega\left(x_+\left(b^k\right)\right)\right)^{n_\omega^k} > 0$, then $x_+\left(b^k\right) \to \bar{x}$ and hence $G_\omega\left(x_+\left(b^k\right)\right) \to 1$. Fortunately, bids for which $\lim \left(G_\omega\left(x_+\left(b^k\right)\right)\right)^{n_\omega^k} = 0$ can be neglected in the characterization proof (the winning bid is strictly higher than b^k with probability 1).

Proof of Lemma 12. Part (i) is immediate. Part (iii) follows from Part (i) and (ii). So, we show Part (ii). For this, it is sufficient to show (shifting the counting integer by 1 to simplify the expressions below)

$$\lim \frac{\pi_\omega(b^k;\beta^k,n_\omega^k+1)}{\pi_\omega(b^k;\beta^k,n_\omega^k+m+1)} = 1.$$

From Lemma 4,

$$\frac{\pi_\omega(b^k;\beta^k,n_\omega^k+1)}{\pi_\omega(b^k;\beta^k,n_\omega^k+m+1)} = \frac{\int_{x_-^k}^{x_+^k} (G_\omega(x))^{n_\omega^k} g_\omega(x)\,dx}{\int_{x_-^k}^{x_+^k} (G_\omega(x))^{n_\omega^k+m} g_\omega(x)\,dx}.$$

The claim is now immediate if $x_-^k \to \bar{x}$ since

$$\frac{1}{G_\omega\left(x_+^k\right)^m} \leq \frac{\int_{x_-^k}^{x_+^k} (G_\omega(x))^{n_\omega^k} g_\omega(x)\,dx}{\int_{x_-^k}^{x_+^k} (G_\omega(x))^{n_\omega^k+m} g_\omega(x)\,dx} \leq \frac{1}{G_\omega\left(x_-^k\right)^m}, \tag{61}$$

and $G_\omega\left(x_+^k\right) \to 1$. Otherwise, we can choose some $\varepsilon > 0$ with $x_-^k < \bar{x} - \varepsilon$ for all k. Observe that

$$\lim \frac{\int_{\bar{x}-\varepsilon}^{x_+^k} (G_\omega(x))^{n_\omega^k} g_\omega(x)\,dx}{\int_{x_-^k}^{x_+^k} (G_\omega(x))^{n_\omega^k} g_\omega(x)\,dx} = 1.$$

The claim now follows using the previous bounds (61) because

$$\lim \frac{\pi_\omega(b^k;\beta^k,n_\omega^k+1)}{\pi_\omega(b^k;\beta^k,n_\omega^k+m+1)} = \lim \frac{\int_{\bar{x}-\varepsilon}^{x_+^k} (G_\omega(x))^{n_\omega^k} g_\omega(x)\,dx}{\int_{\bar{x}-\varepsilon}^{x_+^k} (G_\omega(x))^{n_\omega^k+m} g_\omega(x)\,dx},$$

and because we can choose ε arbitrarily small such that $G_\omega(\bar{x}-\varepsilon) \cong 1$. ∎

Given Lemma 12, the proof of Proposition 3 is identical to the proof of Theorem 1.

C Proof of Intermediate Results from the Proof of Theorem 2 (Strategic Solicitation)

C.1 Proof of Lemma 3: Existence and Uniqueness of $r^*(\rho, \overline{g})$

The proof relies on two lemmas. Define

$$J(r;\rho,\overline{g}) \triangleq \int_0^1 (x-\frac{1}{\overline{g}})x^{\frac{1}{\overline{g}r-1}}\frac{\ln x}{(1+x\rho\overline{g}r)^2}dx. \tag{62}$$

Lemma 13 *For the function defined in (62), for any $\rho \in (0,1)$ and $\overline{g} > 1$:*

1. *There is a unique number $r^* = r^*(\rho,\overline{g}) \in (\frac{1}{\overline{g}},\infty)$ s.t. $J(r^*) = 0$.*
2. *$J(r) < 0$ for $r \in (\frac{1}{\overline{g}}, r^*)$ and $J(r) > 0$ for $r \in (r^*,\infty)$.*

Proof of Lemma 13:
Claim 1: For each $\overline{g} > 1$, there exists an $r' \in (\frac{1}{\overline{g}},\infty)$ (close to $\overline{g}^{-1}$) such that $J(r') < 0$.
Proof: Write

$$\begin{aligned} J(r) = \int_0^{\frac{1}{\overline{g}}} \left(\frac{1}{\overline{g}}-x\right)\left\{(-\ln(x))(x)^{\left(\frac{1}{\overline{g}r-1}\right)}(1+\rho\overline{g}rx)^{-2}\right\}dx \\ -\int_{\frac{1}{\overline{g}}}^1 \left(x-\frac{1}{\overline{g}}\right)\left\{(-\ln(x))(x)^{\left(\frac{1}{\overline{g}r-1}\right)}(1+\rho\overline{g}rx)^{-2}\right\}dx. \end{aligned}$$

The term in the brackets $\{\cdots\}$ is always nonnegative and therefore both integrals are positive.

Let $\sigma \triangleq \frac{1}{\overline{g}r-1}$. The first integral is

$$\begin{aligned}
&\int_0^{\frac{1}{\bar{g}}}\left(\frac{1}{\bar{g}}-x\right)\left\{\left(-\ln(x)\right)(x)^{\left(\frac{1}{\bar{g}r-1}\right)}\frac{1}{\left(1+\rho\bar{g}rx\right)^2}\right\}dx\\
&\leq\frac{1}{\bar{g}}\int_0^{\frac{1}{\bar{g}}}\left(-\ln(x)\right)x^{\sigma}dx\\
&=\left(\frac{1}{\bar{g}}\right)^{\sigma+2}\left(\frac{1}{\sigma+1}\right)^2\left(1-(\sigma+1)\ln\frac{1}{\bar{g}}\right),
\end{aligned}$$

where the equality is derived by integration-by-parts.

Thus, the first integral vanishes to zero at a rate of at least $\left(\bar{g}^{-1}\right)^{\sigma}$ as σ approaches ∞ (or equivalently, $r\to\bar{g}^{-1}$).

The second integral is

$$\begin{aligned}
&\int_{\frac{1}{\bar{g}}}^{1}\left(x-\frac{1}{\bar{g}}\right)\left\{-\ln(x)(x)^{\left(\frac{1}{\bar{g}r-1}\right)}\left(1+\rho\bar{g}rx\right)^{-2}\right\}dx\\
&=\int_{\frac{1}{\bar{g}}}^{1}\left(x-\frac{1}{\bar{g}}\right)\left\{\left(-\ln(x)\right)x^{\sigma}\left(1+\rho\frac{\sigma+1}{\sigma}x\right)^{-2}\right\}dx\\
&\geq\left(1+\rho\frac{\sigma+1}{\sigma}\right)^{-2}\int_{\frac{1}{\bar{g}}}^{1}\left(x-\frac{1}{\bar{g}}\right)\left(-\ln(x)\right)x^{\sigma}dx\\
&=\left(\frac{\sigma}{\sigma+\rho(\sigma+1)}\right)^2\left(\frac{1}{(\sigma+2)^2}-\frac{\bar{g}^{-1}}{(\sigma+1)^2}+\frac{\left(-\ln\left(\bar{g}^{-1}\right)\right)\left(\bar{g}^{-1}\right)^{\sigma+2}}{(\sigma+2)(\sigma+1)}+\frac{1}{\bar{g}}\frac{\left(\bar{g}^{-1}\right)^{\sigma+1}}{(\sigma+1)^2}-\frac{\left(\bar{g}^{-1}\right)^{\sigma+2}}{(\sigma+2)^2}\right),
\end{aligned}$$

where the second equality follows by integration-by-parts.

Thus, either the second integral stays positive or it vanishes at a rate of at most σ^{-2} as σ approaches ∞ (or equivalently, $r\to\bar{g}^{-1}$).

To sum up, $J(r)<0$ for $r\to\bar{g}^{-1}$. ∎

Claim 2: For sufficiently large r, $J(r)>0$.

Proof: We show that $\lim_{r\to\infty}r^2J(r)=\infty$. Let $\xi(x,r)$ denote the integrand of $r^2J(r)$. That is,

$$\xi(x,r)\equiv\left(x-\frac{1}{\bar{g}}\right)\ln(x)\,x^{\frac{1}{\bar{g}r-1}}\left(\frac{r}{1+\rho\bar{g}rx}\right)^2.$$

Observe that $\xi(x,r)$ is nondecreasing in r on the domain $x\in(0,\bar{g}^{-1})$, and is non-increasing in r on the domain $x\in(\bar{g}^{-1},1)$. Therefore, by the monotone convergence theorem,

$$\lim_{r\to\infty} r^2 J(r) \equiv \lim_{r\to\infty} \int_0^1 \xi(x,r)\,dx = \int_0^1 \lim_{r\to\infty} \xi(x,r)\,dx = \frac{1}{(\rho\overline{g})^2} \int_0^1 \left(x^{-1} - \frac{1}{\overline{g}}x^{-2}\right) \ln(x)\,dx$$
$$= \frac{1}{(\rho\overline{g})^2} \left[\int_0^{\overline{g}^{-1}} \left(x^{-1} - \frac{1}{\overline{g}}x^{-2}\right) \ln(x)\,dx + \int_{\overline{g}^{-1}}^1 \left(x^{-1} - \frac{1}{\overline{g}}x^{-2}\right) \ln(x)\,dx\right].$$

Now, letting $a \in (0, \overline{g}^{-1})$,

$$\int_0^{\overline{g}^{-1}} \left(x^{-1} - \frac{1}{\overline{g}}x^{-2}\right) \ln(x)\,dx \geq \lim_{a\to 0} \int_a^{\overline{g}^{-1}} \left(x^{-1} - \frac{1}{\overline{g}}x^{-2}\right) \ln(x)\,dx$$
$$= \lim_{a\to 0} \left[\frac{1}{2}\left(\left(\ln\left(\overline{g}^{-1}\right)\right)^2 - (\ln(a))^2\right) + \frac{1}{\overline{g}}\left[\overline{g}\left(\ln\left(\overline{g}^{-1}\right)+1\right) - a^{-1}\left(1+\ln(a)\right)\right]\right] = \infty,$$

while $\int_{\overline{g}^{-1}}^1 \left(x^{-1} - \frac{1}{\overline{g}}x^{-2}\right) \ln(x)\,dx$ is obviously bounded. Therefore, $\lim_{r\to\infty} r^2 J(r) = \infty$ hence $J(r) > 0$ for large enough r. ∎

Claims 1 and 2 together with the continuity of $J(r)$ in r establish the existence of $r > 1/\overline{g}$ such that $J(r) = 0$.

Claim 3: Fix a $\overline{g} > 1$. For $r > \overline{g}^{-1}$, if $J(r;\rho,\overline{g}) = 0$, then $J_r(r;\rho,\overline{g}) > 0$.

Proof: By hypothesis,

$$J(r;\rho,\overline{g}) \equiv \int_0^1 \left(x - \frac{1}{\overline{g}}\right) x^{\frac{1}{\overline{g}r-1}} \frac{\ln x}{(1+\rho\overline{g}rx)^2}\,dx = 0.$$

Since $x^{\frac{1}{\overline{g}r-1}} \frac{\ln x}{(1+\rho\overline{g}rx)^2} < 0$ for all $x \in (0,1)$, the integrand is positive for all $x \in (0, \frac{1}{\overline{g}})$ and is negative for all $x \in (\frac{1}{\overline{g}}, 1)$. Therefore, at any $r > \overline{g}^{-1}$ that satisfies $J(r) = 0$,

$$\int_0^{\frac{1}{\overline{g}}} \left(x - \frac{1}{\overline{g}}\right) x^{\frac{1}{\overline{g}r-1}} \frac{\ln x}{(1+\rho\overline{g}rx)^2}\,dx = -\int_{\frac{1}{\overline{g}}}^1 \left(x - \frac{1}{\overline{g}}\right) x^{\frac{1}{\overline{g}r-1}} \frac{\ln x}{(1+\rho\overline{g}rx)^2}\,dx > 0.$$

Consider the function $r^2 J(r)$ and observe that

$$\frac{dr^2 J(r)}{dr} = r \int_0^1 \left(x - \frac{1}{\overline{g}}\right) x^{\frac{1}{\overline{g}r-1}} \frac{\ln x}{(1+\rho\overline{g}rx)^2} \left[\frac{-\overline{g}r\ln x}{(\overline{g}r-1)^2} + \frac{2}{(1+\rho\overline{g}rx)}\right] dx.$$

The integrand is equal to the integrand of $J(r)$ times the term $\left[\frac{-\overline{g}r\ln x}{(\overline{g}r-1)^2} + \frac{2}{(1+\rho\overline{g}rx)}\right]$ which is nonnegative and decreasing in x. Therefore, at r such that $J(r) = 0$, the positive part over $(0, \frac{1}{\overline{g}})$ is weighted more heavily than the negative part over $(\frac{1}{\overline{g}}, 1)$ implying $\frac{dr^2 J(r)}{dr} > 0$. Now, at r such that $J(r) = 0$, $sgn\left(J_r(r)\right) = sgn\left(\frac{dr^2 J(r)}{dr}\right)$.

Therefore, $J_r(r) > 0$ as required. $\square$

Claim 3 concludes the proof of the Lemma, since $J_r(r) > 0$ at any r such that $J(r) = 0$, there can be only one such r. $\blacksquare$

Lemma 14 *If r satisfies equation (17), then $J(r;\rho,\overline{g}) = 0$.*

Proof of Lemma 14: Using (5) to spell out $\Phi_\omega(p|r)$ and rearranging, we have

$$\begin{aligned}
& -\int_{v_\ell}^{\bar{p}} \Phi_h(p|r)\ln\Phi_h(p|r)\,dp + r\int_{v_\ell}^{\bar{p}} \Phi_\ell(p|r)\ln\Phi_\ell(p|r)\,dp \\
= & -\frac{\overline{g}r}{\overline{g}r-1}\int_{v_\ell}^{\bar{p}}\left(\frac{1}{\rho\overline{g}r}\frac{p-v_\ell}{v_h-p}-\frac{1}{\overline{g}}\right)\left(\frac{1}{\rho\overline{g}r}\frac{p-v_\ell}{v_h-p}\right)^{\frac{1}{\overline{g}r-1}}\ln\left(\frac{1}{\rho\overline{g}r}\frac{p-v_\ell}{v_h-p}\right)dp
\end{aligned}$$

Changing the integration variable by substituting for p the function $\psi(x) = \frac{v_\ell + x\rho\overline{g}rv_h}{1+x\rho\overline{g}r}$ we get

$$\begin{aligned}
& -\int_{v_\ell}^{\bar{p}} \Phi_h(p|r)\ln\Phi_h(p|r)\,dp + r\int_{v_\ell}^{\bar{p}} \Phi_\ell(p|r)\ln\Phi_\ell(p|r)\,dp \qquad (63)\\
= & -\frac{\overline{g}r}{\overline{g}r-1}\int_{\psi^{-1}(v_\ell)}^{\psi^{-1}(\bar{p})}\left(\frac{1}{\rho\overline{g}r}\frac{\psi(x)-v_\ell}{v_h-\psi(x)}-\frac{1}{\overline{g}}\right)\left(\frac{1}{\rho\overline{g}r}\frac{\psi(x)-v_\ell}{v_h-\psi(x)}\right)^{\frac{1}{\overline{g}r-1}}\ln\left(\frac{1}{\rho\overline{g}r}\frac{\psi(x)-v_\ell}{v_h-\psi(x)}\right)\psi'(x)dx \\
= & -\frac{\overline{g}r}{\overline{g}r-1}\int_0^1\left(x-\frac{1}{\overline{g}}\right)x^{\frac{1}{\overline{g}r-1}}\ln(x)\frac{r\overline{g}\rho(v_h-v_\ell)}{(1+rx\overline{g}\rho)^2}dx = -\frac{(\overline{g}r)^2}{\overline{g}r-1}\rho(v_h-v_\ell)J(r).
\end{aligned}$$

Now, this and (17) imply $J(r) = 0$. $\blacksquare$

C.2 Proof of Proposition 5 (Bounded Number of Bidders)

Let $\bar{F}_\omega = \lim F_\omega^k$, choosing a convergent subsequence if necessary (Helly's selection theorem guarantees the existence of such a subsequence). We will show that for any such subsequence, $\bar{F}$ is concentrated on some $C \leq \mathbb{E}(v)$.

If n_ω^k is bounded, then $\lim n_\omega^k s^k = 0$. Hence, there is a C_ω such that $\bar{F}$ is concentrated on some C_ω, since otherwise Lemma 2 would imply $\lim n_\omega^k s^k > 0$.[31] It remains to be shown that $C_h = C_\ell = C$. Individual rationality then implies $C \leq \mathbb{E}(v)$.

Note that Lemma 2 does not require β^k to be monotone. This is useful because it may be that $n_\omega^k = 1$ for some state, and, hence, we cannot invoke Proposition 1 to argue that β^k must be monotone non-decreasing.

[31] For notational simplicity, the proof is for pure solicitation strategies. It extends immediately to mixed solicitation strategies.

Case 1. Suppose that $\lim n_h^k < \infty$ and $\lim n_\ell^k < \infty$. Then, as was just argued, in each state, $\bar{F}_\omega$ is concentrated on some C_ω. However, $\overline{g} \equiv \frac{g_h(\bar{x})}{g_\ell(\bar{x})} < \infty$ and $n_\omega^k < \infty$ imply that $\bar{F}_h$ and $\bar{F}_\ell$ are mutually absolutely continuous. Hence, $C_\ell = C_h = C$. (This case includes the trivial equilibrium with $C = 0 \leq v_\ell$ and $n_h^k = n_\ell^k = 1$.)

Case 2. Suppose $\lim n_\ell^k = \infty$ and $\lim n_h^k = m < \infty$ for some $m \geq 1$.[32] As observed, $\bar{F}_h$ is concentrated on C_h. To start, by $n_h^k \to m < \infty$ and the bounded likelihood ratio, whenever $\bar{F}_h(p) = 0$, then $\bar{F}_\ell(p) = 0$, i.e., the lower bound on the support of $\lim \bar{F}_\ell(p)$ is weakly above C_h. If $C_h \geq v_\ell$, then the bidders' individual rationality and the law of iterated expectations rule out that $\bar{F}_\ell(p) < 1$ for any $p > C_h \geq v_\ell$. Hence, if $C_h \geq v_\ell$, then $\bar{F}_\ell$ has mass 1 on C_h as well.

The case $C_h < v_\ell$ can be ruled out. From $\lim n_h^k / n_\ell^k = 0$, a solicited bidder is certain that the state is ℓ, and, hence, $\lim U^k(p + \varepsilon | x, \text{sol}) \geq \bar{F}_\ell(p)(v_\ell - p - \varepsilon)$ for all x, p, and $\varepsilon > 0$. Since feasibility requires $\lim U^k(\beta^k(x) | x, \text{sol}) \to 0$ for all x (as in Lemma 6), it follows that $\bar{F}_\ell(p) = 0$ for all $p < v_\ell$. Also, as before, bidders' individual rationality and the law of iterated expectations rule out that $\bar{F}_\ell(p) < 1$ for any $p > v_\ell$. Thus, $\bar{F}_\ell(p)$ has mass 1 atom on v_ℓ. Hence, by Lemma 2, $n_\ell^k s^k \to 0$. Now, the boundedness of $g_h(x)/g_\ell(x)$ implies that, for large k, a sample of n_ℓ^k bidders (or a bounded multiple of it) in state h would yield a winning bid close to v_ℓ as well[33], while total solicitation costs are close to zero. Thus, the seller's payoff in state h is at least v_ℓ. This contradicts the assumption that $C_h < v_\ell$.

Case 3. (This is the remaining case.) Suppose $\lim n_h^k = \infty$ and $\lim n_\ell^k = m < \infty$ for some $m \geq 1$. Since $\lim n_h^k / n_\ell^k = \infty$, a solicited bidder is certain that the state is h, and, hence, $\lim U^k(p + \varepsilon | x, \text{sol}) \geq \bar{F}_h(p)(v_h - p - \varepsilon)$ for all x, p, and $\varepsilon > 0$. Since feasibility requires $\lim U^k(\beta^k(x) | x, \text{sol}) \to 0$, it follows that $\bar{F}_h = 0$ for all $p < v_h$. So, $\bar{F}_h$ puts mass 1 on v_h. Hence, by Lemma 2, $n_h^k s^k \to 0$.

As before, the boundedness of $g_h(x)/g_\ell(x)$ implies that a sample of n_h^k bidders in sate ℓ would yield a winning bid is close to v_h at nearly zero total solicitation cost. Thus, the seller's payoff is at least v_h in both states. This means that, in the limit, the sum of the bidders' payoffs is strictly negative. Therefore, for k large enough, some bidder must have strictly negative expected payoff. This contradicts individual rationality, ruling out this case.

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[32] Recall that we cannot assume β^k to be non-decreasing.

[33] This argument is so worded to apply to case in which β might not be monotone.

Thus, in all cases that are not ruled out, $\bar{F}_\omega$ has mass 1 on the same C in both states. By individual rationality and the law of iterated expectations, $C \leq \mathbb{E}(v)$. This proves Proposition 5.

C.3 Proof of Proposition 6: Existence of a Partially Revealing Equilibrium

Outline. The proof introduces "constrained equilibria" $\left(\beta^k, \boldsymbol{\eta}^k\right)$ for $\left(s^k, \Delta^k\right) \rightarrow (0,0)$ in which β^k is a bidding equilibrium given $\boldsymbol{\eta}^k$, as before, but $\boldsymbol{\eta}^k$ are required to satisfy additional constraints: (i) there is a lower bound sequence $\underline{n}_\omega^k \rightarrow \infty$ such that $n_\omega^k \geq \underline{n}_\omega^k$; and (ii) $\frac{n_h^k}{n_\ell^k} \geq \bar{r}$ for some $\frac{1}{\bar{g}} < \bar{r} < r^*\left(\rho, \overline{g}\right)$. By Theorem 1", any such a sequence of constrained equilibria $\left(\beta^k, \boldsymbol{\eta}^k\right)_{k=1}^\infty$ must be of the partially revealing variety. The proof then uses two Lemmas. The first establishes that, if a sequence of constrained equilibria $\left(\beta^k, \boldsymbol{\eta}^k\right)_{k=1}^\infty$ exists, then the constraints do not bind for large enough k, so that $\left(\beta^k, \boldsymbol{\eta}^k\right)$ is also an unconstrained equilibrium. It then follows from Part 1 of Theorem 2 that $\lim \frac{n_h^k}{n_\ell^k} = r^*$ and the winning bid distribution converges to $\Phi_\omega\left(\cdot | r^*\right)$. The second lemma establishes (using fairly standard arguments) that, for any sequence $\left(s^k, \Delta^k\right) \rightarrow (0,0)$ with $\Delta^k > 0$ for all k, a constrained equilibrium exists. Thus, together these lemmas imply the existence of a sequence of equilibria such that $\bar{F}_\omega = \Phi_\omega\left(\cdot | r^*\right)$. Note that the grid, $\Delta^k > 0$, is only needed in the second lemma to establish the existence of a constrained equilibrium.

One key element is that it is possible to choose $\bar{r} \in \left(\frac{1}{\bar{g}}, r^*\right)$ such that the constraints do not bind for k large enough. To see that this is so, consider $\bar{r}$ that is very close to $\frac{1}{\bar{g}}$ and let us argue why, over a sequence of constrained equilibria, $\lim \frac{n_h^k}{n_\ell^k} > \bar{r}$. Suppose to the contrary that that the restriction $\frac{n_h^k}{n_\ell^k} \geq \bar{r}$ is binding so that $\lim \frac{n_h^k}{n_\ell^k} = \bar{r}$. Then, since in this case, $\overline{g}\bar{r}$ is near 1, by Lemma 5, the winning bid distributions in the two states would be very similar. Therefore, by Lemma 2 the optimal solicitation would also be very similar, implying $\frac{n_h^k}{n_\ell^k}$ close to 1, in contradiction to $\frac{n_h^k}{n_\ell^k}$ being close to $\frac{1}{\bar{g}}$.

Constrained Equilibrium. Given numbers $\left(\underline{n}_\ell, \underline{r}\right) \geq (0,0)$, we define below a *constrained equilibrium* $(\beta, \boldsymbol{\eta})$ of $\Gamma(s, \Delta)$ in which $\boldsymbol{\eta}$ is optimal subject to the constraints $n_\ell \geq \underline{n}_\ell$ and $\frac{n_h}{n_\ell} \geq \underline{r}$.

By Lemma 1 and its proof, for any bidding strategy $\beta : [\underline{x}, \overline{x}] \rightarrow P_\Delta$, $\mathbb{E}\left[p | \omega; \beta, n\right] -$

ns is either strictly decreasing (e.g., if β is constant) or a strictly concave function in n. In either case, $\underset{n\in\{1,..,N\}}{\arg\max}\mathbb{E}\left[p|\omega;\beta,n\right]-ns$ consist of one or (at most) two adjacent integers. Let $m_{\omega}\left(\beta,s\right)$ be the lower one.[34]

Let D denote the set of probability distributions η over $\{1,...,N\}$. Given the bounds $(\underline{n}_{\ell},\underline{r})\geq(0,0)$, let the correspondence Ψ_2 map the bidding strategy β to the set of optimal solicitation strategies in state ℓ subject to the constraint that at least $\underline{n}_{\ell}$ bidders are solicited:

$$\Psi_2(\beta,s;\underline{n}_{\ell})\triangleq\begin{cases}\{\eta\in D: Support(\eta)\subseteq\underset{n'\in\{1,..,N\}}{\arg\max}\mathbb{E}\left[p|\ell;\beta,n'\right]-n's\} & \text{if} \quad m_{\ell}\left(\beta,s\right)\geq\underline{n}_{\ell},\\ \{\eta\in D\mid\eta\left(\underline{n}_{\ell}\right)=1\} & \text{o/w}\end{cases}$$

The second line of Ψ_2 deals with the case in which the lower bound $\underline{n}_{\ell}$ binds. Since $\mathbb{E}\left[p|\omega;\beta,n\right]-ns$ is either strictly decreasing in n or strictly concave, if the constraint $n\geq\underline{n}_{\ell}$ binds, then the optimal constrained solicitation is exactly $\underline{n}_{\ell}$.

Similarly, let $\underline{n}_h\left(\beta,s\right)=\underline{r}\max\left\{m_{\ell}\left(\beta,s\right),\underline{n}_{\ell}\right\}$ and define the correspondence $\Psi_3(\beta,s;\underline{n}_{\ell},\underline{r})$

$$\Psi_3(\beta,s;\underline{n}_{\ell},\underline{r})\triangleq\begin{cases}\{\eta\in D: Support(\eta)\subseteq\underset{n'\in\{1,..,N\}}{\arg\max}\mathbb{E}\left[p|h;\beta,n'\right]-n's\} & \text{if} \quad m_h\left(\beta,s\right)\geq\underline{n}_h\left(\beta,s\right),\\ \{\eta\in D\mid\eta\left(\lceil\underline{n}_h\rceil\right)=1\} & \text{o/w}\end{cases}$$

Now, $(\beta,\boldsymbol{\eta})$ is a *constrained equilibrium* if (i) β is a bidding equilibrium of $\Gamma_0\left(N,\boldsymbol{\eta},\Delta\right)$ and (ii) $\boldsymbol{\eta}=(\eta_{\ell},\eta_h)$ is the seller's optimal constrained solicitation, i.e., $\eta_{\ell}\in\Psi_2(\beta,s;\underline{n}_{\ell})$ and $\eta_h\in\Psi_3(\beta,s;\underline{n}_{\ell},\underline{r})$.

A constrained equilibrium $(\beta,\boldsymbol{\eta})$ is also an equilibrium of $\Gamma\left(s,\Delta\right)$ if $m_{\ell}\left(\beta,s\right)\geq\underline{n}_{\ell}$ and $m_h\left(\beta,s\right)\geq\underline{r}m_{\ell}\left(\beta,s\right)$.

Lemma 15 *If $\Delta>0$, $\Gamma\left(s,\Delta\right)$ has a constrained equilibrium.*

As before in the proof of Theorem 1, a constrained equilibrium exists by standard arguments given the single crossing properties established in Lemma 10, following Athey (2001). We state the proof for completeness.

Proof of Lemma 15: Recall that $P_{\Delta}=[0,v_{\ell})\cup\{v_{\ell},v_{\ell}+\Delta,v_{\ell}+2\Delta,...,v_h-\Delta,v_h\}$. As before, using Athey's idea, let $m=\parallel\{v_{\ell},v_{\ell}+\Delta,...,v_h-\Delta,v_h\}\parallel$ and let Σ_{Δ} be

[34] We use $m_{\omega}\left(\beta,s\right)$ rather than n_{ω} which was introduced earlier to denote the lower of the integers in the support of a mixed strategy η_{ω}, since, in the constrained environment considered below, these could be different numbers.

a set of vectors of dimension $m+1$ whose coordinates belong to $[\underline{x}, \overline{x}]$,

$$\Sigma_{\Delta} = \{\sigma = (\sigma_0, \sigma_1, ..., \sigma_m) \in [\underline{x}, \bar{x}]^{m+1} \mid \underline{x} \triangleq \sigma_0 \leq \sigma_1 \leq ... \leq \sigma_m \triangleq \bar{x}\},$$

where σ determines a bidding strategy β_σ by $\beta_\sigma(x) = v_\ell + i\Delta$ if $x \in [\sigma_i, \sigma_{i+1})$, $i = 0, ..., m-1$.

Define the correspondence $\Psi = \Psi_1(\sigma', \boldsymbol{\eta}') \times \hat{\Psi}_2(\sigma', \boldsymbol{\eta}') \times \hat{\Psi}_3(\sigma', \boldsymbol{\eta}')$ from $\Sigma_\Delta \times D \times D$ into itself: For any $\sigma' \in \Sigma_\Delta$ and $\boldsymbol{\eta}' \in D \times D$,

$$\begin{aligned}
\Psi_1(\sigma', \boldsymbol{\eta}') &= \{\sigma \in \Sigma_\Delta \mid \beta_\sigma \in \arg\max U(b|x, \text{sol}; \beta_{\sigma'}, \boldsymbol{\eta}')\}, \\
\hat{\Psi}_2(\sigma', \boldsymbol{\eta}') &= \Psi_2(\beta_{\sigma'}, s; \underline{n}_\ell), \\
\hat{\Psi}_3(\sigma', \boldsymbol{\eta}') &= \Psi_3(\beta_{\sigma'}, s; \underline{n}_\ell, \underline{r}).
\end{aligned}$$

Both Σ_Δ and D are closed convex sets. $\hat{\Psi}_2$ and $\hat{\Psi}_3$ are non-empty, convex-valued, and continuous by virtue of being the set of constrained maximizers of a concave problem on a finite convex set (see Lemma 1). Ψ_1 is non-empty, convex-valued, and upper hemi-continuous by the same arguments as in Athey (2001): For given σ', $\beta_{\sigma'}(\underline{x}) \geq v_\ell$, so that by the arguments from Lemma 11, the unconstrained best response has support only on bids $\geq v_\ell$. Moreover, by the single crossing properties of Lemma 10, the unconstrained best response is without loss of generality weakly monotone. Therefore, there exists some $\sigma \in \Sigma_\Delta$ such that β_σ is a best response to σ'. Convex valuedness also follows from Lemma 10. The upper hemi-continuity follows from the theorem of the maximum, given that $U(\cdot|x,\text{sol};\beta_{\sigma'}, \boldsymbol{\eta}')$ is continuous in $\boldsymbol{\eta}'$ and in σ'. It follows that Ψ is convex valued and upper hemi-continuous. By Kakutani's Theorem, Ψ has a fixed point. ∎

Next, we show that, for certain choice of the bounds $(\underline{n}_\ell, \underline{r})$, all the constrained equilibria are, in fact, unconstrained, for sufficiently large k.

Lemma 16 *Consider a sequence of games $\Gamma^B(s^k, \Delta^k)$ and parameters $(\underline{n}_\ell^k, \underline{r})$ such that $(s^k, \Delta^k) \to (0,0)$, $\underline{n}_\ell^k = \frac{1}{\sqrt{s^k}}$ and $\underline{r} \in (\frac{1}{\overline{g}}, r^*(\rho, \overline{g}))$. For any sequence of constrained equilibria $(\beta^k, \boldsymbol{\eta}^k)$ of $\Gamma^B(s, \Delta^k)$ given $(\underline{n}_\ell^k, \underline{r})$: $n_\ell^k > \underline{n}_\ell^k$ for large k, $\lim \frac{n_h^k}{n_\ell^k} = r^*(\rho, \overline{g})$ and $\bar{F}_\omega = \Phi_\omega(\cdot|r^*)$, with Φ_ω defined by (5).*

Proof: Given the sequence of constrained equilibria, let $r = \lim_{k\to\infty} \frac{n_h^k}{n_\ell^k}$, with n_ω^k the lower of the integers in the support of η_ω^k, as before.

Step 1. $\bar{F}_\omega(p) = \Phi_\omega(p|r)$.
Proof of Step 1: The choice of $\underline{n}_\ell^k$ and $\underline{r}$ implies $\min\left\{n_\ell^k, n_h^k\right\} \to \infty$ and

$$\lim\left(n_h^k/n_\ell^k\right) = r \geq \underline{r}. \tag{64}$$

Given (64), Proposition 3 implies that $\bar{F}_\omega(p) = \Phi_\omega(p|r)$, for all p and $\omega = \ell, h$.
□

Step 2. For k sufficiently large, $n_\ell^k > \underline{n}_\ell^k$ and $n_h^k > \underline{r} n_\ell^k$
Proof of Step 2: By (64) and the choice of r, $\lim_{k\to\infty} \frac{n_h^k}{n_\ell^k} > \frac{1}{g}$. The argument from the proof of Proposition 4) implies that $\lim_{k\to\infty} \frac{n_h^k}{n_\ell^k} < \infty$. Hence, Φ_ℓ is not degenerate. As before, let $m_\omega^k = m_\omega\left(\beta^k, s^k\right)$ denote the lowest optimal n for type ω given β^k and s^k. By Lemma 2, m_ℓ^k satisfies

$$\lim_{k\to\infty} m_\ell^k s^k = -\int_0^{v_h} \Phi_\ell(p|r) \ln\left(\Phi_\ell(p|r)\right) dp > 0. \tag{65}$$

Since $\underline{n}_\ell^k s^k = \sqrt{s^k} \to 0$, $\lim m_\ell^k s^k > 0$ implies $\lim \frac{\underline{n}_\ell^k}{m_\ell^k} = 0$, so that $m_\ell^k > \underline{n}_\ell^k$ for sufficiently large k. Thus, $n_\ell^k > \underline{n}_\ell^k$, as claimed.

Suppose to the contrary that $m_h^k \leq \underline{r} m_\ell^k$. Then, the strict concavity of the seller's optimization implies $n_h^k = \underline{r} m_\ell^k$ (ignoring integer constraints). By $\frac{1}{g} < \underline{r} < r^*(\rho, \overline{g})$ and Lemma 13, $J(\underline{r}; \rho, \overline{g}) < 0$.

We have observed in the proof of Lemma 2 that (for any sequence n_h^k and β^k—not only for optimal n_h^k)

$$\lim_{k\to\infty} n_h^k \left(\mathbb{E}\left[p|h; \beta^k, n_h^k + 1\right] - \mathbb{E}\left[p|h; \beta^k, n_h^k\right]\right) = -\int_0^{v_h} \Phi_h(p|r) \ln\left(\Phi_h(p|r)\right) dp.$$

This, (65) and $n_h^k = \underline{r} m_\ell^k$ imply

$$\begin{aligned}
&\lim_{k\to\infty} n_h^k \left(\mathbb{E}\left[p|h; \beta^k, n_h^k + 1\right] - \mathbb{E}\left[p|h; \beta^k, n_h^k\right]\right) - \lim_{k\to\infty} n_h^k s^k \\
= \; & -\int_0^{v_h} \Phi_h(p|\underline{r}) \ln\left(\Phi_h(p|\underline{r})\right) dp + \underline{r} \int_0^{v_h} \Phi_\ell(p|\underline{r}) \ln\left(\Phi_\ell(p|\underline{r})\right) dp.
\end{aligned}$$

From Equation (63) in the proof of Lemma 3 the sign of the last expression is the same as $sign\left(-J(\underline{r})\right)$. Thus, $J(\underline{r}) < 0$ implies

$$\lim_{k\to\infty} n_h^k \left(\mathbb{E}\left[p|h; \beta^k, n_h^k + 1\right] - \mathbb{E}\left[p|h; \beta^k, n_h^k\right]\right) > \lim_{k\to\infty} n_h^k s^k.$$

Hence, for sufficiently large k,

$$\mathbb{E}\left[p|h;\beta^k,n_h^k+1\right]-\mathbb{E}\left[p|h;\beta^k,n_h^k\right]>s^k.$$

That is, at n_h^k sampling an additional bidder is strictly profitable for type h. Therefore, $m_h^k>\underline{r}m_\ell^k$, as claimed. $\square$

Step 3. $\lim n_h^k/n_\ell^k=\lim n_h^k/n_\ell^k=r^*\left(\rho,\overline{g}\right)$.

Proof of Step 3: By Step 2, n_h^k and n_ℓ^k are both unconstrained optimal given β^k. Hence, $(\beta^k,\boldsymbol{\eta}^k)$ is an (unconstrained) equilibrium for k large enough. Therefore, Lemma 3 implies that $\lim\frac{n_h^k}{n_\ell^k}=r^*\left(\rho,\overline{g}\right)$. $\square$

Steps 1 and 3 together establish the lemma. ∎

Lemma 15 implies that $\Gamma\left(s^k,\Delta^k\right)$ has a constrained equilibrium whenever $\Delta^k>0$. Lemma 16 implies that, for suitably chosen $\left(\underline{n}_\ell^k,\underline{r}\right)$ and for sufficiently small s_k and $\Delta_k\geq 0$, all constrained equilibria $(\beta^k,\boldsymbol{\eta}^k)$ are also (unconstrained) equilibria of $\Gamma\left(s^k,\Delta^k\right)$ and over any such sequence $\bar{F}_\omega=\Phi_\omega\left(\cdot|r\right)$. Therefore, there exists a sequence of equilibria $(\beta^k,\boldsymbol{\eta}^k)$ for $\Gamma\left(s^k,\Delta^k\right)$ such that $\bar{F}_\omega=\Phi_\omega\left(\cdot|r\right)$. ∎

Comment: The proof of Lemma 16 does not require $\Delta>0$. Thus, if without the grid ($\Delta=0$), there exists a sequence of constrained equilibria, then there also exists a sequence of equilibria that converges to the partially revealing outcome of Theorem 2.