Stock Price Cycles and Business Cycles

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Abstract

We present a simple model that quantitatively replicates the behavior of stock prices *and* business cycles in the United States. The business cycle model is standard, except that it features extrapolative belief formation in the stock market, in line with the available survey evidence. Extrapolation amplifies the price effects of technology shocks and - in response to a series of positive technology surprises - gives rise to a large and persistent boom and bust cycle in stock prices. Boom-bust dynamics are more likely when the risk-free interest rate is low because low rates strengthen belief-based amplification. Stock price cycles transmit into the real economy by generating inefficient price signals for the desirability of new investment. The model thus features a 'financial accelerator', despite the absence of financial frictions. The financial accelerator causes the economy to experience persistent periods of over- and under-accumulation of capital.

JEL Class.No.: E32, E44, G12

1   Introduction

We combine a standard real business cycle model with extrapolative belief formation in the stock market and show that the resulting model quantitatively replicates key data

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moments capturing the behavior of business cycles and stock prices in the United States. The simplicity of the setup is remarkable in light of the long quest to develop modeling frameworks that can simultaneously replicate the behavior of business cycles and stock prices.

The main quantitative challenge consists in matching the huge volatility differences between the real and the financial side of the economy: while the business cycle is relatively smooth, especially when considering private consumption, stock prices are very volatile and display cycles that are orders of magnitude larger than the business cycle. We show how this quantitative tension can be resolved using extrapolative stock price beliefs that are quantitatively plausible in light of the available survey evidence.

Our model predicts that periods in which productivity grows persistently above average, as for instance during the 1990's in the United States, trigger persistent booms in stock prices, investment and hours worked. Following the reversal of such booms, the economy may enter a bust period and persistently undershoot its long-run growth trend. And once the economy recovers from the bust, there is an increased likelihood of another boom-bust cycle taking place, such that boom-bust cycles tend to come in clusters. Finally, the model predicts that periods with low risk-free interest rates, as recently experienced in a number of advanced economies[1], are periods in which boom-bust cycles emerge with higher likelihood.

Figure 1 illustrates that the U.S. economy recently experienced a series of sizable stock price boom and bust cycles. Over the thirty year period depicted in the figure, the S&P500 index features three large price run-ups and - thus far - two large price reversals. Both reversals coincided with economic recessions, with stock prices dropping each time by almost 50% from their prior peak value.[2] Similar medium-term price run-ups and reversals can be observed in the stock markets of other advanced economies.

Figure 2 presents an alternative approach for quantitatively capturing U.S. stock price cycles. It scales stock prices by dividends and depicts the empirical distribution of the quarterly price-dividend (PD) ratio of the S&P500.[3] The empirical PD distribution has a lot of mass around values between 100 and 150, but displays a long right tail that covers quarterly PD values above 300. The positive skewness and the large support of the empirical PD distribution capture the presence of occasional stock price run-ups and reversals, i.e., periods in which prices grow considerably faster (or slower) than dividends.

Key to the empirical success of our model and to the ability to generate stock price cycles of the kind present in the data is a departure from the rational expectations hypothesis (REH) in the stock market. This departure is motivated by data on investor surveys, which clearly show an extrapolative pattern in investors' expectations about future

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[1]See Holsten, Laubach and Williams (2017).

[2]Figure 1 depicts the nominal value of the S&P500. Similar conclusions emerge if one deflates the nominal value by the consumer price index.

[3]The quarterly PD ratio is defined as the end-of-quarter price over a deseasonalized measure of quarterly dividend payouts, see Appendix A.1 for details.
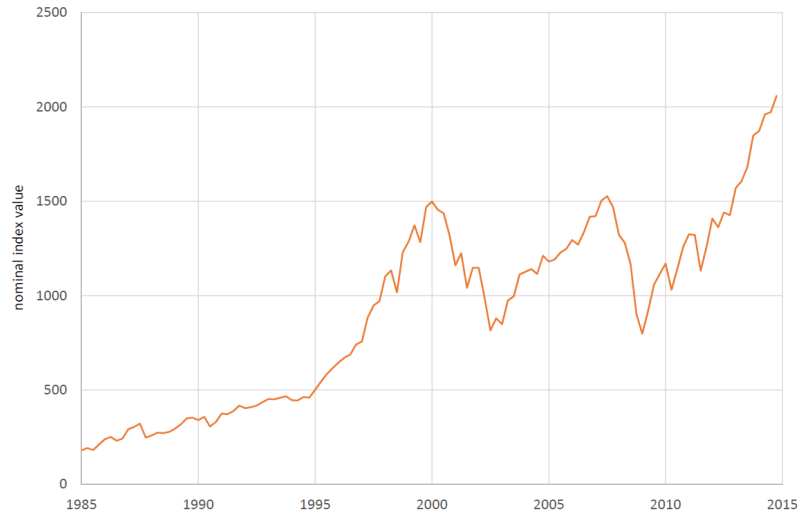
Figure 1. Price cycles in the S&P 500 (Q1:1985-Q4:2014)
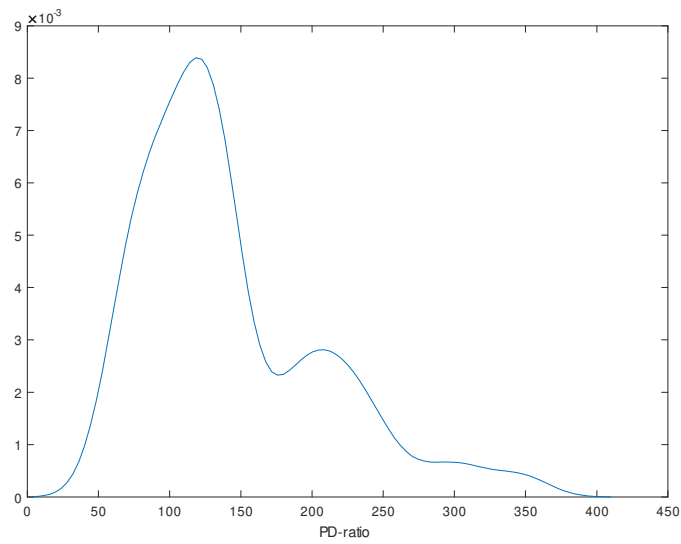
Figure 2. Postwar distribution of the quarterly PD ratio of the S&P 500 (kernel density estimate, Q1:1955-Q4:2014)

capital gains: subjectively expected capital gains (or returns) are higher following high realized capital gains (or returns) in the stock market and in times of high price-dividend ratios (Vissing-Jorgensen (2003), Bacchetta, Mertens, and Wincoop (2009), Malmendier and Nagel (2011) and Greenwood and Shleifer (2014)). This expectations pattern can be parsimoniously captured by modeling investors as subjective Bayesians who filter the long-term trend component of capital gains from observed capital gains.[4]

Subjective components in stock price expectations give rise to speculative mispricing of stocks, compared to a setting in which investors hold rational stock price expectations. This mispricing of stocks has real consequences because it affects agents' optimal choices for investment, consumption and hours worked.

Consider, for instance, a situation in which a sequence of positive technology shocks triggers a sequence of positive capital gain surprises. Since investors filter from observed capital gains, these capital gain surprises cause them to become unduly optimistic about the long-run component of capital gains, in line with what the survey data suggests. Optimistic capital gain expectations push up prices even higher, thereby generate additional capital gains. The mutual reinforcement between capital gain expectations and realized capital gains sets in motion a belief-driven stock price boom. Increasing stock prices signal to capital producers that new investment is increasingly profitable and cause them to optimally expand investment.[5] The resulting positive association between asset price increases and investment increases is reminiscent of the investment booms associated with the U.S. tech stock boom in the late 1990s or the U.S. housing boom at the beginning of the new millennium.[6]

Eventually, the economy experiences a 'Minsky moment' in which belief-driven stock price booms come to an end, then reverse and ultimately lead to a stock price crash in which stock prices, investment and hours worked fall. Key to generating a stock price reversal is the fact that the ever increasing capital gain expectations during the boom phase become eventually too optimistic relative to capital gain outcomes. This happens either because the capital stock expands sufficiently rapidly or because the *increase* in optimism is too weak to generate the high capital gains that investors expect. At this point, outcomes fall short of agents optimistic expectations, which – absent further shocks – sets in motion a stock price reversal.

We show that the reversal can display sufficient momentum to cause the economy

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[4]Beside deviations from rational stock price expectations, which are quantitatively disciplined by survey evidence, agents in our model are otherwise standard: (1) they hold rational expectations about all other decision-relevant variables, and (2) they make state-contingent plans to maximize utility given their constraints and the beliefs they entertain about variables beyond their control, i.e., agents are 'internally rational' in the sense of Adam and Marcet (2011).

[5]As should be clear, it is important for this argument that the supply of new capital is not fully elastic, e.g., due to decreasing returns to scale in the production or due to adjustment frictions.

[6]See Adam and Woodford (2018) for a model in which speculative mispricing in housing markets distorts the supply of new houses.

to persistently fall below its balanced growth path. The economy then transits from a situation with an excessively large capital stock during the boom to one where the capital stock drops persistently below its balanced growth path value, with hours worked and investment also persistently depressed. A belief-driven boom may thus contain the seeds of a future recession.

The interaction between belief-driven speculative mispricing of stocks and investment activity gives rise to a new form of financial accelerator that amplifies and propagates shocks to the real economy and that results in a misallocation of resources from the viewpoint of a social planner with fully rational expectations. Yet, unlike in standard models of the financial accelerator, e.g., Bernanke, Gertler, and Gilchrist (1999) or Kiyotaki and Moore (1997), the misallocation due to asset price fluctuations does not rely on the presence of financial frictions. Furthermore, while models featuring financial frictions typically give rise to under-investment only, the belief-based financial accelerator proposed in the present paper can generate both under- and over-investment.

While our model gives rise to rich business cycle and stock price dynamics, it is sufficiently simple to allow for the analytic derivation of a range of results that explain the mechanisms through which the model achieves its good quantitative performance. We can also prove analytically that there is a globally unique equilibrium with extrapolative stock price beliefs and rational expectations about the remaining variables. Finally, the model is sufficiently simple to permit estimation of the fully nonlinear model using the Simulated Method of Moments (SMM). Relying on the non-linear model turns out to be important for obtaining the correct moment implications in the presence of large stock price fluctuations.

The estimated model reveals that the empirical improvements associated with a departure from the assumption of rational stock price expectations are large and economically significant, both along the business cycle dimension and – even more importantly – along the stock price dimension. Under fully rational expectations, the model spectacularly fails along the stock price dimension and produces only tiny amounts of stock price volatility. It also fails in fully replicating business cycle moments, as it produces too little volatility for investment and hours worked, unless one augments the model by investment-specific productivity shocks. In a sense, investment-specific shocks substitute for the financial accelerator effects that would be operating in the presence of plausibly sized stock price fluctuations.

Under subjective stock price beliefs, the estimated model performs quantitatively well in terms of matching business cycle and stock price moments and does not need to incorporate investment-specific technology shocks. It features reasonably-sized shocks to total factor productivity and generates a stable risk-free interest rate. The most notable dimension along which the model falls short of fully matching the data is the equity premium: the model-implied (unlevered) equity premium reaches only about one third of the empirically observed premium.

The paper is structured as follows. Section 2 reviews the related literature and sec-

tion 3 summarizes key facts about U.S. business cycles, stock prices and their interaction. Section 4 describes the real business cycle model and section 5 introduces subjective price beliefs. Section 6 summarizes the equilibrium conditions and explains how the model gives rise to a financial accelerator. Section 7 assesses the quantitative performance of the estimated model under subjective and rational stock price beliefs. In section 8 we provide analytical insights into the mechanisms of belief-driven boom-bust dynamics. The effects of sustained technology surprises, of low real interest rate and the phenomenon of repeat cycles are all discussed in section 9. Section 10 presents additional model implications and section 11 briefly discusses welfare issues. A conclusion summarizes and discusses the outlook for future work.

2   Related Literature

Most explanations for business cycles and stock price behavior in the existing literature maintain the rational expectations (RE) hypothesis. In contrast to the approach advanced in the present paper, such explanations are inconsistent with the patterns of capital gain (or return) expectations, as available from surveys of stock market participants.[7]

The existing set of RE explanations can be broadly classified into two approaches.[8] The first combines preferences featuring a low elasticity of intertemporal substitution (EIS) with adjustment frictions. The low elasticity of intertemporal substitution is typically generated via habit preferences (Jermann (1998); Boldrin, Christiano, and Fisher (2001); Uhlig (2007); Jaccard (2014)), but sometimes via recursive preferences featuring a low EIS parameter (Guvenen (2009)). A low EIS creates a strong desire for intertemporal consumption smoothing, while adjustment frictions prevent such smoothing from fully taking place. For agents to be willing to accept the observed moderate consumption fluctuations, asset prices then need to adjust strongly in equilibrium. Since labor supply adjustments represent one possible adjustment margin in business cycle models, flexible adjustments in the number of hours worked need to be prevented.[9] EIS-based explanations therefore include some form of labor market frictions (adjustment frictions, inflexible labor supply through preferences, real wage frictions). In fact, labor market frictions become central for explaining stock price behavior. To maximally distinguish our setup from this strand of the literature, we consider a frictionless labor market and households with infinite Frisch elasticity of labor supply.

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[7]Adam, Marcet, and Beutel (2017) show that the patterns of expected returns/capital gains from survey data are inconsistent with the RE. Adam, Matveev, and Nagel (2018) show that the failure of the RE hypothesies cannot be explained away by postulating that survey respondents report risk-adjusted expectations.

[8]See appendix A.2 for a more detailed discussion of the rational expectations literature.

[9]Otherwise the high desire to smooth intertemporal consumption fluctuations leads to a strong adjustment in hours worked and a very smooth consumption profile, which in turn would largely eliminate asset price fluctuations.

A second explanation reconciling smooth business cycles with volatile stock prices relies on specifying recursive preferences in conjunction with an additional source of exogenous uncertainty, either long-run growth risk (Croce (2014)) or disaster risk (Gourio (2012)). Such shocks have large pricing implications under recursive preferences, provided the coefficient of risk aversion is larger than the inverse of the EIS. The presence of such shocks then generates a large equity premium in the presence of realistic consumption dynamics, while time variation in the equity premium leads to substantial volatility in stock prices and returns.[10] To maximally distinguish our setup from this part of the literature, we consider time-separable consumption preferences and standard productivity shocks as the only exogenous source of random variation.

The present paper is also related to a growing literature that introduces subjective belief components into business cycle models. Eusepi and Preston (2011) study a setting where agents are learning about the behavior of wages and rental rates and show how this can improve business cycle performance. Angeletos, Collard, and Dellas (2018) introduce confidence shocks in the form of autonomous movements in higher-order expectations. Bhandari, Borovička, and Ho (2017) consider households with time-varying ambiguity aversion. None of these papers considers stock price implications.

The paper is also related to Barberis, Greenwood, Jin, and Shleifer (2014), who study the stock price effects of subjective dividend beliefs, and to Adam, Marcet, and Nicolini (2016) and Adam, Marcet, and Beutel (2017), who study the stock price effects of subjective price beliefs. While these papers consider endowment economies, the present paper features endogenous consumption, labor and investment choices, as well as capital accumulation over time. It thus allows for meaningful interactions between the real and the financial side of the economy.

Hirshleifer, Li, and Yu (2015) consider a production economy with recursive preferences in which agents over-extrapolate recent productivity observations. Extrapolation endogenously generates long-run variations in perceived technology growth and thus consumption growth, which allows the model to generate a sizable equity premium and about half of the observed volatility of stock returns. Since hours worked are assumed to be constant, the model falls short of fully capturing business cycle dynamics.

Bordalo, Gennaioli, and Shleifer (2018) present a simple production model in which diagnostic expectations of investors give rise to extrapolative forecasting behavior. They show how this allows to qualitatively replicate important features of credit cycles and credit spreads, but do not explore the model's quantitative predictions with regard to the business cycle or stock prices. Winkler (2018) considers a rich DSGE model featuring

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[10]Labor market frictions or the elasticity of labor supply are usually not discussed in this strand of literature. While they seem less central than for the low-EIS-based explanations, flexible labor supply in a frictionless labor market is still a powerful tool for insuring against consumption fluctuations. The fact that the asset pricing models in this second strand of literature tend to generate too little volatility of hours worked suggests that the chosen preference specifications may make labor supply insufficiently flexible.

borrowing frictions, price and wage rigidities, limited stock market participation and a rich set of deviations from the rational expectations hypothesis. We work with a simple real business cycle model and limit RE deviations to stock price expectations. This allows showing that subjective stock price expectations alone are sufficient to reconcile real business cycle models with the empirically observed behavior of stock prices.[11]

3   Stock Prices and Business Cycles: Key Facts

This section presents key data moments characterizing U.S. business cycles and stock price behavior. We consider quarterly U.S. data for the period Q1:1955-Q4:2014. The start date of the sample is determined by the availability of the aggregate hours worked series. Details of the data sources are reported in Appendix A.1.

Table 1 presents a standard set of business cycle moments for output (Y), consumption (C), investment (I) and hours worked (H).[12] These quantities have been divided by the working age population so as to take into account demographic changes in the U.S. population over the sample period. The second to last column in Table 1 reports the data moment and the last column the standard deviation of the estimated moment. We will use the latter in our simulated methods of moments estimation and for computing t-statistics.[13]

The picture that emerges from Table 1 is a familiar one: output fluctuations are relatively small, consumption is considerably less volatile than output, while investment is considerably more volatile; hours worked are roughly as volatile as output. Consumption, investment and hours all correlate strongly with output. A major quantitative challenge will be to simultaneously replicate the relative smoothness of the business cycle with the much larger fluctuations in stock prices to which we turn next.

Table 2 presents a standard set of moments characterizing U.S. stock price behavior. The first three moments summarize the behavior of the PD ratio:[14] the average PD ratio is large and implies a dividend yield of just 0.66% per quarter. The PD ratio is also very volatile: the standard deviation of the PD ratio is more than 40% of its mean value and fluctuations in the PD ratio are very persistent, as documented by the high quarterly

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[11] Our simpler setup can be solved fully nonlinearly, which we find to be important for evaluating the quantitative potential of subjective stock price expectations.

[12] As is standard in the business cycle literature, we compute business cycle moments based on logged and then HP-filtered variables (smoothing parameter of 1600). All other data moments will rely on unfiltered (level) data. We HP filter model variables when comparing to filtered moments in the data and use unfiltered model moments otherwise.

[13] The reported standard deviations are computed by combining Newey-West estimators with the delta method.

[14] The PD ratio is defined as the end-of-quarter stock price divided by dividend payments over the quarter. Following standard practice, dividends are deseasonalized by averaging dividends over the last four quarters.

Table 1

Business cycle moments

U.S., quarterly real values, Q1:1955-Q4:2014

	Symbol	Data moment	Std. dev. data moment
Std. dev. of output	$\sigma(Y)$	1.72	0.25
Relative std. dev. of consumption	$\sigma(C)/\sigma(Y)$	0.61	0.03
Relative std. dev. of investment	$\sigma(I)/\sigma(Y)$	2.90	0.35
Relative std. dev. of hours worked	$\sigma(H)/\sigma(Y)$	1.08	0.13
Correlation output and consumption	$\rho(Y,C)$	0.88	0.02
Correlation output and investment	$\rho(Y,I)$	0.86	0.03
Correlation output and hours worked	$\rho(Y,H)$	0.75	0.03

Table 2

Key moments of stock prices, risk-free rates and dividends,

U.S., quarterly real values, Q1:1955-Q4:2014

	Symbol	Data moment	Std. dev. data moment
Average PD ratio	$E[P/D]$	152.3	25.3
Std. dev. PD ratio	$\sigma(P/D)$	63.39	12.39
Auto-correlation PD ratio	$\rho(P/D)$	0.98	0.003
Average equity return (%)	$E[r^e]$	1.87	0.45
Std. dev. equity return (%)	$\sigma(r^e)$	7.98	0.35
Average risk-free rate (%)	$E[r^f]$	0.25	0.13
Std. dev. risk-free rate (%)	$\sigma(r^f)$	0.82	0.12
Std. dev. dividend growth (%)	$\sigma(D_{t+1}/D_t)$	1.75	0.38

auto-correlation of the PD ratio. Table 2 also reports the average real stock return, which is high and equals almost 2% per quarter. Stock returns are also very volatile: the standard deviation of stock returns is about four times its mean value. This contrasts with the behavior of the short-term risk-free interest rate documented in table 2. The risk-free interest rate is very low and very stable. The standard deviation of the risk-free interest rate in table 2 is likely even overstated, as we used ex-post realized inflation rates to transform nominal safe rates into a real rate. Table 2 also reports the standard deviation of dividend growth. Dividend growth is relatively smooth, especially when compared to the much larger fluctuation in equity returns. This fact is hard to reconcile with the large observed fluctuations in stock prices (Shiller, 1981).

Table 3 presents data moments that link the PD ratio to business cycle variables. It shows that stock prices are pro-cyclical: (1) the PD ratio correlates positively with hours

Table 3

Comovement of stock prices with real variables and survey expectations

U.S., quarterly real values, Q1:1955-Q4:2014, survey data: Q2:1998-Q4:2007

Correlations	Symbol	Data moment	Std. dev. data moment
Hours & PD ratio	$\rho(H, P/D)$	0.51	0.17
Investment-output & PD ratio	$\rho(I/Y, P/D)$	0.19	0.31
Survey expect. & PD ratio	$\rho(E^{\mathcal{P}}[r^e], P/D)$	0.79	0.07

worked; (2) stock prices also correlate positively with the investment to output ratio, but the correlation is surprisingly weak and also estimated very imprecisely. Table 4 below shows why this is the case: the investment to output ratio correlates positively with the PD ratio over the second half of the sample period (1985-2014), i.e., in the period with large stock price cycles, but negatively in the first half of the sample period (1955-1984).[15] Table 4 also shows that the overall investment to output ratio correlates much more strongly with the PD ratio if one excludes residential investment and investment in non-residential structures. Since our model does not feature real estate investment, we shall use the sample correlation for this reduced investment concept, i.e., 0.58, to evaluate the model. The negative correlation in the first half of the sample period, however, appears to be a robust feature of the data.

Table 3 also reports the correlation of the PD ratio with the one-year-ahead expected real stock market return of private U.S. investors. It shows that investors are optimistic about future holding period returns when the PD ratio is high already.

Table 4

Stock prices and investment: alternative measures and sample periods

U.S., quarterly real values, Q1:1955-Q4:2014 and subsamples

$corr(I/Y, P/D)$	1955-2014	1955-1984	1985-2014
Fixed investment	0.19	-0.64	0.40
Fixed investment, less residential inv. and nonresidential structures:	0.58	-0.66	0.77

4 Asset Pricing in a Production Economy

We build our analysis on a stripped-down version of the representative agent model of Boldrin, Christiano, and Fisher (2001). This model features a consumption goods

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[15]The correlation of the hours worked series with the PD ratio is positive in the first and second half of the sample period.

producing sector and an investment goods producing sector. Both sectors produce output using a neoclassical production function with capital and labor as input factors. Output from the investment goods sector can be invested to increase the capital stock.

We deviate from Boldrin, Christiano, and Fisher (2001) by using a standard time-separable specification for consumption preferences instead of postulating consumption habits. In addition, we remove all labor market frictions and make hours worked perfectly flexible. We furthermore simplify the setup by specifying an exogenous capital accumulation process in the investment goods sector, in line with a balanced growth path solution. This helps with analytical tractability of the model, but also insures that the supply of new capital goods is sufficiently inelastic, so that the model has a chance of producing large and persistent swings in stock prices.[16]

4.1   Production Technology

There are two sectors, one producing a perishable consumption good (consumption sector), the other producing an investment good that can be used to increase the capital stock in the consumption sector (investment sector). The representative firm in each sector hires labor and rents capital, so as to produce its respective output good according to standard Cobb-Douglas production functions,

$$Y_{c,t} = K_{c,t}^{\alpha_c} \left(Z_t H_{c,t} \right)^{1-\alpha_c}, \qquad Y_{i,t} = K_{i,t}^{\alpha_i} \left(Z_t H_{i,t} \right)^{1-\alpha_i}, \tag{1}$$

where $K_{c,t}$, $K_{i,t}$ denote capital inputs and $H_{c,t}$, $H_{i,t}$ labor inputs in the consumption and the investment sector, respectively, and $\alpha_c \in (0,1)$ and $\alpha_i \in (0,1)$ the respective capital shares in production. Z_t is an exogenous labor-augmenting level of productivity and the only source of exogenous variation in the model. Productivity follows

$$Z_t = \gamma Z_{t-1} \varepsilon_t, \qquad \ln \varepsilon_t \sim ii\mathcal{N} \left(-\frac{\sigma^2}{2}, \sigma^2 \right), \tag{2}$$

with $\gamma \geq 1$ denoting the mean growth rate of technology and $\sigma > 0$ the standard deviation of log technology growth.

Labor is perfectly flexible across sectors, but capital is sector-specific. The output of investment goods firms increases next period's capital in the consumption goods sector, so that

$$K_{c,t+1} = (1 - \delta_c) K_{c,t} + Y_{i,t}, \tag{3}$$

where $\delta_c \in (0,1)$ denotes the depreciation rate. Capital in the investment goods sector evolves according to

$$K_{i,t+1} = (1 - \delta_i) K_{i,t} + X_t, \tag{4}$$

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[16]Capital prices in Boldrin, Christiano, and Fisher (2001) fail to display large and persistent fluctuations, instead display volatile but negatively autocorrelated returns.

where X_t denotes investment in new capital in the investment goods sector and $\delta_i \in (0,1)$ the capital depreciation rate.

We let new investment in the capital good sector (X_t) be given by an exogenous endowment process. In particular, we assume that $K_{i,t+1} = Z_t$, which insures that the model remains consistent with balanced growth, while capital good production that deviates from the balanced growth path is subject to decreasing returns to scale. The latter feature is key for being able to generate persistent price fluctuations in the price of consumption capital around the balanced growth path. It also considerably simplifies the analytic derivations.

With this simplifying assumption, the production setup in the investment sector is isomorphic to having a decreasing returns to scale formulation of the form

$$Y_{i,t} = Z_t \left(\varepsilon_t \right)^{-\alpha_i} \left(H_{i,t} \right)^{1-\alpha_i} . \tag{5}$$

We prefer working with the constant returns formulation (1), capital depreciation (4) and exogenous investment, as this allows us to define capital values in both sectors in a symmetric fashion.

Appendix A.3 shows that the production setup in the investment sector is also isomorphic to a formulation with a linear production technology for new investment goods where capital adjustment frictions determine how new investment goods can be transformed into installed capital. The parameter α_i then captures the curvature in the adjustment cost function. Appendix A.3 also shows that the model can then account for the fact that the relative price of investment is countercyclical (Greenwood, Hercowitz, and Krusell, 2000; Fisher, 2006), despite the absence of investment-specific technology shocks. The main body of the paper, however, focuses on the price of installed capital and the formulation in equation (1), as this simplifies the exposition and makes the approach for pricing investment capital most transparent.

4.2 Households

Households are internally rational in the sense of Adam and Marcet (2011), i.e., they maximize utility but do not necessarily hold rational expectations about all variables beyond their control. Each period, the representative household chooses consumption $C_t \geq 0$, hours worked $H_t \geq 0$, the end-of-period capital stocks $K_{c,t+1} \geq 0$ and $K_{i,t+1} \geq 0$ to maximize

$$E_0^{\mathcal{P}} \left[\sum_{t=0}^{\infty} \beta^t \left(\ln C_t - H_t \right) \right], \tag{6}$$

where the operator $E_0^{\mathcal{P}}$ denotes the agent's expectations in some probability space $(\Omega, \mathcal{S}, \mathcal{P})$. Here, Ω is the space of realizations, $\mathcal{S}$ the corresponding σ-algebra, and $\mathcal{P}$ a subjective

probability measure over $(\Omega, \mathcal{S})$. As usual, the probability measure $\mathcal{P}$ is a model primitive and given to agents.[17] The special case with rational expectations is nested in this specification, as explained below.

Household choices are subject to the flow budget constraint

$$C_t + K_{c,t+1}Q_{c,t} + K_{i,t+1}Q_{i,t} = W_t H_t + X_t Q_{i,t} + K_{c,t}\left((1-\delta_c)Q_{c,t} + R_{c,t}\right) \qquad (7)$$
$$+ K_{i,t}\left((1-\delta_i)Q_{i,t} + R_{i,t}\right),$$

for all $t \geq 0$, where $Q_{c,t}$ and $Q_{i,t}$ denote the prices of consumption-sector and investment-sector capital, respectively, and $R_{c,t}$ and $R_{i,t}$ the rental rates earned by renting out capital to firms in the consumption and investment sector, respectively; W_t denotes the wage rate and X_t the endowment of new investment-sector capital.

To allow for subjective price beliefs, we shall consider an extended probability space relative to the case with rational expectations. In its most general form, households' probability space is spanned by all external processes, i.e. by all variables that are beyond their control. These are given by the process $\{Z_t, X_t, W_t, R_{c,t}, R_{i,t}, Q_{c,t}, Q_{i,t}\}_{t=0}^{\infty}$, so that the space of realizations is

$$\Omega := \Omega_Z \times \Omega_X \times \Omega_W \times \Omega_{R,c} \times \Omega_{R,i} \times \Omega_{Q,c} \times \Omega_{Q,i}, \qquad (8)$$

where $\Omega_{\mathcal{X}} = \prod_{t=0}^{\infty}\mathbb{R}$ with $\mathcal{X} \in \{Z, X, W, R_c, R_i, Q_c, Q_i\}$. Letting $\mathcal{S}$ denote the sigma-algebra of all Borel subsets of Ω, beliefs will be specified by a well-defined probability measure $\mathcal{P}$ over $(\Omega, \mathcal{S})$. Letting Ω^t denote the set of all partial histories up to period t, households' decision functions can then be written as

$$(C_t, H_t, K_{c,t+1}, K_{i,t+1}) : \Omega^t \longrightarrow \mathbb{R}^4. \qquad (9)$$

The household chooses the functions (9) to maximize (6) subject to the constraints (7).

In the special case with rational expectations, $(X, W, R_c, R_i, Q_c, Q_i)$ are typically redundant elements of the probability space Ω, because households are assumed to know that these variables can at time $t \geq 0$ be expressed as *known deterministic equilibrium functions* of the history of fundamentals Z^t.[18] Without loss of generality, one can then exclude these elements from the probability space and write:

$$(C_t, H_t, K_{c,t+1}, K_{i,t+1}) : \Omega_Z^t \longrightarrow \mathbb{R}^4,$$

where $\Omega_Z^t = \prod_{s=0}^{t}\mathbb{R}$ is the space of all realizations of $Z^t = (Z_0, Z_1, ..., Z_t)$. This routinely performed simplification implies that households perfectly know how the markets determine the excluded variables as a function of the history of shocks. By introducing subjective beliefs, we will step away from this assumption.

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[17]The fact that there is a representative household is assumed not to be common knowledge among households.

[18]This assumes that there are no sunspot fluctuations in the rational expectations equilibrium.

To insure that the household's maximization problem remains well-defined in the presence of the kind of subjective price beliefs introduced below, we impose additional capital holding constraints of the form $K_{c,t+1} \leq \overline{K}_{c,t+1}$ and $K_{i,t+1} \leq \overline{K}_{i,t+1}$, for all $t \geq 0$, where the bounds $\left(\overline{K}_{c,t+1}, \overline{K}_{i,t+1}\right)$ are assumed to increase in line with the balanced growth path and are assumed sufficiently large, such that they never bind in equilibrium. The bounds also need to be sufficiently tight such that the transversality condition holds.[19] Importantly, the precise choice of these bound does not affect equilibrium outcomes.

4.3 Competitive Equilibrium

The competitive equilibrium of an economy in which households hold subjective beliefs is defined as follows:

Definition 1. For given initial conditions $(K_{c,-1}, K_{i,-1})$, a competitive equilibrium with subjective household beliefs $\mathcal{P}$ consists of allocations $\{C_t, H_t, H_{c,t}, H_{i,t}, K_{c,t+1}, K_{i,t+1}\}_{t=0}^{\infty}$ and prices $\{Q_{c,t}, Q_{i,t}, R_{c,t}, R_{i,t}, W_t\}_{t=0}^{\infty}$, all of which are measurable functions of the process $\{Z_t\}_{t=0}^{\infty}$, such that for all partial histories $Z^t = (Z_0, Z_1, ...Z_t)$ and all $t \geq 0$, prices and allocations are consistent with

1. profit maximizing choices by firms,

2. the subjective utility maximizing choices for households decision functions (9), and

3. market clearing for consumption goods ($C_t = Y_{c,t}$), hours worked ($H_t = H_{c,t} + H_{i,t}$), and the two capital goods (equations (3) and (4)).

The equilibrium requirements are weaker than what is required in a competitive rational expectations equilibrium, because household beliefs are not restricted to be rational. For the special case where $\mathcal{P}$ incorporates rational expectations, the previous definition defines a standard competitive rational expectations equilibrium.

4.4 Connecting Model Variables to Data Moments

To be able to compare our model to the data, we need to define real investment and output and various stock market variables (stock prices, dividends). We follow Boldrin, Christiano, and Fisher (2001) and define investment I_t as the quantity of capital produced ($Y_{i,t}$), valuing it at the real steady-state price of consumption capital (Q_c^{ss}), so that fluctuations in the price of capital do not contribute to fluctuations in real investment:

$$I_t = Q_c^{ss} Y_{i,t}.$$

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[19]Appendix D.2 presents an example for bounds that satisfy both requirements simultaneously.

Output is then defined as $Y_t = C_t + I_t$.

To define stock prices and dividends, we consider a setup where investment-sector and consumption-sector capital can be securitized via shares and where securitization (and its undoing) are cost-free. The absence of arbitrage opportunities then implies that the price of shares is determined by the price of the capital it securitizes. Considering a representative consumption-sector share and a representative investment-sector share, the only free parameter in this extended setup is the dividend policy of stocks, which is well-known to be indeterminate (Miller and Modigliani, 1961). To obtain a parsimonious setting, we consider a time-invariant profit payout share $p \in (0, 1)$: a share p of rental income/profits per share is paid out as dividends in both sectors each period, with the remaining share $1 - p$ being reinvested in the capital stock that the share securitizes.[20] Appendix A.4 shows how stock prices P_t and dividends D_t are then defined. The sectoral price dividend ratio $P_{s,t}/D_{s,t}$ $(s = c, i\,)$ is then an affine function of the sectoral capital price to rental rate ratio:

$$\frac{P_{s,t}}{D_{c,t}} = \frac{1-p}{p} + \frac{1-\delta_s}{p}\frac{Q_{s,t}}{R_{s,t}}. \tag{10}$$

For reasonable payout ratios p, the constant $(1-p)/p$ tends to be small, so that the price dividend ratio is essentially proportional to the capital price to rental rate ratio, with the payout ratio p determining the factor of proportionality.

5   Subjective Price Optimism/Pessimism

We consider two alternative specifications for the beliefs $\mathcal{P}$: a standard setting in which *all* expectations are rational and an alternative setting that allows for subjective beliefs about future capital prices $(Q_{c,t+j}, Q_{i,t+j})$ but maintains rational expectations about all remaining variables $(Z_{t+j}, X_{t+j}, W_{t+j}, R_{c,t+j}, R_{i,t+j})$.[21]

It is well known that the asset pricing implications of the model under fully rational expectations are strongly at odds with the data. It is nevertheless useful to consider a setting with fully rational expectations (RE), as this allows highlighting the empirical improvements achieved by introducing subjective price beliefs. The rational expectations outcome also represents an important normative benchmark, as it is efficient.

We maintain the assumption of rational expectations about variables other than prices to illustrate that a single deviation from the standard paradigm is sufficient to jointly replicate stock price and business cycle behavior. Furthermore, investor expectations about future stock prices can be observed relatively easily from investor surveys, which

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[20]To make sure that this replicates aggregate investment, we allow new shares to be issued or outstanding shares to be repurchased.

[21]By rational expectations about the processes Z, X, W, R_c and R_i we mean that agents know (1) the distribution of the fundamental process Z and (2) how its histories Z^t are mapped into X_t, W_t, $R_{c,t}$ and $R_{i,t}$ in equilibrium.

Figure 3. Empirical fit of the Kalman filter model ($g = 0.019$).

allows disciplining the subjective belief choice. Observing beliefs about the other variables is a considerably harder task.

We wish to specify a simple – yet empirically plausible – specification for subjective price beliefs. Adam, Marcet, and Beutel (2017) show that this can be achieved by assuming that agents perceive capital prices $Q_{s,t}$ ($s = c, i$) to evolve according

$$\ln Q_{s,t} = \ln Q_{s,t-1} + \ln \beta_{s,t} + \ln \varepsilon_{s,t}, \tag{11}$$

where $\ln \varepsilon_{s,t} \sim ii\mathcal{N}(-\sigma_\varepsilon^2/2, \sigma_\varepsilon^2)$ denotes a transitory shock to price growth and $\ln \beta_{s,t}$ a persistent price growth component. The persistent component evolves according to

$$\ln \beta_{s,t} = \ln \beta_{s,t-1} + \ln \nu_{s,t}, \tag{12}$$

where $\ln \nu_{s,t} \sim ii\mathcal{N}(-\sigma_\nu^2/2, \sigma_\nu^2)$ denotes the innovation to the persistent component.[22] The innovations $(\varepsilon_{s,t}, \nu_{s,t})$ are independent of each other and also independent of technology shocks ε_t.

Agents observe the capital prices Q_s but not the shocks $(\varepsilon_{s,t}, \nu_{s,t})$.[23] To forecast capital prices, agents must thus estimate the persistent price growth components $\ln \beta_{s,t}$

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[22] The fact that the perceived growth rate of capital prices is non-stationary is not important for any of our results, because only subjectively expected capital prices in the next period matter for equilibrium outcomes. Online appendix D.3 shows how perceived price dynamics can be adjusted, so as to make the *level* of capital prices stationary, while generating only vanishing changes to the expected capital price tomorrow, relative to the one implied by equations (11) and (12).

[23] For this reason, the shocks $(\varepsilon_{s,t+1}, \nu_{s,t+1})$ are not defined on the probability space Ω.

from observed price data. Letting $\ln \beta_{s,t-1} \sim \mathcal{N}(\ln m_{s,t-1}, \sigma^2)$ denote the prior belief based on information up to period $t-1$ and σ the steady-state Kalman filter uncertainty,[24] the Kalman filter implies that posterior beliefs after observing the new capital price $Q_{s,t}$ is given by $\ln \beta_{s,t} \sim \mathcal{N}(\ln m_{s,t}, \sigma^2)$, where

$$\ln m_{s,t} = \ln m_{s,t-1} - \frac{\sigma_v^2}{2} + g\left(\ln Q_{s,t} - \ln Q_{s,t-1} + \frac{\sigma_\varepsilon^2 + \sigma_v^2}{2} - \ln m_{s,t-1}\right), \qquad (13)$$

and where the Kalman gain is

$$g = \frac{\sigma^2}{\sigma^2 + \sigma_\varepsilon^2}. \qquad (14)$$

Agents' beliefs can thus be parsimoniously summarized by their posterior mean beliefs about price growth $(m_{c,t}, m_{i,t})$. These mean beliefs capture *agents' degree of optimism* about future capital gains and equation (13) shows how agents' extrapolate past capital gain observations into the future whenever $g > 0$. Bordalo, Gennaioli, and Shleifer (2018) explain how extrapolation can alternatively be obtained by postulating that agents hold diagnostic expectations. Nagel and Xu (2018) show how learning from experience by individual agents generates fading memory for the representative agent and thereby gives rise to the kind of perpetual learning present also in equation (13).

To avoid simultaneous determination of price beliefs and prices, we also follow Adam, Marcet, and Beutel (2017) and use a slightly modified information setup in which agents receive in period t information about the lagged transitory component $\ln \varepsilon_{s,t-1}$. The modification causes the updating equation (13) to contain only lagged price growth and no variance correction terms. Capital gain beliefs then evolve according to

$$\ln m_{s,t} = \ln m_{s,t-1} + g\left(\ln Q_{s,t-1} - \ln Q_{s,t-2} - \ln m_{s,t-1}\right) + g \ln \varepsilon_{s,t}^1, \qquad (15)$$

where $\ln \varepsilon_{s,t}^1 \sim ii\mathcal{N}(\frac{-\sigma_\varepsilon^2}{2}, \sigma_\varepsilon^2)$ is a time t innovation to agents' information set (unpredictable using information available to agents up to period $t-1$), which captures the information that agents receive about $\ln \varepsilon_{s,t-1}$ in period t. When simulating the model, we always set $\ln \varepsilon_{s,t}^1 = 0$.[25]

With this slight modification, agents' capital gain expectations are given by

$$E_t^{\mathcal{P}}\left[\frac{Q_{s,t+1}}{Q_{s,t}}\right] = m_{s,t} \qquad (16)$$

and $m_{s,t}$ evolves according to equation (15).

To prevent equation (15) from generating subjective price beliefs $m_{s,t}$ that are so optimistic that they imply infinite utility (in subjective terms), we follow Adam, Marcet,

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[24]We have $2\sigma^2 \equiv -\sigma_\nu^2 + \sqrt{(\sigma_\nu^2)^2 + 4\sigma_\nu^2\sigma_\varepsilon^2}$.

[25]We also exclude $\varepsilon_{s,t}^1$ from the subjective probability space, as beliefs about future values of $m_{s,t}$ do not matter for the policy functions in our model.

and Nicolini (2016) and impose a projection facility that dampens upward belief revisions beyond a certain point of optimism. The projection facility will bind only rarely in our simulations and can be interpreted as an approximate implementation of a Bayesian updating scheme where agents have a normal prior about $\ln \beta_{s,t}$ but where the support of the prior is truncated above. Details of the projection facility are spelled out in appendix A.5.

Despite their simplicity, the Kalman filter equations (15) and (16) capture the empirical dynamics of investor survey beliefs surprisingly well. This is illustrated in figure 3, which depicts the expected price growth rates implied by the Kalman filter model, together with the median price growth forecast for the S&P500 from the UBS survey.[26] The predictions of the Kalman filter model are obtained by feeding the historical price growth rates of the S&P500 in to it, using for the Kalman gain the value $g = 0.019$ obtained when estimating the model in section 7, where the estimation does *not* rely on survey information.[27]

Figure 3 shows that the subjective price beliefs implied by equations (15) and (16) provide - for the estimated model parameters - a very good fit to the survey data.[28]

6 Equilibrium Conditions and Financial Accelerator

This section derives the set of equations characterizing the competitive equilibrium. These equations hold independently of the assumed belief structure and deliver a unique equilibrium outcome under both considered belief settings. The equilibrium conditions illustrate how the model can give rise to a smooth consumption process in the presence of volatile stock prices. They also show how the presence of subjective price beliefs can generate price volatility and a 'financial accelerator' effect.

From the household's first-order conditions, we get[29]

$$C_t = W_t, \tag{17}$$

$$Q_{c,t} = \beta E_t^{\mathcal{P}} \left[\frac{W_t}{W_{t+1}} \left((1 - \delta_c) Q_{c,t+1} + R_{c,t+1} \right) \right], \tag{18}$$

$$Q_{i,t} = \beta E_t^{\mathcal{P}} \left[\frac{W_t}{W_{t+1}} \left((1 - \delta_i) Q_{i,t+1} + R_{i,t+1} \right) \right], \tag{19}$$

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[26]The UBS survey reports subjective return expectations, i.e., $E_t^{\mathcal{P}}[R_{t+1}] = E_t^{\mathcal{P}}[(P_{t+1} + D_{t+1})/P_t]$. We transform subjective return expectations into subjective price growth expectations $E_t^{\mathcal{P}}[P_{t+1}/P_t]$ by subtracting $E_t^{\mathcal{P}}[D_{t+1}/P_t] = D_t/P_t \cdot E_t^{\mathcal{P}}[D_{t+1}/D_t]$ from the return forecast. In the latter expression, we approximate the unknown dividend growth expectations by the sample average of D_{t+1}/D_t.

[27]We set the initial value $m_{Q1:1955} = 1$ and the shocks $\ln \varepsilon_{s,t}^1 = 0$ for all t.

[28]This result is robust to using the survey mean instead of the survey median or to converting nominal variables into real ones.

[29]The household budget constraint holds automatically, because we keep all market clearing conditions in the set of equations characterizing equilibrium.

for all $t \geq 0$, where $E_t^{\mathcal{P}}[\cdot] = E_t[\cdot]$ for the case with rational expectations. The first equation is due to our log-linear specification of household utility and shows that wages will be smooth, whenever consumption is a smooth process. The last two equations show how optimism/pessimism about future capital prices (high or low values for $E_t^{\mathcal{P}} Q_{c,t+1}$ or $E_t^{\mathcal{P}} Q_{i,t+1}$) affect – ceteris paribus – current capital prices. Variations in subjective price expectations can thus contribute to increasing the variability of capital prices (and thus stock prices) relative to a setting with rational expectations. Since realized capital prices feed back into agents' beliefs, see equation (15), the additional price volatility coming from subjective belief variations can be very large, as we illustrate in the next section.

Household optimality also require the transversality condition to hold:[30]

$$\lim_{t \to \infty} \beta^t E^{\mathcal{P}} \left[\frac{1}{W_t} \left(K_{c,t+1} Q_{c,t} + K_{i,t+1} Q_{i,t} \right) \right] = 0. \tag{20}$$

Appendix D.3 provides conditions under which the previous condition is satisfied. We verify these conditions for our estimated model parameters.

The first-order conditions of consumption-sector firms deliver

$$W_t = \frac{(1 - \alpha_c) Y_{c,t}}{H_{c,t}}, \tag{21}$$

$$R_{c,t} = \frac{\alpha_c Y_{c,t}}{K_{c,t}}, \tag{22}$$

and the optimality conditions of investment-sector firms

$$W_t = (1 - \alpha_i) Q_{c,t} K_{i,t}^{\alpha_i} Z_t^{1-\alpha_i} H_{i,t}^{-\alpha_i}, \tag{23}$$

$$R_{i,t} = \alpha_i Q_{c,t} K_{i,t}^{\alpha_i - 1} Z_t^{1-\alpha_i} H_{i,t}^{1-\alpha_i}. \tag{24}$$

The market clearing conditions are given by

$$C_t = Y_{c,t}, \tag{25}$$

$$H_t = H_{c,t} + H_{i,t}, \tag{26}$$

$$K_{c,t+1} = (1 - \delta_c) K_{c,t} + Y_{i,t}, \tag{27}$$

$$K_{i,t+1} = (1 - \delta_i) K_{i,t} + X_t. \tag{28}$$

For the case with subjective beliefs, capital price expectations are given by

$$E_t^{\mathcal{P}}[Q_{s,t+1}] = m_{s,t} Q_{s,t} \qquad \text{for } s = c, i, \tag{29}$$

where $m_{s,t}$ evolves according to (15), otherwise capital price expectations are rational.

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[30] See e.g. Kamihigashi (2005) for the formulation of transversality conditions in stochastic problems with inequality constraints.

Smooth Consumption. Equations (17), (21) and (25) imply that hours worked in the consumption sector are constant over time:

$$H_{c,t} = 1 - \alpha_c. \tag{30}$$

This is a result of our log-linear utility specification and is important for two reasons.

First, it implies that consumption variations are exclusively driven by productivity changes and by the dynamics of capital accumulation in the consumption sector.[31] Since productivity shocks tend to be small and capital accumulation a slowly moving process, it gives the model a chance to replicate the observed smoothness of the aggregate consumption series. Clearly, this would still be approximately the case, if one deviated slightly from the exact log-linear utility specification.

Second, the fact that $H_{c,t}$ is constant, allows expressing the future wage W_{t+1} showing up in the capital pricing equations (18) and (19) as a function of current variables and future exogenous variables. In particular, combining equations (1), (21) and (30) for period $t + 1$, we obtain

$$W_{t+1} = ((1 - \alpha_c) Z_{t+1})^{1-\alpha_c} K_{c,t+1}^{\alpha_c}. \tag{31}$$

The log-linear preference specification thus allows for a simple and fast computation of inverse wage expectations $E_t [1/W_{t+1}]$ as a function of the time t capital choice $K_{c,t+1}$, without having to explicitly solve the nonlinear household optimization problem. This feature is crucial for being able to efficiently solve for the equilibrium dynamics of the non-linear model under subjective price expectations and allows us to estimate the nonlinear model using the Simulated Method of Moments.[32]

Financial Accelerator. From equation (23) follows that hours worked in the investment sector are given by

$$H_{i,t} = K_{i,t} Z_t^{\frac{1-\alpha_i}{\alpha_i}} \left((1 - \alpha_i) \frac{Q_{c,t}}{W_t} \right)^{\frac{1}{\alpha_i}}, \tag{32}$$

which shows that high prices for consumption sector capital ($Q_{c,t}$) induce – ceteris paribus – high labor demand by firms in the investment sector and thus drive up investment. Since investment sector firms produce new consumption sector capital, a high value for $Q_{c,t}$ signals that it pays to expand production, even in the presence of decreasing returns to scale.[33] Since $Q_{c,t}$ is potentially affected by subjective capital price expectations,

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[31] This follows from the production function for $Y_{c,t}$, in equation (1) and the market clearing condition (25), which together with equation (30) imply $C_t = (1 - \alpha_c)^{1-\alpha_c} Z_t^{1-\alpha_c} (K_{c,t})^{\alpha_c}$.

[32] See appendix B for further details on the equilibrium computation. It still takes about one week to estimate the model parameters, despite parallelization efforts.

[33] There are decreasing returns in the short-run because capital in the investment sector is predetermined within the period. In the long-run, investment sector capital grows in line with the balanced growth path.

as discussed before, belief-driven fluctuations in the price of consumption sector capital give rise to fluctuations in hours worked and in investment. We thus have a financial accelerator that translates belief-driven price fluctuations into variations of real variables. Importantly, the accelerator is such that it gives rise to positive comovement between capital prices, hours worked and investment, in line with the empirical evidence presented in section 3.

While capital prices vary very little under fully rational expectations, they can persistently deviate from their rational expectations value under our subjective belief setup. These deviations can take the form of over-valuations, as well as under-valuations, relative to a setup with fully rational expectations. Therefore, unlike standard forms of the financial accelerator, which rely on collateral constraints and generate under-investment to various degrees, the accelerator in the present model can generate under-investment as well as over-investment. Over-investment occurs whenever $Q_{c,t}$ is persistently above its rational expectations value.

Equilibrium Uniqueness. Appendix B shows that the equations derived in the present section deliver a unique equilibrium outcome under the two belief specifications introduced in the previous section. It also shows how equilibrium outcomes can be computed numerically and why the linear disutility of labor in household preferences is key for being able to do so efficiently in the presence of subjective price beliefs.

7 Quantitative Model Performance

This section reports the outcomes from estimating the rational expectations (RE) and subjective belief model using the simulated methods of moments. It documents the ability of the subjective belief model to simultaneously replicate business cycle and stock price moments. It also illustrates the vast empirical improvements associated with introducing subjective price beliefs relative to a model with fully rational expectations.

Section 7.1 describes the estimation approach and reports the estimated parameter values. Section 7.2 compares the model moments to the data. We will explain in section 8 why the subjective belief model performs so much better than its rational expectations counterpart.

7.1 Estimation Procedure and Parameter Estimates

The RE model features eight model parameters $(\beta, \alpha_c, \alpha_i, \delta_c, \delta_i, \gamma, \sigma, p)$ that need to be estimated. The subjective belief model additionally features the Kalman gain parameter (g).

To put both models on equal footing in terms of the number of parameters, we will consider also an augmented RE model that additionally features investment-specific technology shocks. The variance σ_i^2 of these shocks is then the ninth model parameter of the

augmented RE model.[34] The introduction of investment-specific technology shocks is also motivated by the fact that – absent such shocks – the RE model has difficulties with replicating the volatility of hours worked and investment. The subjective belief model never features investment-specific technology shocks.

We estimate parameters using the simulated method of moments using a diagonal weighting matrix consisting of the data-implied variances of the target moments. For the subjective belief model, the set of targets include the seven business cycle moments listed in table 1 and all asset pricing moments listed in table 2, except for the mean and standard deviation of the risk-free interest rate. We exclude the risk-free rate moments from the set of targeted moments because the subjective belief model has difficulties with fully matching the equity premium.

For the rational expectations model, we furthermore exclude the equity return moments ($E[r^e]$, $\sigma(r^e)$) and the autocorrelation of the PD ratio ($\rho(P/D)$). Including these variables as estimation targets significantly deteriorates the model fit along the business cycle dimension, without noticeably improving the fit for the excluded financial moments.[35]

When estimating the learning model, we impose the additional restriction that the impulse responses of capital prices to technology shocks display exponential decay.[36] We do so to avoid that the estimation selects parameter values that would imply deterministic equilibrium cycles. Clearly, imposing this additional restriction can only deteriorate the fit with the target moments.

Table 5 presents the estimation outcome. For the RE models, all estimated parameter are in line with standard values in the literature, except for the payout ratio (p), which has no direct counterpart in the literature.[37] Since the role of the payout ratio is largely limited to acting as a scaling factor for the price-dividend ratio, see equation (10), we will not comment further on the estimated values for p.

The RE model with investment-specific shocks delivers very similar parameter estimates as the simpler RE model. The main difference is that the estimated standard deviation of standard technology shocks (σ) drops by around 20% when allowing for investment-specific shocks (σ_i).

The subjective belief model delivers estimates that are mostly in line with those of its RE counterparts. The most noticeable differences are the slightly lower quarterly capital depreciation rates and the larger estimate for the parameter α_i. The latter may appear

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[34]The augmented RE model simply replaces $Y_{i,t}$ from equation (1) by $Y_{i,t} = \varepsilon_t^I K_{i,t}^{\alpha_i} \left(Z_t H_{i,t} \right)^{1-\alpha_i}$, where $\ln \varepsilon_t^I \sim ii\mathcal{N}(-\frac{\sigma_i^2}{2}, \sigma_i^2)$ denote the investment-specific technology shocks.

[35]We keep the level and standard deviation of the PD ratio as estimation targets, as otherwise the payout ratio p would not be identified. Also excluding these two PD moments from the set of target moments has almost no impact on the remaining estimated parameters.

[36]We impose a minimum exponential decay rate of around 4.5% per year, see appendix D.5 for details.

[37]It is difficult to compare p to empirical estimates of the profit payout ratio because we do not consider financial leverage within our model.

Table 5

Estimated parameters

Parameter	Subjective Belief Model	RE Model	RE Model with inv. shocks
β	0.996	0.997	0.997
α_c	0.36	0.36	0.36
α_i	0.73	0.35	0.35
δ_c	0.010	0.020	0.010
δ_i	0.014	0.030	0.030
γ	1.004	1.005	1.003
σ	0.015	0.018	0.015
p	0.350	0.195	0.337
g	0.019	–	–
σ_i	–	–	0.015

implausibly high, when literally interpreted as the capital share in the investment sector. Yet, the parameter α_i should be interpreted as capturing the combined effects of the capital share in production and of adjustment costs associated with installing produced investment goods, as discussed in section 4.1. In fact, appendix A.3 shows that the level of adjustment costs required to rationalize the estimated value of α_i is below the levels typically assumed in the literature.

The estimated Kalman gain g in the subjective belief model is in line with other estimates in the literature.[38] The estimated value is also plausible on the grounds that the belief updating equation (15) then successfully tracks the dynamics of the UBS survey expectations, see figure 3. Overall, the estimated parameters in table 5 appear reasonable on a priori grounds.

7.2   Quantitative Results

Table 6 reports the data moments discussed in section 3 together with an estimate of the standard deviation of the data moments (column 2), as well as the moments implied by the three estimated models (columns 3-5).[39] An asterisk behind a model moment indicates that the corresponding data moment has been targeted in the estimation.

The subjective belief model fits all the business cycle moments very well. None of the model moments deviates more than two standard deviations from the data moment. In

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[38] Adam, Beutel, Marcet, and Merkel (2016) estimate gains for different groups of investors and find estimates ranging from 0.018 to 0.032.

[39] The reported model moments are obtained from simulating the models for 10000 periods. We choose balanced growth path values as initial value and discard the first 500 observations when computing moments.

addition, the subjective belief model replicates an important set of financial moments. It matches the behavior of the PD ratio by replicating its mean, $E[P/D]$, by generating a high value for its standard deviation, $\sigma(P/D)$, and by producing an auto-correlation, $\rho(P/D)$, close to that in the data. Furthermore, it generates volatile stock returns, $\sigma(r^e)$, replicating about 90% of the empirically observed volatility, while simultaneously giving rise to a very stable risk-free interest rate, $\sigma(r^f)$.[40] Finally, it does so by generating a standard deviation for dividend growth that is within 2 standard deviations of its data moment.

The fit in terms of matching financial moments is, however, not perfect. In particular, the subjective belief model has difficulties in matching the mean stock return ($E[r^e]$) and the mean risk-free interest rate ($E[r^f]$). In fact, the subjective belief model generates only around 30% of the historically observed equity premium. The ability to generate a sizable equity premium is nevertheless surprising. The source of the equity premium in the model is subjective return pessimism, which gives rise to an ex-post equity premium.[41]

Importantly, the subjective belief model also manages to generate a positive correlation between the PD-ratio and hours worked (H), the investment ratio (I/Y), and subjectively expected returns($E^{\mathcal{P}}[r^e]$), despite the fact that none of these moments have been targeted in the estimation. The first two correlations are positive due to the financial accelerator effect. The latter is positive because stock price booms are generated by subjective return optimism, as discussed in section 8 below.

Overall, the subjective belief model performs surprisingly well in terms of matching business cycle moments, stock price moments, as well as moments characterizing the interaction between real and financial variables. The equity premium, however, can be replicated only partially.

The quantitative performance contrasts strongly with that of the RE models. The RE model without investment-specific shocks not only grossly fails in matching the volatility of stock prices ($\sigma(P/D), \sigma(r^e)$), but also has great difficulties in matching business cycle moments: it falls significantly short of replicating the relatively large volatility of investment ($\sigma(I)/\sigma(Y)$) and hours worked ($\sigma(H)/\sigma(Y)$).

The failure to generate sufficiently volatile investment and hours worked is in fact related to the failure to generate sufficiently volatile stock prices. By generating more realistic stock market volatility, the subjective belief model sets in motion the financial

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[40] The volatility of the risk-free interest rate in the data is probably slightly overstated as it uses ex-post realized inflation rates to transform nominal rates into real rates.

[41] Return pessimism emerges because of an asymmetry in the interaction between prices and price expectations. It can be understood by considering the simplified pricing equation (34). The hyperbolic relationship between capital prices ($Q_{c,t}$) and beliefs ($m_{c,t}$) in this equation implies that prices are considerably more sensitive to belief adjustments when beliefs are optimistic, i.e., close to $1/B$, whereas prices are relatively insensitive to belief changes when beliefs are pessimistic, i.e., far below $1/B$. As a result, price and belief adjustments are slow when agents are pessimistic, but fast when agents are optimistic. Periods with pessimism thus last longer than periods with optimism. This gives rise to a pessimistic average bias and an ex-post return premium.

Table 6

Empirical model fit

	Data	Subjective Belief	RE Model	RE Model
	(std.dev.)	Model		with inv. shocks
Business Cycle Moments				
$\sigma(Y)$	1.72 (0.25)	1.83*	1.90*	1.85*
$\sigma(C)/\sigma(Y)$	0.61 (0.03)	0.67*	0.75*	0.66*
$\sigma(I)/\sigma(Y)$	2.90 (0.35)	2.90*	1.88*	2.79*
$\sigma(H)/\sigma(Y)$	1.08 (0.13)	1.06*	0.31*	0.56*
$\rho(Y,C)$	0.88 (0.02)	0.84*	0.98*	0.86*
$\rho(Y,I)$	0.86 (0.03)	0.89*	0.97*	0.90*
$\rho(Y,H)$	0.75 (0.03)	0.70*	0.89*	0.80*
Financial Moments				
$E[P/D]$	152.3 (25.3)	150.0*	174.6*	166.0*
$\sigma(P/D)$	63.39 (12.39)	44.96*	7.00*	8.28*
$\rho(P/D)$	0.98 (0.003)	0.97*	0.96	0.95
$E[r^e]$	1.87 (0.45)	1.25*	0.77	0.57
$\sigma(r^e)$	7.98 (0.35)	7.07*	0.16	0.16
$E[r^f]$	0.25 (0.13)	0.78	0.77	0.58
$\sigma(r^f)$	0.82 (0.12)	0.06	0.09	0.06
$\sigma(D_{t+1}/D_t)$	1.75 (0.38)	2.46*	1.19*	1.69*
Other Moments				
$\rho(H,P/D)$	0.51 (0.17)	0.79	-0.97	-0.95
$\rho(I/Y,P/D)$	0.58 (0.19)	0.69	-0.97	-0.94
$\rho(E^{\mathcal{P}}[r^e],P/D)$	0.79 (0.07)	0.52	-0.99	-0.98

Notes: Model moments marked with an asterisk have been targeted in the estimation. The label of the moments symbols can be found in tables 1, 2 and 3. Financial return moments are expressed in quarterly rates of return. Similarly, the P/D ratio is defined as the price over quarterly dividend payments.

accelerator effect, as discussed in section 6, which causes hours worked and investment to become more volatile.

The performance of the RE model along the business cycle dimension can be slightly improved by adding investment-specific technology shocks. These shocks can partly substitute for the missing stock price volatility and allow the RE model to generate more volatile investment dynamics, even if the volatility of hours worked still falls short of that in the data.

Yet, the RE model with investment shocks still severely underpredicts the volatility of the PD-ratio ($\sigma(P/D)$) and of stock returns ($\sigma(r^e)$), and it produces virtually no equity premium. The RE models also generates the wrong sign for the correlation between the PD-ratio and hours worked (H), investment (I), and the model-implied expected stock returns ($E[r^e]$), see the last three entries in columns 4 and 5 of the table.

Importantly, the failure of the RE model to replicate important financial moments can not be remedied by including financial moments into the set of targeted moments. In fact, there is no significant improvement along the financial dimension, even when making the stock return moments $E[r^e]$ and $\sigma(r^e)$ the *only* estimation targets of the RE model.[42]

We can conclude from the results of table 6 that the subjective belief model fits the real and financial moments surprisingly well and that it generates large and significant improvements over the RE model. As we document in the next section, key to the improved performance is the ability to generate volatile capital prices via belief dynamics.

Online appendix D.6 runs additional horse races between our subjective belief model and the rational expectations model of Boldrin, Christiano and Fisher (2001) and the endowment model with subjective beliefs of Adam, Marcet and Beutel (2017). The online appendix shows that the subjective belief model performs well also in comparison to these models.

8 Boom-Bust Cycles and Belief-Driven Propagation

This section shows that the subjective believe model generates stock price volatility by giving rise to stock price boom and bust cycles. These cycles are associated with boom-bust cycles in investment, hours worked and output and with persistent over-accumulation and under-accumulation of capital. The ability to generate such cycles is key for the quantitative success of the model, as documented in the previous section.

Stock price cycles emerge because subjective belief dynamics generate strong endogenous propagation. We illustrate this below by considering the dynamic effects of a pure 'optimism shock', i.e., of an exogenous positive shock to capital gain expectations. Section 9 will show how a sequence of positive technology surprises triggers the belief-driven

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[42]The model moments for $E[r^e]$ and $\sigma(r^e)$ are then 0.74% and 0.50%, respectively, instead of 0.77% and 0.16%, as reported in table 6, i.e., remain very far away from the data moments of these variables. There is furthermore a considerable deterioration in terms of fitting business cycle moments.

propagation associated with such an 'optimism shock'.

We start by considering the price of consumption sector capital. Appendix A.6 derives the following result:

Proposition 1. *Under subjective price expectations, consumption sector capital prices satisfy*

$$Q_{c,t} = \frac{\beta \alpha_c \left(1 - \alpha_c\right)^{1-\alpha_c}}{1 - \beta \left(1 - \delta_c\right) e \left(\frac{K_{c,t}}{K_{c,t+1}}\right)^{\alpha_c} m_{c,t}} \left(\frac{Z_t}{K_{c,t+1}}\right)^{1-\alpha_c} \left(\frac{K_{c,t}}{K_{c,t+1}}\right)^{\alpha_c}, \tag{33}$$

where $e \equiv E_t[(\gamma \varepsilon_{t+1})^{\alpha_c - 1}] > 0$.

Equation (33) reveals that time variation in $Q_{c,t}$ is driven by time variation in three key variables: (i) the capital gain beliefs $m_{c,t}$, (ii) the growth rate of the capital stock $K_{c,t+1}/K_{c,t}$, and (iii) the capital to technology ratio $K_{c,t+1}/Z_t$.

Since the capital stock tends to move slowly over time, one can approximate the capital price dynamics by setting the latter two variables equal to their deterministic balanced growth path values. (We discuss the additional effects due to changes in the capital stock below). Equation (33) then simplifies to

$$Q_{c,t} = \frac{A}{1 - B \cdot m_{c,t}}, \tag{34}$$

where A is a positive constant and

$$B \equiv \frac{\beta}{\gamma} \left(1 - \delta_c\right) E_t \left[(\varepsilon_{t+1})^{\alpha_c - 1}\right] < 1.$$

In this simplified setting, the capital price varies solely due to variations in beliefs. In fact, combining simplified pricing equation (34) with the belief updating equation (15) delivers a second order difference equation that uniquely determines the dynamics of beliefs and thereby the (approximate) dynamics of consumption sector capital prices.

Figure 4 illustrates these dynamics using a 2-dimensional phase diagram for the variables $(m_{c,t}, m_{c,t-1})$.[43] The two solid black lines indicate the points in the space along which the dynamics imply $m_{c,t} = m_{c,t-1}$ and $m_{c,t+1} = m_{c,t}$, respectively. The blue arrows in the figure indicate the directions in which the belief variables adjust from one period to the next, when being away from those two lines. They show that beliefs have a tendency to move counterclockwise around the perfect foresight balanced growth path point $(m_{c,t}, m_{c,t-1}) = (1, 1)$, which is marked by a black dot in the figure.

Since the Kalman gain g in the belief updating equation (15) is small, in line with what the survey data suggests, beliefs will tend to move – for the most part – slowly around the balance growth path outcome.[44] This is indicated by the red dotted line, which illustrates

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[43] See appendix D.4 for details on how the phase diagram can be constructed.

[44] Absent further shocks, beliefs would over time locally converge to the point $(m_{c,t}, m_{c,t-1}) = (1, 1)$.
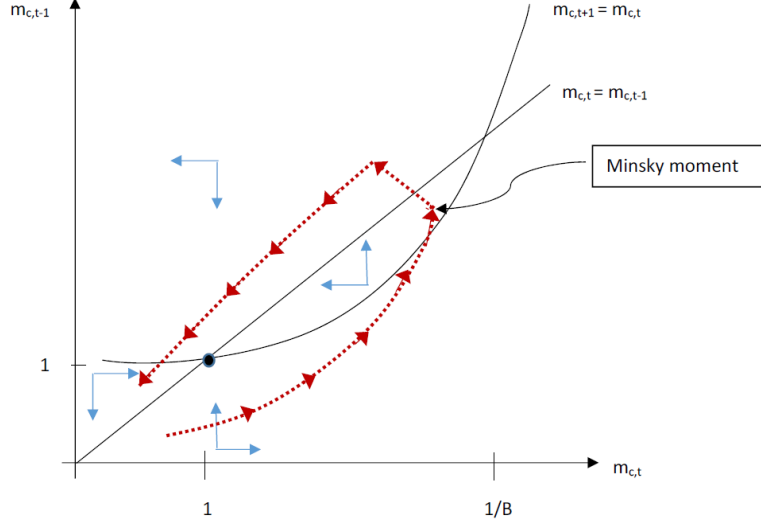
Figure 4. Boom and bust cycles in capital gain beliefs $(m_{c,t})$

how beliefs evolve when starting from an initial belief below the balanced growth value $(m_{c,t} < 1)$ and a situation in which beliefs have been rising $(m_{c,t} > m_{c,t-1})$. It shows that beliefs keep rising further over time and well beyond the point $(m_{c,t} = 1)$. This is so because the mere fact that beliefs increase generates additional capital gains, see equation (34), which cause upward revisions in beliefs beyond the balanced growth path outcome.

If the increase in beliefs becomes too weak, i.e., if the belief pair $(m_{c,t}, m_{c,t-1})$ approaches the 45 degree line from below, then $m_{c,t}$ starts reverting direction. At this point, the economy experiences a 'Minsky moment': beliefs are high but stop increasing further, so that realized capital gain outcomes disappoint.[45] Beliefs will then jump in a discrete step across the 45 degree line and enter a period of persistent reversal, with the potential to undershoot the rational expectations value, as indicated in the figure.

We can now discuss the additional pricing effects associated with variations in $K_{c,t+1}/K_{c,t}$ and $K_{c,t+1}/Z_t$ in equation (33). Combining equation (32), which captures the financial accelerator effect, with equation (31) for period t, one obtains

$$H_{it} = \overline{C}_t \cdot \left(\frac{Q_{c,t}}{(K_{c,t}/Z_t)^{\alpha_c}} \right)^{\frac{1}{\alpha_i}}, \tag{35}$$

where $\overline{C}_t \equiv \frac{K_{i,t}}{Z_t} \left(\frac{(1-\alpha_i)}{(1-\alpha_c)^{1-\alpha_c}} \right)^{\frac{1}{\alpha_i}}$ is constant in the absence of productivity shocks. Since

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[45]With $m_{c,t} \leq m_{c,t-1}$, it follows from equation (34) that $Q_{c,t}/Q_{c,t-1} \leq 1$, so that outcomes disappoint whenever $m_{c,t} > 1$.
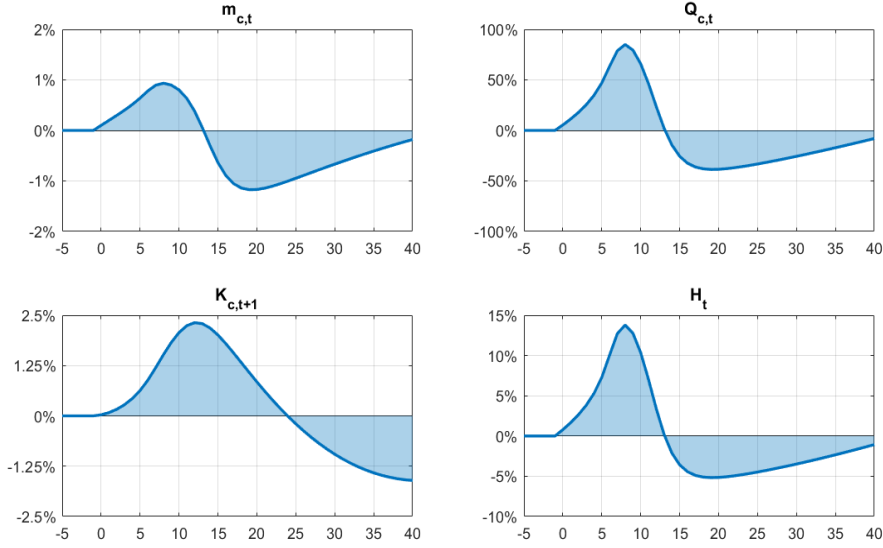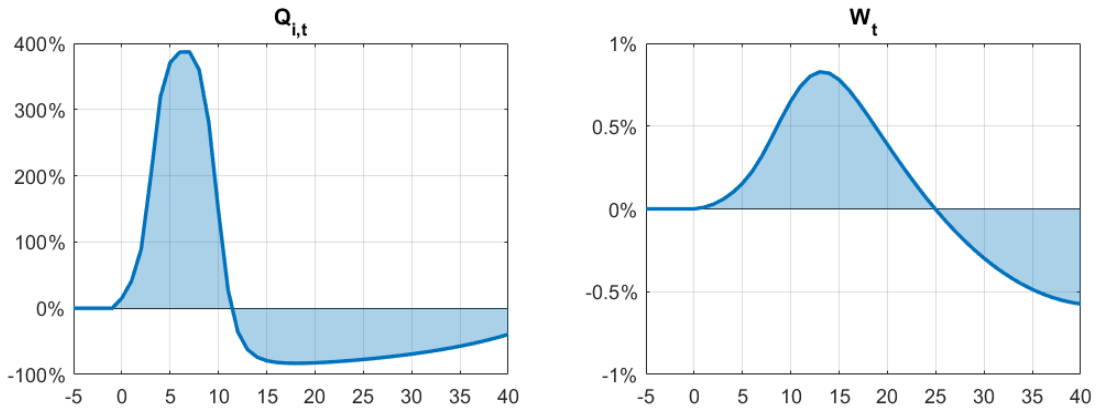
Figure 5. Impulse response to a +10bps shock to expected capital gains $(m_{c,t})$ in consumption-sector capital. (Variables are normalized relative to the deterministic balanced growth path value emerging in the absence of the shock.)

$K_{i,t}$ and Z_t are exogenous processes, it follows from the production function for $Y_{i,t}$ that variations in $H_{i,t}$ capture all the variation of investment. It thus follows from equation (35) that investment expands during a belief-driven run-up in capital prices, provided capital prices $(Q_{c,t})$ increase faster than the capital stock $((K_{c,t})^{\alpha_c})$. This is the case in our estimated model because a high value for α_i implies that the capital stock adjusts only slowly over time.

A persistent run-up in capital prices is then associated with a persistent run-up in investment, causing prices and investment to commove. It also causes the ratio $K_{c,t+1}/Z_t$ to slowly increase over time and similarly causes an increase in the ratio $K_{c,t+1}/K_{c,t}$. It follows from equation (33) that both of these effects dampen capital price increase, thus help with eventually bringing the boom to an end.

Figure 5 illustrates the belief-driven propagation using the estimated parameters from section 7. It depicts the impulse responses to an exogenous one-time 10 basis point increase in the capital gain expectations for consumption sector capital $(m_{c,t})$.[46] The initial increase in optimism and the associated small increase in the capital price is followed by eight quarters of further increases in optimism and capital prices. The increased capital price leads to an increase in hours worked and in the stock of consumption-sector capital,

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[46]The figure normalizes all variables relative to the deterministic balanced growth path values that would emerge in the absence of the shock.

Figure 6. Impulse response to a +10bps shock to expected capital gains $(m_{c,t})$ in consumption-sector capital. (Variables are normalized relative to the deterministic balanced growth path value emerging in the absence of the shock.)

as discussed above. These reactions eventually dampen the price increase, so that capital increases fall short of expectation: prices and beliefs then start to revert direction and fall persistently below their balanced growth path values, before slowly recovering. As capital prices become depressed, investment falls and consumption-sector capital persistently undershoots its balanced growth path value. The boom thus contains the seeds of a future recession.

Figure 6 illustrates how the dynamics in the consumption sector spill over into the investment sector. To understand the channels through which this happens, note that equations (19) and (29) deliver the following relationship:

$$Q_{i,t} = \frac{\beta E_t \left[\frac{W_t}{W_{t+1}} R_{i,t+1} \right]}{1 - \beta (1 - \delta_i) E_t \left[\frac{W_t}{W_{t+1}} \right] m_{i,t}}. \tag{36}$$

Since wages increase during the price boom in consumption sector capital, the drop of $E[W_t/W_{t+1}]$ exerts downward pressure on the price of investment-sector capital, via a standard discount factor effect. Equation (24) shows, however, that (rationally) expected rental rates $R_{i,t+1}$ increase strongly as the price of consumption-sector capital and hours worked increase. The overall price effect at time zero is thus positive. This positive initial effect then propagates via the belief updating mechanism (15) and the positive dependence of future capital prices on future beliefs $m_{i,t+1}$, see equation (36), into a persistent boom for the price of investment-sector capital.

As a result, the capital prices in the two sectors thus have a tendency to commove over time. This is further illustrated in figure 7 which depicts the impulse responses of the
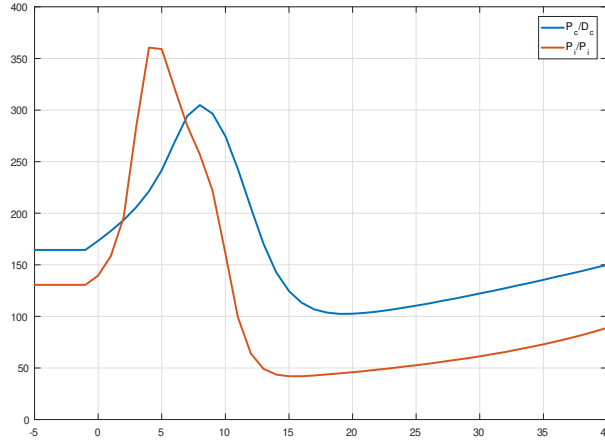
Figure 7. Impulse response to a +10bps shock to expected capital gains in consumption-sector capital $(m_{c,t})$

PD ratio in the capital and investment sector. While the PD ratios approximately double in response to the considered optimism shock, the response of real quantities is orders of magnitude smaller. Consumption, for instance, increases in line with the increase in the stock of consumption sector capital and moves by less than $\pm 1\%$ over the boom-bust cycle.[47] The relatively muted response of real variables in the presence of a very large stock price cycle allows the model to reconcile the relative smoothness of the business cycle with the observed high volatility of stock prices.

# 9 Technology Booms, Real Rates and Repeat Cycles

This section shows how repeated positive technology surprises trigger the endogenous belief propagation described in the previous section and give rise to boom-bust dynamics. It also shows how low safe real interest rates strengthen belief propagation, thereby making the occurrence of boom-bust cycles more likely. Finally, it illustrates that following a boom-bust cycle, further cycles are more likely to emerge.

The Effect of Repeated Positive Technology Surprises. The top panel of figure 8 depicts the impulse response of the capital price in the consumption sector to a series of 4 and 8 positive technology shocks of two standard deviations each.[48] While a series of

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[47] Hours worked in the consumption sector are constant and the elasiticity of consumption with respect to changes in the capital stock is equal to the capital share in consumption sector production, which is estimated to be 0.36.

[48] The figure use the estimated model from section 7 and initializes state variables at the ergodic mean

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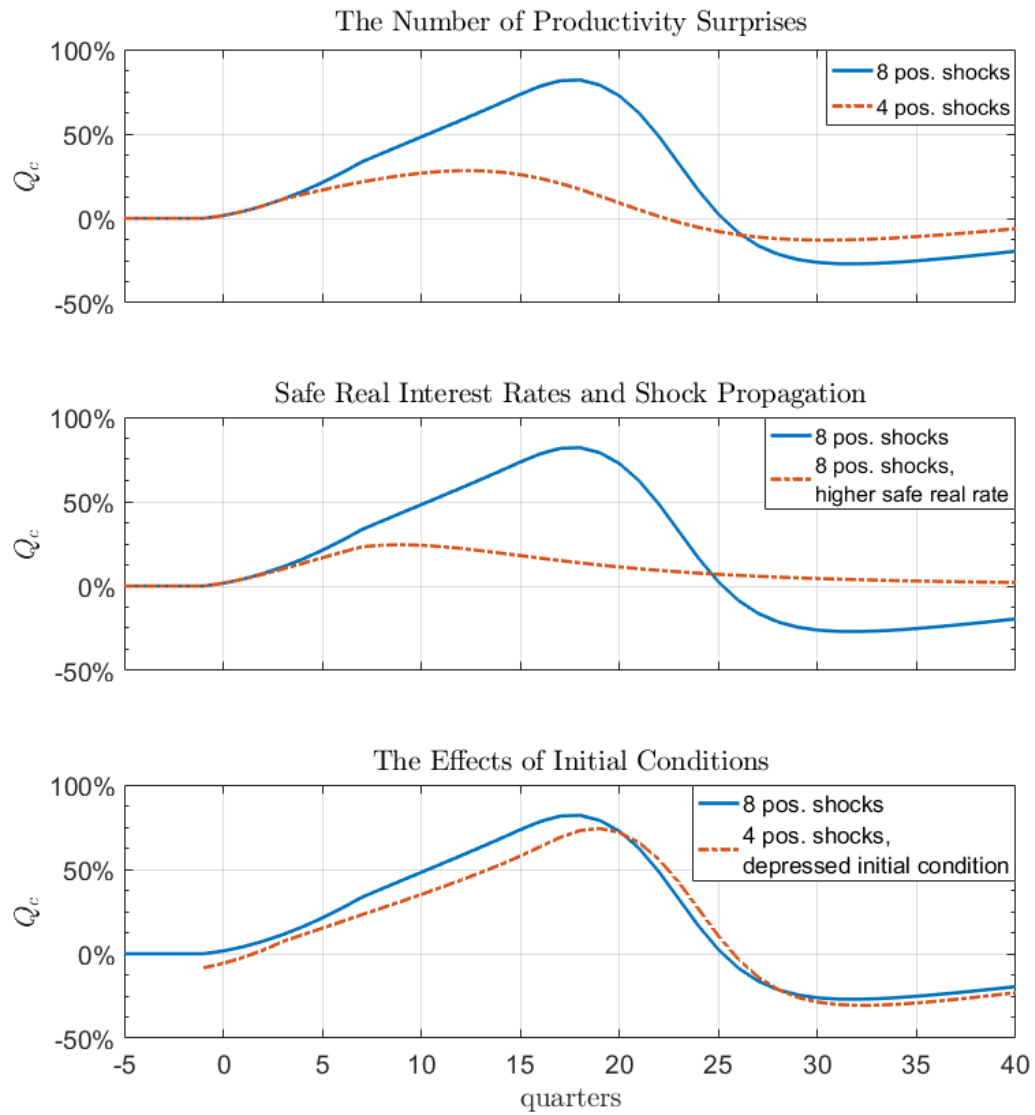
Figure 8. Capital Price Responses: The Effect of the Number of Shocks, Real Interest Rates and Initial Conditions

4 shocks triggers a sizable response, the response following 8 shocks is about three times as large. This non-linearity in the number of shocks is due to the fact that a long series of shocks triggers the endogenous belief propagation described in the previous section, while a shorter series fails to do so to the same extent.

To see why this is the case, notice that positive productivity shocks increase the expected rental rate on capital,[49] thereby give rise to an increase in capital prices today, even when holding capital gain beliefs constant. These realized capital gains lift capital gain expectations upward. For a sufficiently long series of positive productivity shocks, capital gain expectations increase sufficiently strongly to trigger the self-reinforcing endogenous asset price increases described in the previous section. The model thus tightly links persistent productivity surprises with boom and bust cycles.

The situation differs when there are fewer shocks. Capital gain beliefs then increase by less, so that $B \cdot m_{c,t}$ in equation (34) remains sufficiently far below one, so as to not trigger strong self-reinforcing dynamics: beliefs quickly enter the reversal area below the 45 degree line in figure 4. In fact, the impulse response to a single one standard deviation technology shock in the subjective belief model is very similar to that predicted by the model under rational price expectations, as we illustrate in appendix C.

Boom-Bust Cycles and the Real Interest Rate. The central panel in figure 8 depicts impulse responses to a series of 8 positive productivity shocks. It depicts it once for the estimated model and once for a model where we lower the time discount factor (β) from 0.996 to 0.99, so that the safe real interest rate increases from 0.8% to 1.4%.[50] With a higher safe rate, the capital price response becomes considerably more muted: prices monotonically decrease back to steady state after the shocks have ended. The source of this dampening effect can again be understood from the simplified capital pricing equation (34).[51] A lower value for β reduces the value of B in this equation and thereby moves Bm_t further away from one. Revisions in capital gain expectations (m_t), triggered by the series of positive productivity shocks, then generate smaller movements in capital prices. Once prices become less sensitive to belief revision, endogenous propagation, which operates via the mutual feedback between belief revisions and price changes, is dampened. As a result, the economy becomes more stable when safe interest rates are higher.

Repeated Boom-Bust Cycles. The lower panel in figure 8 presents impulse responses for different starting values and different number of shocks. One impulse response depicts the standard response following 8 positive productivity shocks and when starting the economy at its ergodic mean for the state variables. This is the same response as shown in the upper two panels of figure 8. The second impulse response in the lower panel

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of the stationary distribution.

[49]This follows from equation (21) and the fact that output in the consumption sector permanently increases following the shock, while hours worked remain constant.

[50]Safe real rates are almost constant, as can be seen from the results in table 6.

[51]The exact pricing equation (33) delivers the same insights.

uses different starting values: its start from a situation where capital gain expectations $(m_{c,t})$ and lagged capital prices $(Q_{c,t-1})$ and the beginning-of-period capital stock $(K_{c,t})$ are depressed, i.e., at their 33% quantile value.[52] Capital gain expectations are then 20 basis points below their ergodic mean value, a situation that emerges towards the end of the bust phase of a boom-bust cycle, see for instance figure 5. Starting from this initial value, we then subject the economy to 4 positive shocks (of 2 standard deviation size). The lower panel in figure 8 shows that with this initial condition, the 4 positive shocks deliver nearly the same price response as when starting the economy at the ergodic mean and subjecting it to 8 surprises. The top panel in figure 8 shows that 4 surprises deliver a much smaller price response, when starting the economy in its ergodic mean.

This shows how depressed initial conditions give rise to a heightened sensitivity to another boom-bust cycle. The sensitivity is increased because there is already upward momentum of prices and beliefs when starting from a depressed initial situation. This can be seen from the belief dynamics depicted in figure 4, which shows that from certain depressed initial conditions (in the lower left corner of the figure, below the 45 degree line) no shocks at all would be needed to trigger another boom-bust cycle. This model can thus give rise to a pattern of repeated boom-bust cycles, as experienced over the past decades in the United States, see figure 1.

10   Additional Evidence on Model Performance

Figure 9 further illustrates the dynamics of the estimated subjective belief model by depicting simulated paths for output and technology and the PD-ratio for 200 quarters.[53] The lower panel shows that the aggregate PD-ratio occasionally displays large and persistent stock price run-ups and reversals, largely in line with the empirical behavior of the PD ratio. Interestingly, all three price run-ups in the lower panel of figure 9 are preceded by a sequence of positive technology surprises that cause technology to increase faster than average over a number of periods, see the upper panel.[54] Moreover, there is evidence of clustering of boom-bust cycles, as three of the four price peaks occur between quarters 45 and 100.

The occasional price run-ups and reversals allows the subjective belief model to replicate the skewness of the empirical PD distribution. This is illustrated in Figure 10, which plots the kernel density of the empirical P/D distribution together with the distribution of the estimated subjective belief model. It shows that the model generates a right tail that

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[52]The inital values for beliefs and capital prices in the investment sector $(m_{i,t}, Q_{i,t})$ do not affect the impulse response dynamics for $Q_{c,t}$.

[53]The figure detrends log output and log technology by removing a deterministic linear time trend with slope equal to the mean log growth rate of technology, $\log \gamma - \frac{\sigma^2}{2}$.

[54]Due to detrending, the expected growth rate of technology is zero, so that an upward move represents a positive surprise.
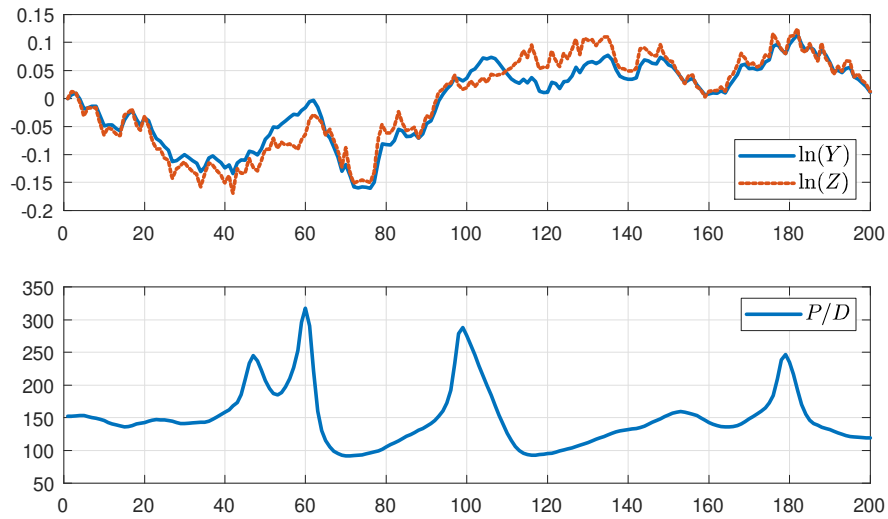
Figure 9. Simulated equilibrium paths for output, technology, and the price-dividend ratio
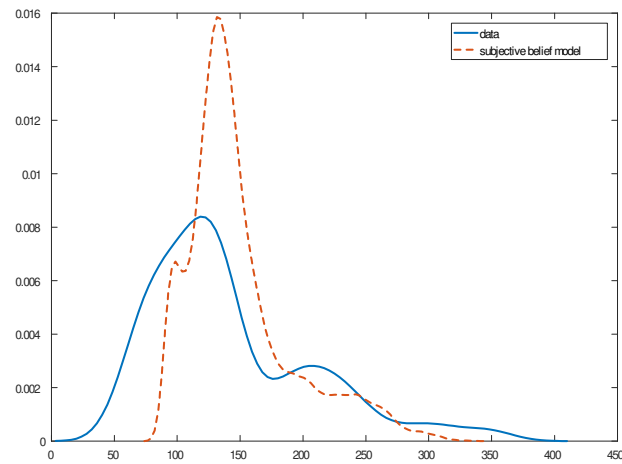
Figure 10. Unconditional density of PD ratio: subjective belief model versus data (not targeted in estimation, kernel estimates)

is remarkably close to the one observed in the data. However, the model has more mass around the mode, as very small PD ratios are less likely than in the data. The reason for this asymmetry is again the fact that belief revisions have much larger effects on prices in states of optimism than in states of pessimism, as is easily seen from equations (33) and (36). For this reason, additional negative shocks in a price bust do not drive down prices by as much as additional positive shocks drive prices up in a boom. The lack of mass in the left tail of the distribution also explains why we cannot match the full scale of the volatility of the PD ratio in table 5.

The PD distributions implied by the two estimated RE models from section 7 are depicted in online appendix D.7. They show that the RE models fail to generate a skewed PD distribution.

Table 7
The effects of shutting down subjective price beliefs

	Data	Subjective Belief Model	REE Implied by Subj. Belief Model
Business Cycle Moments			
$\sigma(Y)$	1.72 (0.25)	1.83	1.60
$\sigma(C)/\sigma(Y)$	0.61 (0.03)	0.67	0.89
$\sigma(I)/\sigma(Y)$	2.90 (0.35)	2.90	1.59
$\sigma(H)/\sigma(Y)$	1.08 (0.13)	1.06	0.12
$\rho(Y,C)$	0.88 (0.02)	0.84	0.96
$\rho(Y,I)$	0.86 (0.03)	0.89	0.91
$\rho(Y,H)$	0.75 (0.03)	0.70	0.70
Financial Moments			
$E[P/D]$	152.3 (25.3)	150.0	199.7
$\sigma(P/D)$	63.39 (12.39)	44.96	8.99
$\rho(P/D)$	0.98 (0.003)	0.97	0.99
$E[r^e]$	1.87 (0.45)	1.25	0.68
$\sigma(r^e)$	7.98 (0.35)	7.07	0.19
$E[r^f]$	0.25 (0.13)	0.78	0.68
$\sigma(r^f)$	0.82 (0.12)	0.06	0.06
$\sigma(D_{t+1}/D_t)$	1.75 (0.38)	2.46	0.92

11 Welfare Implications of Belief-Driven Booms

We now analyze how the presence of subjective beliefs affects welfare. Table 7 compares moments of the estimated subjective belief model from section 7 to the ones implied by the same models parameters, when replacing subjective price beliefs by rational price

expectations. Since the RE outcome is efficient, the latter represents a natural welfare benchmark. Table 7 shows that with rational price expectations, the volatility of output, hours worked and investment all fall, while the volatility of consumption rises. Not surprisingly, the standard deviation of the PD ratio decreases by more than three quarters and the standard deviation of stock returns falls even more dramatically.

Despite the large volatility reduction, the welfare gains associated with eliminating subjective beliefs turn out to be small. Indeed, we find that the utility gain associated with replacing subjective price beliefs by rational price beliefs amount to 0.29% of consumption per period, when using the distribution of ex-post realized outcomes to evaluate welfare in the subjective belief model.

The welfare gains arise mainly through changes in the *mean* levels of consumption and work. With rational price expectations, consumption is on average higher and hours worked lower than under subjective price expectations. This is possible because asset prices correctly signal the value of investment under rational expectations, so that agents make more efficient investment choices. With subjective price beliefs, instead, investment is often triggered by belief-driven price signals.

12   Conclusions

A simple real business cycle model featuring extrapolative stock price expectations, in line with the survey evidence, can produce realistic amounts of stock price volatility and realistic business cycle dynamics.

The model delivers radically different messages than traditional RE modeling approaches that search for appropriate discount factor specifications to explain stock price volatility: (1) a large part of the observed stock price volatility, investment volatility and volatility in hours worked is inefficient and the result of belief-driven boom and bust dynamics; (2) stock price booms and the associated output booms are triggered by repeated positive technology surprises and are more likely to occur when the safe interest rate is low; (3) boom-bust cycles tend to occur in clusters; and (4) belief-driven boom-bust cycles can lead to a persistent over-accumulation of capital that is followed by a prolonged recessionary period with under-accumulation, so that booms can carry in them the seeds of a future recession.

The welfare costs of belief-driven boom-bust cycles, however, turn out to be moderate. This is partly due to the fact that – within our representative agent framework – belief-driven fluctuations generate only intertemporal distortions for investment. Several aspects that are absent from the present model may potentially overturn this finding. The welfare effects of price cycles could become larger, for instance, in a setting with heterogeneous agents, where trading activity may give rise to wealth redistribution across boom and bust cycles. Likewise, in a richer model with a cross-section of economic sectors, belief-driven booms may distort the investment decisions across sectors and across

time. Finally, if asset prices play an important role for the balance sheet dynamics of a financially constrained sector, then fluctuations in asset prices may have larger effects on allocations than the present model suggests. Exploring the welfare relevance of these channels appears to be an interesting avenue for future research.

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A   Appendix

A.1   Data Sources

Data on Macro Aggregates Data series on macro aggregates and related variables
have all been downloaded from the Federal Reserve Economic Data (FRED) database
maintained by the federal reserve bank of St. Louis (https://fred.stlouisfed.org). All time
series refer to the United States, are at quarterly frequency and cover the sample period
Q1:1955 to Q4:2014. Data on output, consumption and investment is from the National
Income and Product Accounts of the BEA. We measure nominal output by gross domestic
product (FRED Code "GDP"), nominal consumption by personal consumption expendit-
ures in nondurable goods ("PCND") and services ("PCESV") and nominal investment
by fixed private investment ("FPI"). Nominal values of investment subcomponents for
Table 4 are private residential fixed investment ("PRFI") and private nonresidential fixed
investment in structures ("B009RC1Q027SBEA"), equipment ("Y033RC1Q027SBEA")
and intellectual property products ("Y001RC1Q027SBEA"). These nominal series have
been deflated by the consumer price index for all urban consumers ("CPIAUCSL") from
the BLS, which is consistent with the deflating procedure used for stock prices and in-
terest rates by Adam, Marcet, and Beutel (2017) and us. Hours worked are based on
an index of nonfarm business sector hours ("HOANBS") published by the BLS. Working
age population is based on data published by the OECD ("LFWA64TTUSQ647N").

Stock Prices, Interest Rates and Investor Expectations We use identical data
sources as in Adam, Marcet, and Beutel (2017) and refer to their data appendix for
details. They use 'The Global Financial Database' to obtain data on stock prices and
interest rates until Q1:2012. We extend their stock price and interest rate data to Q4:2014
using identical data sources as they do. We use the UBS survey as our benchmark survey
source in table 3.

   We compute dividends based on price and total return index data of the SP 500 index
by the procedure outlines there, resulting in a dividend series $\{D_t\}$, where t runs through
all quarters from Q1:1955 to Q4:2014. Given the price series from the price index $\{P_t\}$,
where P_t is the closing price of the last trading day in quarter t, we define the price-
dividend ratio as the ratio P_t/D_t of the end-of-quarter closing price P_t and the within
quarter dividend D_t.

A.2   Related Rational Expectations Literature: An Overview

This appendix discusses the rational expectations literature studying business cycle and
stock price behavior. To the extent that papers in this literature produce empirically
realistic stock price behavior, i.e., expected stock returns that are counter-cyclical, the

implications of these models are inconsistent with the pro-cyclical behavior of survey return expectations.[55]

Early modeling approaches rely on a combination of habit preferences and adjustment frictions to generate high stock price volatility and plausible business cycle dynamics (Jermann, 1998; Boldrin, Christiano, and Fisher, 2001; Uhlig, 2007). Habit preferences create a low elasticity of intertemporal substitution (EIS) and thereby a strong desire to smooth consumption over time. As a result, even small (business-cycle sized) consumption fluctuations give rise to volatile stock prices. Aversion against intertemporal consumption substitution, however, generates volatility in all assets and thus a counterfactually high volatility for the risk-free interest rate. A notable exception is Uhlig (2007) who considers a setting with external habits. External habits create strong fluctuations in risk aversion[56] and thereby can give rise to an empirically plausible Sharpe ratio and a stable risk-free interest rate.[57] Agents in these models have strong desire to intertemporally smooth consumption, but are prevented from doing so, as the model otherwise produces insufficient stock price volatility. This is achieved by introducing labor market frictions in various forms: Jermann (1998) assumes labor supply to be fully inelastic, Boldrin, Christiano, and Fisher (2001) introduce timing frictions that force households to choose labor supply in advance, and Uhlig (2007) introduces real wage rigidity that leads to labor supply rationing following negative productivity shocks.

Models with limited stock market participation have subsequently been employed to jointly model business cycle dynamics and stock price behavior.[58] In these settings, a limited set of agents has access to the stock market and in addition insures the consumption of non-participating agents via various other contracts. An early example is Danthine and Donaldson (2002), who consider shareholders and hand-to-mouth workers. Shareholders optimally offer workers a labor contract that insures workers and that gives rise to 'operating leverage' for shareholders. The cash flows of shareholders thus become more volatile and procyclical, which gives rise to an equity premium and volatile stock returns, albeit at the cost of creating too much volatility for shareholders' consumption.[59]

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[55] See Adam, Marcet, and Beutel (2017).

[56] See Boldrin, Christiano, and Fisher (1997) for a discussion of the differing risk aversion implications of internal and external habit specifications.

[57] It remains unclear whether the model seperately matches the volatility of stock returns and the size of the equity premium, as only the Sharpe ratio is reported.

[58] As mentioned in Guvenen (2009), stock market participation increased substantially in the U.S. during the 1990's. From 1989 to 2002 the number of households who owned stocks increased by 74%, with half of U.S. households owing stocks by the year 2002.

[59] Table 6 in Danthine and Donaldson (2002), which displays the specification with the best overall empirical fit, shows that shareholder's consumption volatility is about 10 times as large as aggregate consumption volatility. Guvenen (2009) reviews the empirical evidence on stockholders' relative consumption volatility and concludes that stockholders' consumption is about 1.5-2 times as volatile as non-stockholders' and thus - in relative terms - even less high when compared to aggregate consumption volatility

Guvenen (2009) considers a model in which all agents participate in the bond market but only some in the stock market. If stock market participants have a higher EIS than non-participants, then the former optimally insure the latter against income fluctuations via bond market transactions, thereby channeling most labor income risk to stock market participants. As a result, their consumption is strongly procyclical and gives rise to both a high equity premium and high volatility of returns. The model assumes the EIS to be in absolute terms low - even for shareholders - thus generates additional stock price volatility through the same channels as habit models, but generates a more stable risk-free rate. The model performs quantitatively very well along the financial dimension, while on the business cycle dimension consumption tends to be too volatile and investment and hours too smooth.

Tallarini (2000), Gourio (2012), Croce (2014) and Hirshleifer, Li, and Yu (2015) discuss the asset pricing predictions of the real business cycle models under Epstein-Zin preferences (Epstein and Zin, 1989), assuming that the coefficient of risk aversion is larger than the inverse of the EIS. Tallarini (2000) shows that increasing risk aversion - while keeping the EIS fixed and equal to one - barely affects business cycle dynamics, but has substantial effects on the price of risk. This allows the model to generate a high Sharpe ratio, although it considerably undershoots the equity premium and the volatility of stock returns. Gourio (2012) considers preferences with a larger EIS and moderate risk aversion and enriches the model by time-varying disaster risk.[60] Whereas constant disaster risk has little effect on the model dynamics, time-variation in disaster risk combined with preferences for early resolution of uncertainty generate a high equity premium and high return volatility. At the same time, the business cycle dynamics remain in line with the data, provided the disaster does not realize. Croce (2014) considers a production economy with Epstein-Zin preferences and long-run productivity growth risk. The model matches well the equity premium for levered equity returns and produces a low and stable risk-free rate. It also matches business cycle dynamics, but falls slightly short of fully replicating the volatility of stock returns.

The present paper is also related to the literature on rational stock market bubbles, as for instance derived in classic work by Froot and Obstfeld (1991). While rational bubbles provide an alternative approach for generating stock market volatility, rational bubbles are inconsistent with empirical evidence along two important dimensions. First, the assumption of rational return expectations is strongly at odds with survey measures of return expectations, as mentioned above. Second, Giglio, Maggiori, and Stroebel (2016) show that there is little evidence supporting the notion that violations of the transversality condition drive asset price fluctuations, unlike suggested by the rational bubble hypothesis.

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[60] A disaster is a potentially persistent event in which the economy experiences in each disaster period a negative productivity shock and a large capital depreciation shock.

A.3 Isomorphic Specification with Capital Adjustment Costs

This appendix presents a model version with a linear production technology in the investment sector and capital adjustment costs that is isomorphic to the model in the main text. It distinguishes between the price of investment goods and the price of installed capital and is able to account for the fact that the relative price of investment is countercyclical in the data (Greenwood, Hercowitz, and Krusell, 2000; Fisher, 2006).

We replace the production function for investment goods in equation (1) by a linear labor-only production function

$$Y_{ig,t} = Z_t H_{ig,t}, \tag{37}$$

where the subscript 'ig' stands for investment goods. Output produced by the investment sector consist of raw investment goods that are not immediately productive. To become productive as capital in the consumption sector, investment goods must be transformed into capital goods. This transformation is performed by an additional capital goods sector. This sector consists of $K_{i,t}$ firms, each of which operates a decreasing-returns-to-scale production technology[61]

$$y_{i,t} = \phi(y_{ig,t}),$$

that transforms $y_{ig,t}$ units of investment goods into $y_{i,t}$ units of capital goods. ϕ is a concave function that describes the degree of adjustment costs. The aggregate output

$$Y_{i,t} = y_{i,t} K_{i,t}$$

enters the capital accumulation equation for K_c (equations (3) and (27)) in exactly the same way as in our baseline model.

Each period, a fraction δ_i of capital goods firms receive a shock that makes their transformation technology obsolete. Also, at the end of each period, new capital goods firms are created by households according to the exogenous process X_t in the main text, such that the number of capital goods firms at the end of the period is $K_{i,t+1} = Z_t$. Household can trade shares of capital sector firms at a price $Q_{i,t}$ per unit of capital K_i in period t.

Investment good firms choose labor input $H_{ig,t}$ to maximize profits

$$Q_{ig} Z_t H_{ig,t} - W_t H_{ig,t},$$

where Q_{ig} is the price of investment goods, which is the model counterpart to the relative price of investment measured in the data. In equilibrium, investment good firms make zero profits and their choices imply

$$Q_{ig,t} = \frac{W_t}{Z_t}. \tag{38}$$

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[61]We choose to index the capital goods sector by i instead of k (and use ig instead of i for the investment sector) to make the mapping between variables in the main text and variables in this alternative specification more transparent. With this naming convention, all equations stated in the main text remain valid in this alternative model specification.

This relationship implies that the relative price of investment will be countercyclical, whenever the wage is less procyclical than productivity Z_t or even countercyclical. This is in fact the case for our estimated model in section 7 and appears to be a robust feature of the model for any reasonable alternative calibration. The presence of investment-specific technology shocks is not required to generate this cyclical pattern.

Capital goods firms purchase raw investment goods at price $Q_{ig,t}$ and sell their output to households at price $Q_{c,t}$. In order to maximize profits

$$Q_{c,t}\phi(y_{ig,t}) - Q_{ig,t}y_{ig,t}$$

their quantity choice $y_{ig,t}$ has to satisfy the first-order condition

$$\frac{Q_{c,t}}{Q_{ig,t}} = \frac{1}{\phi'\left(\frac{Y_{ig,t}}{K_{i,t}}\right)}, \tag{39}$$

where the firm-specific choice $y_{ig,t}$ is already replaced by its representation $\frac{Y_{ig,t}}{K_{i,t}}$ in terms of aggregates (from investment goods market clearing). Equation (39) is the familiar Tobin's Q condition for optimal investment.

Each period, profits of capital goods firms are paid out to households. We denote profits per firm by $R_{i,t}$, because they corresponds to the investment sector rental rate in our baseline model. Using again $y_{ig,t} = Y_{ig,t}/K_{i,t}$, profits are given by

$$R_{i,t} = Q_{c,t}\phi\left(\frac{Y_{ig,t}}{K_{i,t}}\right) - Q_{ig,t}\frac{Y_{ig,t}}{K_{i,t}}. \tag{40}$$

Equations (38), (39) and (40) jointly replace equations (23) and (24) in the baseline model and the production function (37) replaces the investment-sector part of (1). All other equations collected in section 6 remain valid and are – together with the three conditions derived here and the alternative production function – sufficient to characterize the equilibrium in this alternative specification. The model presented here is thus identical to the baseline model, if (37), (38), (39) and (40) are equivalent to the Y_i equation in (1) and equations (23) and (24).[62] Obviously, this depends on the choice of the adjustment cost function ϕ. We show next that a standard adjustment cost function ϕ achieves this equivalence.

Substituting equation (37), (38) into (39) and (40) yields after some minor rearrangements

$$W_t = Q_{c,t}\phi'\left(\frac{Z_t H_{ig,t}}{K_{i,t}}\right) Z_t \tag{41}$$

$$R_{i,t} = Q_{c,t}\phi\left(\frac{Z_t H_{ig,t}}{K_{i,t}}\right) - \frac{W_t H_{ig,t}}{K_{i,t}} \tag{42}$$

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[62]More precisely: have equivalent implications for all variables other than $Q_{ig,t}$ for which no counterpart exists in the main text.

Comparing equation (41) with equation (23) and imposing $H_{ig,t} = H_{i,t}$, we get

$$\phi'\left(\frac{Z_t H_{i,t}}{K_{i,t}}\right) = (1 - \alpha_i)\left(\frac{Z_t H_{i,t}}{K_{i,t}}\right)^{-\alpha_i}.$$

Integrating this equation implies the functional form

$$\phi(\iota) = \iota^{1-\alpha_i}. \tag{43}$$

Given this equation, it is easy to see that equation (42) is then also identical to equation (24). In addition, aggregate output of the capital goods sector in equilibrium is

$$Y_{i,t} = \phi\left(\frac{Z_t H_{i,t}}{K_{i,t}}\right) K_{i,t} = (Z_t H_{i,t})^{1-\alpha_i} K_{i,t}^{\alpha_i}$$

which corresponds exactly to the investment-sector production function of the model in the main text (equation (1)).

Consequently, under the specific adjustment cost specification (43) the two models are isomorphic. α_i should then not be interpreted as a capital share in production, but rather as a curvature parameter of the capital transformation function. In the literature, a popular adjustment specification for ϕ is[63]

$$\phi(\iota) = \frac{a_1}{1 - 1/\xi}\iota^{1-1/\xi} + a_2,$$

where ξ is a curvature parameter and a_1 and a_2 are usually fixed by imposing that the balanced growth path of the economy should be identical to the one of an economy without adjustment costs (in particular, Tobin's Q is 1 along the balanced growth path). Equation (43) is a special case of this general functional form used in the literature. It differs from the literature in sofar as it does not satisfy the steady-state restriction. The important parameter, however, is the curvature parameter ξ and using the definition $\xi = 1/\alpha_i$, our adjustment cost specification here is a special case of the standard specification in the literature. With this reinterpretation of the capital share α_i in capital goods production, our estimated parameter value of $\alpha_i = 0.73$ translates into a ξ parameter of $\xi = 1.37$ which does not seem implausibly low relative to the values typically chosen in the literature (e.g. $\xi = 0.23$ in Jermann (1998) and Boldrin, Christiano, and Fisher (2001), $\xi = 0.4$ in Guvenen (2009)).

A.4 Stock Prices and Dividends as Functions of Capital Prices, Rental Rates and Profit Payout Rates

This appendix provides details on the definition of stock prices and dividends in the model. We start with the definition of sectoral prices and dividends. We describe the

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[63]Compare e.g. Jermann (1998), Boldrin, Christiano, and Fisher (2001) and Guvenen (2009).

setting for the consumption sector in greater detail. The setup for the investment sector is identical up to an exchange of subscripts. Let $k_{c,t}$ denote the units of (beginning-of-period t) capital held per unit of shares issued in the consumption sector. The capital is used for production and earns a rental income/profit of $k_{c,t}R_{c,t}$. Given the payout ratio $p \in (0,1)$, dividends per share are given by

$$D_{c,t} = pk_{c,t}R_{c,t}.$$

Retained profits are reinvested to purchase $(1-p)\,k_{c,t}R_{ct}/Q_{c,t}$ units of new capital per share.[64] The end-of-period capital per share then consists of the depreciated beginning-of-period capital stock and purchases of new capital from retained profits. The end-of-period share price $P_{c,t}$ is thus equal to[65]

$$P_{c,t} = (1-\delta)k_{c,t}Q_{c,t} + (1-p)k_{c,t}R_{c,t},$$

and the end-of-period PD ratio is given by

$$\frac{P_{c,t}}{D_{c,t}} = \frac{1-\delta_c}{p}\frac{Q_{c,t}}{R_{c,t}} + \frac{1-p}{p}.$$

Since the last term is small for reasonable payout ratios p, the end-of-period PD ratio is approximately proportional to the capital price over rental price ratio $(Q_{c,t}/R_{c,t})$. Moreover, as is easily verified, the equity return per unit of stock $R_{c,t}^e = (P_{c,t} + D_{c,t})/P_{c,t-1}$ is equal to the return per unit of capital $R_{c,t}^k = ((1-\delta_c)Q_{c,t} + R_{c,t})/Q_{c,t-1}$.

Given sectoral stock prices and PD ratios, we can define the aggregate PD ratio using a value-weighted portfolio of the sectoral investments. Let

$$w_{c,t-1} = \frac{Q_{c,t-1}K_{c,t}}{Q_{c,t-1}K_{c,t} + Q_{i,t-1}K_{i,t}}$$

denote the end-of-period $t-1$ value share of the consumption sector. The value share of the investment sector is then $1 - w_{c,t-1}$. A portfolio with total value P_{t-1} at the end of period $t-1$ and value shares $w_{c,t-1}$ and $1 - w_{c,t-1}$ in the consumption and investment-sector, respectively, must contain $w_{c,t-1}P_{t-1}/P_{c,t-1}$ consumption shares and $(1 - w_{c,t-1})\,P_{t-1}/P_{i,t-1}$ investment shares. The end-of-period t value of this portfolio is then given by

$$P_t = \frac{w_{c,t-1}P_{t-1}}{P_{c,t-1}}P_{c,t} + \frac{(1 - w_{c,t-1})\,P_{t-1}}{P_{i,t-1}}P_{i,t} \tag{44}$$

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[64]In case the aggregate capital supply differs from capital demand implied by the existing number of shares, new shares are created or existing shares repurchased to equilibrate capital demand and supply.

[65]We compute end-of-period share prices, because this is the way prices have been computed in the data.

and period t dividend payments for this portfolio are

$$D_t = \frac{w_{c,t-1}P_{t-1}}{P_{c,t-1}}D_{c,t} + \frac{(1 - w_{c,t-1})\,P_{t-1}}{P_{i,t-1}}D_{i,t}. \tag{45}$$

Using the previous two equations, the aggregate PD ratio can be expressed as a weighted mean of the sectoral PD ratios where the weights are given by the share of portfolio dividends coming from each sector:

$$\frac{P_t}{D_t} = \frac{w_{c,t-1}\frac{D_{c,t}}{P_{c,t-1}}}{w_{c,t-1}\frac{D_{c,t}}{P_{c,t-1}} + (1 - w_{c,t-1})\frac{D_{i,t}}{P_{i,t-1}}}\frac{P_{c,t}}{D_{ct}} + \frac{(1 - w_{c,t-1})\frac{D_{i,t}}{P_{i,t-1}}}{w_{c,t-1}\frac{D_{c,t}}{P_{c,t-1}} + (1 - w_{c,t-1})\frac{D_{i,t}}{P_{i,t-1}}}\frac{P_{i,t}}{D_{i,t}}.$$

Note that the PD ratio is independent of the initial portfolio value P_{t-1}. Aggregate dividend growth can similarly be expressed using equations (45) and (44) as a weighted average of the sectoral dividend growth rates.

A.5   Details of the Projection Facility

Following Adam, Marcet, and Nicolini (2016), we modify the belief updating equation (15) to

$$\ln m_{s,t} = w_{s,t}\left(\ln m_{s,t-1} + g\left(\ln Q_{s,t-1} - \ln Q_{s,t} - \ln m_{s,t-1}\right) + g\ln \varepsilon_{s,t}^1\right)$$

where $w_{s,t}(\cdot)$ is a differentiable function satisfying $w_{s,t}(x) = x$ for $x \leq \underline{m}_{s,t}$ and $w_{s,t}(x) \leq \overline{m}_{s,t}$ for all x, with $\overline{m}_{s,t} > \underline{m}_{s,t}$. Beliefs are thus bounded below $\overline{m}_{s,t}$, but evolve as described in the main text as long as they remain below $\underline{m}_{s,t}$. Following Adam, Marcet, and Nicolini (2016) we consider the function

$$w_{s,t}(x) = \begin{cases} x & \text{if } x \leq \underline{m}_{s,t} \\ \underline{m}_{s,t} + \frac{x - \underline{m}_{s,t}}{x + \overline{m}_{s,t} - 2\underline{m}_{s,t}}(\overline{m}_{s,t} - \underline{m}_{s,t}) & \text{if } \underline{m}_{s,t} < x. \end{cases} \tag{46}$$

and calibrate the critical values $\left(\underline{m}_{s,t}, \overline{m}_{s,t}\right)$ in both sectors $s = c, i$ such that $\underline{m}_{s,t}$ is the degree of optimism that implies a quarterly PD ratio of 250 and $\overline{m}_{s,t}$ is the degree of optimism implying a PD ratio of 500. The critical PD values of 250 and 500 are taken from Adam, Marcet, and Nicolini (2016).

We now explain how these critical values can be computed. The Euler equation for capital in sector s is

$$Q_{s,t} = \beta\left(1 - \delta_s\right)E_t^{\mathcal{P}}\left[\frac{W_t}{W_{t+1}}Q_{s,t+1}\right] + \beta E_t^{\mathcal{P}}\left[\frac{W_t}{W_{t+1}}R_{s,t+1}\right]$$

$$= \beta\left(1 - \delta_s\right)E_t^{\mathcal{P}}\left[\frac{W_t}{W_{t+1}}\right]m_{s,t}Q_{s,t} + \beta E_t^{\mathcal{P}}\left[\frac{W_t}{W_{t+1}}R_{s,t+1}\right]$$

implying

$$Q_{s,t} = \frac{\beta E_t^{\mathcal{P}} \left[\frac{W_t}{W_{t+1}} R_{s,t+1} \right]}{1 - \beta \left(1 - \delta_s \right) E_t^{\mathcal{P}} \left[\frac{W_t}{W_{t+1}} \right] m_{s,t}}$$

and thus the price-dividend ratio is

$$\frac{P_{s,t}}{D_{s,t}} = \frac{1 - \delta}{p} \frac{Q_{s,t}}{R_{s,t}} + \frac{1 - p}{p} = \frac{1}{p} \frac{(1 - \delta) \beta E_t^{\mathcal{P}} \left[\frac{W_t}{W_{t+1}} \frac{R_{s,t+1}}{R_{s,t}} \right]}{1 - \beta \left(1 - \delta \right) E_t^{\mathcal{P}} \left[\frac{W_t}{W_{t+1}} \right] m_{s,t}} + \frac{1 - p}{p}. \qquad (47)$$

The value $\underline{m}_{s,t}$ is the value for $m_{s,t}$ in the preceding equation that causes the PD to be equal to 250; likewise, $\overline{m}_{s,t}$ is the value that causes the PD ratio to be equal to 500.[66] Since the expectations of W_{t+1} and $R_{s,t+1}$ both depend on $K_{c,t+1}$ (this follows from equations (21) and (22) and the fact that $Y_{c,t+1} = K_{c,t+1}^{\alpha_c} \left(Z_{t+1}(1 - \alpha) \right)^{1 - \alpha_c}$), solving for the values $\left(\underline{m}_{c,t}, \overline{m}_{c,t} \right)$ thus requires simultaneously solving for the optimal investment $Y_{i,t}$ in period t. Since the capital stock dynamics in the investment sector are exogenous, $\underline{m}_{i,t}, \overline{m}_{i,t}$ can be computed once the new belief $m_{c,t}$ that incorporates the projection facility has been determined.

A.6   Proof of Proposition 1

Given the assumption of rational dividend and wage expectations and given the assumed subjective price beliefs (29), we obtain from equation (18)

$$Q_{c,t} = \beta E_t \left[\frac{W_t}{W_{t+1}} \right] \left(1 - \delta_c \right) m_{c,t} Q_{c,t} + \beta E_t \left[\frac{W_t}{W_{t+1}} R_{c,t+1} \right]. \qquad (48)$$

Using the equilibrium relationships (17), (22) and (25) allows expressing the discounted expected rental rate (the last term in equation (48)) using period t variables:

$$E_t \left[\frac{W_t}{W_{t+1}} R_{c,t+1} \right] = \alpha_c \frac{W_t}{K_{c,t+1}} \qquad (49)$$

Substituting the previous expression into equation (48), solving for $Q_{c,t}$ and using equation (31) to substitute W_t and W_{t+1}, delivers (33), where $e \equiv E_t[(\gamma \varepsilon_{t+1})^{\alpha_c - 1}] > 0$.

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[66] In addition, we do not allow $\overline{m}_{c,t}$ to exceed the theoretical upper bound on beliefs for which uniqueness has been proven in Appendix D.1.3 (see Lemma 4), irrespective of the implied PD ratio. This is done to insure equilibrium uniqueness, but is a purely theoretical concern. In practice, we encountered not a single case in which this upper bound was binding in our numerical simulations.

B Equilibrium Uniqueness and Recursive Dynamics

This appendix discusses the recursive structure of equilibrium dynamics and shows that the equilibrium is unique under our belief setups.[67]

With fully rational expectations, the setup is standard. Household decision functions depend only on the history of fundamental shocks Z^t. Due to the Markov structure of these shocks, the household decision functions and thus the model solution can be described by recursive time-invariant functions that depend on the current productivity shock and the beginning-of-period capital stocks only, $(Z_t, K_{c,t}, K_{i,t})$. We standardly solve for the nonlinear rational expectations equilibrium using global approximation methods.[68] As is well known, the rational expectations equilibrium is unique.[69]

The situation is slightly more complicated for our setup with subjective capital price beliefs. Households' probability space is then given by

$$\widetilde{\Omega} \equiv \Omega_Z \times \Omega_{Q,c} \times \Omega_{Q,i},$$

where $\Omega_{Q,s}$ denotes the history of capital prices ($s = i, c$). The household probability space is thus larger than under rational expectations. Due to our parsimonious belief formulation, it is still considerably simpler than the general specification in equation (8). We can thus consider decision functions of the form

$$(C_t, H_t, K_{c,t+1}, K_{i,t+1}) : \widetilde{\Omega}^t \longrightarrow \mathbb{R}^4. \tag{50}$$

and let $\widetilde{\mathcal{P}}$ denote the households' subjective probability beliefs over this reduced probability space $\widetilde{\Omega}$. Given the specific assumption about subjective price beliefs made in section 5, we can furthermore summarize the history of capital prices by the posterior mean beliefs $(m_{c,t}, m_{i,t})$, so that household decision functions are time-invariant functions of the extended set of state variables $(Z_t, K_{c,t}, K_{i,t}, m_{c,t}, m_{i,t})$.

Under the subjective belief setup, the state S_t describing the aggregate economy at time t is given by the vector $S_t = (Z_t, K_{c,t}, K_{i,t}, m_{c,t}, m_{i,t}, Q_{c,t-1}, Q_{i,t-1})$. It includes the lagged capital prices $(Q_{c,t-1}, Q_{i,t-1})$, because these are required for describing the evolution of beliefs over time, see equation (15). The equilibrium dynamics can then be described by a nonlinear state transition function $G(\cdot)$ that maps current states and future technology into future states

$$S_{t+1} = G(S_t, Z_{t+1}),$$

together with an outcome function $F(\cdot)$ that maps these states into economic outcomes for the remaining variables

$$(W_t, R_{c,t}, R_{i,t}, Y_t, C_t, I_t, H_t) = F(S_t).$$

---

[67] Online appendix D.1 offers a more detailed discussion.

[68] This requires a standard transformation of variables, so as to render them stationary.

[69] The market allocation is equivalent to the solution of a social planning problem. The latter features a concave objective function and a convex set of constraints.

We endow households with beliefs about future values $(W_{t+j}, R_{c,t+j}, R_{i,t+j})$ consistent with these equilibrium mappings.[70] This requires solving for a fixed point, as the equilibrium mappings depend on the solution to the household problem and the solution to the household problem on the assumed equilibrium mappings. For our subjective belief specification, solving for the fixed point at a speed that would allow for formal estimation of the model is generally not feasible, as the state vector S_t contains seven state variables. It is, however, feasible for our linear disutility of labor specification for household preferences, which considerably simplify the problem.

The following proposition shows that there exists a unique equilibrium with subjective price beliefs of the kind we have specified. As we discuss below, the proof of the proposition also provides an approach for efficiently simulating the subjective belief model.

Proposition 2. *There are unique functions G and F and a unique measure $\tilde{\mathcal{P}}$ on $\tilde{\Omega}$, such that*

1. *$\tilde{\mathcal{P}}$ describes the joint beliefs of households about technology and capital prices as defined in Section 5*

2. *G is a state transition function, $S_{t+1} = G(S_t, Z_{t+1})$, and F an outcome function, $(W_t, R_{c,t}, R_{i,t}, Y_t, C_t, I_t, H_t) = F(S_t)$, such that S_t, S_{t+1} and $(W_t, R_{c,t}, R_{i,t}, Y_t, C_t, I_t, H_t)$ are consistent with*

   (a) *All equilibrium conditions (see Section 6)*

   (b) *Households' belief updating equations (equation (15) for $s \in \{c, i\}$)[71]*

The proof of proposition 2 can be found in appendix D.1.3. It provides an explicit characterization of the mappings G and F and determines these functions up to a static nonlinear equation system in three unknown variables (lemma 4 in the appendix).[72]

# C   Impulse Responses: Subjective Beliefs versus RE

This appendix reports impulse responses to a single positive technology shock of a size of one standard deviation. Figure 11 reports the responses using the parameters of the estimated subjective belief model from section 7, once for the setting with subjective price expectations once using rational price expectations. It shows that these responses are surprisingly similar.

---

[70] Appendix D.1.1 shows how the beliefs $\mathcal{P}$ over Ω can be constructed from the beliefs $\tilde{\mathcal{P}}$ over $\tilde{\Omega}$ and the mappings F and G.

[71] With projection adjustments as described in Appendix A.5, where necessary.

[72] Our numerical solution approach repeatedly solves this static equation system to simulate the evolution of the economy over time. The linear disutility of labor in household preference is key to insure that equilibrium dynamics under subjective beliefs have this property.
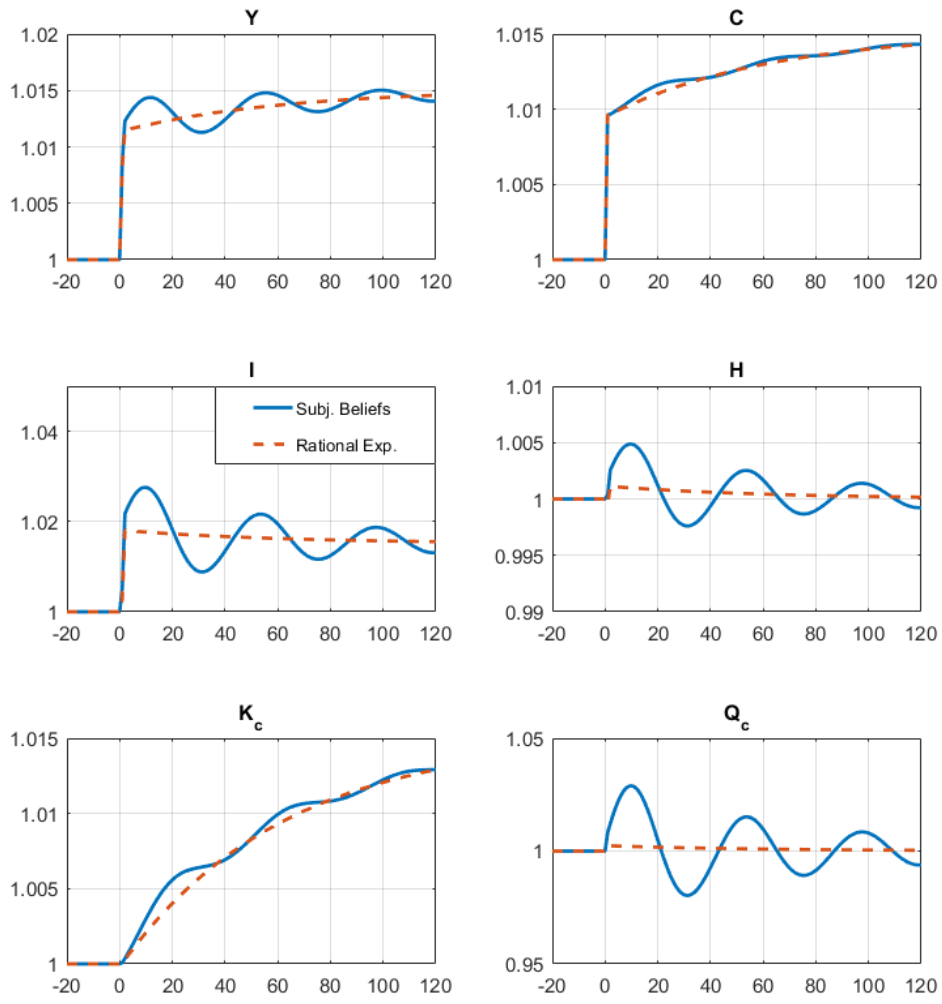
Figure 11. Impulse responses to a single +1 std. dev. technology shock

D   Online Appendix - Not for Publication

D.1   Details on the Representation of Beliefs and Equilibrium Existence

This appendix provides further detailed information on the subjective belief specification, the construction of equilibrium under subjective beliefs, and the proof of proposition 2.

D.1.1   Belief Representations on Ω and $\tilde{\Omega}$

To simplify the exposition, we first introduce some notation. It will also be used in Sections D.1.2 and D.1.3.

Notation   For each random (upper case) symbol A, denote elements of Ω_A by a (these are real sequences) and their t-th component by a_t (these are real numbers).[73] For each random sequence $\{A_t\}_{t=0}^{\infty}$, write shorter just A. On the domain $\tilde{\Omega} = \Omega_Z \times \Omega_{Q,c} \times \Omega_{Q,i}$ we define the random variables (sequences) $\tilde{Z}$, $\tilde{Q}_c$ and $\tilde{Q}_i$ as projections on the first, second and third component, respectively,

$$\tilde{Z}(z, q_c, q_i) = z, \qquad \tilde{Q}_c(z, q_c, q_i) = q_c, \qquad \tilde{Q}_i(z, q_c, q_i) = q_i.$$

Similarly, we define the random variables Z, X, W, R_c, R_i, Q_c, Q_i as projections from the domain $\Omega = \Omega_Z \times \Omega_X \times \Omega_W \times \Omega_{R,c} \times \Omega_{R,i} \times \Omega_{Q,c} \times \Omega_{Q,i}$ to the respective factor.[74] We make the difference between $\tilde{Z}$ (defined on $\tilde{\Omega}$ consisting of typical elements $\tilde{\omega} = (z, q_c, q_i)$) and Z (defined on Ω consisting of typical elements $\omega = (z, x, w, r_c, r_i, q_c, q_i)$) explicit to avoid possible confusion arising in the arguments below. In the main text, we regularly do not make these distinctions and use the same symbols, whenever a variable has the same interpretation, no matter on which space it is defined and whether it is a random variable or a realization.

We regularly have to work with the big vector $(Z, X, W, R_c, R_i, Q_c, Q_i)$ of random sequences on Ω.[75] We will use the short-hand notation O ("observables") for this vector.

Furthermore, we use the following two conventions for the outcome mapping F : $s_t \mapsto (w_t, r_{c,t}, r_{i,t}, y_t, c_t, i_t, h_t)$ (where $s_t = (z_t, k_{c,t}, k_{i,t}, m_{c,t}, m_{i,t}, q_{c,t-1}, q_{i,t-1})$) introduced in section B. We let F_W, $F_{R,c}$, $F_{R,i}$ etc. denote the components of the function F. In a slight abuse of notation, we let F also denote the mapping from the full state sequence $s = \{s_t\}_{t=0}^{\infty}$ into the full outcome sequence $(w, r_c, r_i, y, c, i, h) = \{(w_t, r_{c,t}, r_{i,t}, y_t, c_t, i_t, h_t)\}_{t=0}^{\infty}$ and similarly for the component functions F_W, $F_{R,c}$, $R_{R,i}$ etc.

---

[73]Because the variables m_c and m_i are lower case in the model, those symbols can denote both random variables and realizations. This ambiguity should not lead to any confusion below.

[74]E.g. $Z(z, x, w, r_c, r_i, q_c, q_i) = z$.

[75]As a mapping, this vector just agrees with the identity id_Ω, but this notation conceals the economic interpretation of the random variables.

Mapping a Recursive Evolution into a Sequence Representation Consider a fixed initial state $s_0 = (z_0, k_{c,0}, k_{i,0}, m_{c,0}, m_{i,0}, q_{c,-1}, q_{i,-1})$ and a measurable function G describing a recursive state evolution as in the main text.[76] For any sequence $z \in \Omega_Z$ such that z_0 is consistent with s_0, the recursion

$$s_{t+1} = G(s_t, z_{t+1}) \qquad t = 0, 1, 2, \ldots$$

defines then a unique sequence $s = \{s_t\}_{t=0}^{\infty}$ in $\Sigma := \prod_{t=0}^{\infty} \mathbb{R}^7$ (Σ is the space of all state sequences). The procedure just outlined defines therefore a function $H : \Omega_Z \to \Sigma$ which maps technology sequences z into state sequences s. Obviously, if G is measurable and Σ is endowed with the usual product σ-field, then H is also a measurable function. Call H the sequence representation associated with G, given the initial state s_0.[77]

Consistency of Beliefs with an (Equilibrium) Evolution The following definition clarifies the notion in the main text that beliefs about wages and rental rates be consistent with the equilibrium mappings G and F. The definition can be formulated for arbitrary mappings G and F that do not necessarily need to correspond to the equilibrium mappings of the model, but obviously we are only interested in equilibrium mappings later on.

Definition 2. For a given measurable state evolution G and a given measurable outcome function F, we say that a measure $\mathcal{P}$ on Ω implies beliefs about (W, R_c, R_i) consistent with mappings G and F, if

$$W = F_W \circ H \circ Z, \qquad R_c = F_{R,i} \circ H \circ Z, \qquad R_i = F_{R,i} \circ H \circ Z$$

$\mathcal{P}$-a.s. Here, H is the sequence representation associated with G.

Constructing Consistent Beliefs from G, F and $\tilde{\mathcal{P}}$ The following lemma is the main result of this section and provides the justification why it is sufficient to work with the smaller probability space $\tilde{\Omega}$ instead of Ω.[78]

Lemma 1. *For any given measure $\tilde{\mathcal{P}}$ on $\tilde{\Omega}$ and measurable G and F, there is a unique measure $\mathcal{P}$ on Ω with the following properties:*

1. *The distribution of (Z, Q_c, Q_i) under $\mathcal{P}$ equals $\tilde{\mathcal{P}}$;*

2. *The joint distribution of (Z, X) under $\mathcal{P}$ is consistent with the exogenous relationship between Z and X;*

---

[76] The function G is allowed to be arbitrary here, but should have the same domain and codomain as in Section 5. There is no need for it to conform with any notion of equilibrium or model consistency.

[77] From now on we consider a fixed initial state throughout without explicitly mentioning this anymore.

[78] Note that properties 2 and 3 in the lemma formalize the verbal notion of rational expectations about Z, X, W, R_c and R_i given in footnote 21 in the main text, provided F and G are the equilibrium mappings of the model.

3. $\mathcal{P}$ *implies beliefs about* (W, R_c, R_i) *that are consistent with the mappings* G *and* F.

Proof. We give an explicit construction of the measure $\mathcal{P}$ as the distribution of a set of suitable random variables on $\tilde{\Omega}$ under $\tilde{\mathcal{P}}$. First, the capital accumulation equation (28) for investment-sector capital and the assumption $K_{i,t+1} = Z_t$ imply

$$Z_t = (1 - \delta) Z_{t-1} + X_t \Rightarrow X_t = (Z_t - (1 - \delta_i) Z_{t-1}), \tag{51}$$

The second requirement that the joint distribution of (Z, X) under $\mathcal{P}$ be consistent with the exogenous relationship between those variables means that equation (51) has to hold $\mathcal{P}$-a.s. for all t. We thus define a random variable $\tilde{X} : \tilde{\Omega} \to \Omega_X$ in a way that is consistent with the analog equation (51) on the $\tilde{\Omega}$ domain:[79]

$$\tilde{X}_t := \left(\tilde{Z}_t - (1 - \delta_i) \tilde{Z}_{t-1} \right).$$

Next, let the function $H : \Omega_Z \to \Sigma$ be the sequence representation associated with G. As for X, we simply define random variables for wages, $\tilde{W} : \tilde{\Omega} \to \Omega_W$, and rental rates, $\tilde{R}_c : \tilde{\Omega} \to \Omega_{R,c}$, $\tilde{R}_i : \tilde{\Omega} \to \Omega_{R,i}$, on the domain $\tilde{\Omega}$ in a way that they satisfy the analog of the consistency condition for the probability space $(\tilde{\Omega}, \tilde{\mathcal{S}}, \tilde{\mathcal{P}})$, namely

$$\tilde{W} = F_W \circ H \circ \tilde{Z}, \qquad \tilde{R}_c = F_{R,i} \circ H \circ \tilde{Z}, \qquad \tilde{R}_i = F_{R,i} \circ H \circ \tilde{Z}.$$

With these definitions the random (observables) vector

$$\tilde{O} := (\tilde{Z}, \tilde{X}, \tilde{W}, \tilde{R}_c, \tilde{R}_i, \tilde{Q}_c, \tilde{Q}_i)$$

is a measurable mapping from $\tilde{\Omega}$ to Ω and thus its distribution defines a measure $\mathcal{P}$ on Ω. We claim that this measure satisfies the three conditions in the assertion and is the only measure to do so:

1. Z is the projection defined by $Z(z, x, w, r_c, r_i, q_c, q_i) = z$, so $Z(\tilde{O}) = \tilde{Z}$ and a similar argument shows $Q_c(\tilde{O}) = \tilde{Q}_c$ and $Q_i(\tilde{O}) = \tilde{Q}_i$. So we get

$$(Z, Q_c, Q_i)(\tilde{O}) = \left(\tilde{Z}, \tilde{Q}_c, \tilde{Q}_i \right) = \mathrm{id}_{\tilde{\Omega}},$$

   where id denotes the identity mapping. Because $\mathcal{P}$ is the distribution of $\tilde{O}$ under $\tilde{\mathcal{P}}$, this equation implies that the distribution of (Z, Q_c, Q_i) under $\mathcal{P}$ and the distribution of $\mathrm{id}_{\tilde{\Omega}}$ under $\tilde{\mathcal{P}}$ must be identical. As the latter distribution is $\tilde{\mathcal{P}}$ itself, this proves the first property.

---

[79] For $t = 0$ one must back out $\tilde{Z}_{t-1} = z_{-1}$ from the entry $k_{i,0}$ of the initial state: $z_{-1} = k_{i,0}$.

2. The following equation holds by definition of the random variables X, Z and $\tilde{X}$ (for all $\tilde{\omega} \in \tilde{\Omega}$)

$$
\begin{aligned}
X_t(\tilde{O}) = \tilde{X}_t &= \left(\tilde{Z}_t - (1 - \delta_i) \, \tilde{Z}_{t-1} \right) \\
&= \left(Z_t(\tilde{O}) - (1 - \delta_i) \, Z_{t-1}(\tilde{O}) \right). \\
&= \left[(Z_t - (1 - \delta_i) \, Z_{t-1}) \right] (\tilde{O})
\end{aligned}
$$

As this equation holds on $\tilde{\Omega}$, it must in particular hold $\tilde{\mathcal{P}}$-a.s., so X_t and $(Z_t - (1 - \delta_i) \, Z_{t-1})$ must coincide a.s. with respect to the distribution of $\tilde{O}$ under $\tilde{\mathcal{P}}$, which is exactly $\mathcal{P}$. Hence, equation (51) holds $\mathcal{P}$-a.s.

3. The consistency proof works along the same lines as the proof that (51) has to hold $\mathcal{P}$-a.s. by reducing it to the analogous consistency condition in the tilde space for the tilde variables. The argument is omitted for this reason.

For uniqueness, suppose that $\mathcal{P}'$ is another (arbitrary) measure on Ω such that properties 1-3 are satisfied. Then in particular equation (51) holds $\mathcal{P}'$-a.s. for all t and by the definition of $\tilde{X}$ and $\tilde{Z}$ we obtain for all t

$$
\begin{aligned}
\tilde{X}_t \left(Z, Q_c, Q_i \right) &= \left(\tilde{Z}_t - (1 - \delta_i) \, \tilde{Z}_{t-1} \right) (Z, Q_c, Q_i) \\
&= (Z_t - (1 - \delta_i) \, Z_{t-1}) \\
&= X_t && \mathcal{P}'\text{-a.s.}
\end{aligned}
$$

(here, all equalities except for the last hold even ω-by-ω and the last is just equation (51)). Similarly, from $W = F_W \circ H \circ Z$ $\mathcal{P}'$-a.s. (the consistency condition for wage beliefs under $\mathcal{P}'$) and the definition of $\tilde{W}$ and $\tilde{Z}$ we can conclude

$$
\tilde{W} \left(Z, Q_c, Q_i \right) = F_W \circ H \circ \tilde{Z} \left(Z, Q_c, Q_i \right) = F_W \circ H \circ Z = W \qquad \mathcal{P}'\text{-a.s.}
$$

Identical arguments also yield

$$
\begin{aligned}
\tilde{R}_c \left(Z, Q_c, Q_i \right) &= R_c && \mathcal{P}'\text{-a.s.} \\
\tilde{R}_i \left(Z, Q_c, Q_i \right) &= R_i && \mathcal{P}'\text{-a.s.}
\end{aligned}
$$

Combining those results with the obvious equations $\tilde{Z}(Z, Q_c, Q_i) = Z$, $\tilde{Q}_c(Z, Q_c, Q_i) = Q_c$, $\tilde{Q}_i(Z, Q_c, Q_i) = Q_i$ implies

$$
\mathrm{id}_\Omega = O = \tilde{O} \left(Z, Q_c, Q_i \right) \qquad \mathcal{P}'\text{-a.s.}
$$

But by property 1, the distribution of (Z, Q_c, Q_i) under $\mathcal{P}'$ must be $\tilde{\mathcal{P}}$ and thus the equation shows that the distribution of id_Ω under $\mathcal{P}'$ must be equal to the distribution of $\tilde{O}$ under $\tilde{\mathcal{P}}$, which is by definition $\mathcal{P}$. Hence, $\mathcal{P}' = \mathcal{P}$. $\qquad \square$

D.1.2 Construction and Uniqueness of $\tilde{\mathcal{P}}$

We first construct a measure and show that it has all the properties that any candidate for $\tilde{\mathcal{P}}$ has to have. We then argue why it is the only such measure. First, the following two auxiliary constructions are required. As always, we assume implicitly, that an initial state $s_0 = (z_0, k_{c,0}, k_{i,0}, m_{c,0}, m_{i,0}, q_{c,-1}, q_{i,-1})$ is fixed.

1. Let $\mathcal{P}_Z$ be a measure on Ω_Z that describes the exogenous evolution of Z, i.e. under $\mathcal{P}_Z$ for all $t \geq 1$

$$\ln \varepsilon_t := \ln \left(\frac{Z_t}{\gamma Z_{t-1}} \right)$$

   is i.i.d. normal with mean $-\frac{\sigma^2}{2}$ and variance σ^2 and $Z_0 = z_0$ $\mathcal{P}_Z$-a.s. Clearly, a unique measure with this property exists.

2. For $s \in \{c, i\}$ let $\mathcal{P}_{Q,s}$ be a measure on $\Omega_{Q,s}$ that describes the subjective evolution of Q_s under learning, i.e. under $\mathcal{P}_{Q,s}$ for all $t \geq 1$

$$\ln \varepsilon_t^{Q,s} := \ln \left(\frac{Q_{s,t}}{m_{s,t-1} Q_{s,t-1}} \right)$$

   is i.i.d. normal with mean $-\frac{\sigma_Q^2}{2}$ and variance σ_Q^2 and $\sigma_Q^2 = \sigma_\varepsilon^2$. Here $m_{s,t}$ is recursively defined by equation (15),[80] for $t \geq 1$

$$m_{s,t} = m_{s,t-1} \left(\frac{Q_{s,t-1}}{m_{s,t-1} Q_{s,t-2}} \right)^g .$$

   In addition, $Q_{s,-1} = q_{s,-1}$ $\mathcal{P}_{Q,s}$-a.s. Also, the measure $\mathcal{P}_{Q,c}$ is uniquely defined by these properties.

It is obvious, that $\mathcal{P}_Z$ indeed describes the exogenous evolution of Z as defined in equation (2) and that $\mathcal{P}_{Q,c}$ and $\mathcal{P}_{Q,i}$ represent the marginal distribution of Q_c and Q_i, respectively, under agents' beliefs, if they are to be consistent with the Bayesian learning formulation in Section 5. So any measure $\tilde{\mathcal{P}}$ on $\tilde{\Omega}$ that correctly represents subjective beliefs as defined in Section 5 must imply the marginal measures $\mathcal{P}_Z$, $\mathcal{P}_{Q,c}$ and $\mathcal{P}_{Q,i}$ on Ω_Z, $\Omega_{Q,c}$ and $\Omega_{Q,i}$, respectively. The only issue left to discuss is thus which assumptions about the dependence structure of the processes Z, Q_c and Q_i lead to a valid belief measure $\tilde{\mathcal{P}}$ in line with the assumptions made in Section 5. We claim that only independence does and thus $\tilde{\mathcal{P}}$ must be given by $\tilde{\mathcal{P}} = \mathcal{P}_Z \otimes \mathcal{P}_{Q,c} \otimes \mathcal{P}_{Q,i}$.

---

[80]Note that we ignore here the additional observable innovation present in that equation and set $\ln \varepsilon_{t,s}^1 = 0$, as explained in the main text.

To see this, note that on the extended probability space that includes latent variables in households' filtering problem, agents must think that the five equations

$$\ln Z_t = \ln \gamma + \ln Z_{t-1} + \ln \varepsilon_t$$
$$\ln Q_{c,t} = \ln Q_{c,t-1} + \ln \beta_{c,t} + \ln \varepsilon_{c,t}$$
$$\ln \beta_{c,t} = \ln \beta_{c,t-1} + \ln \nu_{c,t}$$
$$\ln Q_{i,t} = \ln Q_{i,t-1} + \ln \beta_{i,t} + \ln \varepsilon_{i,t}$$
$$\ln \beta_{i,t} = \ln \beta_{i,t-1} + \ln \nu_{i,t}$$

hold with probability 1 for all $t \geq 1$. In addition, as stated in Section 5, $\{\varepsilon_t\}_{t=1}^{\infty}$, $\{\varepsilon_{c,t}\}_{t=1}^{\infty}$, $\{\nu_{c,t}\}_{t=1}^{\infty}$, $\{\varepsilon_{i,t}\}_{t=1}^{\infty}$, $\{\nu_{i,t}\}_{t=1}^{\infty}$ are independent stochastic processes. Thus, also the three processes Z, Q_c and Q_i must be independent.[81]

D.1.3 Proof of Proposition 2

We start with a result that collects important equations and gives an explicit characterization of the function F – up to the presence of the argument $Q_{c,t}$, which is not part of the state S_t.

Lemma 2. *In any equilibrium, irrespective of beliefs, the following equations have to hold*

$$W_t = K_{c,t}^{\alpha_c} Z_t^{1-\alpha_c}(1-\alpha_c)^{1-\alpha_c}$$
$$R_{c,t} = \alpha_c K_{c,t}^{\alpha_c-1} Z_t^{1-\alpha_c}(1-\alpha_c)^{1-\alpha_c}$$
$$R_{i,t} = \alpha_i \left(\frac{1-\alpha_i}{W_t}\right)^{\frac{1-\alpha_i}{\alpha_i}} Z_t^{\frac{1-\alpha_i}{\alpha_i}} Q_{c,t}^{\frac{1}{\alpha_i}}$$
$$Y_t = K_{c,t}^{\alpha_c} Z_t^{1-\alpha_c}(1-\alpha_c)^{1-\alpha_c} + Q_c^{ss} K_{i,t} Z_t^{\frac{1}{\alpha_i}} \left(\frac{(1-\alpha_i)\, Q_{c,t}}{W_t}\right)^{\frac{1-\alpha_i}{\alpha_i}}$$
$$C_t = K_{c,t}^{\alpha_c} Z_t^{1-\alpha_c}(1-\alpha_c)^{1-\alpha_c}$$
$$I_t = Q_c^{ss} K_{i,t} Z_t^{\frac{1}{\alpha_i}} \left(\frac{(1-\alpha_i)\, Q_{c,t}}{W_t}\right)^{\frac{1-\alpha_i}{\alpha_i}}$$
$$H_t = 1 - \alpha_c + K_{i,t} Z_t^{\frac{1-\alpha_i}{\alpha_i}} \left((1-\alpha_i)\frac{Q_{c,t}}{W_t}\right)^{\frac{1}{\alpha_i}}$$

for all $t \geq 0$. In particular, $(W_t, R_{c,t}, R_{i,t}, Y_t, C_t, I_t, H_t)$ is a deterministic function of $(Z_t, K_{c,t}, K_{i,t}, Q_{c,t})$.

Conversely, if these equations hold, and $H_{c,t}$, $H_{i,t}$ are given by equations (30) and (32), then allocations $(C, H, H_c, H_i, K_c, K_i)$ and prices (Q_c, Q_i, R_c, R_i, W) are consistent

---

[81]Formally, this requires a (simple) induction proof over time, which is not explicitly spelled out here.

with all equilibrium conditions (equations (17)-(28)), except for the two Euler equations (equations (18) and (19)) and the two capital accumulation equations (equations (27) and (28)).

Proof. In any competitive equilibrium with subjective beliefs, the equilibrium equations (17)-(28) stated in Section 6 have to hold. Based on these equations and definitions in the model description, the expressions in the assertion can be computed. The wage is given by

$$W_t \overset{(17)}{=} C_t \overset{(25)}{=} Y_{c,t} \overset{(1)}{=} K_{c,t}^{\alpha_c} Z_t^{1-\alpha_c} H_{c,t}^{1-\alpha_c} \overset{(30)}{=} (1-\alpha_c)^{1-\alpha_c} K_{c,t}^{\alpha_c} Z_t^{1-\alpha_c}. \tag{52}$$

Similarly, the rental rate in the consumption sector is

$$R_{c,t} \overset{(22)}{=} \frac{\alpha_c Y_{c,t}}{K_{c,t}} \overset{(1),(30)}{=} \alpha_c (1-\alpha_c)^{1-\alpha_c} K_{c,t}^{\alpha_c-1} Z_t^{1-\alpha_c}.$$

Substituting $H_{i,t}$ as given by equation (32) into the capital first-order condition of investment firms (24) yields for the rental rate in the investment sector

$$R_{i,t} = \alpha_i Q_{c,t} K_{i,t}^{\alpha_i-1} Z_t^{1-\alpha_i} \left(K_{i,t} Z_t^{\frac{1-\alpha_i}{\alpha_i}} \left((1-\alpha_i) \frac{Q_{c,t}}{W_t} \right)^{\frac{1}{\alpha_i}} \right)^{1-\alpha_i}$$

$$= \alpha_i \left(\frac{1-\alpha_i}{W_t} \right)^{\frac{1-\alpha_i}{\alpha_i}} Z_t^{\frac{1-\alpha_i}{\alpha_i}} Q_{c,t}^{\frac{1}{\alpha_i}}.$$

Output is defined as $Y_t = C_t + I_t$, so the asserted output equation follows from the equations for C_t and I_t. The equation for C_t follows from $C_t = W_t$ (17) and the equation for W_t has already been proven. The equation for $I_t = Q_c^{ss} Y_{i,t}$ is obtained by substituting $H_{i,t}$ stated in (32) into the investment-sector production function (1). Finally, the expression for H_t just combines hours in the two sectors (given by (30) and (32)) with labor market clearing (26).

For the second part of the lemma, we just remark that all the equations (17), (21), (22), (23), (24), (25) and (26) have been used in deriving the equations in the first part of the lemma and inverting the arguments used there shows that these equations also necessarily need to hold, if the equations in the lemma and (30) and (32) hold. ☐

The functional relationships in Lemma 2 contain $Q_{c,t}$ as the sole argument that is not contained in the state of time t. $Q_{c,t}$ and $K_{c,t+1}$ are simultaneously determined by the consumption-sector Euler equation (18) and the accumulation equation of consumption-sector capital (27). It has been shown in Proposition 1 how the former equation can be

solved for $Q_{c,t}$ under the assumption of subjective price beliefs, compare equation (33). This equation can be brought in the equivalent form

$$q_{c,t} = \frac{\alpha_c \beta \left(1 - \alpha_c\right)^{1-\alpha_c} z_t^{1-\alpha_c} k_{c,t}^{\alpha_c}}{1 - \beta \left(1 - \delta_c\right) e k_{c,t}^{\alpha_c} m_{c,t} \cdot k_{c,t+1}^{-\alpha_c}} \cdot \frac{1}{k_{c,t+1}}, \tag{53}$$

where we used lower case letters to denote realizations of random variables. Combining the capital accumulation equation (27) with $Y_{c,t} = \frac{I_t}{Q_c}$ and the expression for I_t from Lemma 2, we obtain a second equation has to hold along any equilibrium path as well:

$$k_{c,t+1} = \left(1 - \delta_c\right) k_{c,t} + z_t^{\frac{1}{\alpha_i}} k_{i,t} \left(\frac{\left(1 - \alpha_i\right) q_{c,t}}{w_t}\right)^{\frac{1-\alpha_i}{\alpha_i}}. \tag{54}$$

The following lemma shows that in a situation in which beliefs $m_{c,t}$ are predetermined, i.e., when the projection facility does not apply, there is a unique solution for $(q_{c,t}, k_{c,t+1})$:

Lemma 3. *For any given $(z_t, k_{c,t}, k_{i,t}, m_{c,t})$ with $z_t, k_{c,t}, k_{i,t}, m_{c,t} > 0$, equations (53) and (54) have a unique solution $(q_{c,t}, k_{c,t+1})$ with $q_{c,t}, k_{c,t+1} > 0$. Here, $w_t = k_{c,t}^{\alpha} \left(1 - \alpha\right)^{1-\alpha} z_t^{1-\alpha}$ is a function of $k_{c,t}$ and z_t.*

Proof. Equation (53) expresses $q_{c,t}$ as a function of $k_{c,t+1}$, $q_{c,t} = f(k_{c,t+1})$, equation (54) expresses $k_{c,t+1}$ as a function of $q_{c,t}$, $k_{c,t+1} = g(q_{c,t})$. Clearly, due to $w_t, z_t > 0$ and $\alpha_i < 1$ the function g is strictly increasing on the domain $(0, \infty)$. Furthermore, as long as the denominator on the left of equation (53) is positive, i.e. for $k_{c,t+1} \in (\underline{K}, \infty)$ with

$$\underline{K} := \left(\beta \left(1 - \delta_c\right) e m_{c,t}\right)^{\frac{1}{\alpha_c}} k_{c,t},$$

$f(k_{c,t+1})$ is strictly decreasing in $k_{c,t+1}$.[82] Hence, also the function

$$h : (\underline{K}, \infty) \to \left(\left(1 - \delta_c\right) k_{c,t}, \infty\right), k_{c,t+1} \mapsto g(f(k_{c,t+1}))$$

must be strictly decreasing and thus there is at most one fixed point $k_{c,t+1}^*$ (satisfying $k_{c,t+1}^* = h(k_{c,t+1}^*)$). As any solution $(q_{c,t}, k_{c,t+1})$ to (53) and (54) must satisfy $q_{c,t} = f(k_{c,t+1})$ and $k_{c,t+1} = g(q_{c,t})$, any such $k_{c,t+1}$ must necessarily be a fixed point of h. Thus, there can be at most one (positive) solution to (53) and (54).

Conversely, for any fixed point $k_{c,t+1}^*$ of h, the pair $(q_{c,t}^*, k_{c,t+1}^*) = (f(k_{c,t+1}^*), k_{c,t+1}^*)$ is obviously a (positive) solution to (53) and (54). It is thus left to show that a fixed point always exists. The function h is continuous and the following limit considerations show the existence of a fixed point by the intermediate value theorem:

- $f(k) \to 0$ as $k \to \infty$, so $h(k) \to g(0) = \left(1 - \delta_c\right) k_{c,t}$ as $k \to \infty$, hence for large k $h(k) - k$ is negative

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[82]When looking for a positive solution $(q_{c,t}, k_{c,t+1})$ to (53) and (54), we can restrict attention to $k_{c,t+1} > \underline{K}$, because otherwise $q_{c,t} = f(k_{c,t+1})$ becomes negative.

- $f(k) \to \infty$ as $k \searrow \underline{K}$ and $g(q) \to \infty$ as $q \to \infty$, so $h(k) \to \infty$ as $k \searrow \underline{K}$, hence for small k close to $\underline{K}$, $h(k) - k$ is positive

$\square$

The previous result treats $m_{c,t}$ as given. Yet, when the projection facility described in Appendix A.5 applies, $m_{c,t}$ is not fully predetermined in period t, instead might be projected downward, if it implies a too high PD ratio. Hence, the previous result does not apply in all periods along the equilibrium path. The following technical lemma shows that we also have uniqueness in periods in which the projection facility is active, provided the projection bounds are not too loose:

Lemma 4. *For any given positive values of z_t, $k_{c,t}$, $k_{i,t}$, $m_{c,t}$ consider the three equations (again, $w_t = (1 - \alpha_c)^{1-\alpha_c} k_{c,t}^{\alpha_c} z_t^{1-\alpha_c}$)*

$$q_{c,t} = \frac{\alpha_c \beta \left(1 - \alpha_c\right)^{1-\alpha_c} z_t^{1-\alpha_c} k_{c,t}^{\alpha_c}}{1 - \beta \left(1 - \delta_c\right) e k_{c,t}^{\alpha_c} m_{c,t}^{p} \cdot k_{c,t+1}^{-\alpha_c}} \cdot \frac{1}{k_{c,t+1}} \tag{55}$$

$$k_{c,t+1} = \left(1 - \delta_c\right) k_{c,t} + z_t^{\frac{1}{\alpha_i}} k_{i,t} \left(\frac{\left(1 - \alpha_i\right) q_{c,t}}{w_t}\right)^{\frac{1-\alpha_i}{\alpha_i}} \tag{56}$$

$$m_{c,t}^{p} = \begin{cases} m_{c,t}, & m_{c,t} \leq \underline{m}\left(k_{c,t+1}\right) \\ \underline{m}\left(k_{c,t+1}\right) + \frac{(\overline{m}-\underline{m})(k_{c,t+1})}{m_{c,t}+(\overline{m}-2\underline{m})(k_{c,t+1})} \left(m_{c,t} - \underline{m}\left(k_{c,t+1}\right)\right), & m_{c,t} \geq \underline{m}\left(k_{c,t+1}\right) \end{cases} \tag{57}$$

where the functions $\underline{m}$, $\overline{m}$ are the projection thresholds for m_c as defined in Appendix A.5.[83] If the projection bounds insure that

$$m_{c,t} < \frac{1}{\beta \left(1 - \alpha_c\right) e} \left(\frac{1}{1 - \delta_c}\right)^{1-\alpha_c}, \tag{58}$$

then the equation system has a unique solution $(q_{c,t}, k_{c,t+1}, m_{c,t}^{p})$.

The additional upper bound on beliefs stated in equation (58) is of little practical relevance in our numerical simulations. In all calibrations we consider, the upper bound is approximately equal to $\frac{1}{1-\alpha_c} \approx 1.5$, which implies an expected appreciation in the capital price of 50% within the next quarter. This is an order of magnitude larger than any value m_c ever attained in our numerical simulations.

Proof. Ignore the capital accumulation equation (56) and first consider equations (55) and (57). To transfer the proof from Lemma 3, we need to show that after substituting

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[83]The bound $\underline{m}$, $\overline{m}$ also depend on z_t, w_t and $k_{c,t}$. We suppress this dependency in our notation, as these variables are predetermined.

$m_{c,t}^p$ into the first equation, this still defines a decreasing relationship between $k_{c,t+1}$ and $q_{c,t}$. As $\underline{m}(k_{c,t+1})$ is strictly increasing in $k_{c,t+1}$ (and approaching $-\infty$ as $k_{c,t+1} \to 0$), there is some threshold $\hat{k}$, such that $\underline{m}(k_{c,t+1}) \leq m_{c,t}$ for $k_{c,t+1} \leq \hat{k}$ and $\underline{m}(k_{c,t+1}) \geq m_{c,t}$ for $k_{c,t+1} \geq \hat{k}$. We consider the two cases separately:

1. If $k_{c,t+1} \leq \hat{k}$, then the projection is actually used, so we have

$$m_{c,t}^p = \underline{m} + \frac{\overline{m} - \underline{m}}{m_{c,t} + \overline{m} - 2\underline{m}} \left(m_{c,t} - \underline{m} \right)$$

from the third equation. For notational convenience define the constants

$$A = \frac{1}{\beta \left(1 - \delta_c \right) e k_{c,t}^{\alpha_c}}$$

$$\underline{B} = \frac{\beta \left(1 - \delta_c \right) k_{c,t}}{\left(p \left(1 + \underline{PD} \right) - 1 \right) \beta \left(1 - \delta_c \right) e k_{c,t}^{\alpha_c}} = \frac{\beta \left(1 - \delta_c \right) k_{c,t}}{p \left(1 + \underline{PD} \right) - 1} A$$

$$\overline{B} = \frac{\beta \left(1 - \delta_c \right) k_{c,t}}{\left(p \left(1 + \overline{PD} \right) - 1 \right) \beta \left(1 - \delta_c \right) e k_{c,t}^{\alpha_c}} = \frac{\beta \left(1 - \delta_c \right) k_{c,t}}{p \left(1 + \overline{PD} \right) - 1} A$$

$$C = \beta \left(1 - \delta_c \right) e k_{c,t}^{\alpha_c} = A^{-1}$$

and drop all subscripts for the following argument (k refers to $k_{c,t+1}$, q to $q_{c,t}$ and m to $m_{c,t}$).

Then we have (this follows from equations (46) and (47) for $s = c$)

$$\underline{m} = Ak^{\alpha_c} - \underline{B}k^{\alpha_c - 1}, \qquad \overline{m} = Ak^{\alpha_c} - \overline{B}k^{\alpha_c - 1}$$

and (from equation (55))

$$q = \frac{\text{const}}{1 - Ck^{-\alpha_c}m^p} \frac{1}{k}.$$

Using $m^p = \underline{m} + \frac{\overline{m} - \underline{m}}{m + \overline{m} - 2\underline{m}} \left(m - \underline{m} \right)$, we obtain

$$\frac{\text{const}}{q} = k - Ck^{1 - \alpha_c}m^p$$

$$= k - Ck^{1 - \alpha_c} \left(\underline{m} + \frac{\overline{m} - \underline{m}}{m + \overline{m} - 2\underline{m}} \left(m - \underline{m} \right) \right)$$

$$= k - Ck^{1 - \alpha_c} \Bigg(Ak^{\alpha_c} - \underline{B}k^{\alpha_c - 1}$$

$$+ \frac{Ak^{\alpha_c} - \overline{B}k^{\alpha_c - 1} - Ak^{\alpha_c} + \underline{B}k^{\alpha_c - 1}}{m + Ak^{\alpha_c} - \overline{B}k^{\alpha_c - 1} - 2Ak^{\alpha_c} + 2\underline{B}k^{\alpha_c - 1}} \left(m - Ak^{\alpha_c} + \underline{B}k^{\alpha_c - 1} \right) \Bigg)$$

$$= C\underline{B} - C \left(\underline{B} - \overline{B} \right) \frac{m - Ak^{\alpha_c} + \underline{B}k^{\alpha_c - 1}}{m - Ak^{\alpha_c} + \left(2\underline{B} - \overline{B} \right) k^{\alpha_c - 1}}$$

62

$C\underline{B}$ and $C\left(\underline{B} - \overline{B}\right)$ are positive constants, so q is decreasing in k, if and only if the expression $\frac{m - Ak^{\alpha_c} + \underline{B}k^{\alpha_c - 1}}{m - Ak^{\alpha_c} + \left(2\underline{B} - \overline{B}\right)k^{\alpha_c - 1}}$ is. Using that the derivative of $x \mapsto \frac{u(x)}{u(x) + v(x)}$ is given by $\frac{u'(x)v(x) - v'(x)u(x)}{(u(x) + v(x))^2}$, we find that $\frac{m - Ak^{\alpha_c} + \underline{B}k^{\alpha_c - 1}}{m - Ak^{\alpha_c} + \left(2\underline{B} - \overline{B}\right)k^{\alpha_c - 1}}$ is (strictly) decreasing in k, if and only if[84]

$$\left(-A\alpha_c k^{\alpha_c - 1} + (\alpha_c - 1)\underline{B}k^{\alpha_c - 2}\right)\left(\underline{B} - \overline{B}\right)k^{\alpha_c - 1}$$
$$< (\alpha_c - 1)\left(\underline{B} - \overline{B}\right)k^{\alpha_c - 2}\left(m - Ak^{\alpha_c} + \underline{B}k^{\alpha_c - 1}\right)$$

After expanding the products on both sides and canceling common terms, this inequality simplifies to

$$0 < (\alpha_c - 1)\left(\underline{B} - \overline{B}\right)mk^{\alpha_c - 2} + A\left(\underline{B} - \overline{B}\right)k^{2\alpha_c - 2}$$
$$\Leftrightarrow m < \frac{Ak^{\alpha_c}}{1 - \alpha_c}.$$

Finally, using the definition of A, we obtain the condition (from now on subscripts are added back again for clarity about the timing of variables)

$$m_{c,t} < \frac{1}{\beta\left(1 - \delta_c\right)\left(1 - \alpha_c\right)e}\left(\frac{k_{c,t+1}}{k_{c,t}}\right)^{\alpha_c}$$

This condition is tighter, the smaller is $k_{c,t+1}$. The smallest possible value of $k_{c,t+1}$ given $k_{c,t}$ is $k_{c,t+1} = (1 - \delta_c)k_{c,t}$ (otherwise equation (56) is inconsistent with a positive $q_{c,t}$), so the above condition on $m_{c,t}$ is certainly satisfied, if

$$m_{c,t} < \frac{1}{\beta\left(1 - \alpha_c\right)e}\left(\frac{1}{1 - \delta_c}\right)^{1 - \alpha_c}$$

which is exactly the condition required in the assertion. So as long as $k_{c,t+1} \le \hat{k}$, the third and first equation define a strictly decreasing relationship between $k_{c,t+1}$ and $q_{c,t}$.

2. If $k_{c,t+1} \ge \hat{k}$, then $m_{c,t}^p = m_{c,t}$ does not depend on the level on $k_{c,t+1}$ anymore and thus the third and first equation define a strictly decreasing relationship between $k_{c,t+1}$ and $q_{c,t}$ by arguments made in the proof of Lemma 3.

After substituting the third equation into the first, we have as before two functional relationships $q_{c,t} = f(k_{c,t+1})$ and $k_{c,t+1} = g(q_{c,t})$ with f strictly decreasing and g strictly increasing. The same arguments made in the proof of Lemma 3 guarantee a unique solution. The associated level of $m_{c,t}^p$ can then be computed from the third equation.

$\square$

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[84]Here, $u(x) = m - Ax^{\alpha_c} + \underline{B}x^{\alpha_c - 1} \Rightarrow u'(x) = -A\alpha_c x^{\alpha_c - 1} + (\alpha_c - 1)\underline{B}x^{\alpha_c - 2}$ and $v(x) = \left(\underline{B} - \overline{B}\right)x^{\alpha_c - 1} \Rightarrow v'(x) = (\alpha_c - 1)\left(\underline{B} - \overline{B}\right)x^{\alpha_c - 2}$

We are now in a position to prove proposition 2:

Proof of Proposition 2. Existence and uniqueness of the measure $\tilde{\mathcal{P}}$ has already been established in section D.1.2 of this appendix. It thus only remains to construct the mappings G and F. Throughout the construction that is spelled out below, we only use necessary equilibrium conditions. Furthermore, no construction step admits several choices. Therefore, the construction also yields uniqueness of the mappings G and F.

Suppose $S_t = s_t$ for an arbitrary (fixed) state $s_t = (z_t, k_{c,t}, k_{i,t}, m_{c,t}, m_{i,t}, q_{c,t-1}, q_{i,t-1})$ such that $m_{c,t}$ respects the upper bound of lemma 4. For the following argument it is also useful to define the "reduced state" $\hat{s}_t = (z_t, k_{c,t}, k_{i,t}, m_{c,t}, q_{c,t-1})$ that does not include $m_{i,t}$ and $q_{i,t-1}$. We first show the existence of a unique state transition function $\hat{G}$ for this reduced state. In any equilibrium, the wage is a function of $K_{c,t}$ and Z_t only, see lemma 2. Hence, conditional on $\hat{S}_t = \hat{s}_t$, $W_t = \tilde{F}_W(\hat{s}_t)$ with some deterministic function $\tilde{F}_W$ (whose explicit form is given in lemma 2). Furthermore, in any equilibrium (with beliefs as specified in section 5), equations (55), (56), (57) have to hold along any equilibrium path. Lemma 4 thus implies the existence of a unique vector $(q_{c,t}, k_{c,t+1}, m_{c,t}^p)$, given z_t, $k_{c,t}$, $k_{i,t}$, $m_{c,t}$ and $w_t = \tilde{F}_W(\hat{s}_t)$, which are all uniquely determined by the reduced state $\hat{s}_t$, such that all three equations hold. The equations thus implicitly define three functions

$$\tilde{G}_{Q,c}(\hat{s}_t) = q_{c,t}, \qquad \tilde{G}_{K,c}(\hat{s}_t) = k_{c,t+1}, \qquad \tilde{G}_{m^p,c}(\hat{s}_t) = m_{c,t}^p$$

and the argument given so far implies that along any equilibrium path $q_{c,t}$, $k_{c,t+1}$ and $m_{c,t}^p$ (the value of $m_{c,t}$ after projection) must necessarily be related to $\hat{s}_t$ as described by these three equations.

Next, $m_{c,t+1}$ must satisfy the belief updating equation[85]

$$\ln m_{c,t+1} = \ln m_{c,t}^p + g\left(\ln q_{c,t} - \ln q_{c,t-1} - \ln m_{c,t}^p\right) =: \ln \tilde{G}_{m,c}(\hat{s}_t),$$

where the right-hand side is a function of $\hat{s}_t$, because $q_{c,t} = \tilde{G}_{Q,c}(\hat{s}_t)$ and $m_{c,t}^p = \tilde{G}_{m^p,c}(\hat{s}_t)$ are and $q_{c,t-1}$ is a component of $\hat{s}_t$.

In addition, $K_{i,t+1} = Z_t$ by definition. This implies for realizations conditional on $\hat{S}_t = \hat{s}_t$ that $k_{i,t+1} = z_t =: \tilde{G}_{K,i}(\hat{s}_t)$.

In total, we obtain from the discussion so far that $\hat{s}_{t+1} = (z_{t+1}, k_{c,t+1}, k_{i,t+1}, m_{c,t+1}, q_{c,t})$ must necessarily satisfy

$$\hat{s}_{t+1} = (z_{t+1}, \tilde{G}_{K,c}(\hat{s}_t), \tilde{G}_{K,i}(\hat{s}_t), \tilde{G}_{m,c}(\hat{s}_t), \tilde{G}_{Q,c}(\hat{s}_t)) =: \hat{G}(\hat{s}_t, z_{t+1}).$$

Thus, in any equilibrium, the evolution of the reduced state $\hat{s}_{t+1}$ must be governed by the transition function $\hat{G}$. From our derivation it is also clear that the evolution of $\hat{G}$ is consistent with equations (55), (56), (57), the belief updating equation (15) in the

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[85]See equation (15) for the consumption sector and recall that we set the additional shock to agents' information set equal to zero in all periods

consumption sector and the exogenous evolution of K_i. In particular, $\hat{G}$ is then consistent with the consumption-sector Euler equation (18) and with the capital accumulation equations (27), (28) in both sectors.

Next, we let $\hat{F}$ be the mapping from reduced states $\hat{s}_t$ to outcomes $(w_t, r_{c,t}, r_{i,t}, y_t, c_t, i_t, h_t)$ defined by the formulas in lemma 2, if $q_{c,t}$ is everywhere replaced by $\tilde{G}_{Q,c}(\hat{s}_t)$.[86] Combining this with the obvious state reduction mapping $s_t \mapsto \hat{s}_t$, we can define $F(s_t) := \hat{F}(\hat{s}_t)$.[87] Lemma 2 tells us then that this choice of F is the only possible choice consistent with equilibrium and in turn this F is consistent with all equilibrium equations (except for the two Euler equations and the two capital accumulation equations, which do not matter for F). Consequently, F and $\hat{G}$ together are consistent with all the equilibrium conditions (17)-(28), except for the investment-sector Euler equation (19). In addition, $\hat{G}$ is also consistent with the belief updating equation for consumption-sector capital prices.

To complete the existence proof, it is left to show that $\hat{G}$ can be extended to a full state transition mapping G, such that G is also consistent with the investment-sector Euler equation and the belief updating equation for investment-sector capital prices. First, consider the conditional expectations in equation (36), which is a partially solved version of the investment-sector Euler equation (19) from the main text. These conditional expectations can be written as

$$E_t\left[\frac{W_t}{W_{t+1}}R_{i,t+1}\right] = E\left[\frac{\hat{F}_W(\hat{S}_t)}{\hat{F}_W(\hat{S}_{t+1})}\hat{F}_{R,i}(\hat{S}_{t+1})\Big| S_t, S_{t-1}, \ldots\right]$$

$$= E\left[\frac{\hat{F}_W(\hat{S}_t)}{\hat{F}_W\left(\hat{G}\left(\hat{S}_t, Z_{t+1}\right)\right)}\hat{F}_{R,i}\left(\hat{G}\left(\hat{S}_t, Z_{t+1}\right)\right)\Big|\hat{S}_t\right],$$

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[86] This means,

$$\hat{F}_W(\hat{s}_t) = k_{c,t}^{\alpha_c} z_t^{1-\alpha_c}(1-\alpha_c)^{1-\alpha_c}$$

$$\hat{F}_{R,c}(\hat{s}_t) = \alpha_c k_{c,t}^{\alpha_c-1} z_t^{1-\alpha_c}(1-\alpha_c)^{1-\alpha_c}$$

$$\hat{F}_{R,i}(\hat{s}_t) = \alpha_i \left(\frac{1-\alpha_i}{\hat{F}_W(\hat{s}_t)}\right)^{\frac{1-\alpha_i}{\alpha_i}} z_t^{\frac{1-\alpha_i}{\alpha_i}}\left(\tilde{G}_{Q,c}(\hat{s}_t)\right)^{\frac{1}{\alpha_i}}$$

$$\vdots$$

[87] We keep the separate mapping $\hat{F}$ as an auxiliary device for the extension of $\hat{G}$ to G below.

$$E_t \left[\frac{W_t}{W_{t+1}} \right] = E \left[\frac{\tilde{F}_W(\hat{S}_t)}{\tilde{F}_W(\hat{S}_{t+1})} \mid S_t, S_{t-1}, \ldots \right]$$

$$= E \left[\frac{\tilde{F}_W(\hat{S}_t)}{\tilde{F}_W \left(\hat{G} \left(\hat{S}_t, Z_{t+1} \right) \right)} \mid \hat{S}_t \right],$$

where in each case the second equality follows from the fact that all information in the history S^t not already contained in $\hat{S}_t$ is redundant for predicting Z_{t+1} and $\hat{S}_t$. Hence, both conditional expectations are deterministic functions of the current reduced state $\hat{S}_t$. As these two conditional expectations and $R_{i,t} = \hat{F}_{R,i}(\hat{S}_t)$ are the only relevant variables to compute the projection bounds in the investment sector, compare Appendix A.5, conditional on $\hat{S}_t = \hat{s}_t$, the projected belief in the investment sector is given by $m_{i,t}^p = \tilde{G}_{m^p,i}(\hat{s}_t, m_{i,t})$ with some deterministic function $\tilde{G}_{m^p,i}$. By equation (36), $Q_{i,t}$ must then assume in equilibrium the value (conditional on $\hat{S}_t = \hat{s}_t$ and $m_{i,t}$)

$$q_{i,t} = \frac{\beta E \left[\frac{W_t}{W_{t+1}} R_{i,t+1} \mid \hat{S}_t = \hat{s}_t \right]}{1 - \beta E_t \left[\frac{W_t}{W_{t+1}} \mid \hat{S}_t = \hat{s}_t \right] (1 - \delta) \tilde{G}_{m^p,i}(\hat{s}_t, m_{i,t})}.$$

The right-hand side is a function of $\hat{s}_t$ and $m_{i,t}$ and therefore of the full state s_t. Denote it by $\tilde{G}_{Q,i}(s_t)$. Finally, the belief updating equation for the investment sector defines $m_{i,t+1}$ as a function of the current state,

$$m_{i,t+1} = \tilde{G}_{m,i}(s_t) := \tilde{G}_{m^p,i}(\hat{s}_t, m_{i,t}) \left(\frac{\tilde{G}_{Q,i}(s_t)}{\tilde{G}_{m^p,i}(\hat{s}_t, m_{i,t}) q_{i,t-1}} \right)^g.$$

We can thus define

$$G(s_t, z_{t+1}) := \left(\hat{G}_Z(\hat{s}_t, z_{t+1}), \hat{G}_{K,c}(\hat{s}_t, z_{t+1}), \hat{G}_{K,i}(\hat{s}_t, z_{t+1}), \hat{G}_{m,c}(\hat{s}_t, z_{t+1}), \right.$$

$$\left. \tilde{G}_{m,i}(s_t), \hat{G}_{Q,c}(\hat{s}_t, z_{t+1}), \tilde{G}_{Q,i}(s_t) \right).$$

By construction, G is then consistent with all equations that $\hat{G}$ is, but in addition consistent with the investment-sector Euler equation (19) and with the belief updating equation for m_i. Since F has already been shown to be consistent with all equilibrium conditions, this completes the proof of the proposition. $\qquad \square$

D.2 An Example for Admissible Capital Holding Bounds

This appendix provides an explicit example of capital holding bound processes

$$\{ \overline{K}_{c,t+1}, \overline{K}_{i,t+1} \}_{t=0}^\infty$$

that satisfy the two requirements stated in the end of section 4.2: (1) bounds need to be sufficiently large, such that they never bind in equilibrium, and (2) bounds need to be sufficiently tight such that the transversality condition holds.

For the construction of the example, note that the equilibrium transition function G in appendix D.1.3 was derived under the assumption that capital holding bounds are never binding along the equilibrium path (otherwise, the Euler equations (18), (19) would not hold in all states) and the transversality condition was never used in appendix D.1.3. Therefore, the function G as a mathematical object exists independently of the choice of capital bounds (it just may not correspond to a valid model equilibrium) and so do the implied equilibrium capital processes K_c and K_i as functions of the technology shock Z. It is thus not a circular definition, if we choose

$$\overline{K}_{c,t+1} := 2K_{c,t+1}, \qquad \overline{K}_{i,t+1} := 2K_{i,t+1}.$$

Then, capital bounds trivially never bind in equilibrium, so requirement (1) is satisfied by construction.

It is left to show that these bounds also satisfy requirement (2). The transversality condition is discussed in detail in appendix D.3 below. Lemma 5 there gives sufficient conditions for the transversality condition to hold. For capital bounds, the relevant conditions are conditions 1 and 2 of that lemma. The first condition is clearly satisfied here, because the bounds $\overline{K}_{c,t+1}$, $\overline{K}_{i,t+1}$ are deterministic functions of histories Z^t (because $K_{c,t+1}$ and $K_{i,t+1}$ are). Conditions 2 cannot be verified analytically, as we can characterize capital stocks only up to a system of equations. Instead, we check the condition numerically for our estimated parameter specification.

D.3 The Transversality Condition

Technically, the transversality condition, fails to hold for our simple belief system (11) and (12) in the main text. This said, it holds for a slightly more complicated belief system that can approximate it with arbitrary precision over any finite horizon. This slightly more general belief system is given by

$$\ln Q_{s,t} = \eta_Q \ln Q_{s,t-1} + (1 - \eta_Q) \ln \bar{Q}_s + \ln \beta_{s,t} + \ln \varepsilon_{s,t}, \tag{59}$$

where $\ln \bar{Q}_s$ denotes the (arbitrary but finite) perceived long-run mean of $\ln Q_{s,t}$, and where

$$\ln \beta_{s,t} = \eta_\beta \ln \beta_{s,t-1} + \ln \nu_{s,t}. \tag{60}$$

The additional parameters $\eta_Q, \eta_\beta \in [0, 1]$ capture the degree of persistence in beliefs about the level and growth rate of prices. We show below that the transversality condition holds for this generalized belief system as long as $\eta_Q < 1, \eta_\beta < 1$. For the case $\eta_Q = \eta_\beta = 1$, the generalized belief system collapses to the one in the main text, i.e., to equations (11)

and (12). The latter equations from the main text should thus be interpreted as the limit outcomes of the general belief system (59) and (60) for $\eta_Q, \eta_\beta \to 1$.

The following lemma provides sufficient conditions under which the transversality condition (20) holds:

Lemma 5. *Suppose the following conditions are satisfied:*

1. *According to households beliefs, capital holding bounds $\overline{K}_{c,t+1}$, $\overline{K}_{i,t+1}$ are functions of histories of technology shocks Z^t only, that is for each t there is a (deterministic) function $\bar{f}_t$ such that $\left(\overline{K}_{c,t+1}, \overline{K}_{i,t+1}\right) = \bar{f}_t(Z^t)$ $\mathcal{P}$-a.s.[88]*

2. *The expectations*

$$E^{\mathcal{P}}\left[\frac{\overline{K}_{c,t+1}}{W_t}\right], \qquad E^{\mathcal{P}}\left[\frac{\overline{K}_{i,t+1}}{W_t}\right]$$

   exist and remain bounded as $t \to \infty$.

3. *$\eta_Q, \eta_\beta < 1$ in the subjective price belief system (59) and (60).*

Then the transversality condition (20) is satisfied.

Proof. Under internal rationality, beliefs about own choices must respect individual constraints, so the capital holding constraints $K_{c,t+1} \leq \overline{K}_{c,t+1}$, $K_{i,t+1} \leq \overline{K}_{c,t+1}$ must hold $\mathcal{P}$-a.s. Using this and the fact that $Q_{c,t}, Q_{i,t}, W_t \geq 0$ $\mathcal{P}$-a.s., we can estimate

$$E^{\mathcal{P}}\left[\frac{1}{W_t}\left(K_{c,t+1}Q_{c,t} + K_{i,t+1}Q_{i,t}\right)\right] = E^{\mathcal{P}}\left[\frac{K_{c,t+1}}{W_t}Q_{c,t}\right] + E^{\mathcal{P}}\left[\frac{K_{i,t+1}}{W_t}Q_{i,t}\right]$$

$$\leq E^{\mathcal{P}}\left[\frac{\overline{K}_{c,t+1}}{W_t}Q_{c,t}\right] + E^{\mathcal{P}}\left[\frac{\overline{K}_{i,t+1}}{W_t}Q_{i,t}\right]. \qquad (61)$$

Under assumption 1, $\overline{K}_{c,t+1}$ and $\overline{K}_{i,t+1}$ are functions of Z^t and due to rational wage expectations, also W_t is a function of Z^t. Z^t and Q_s^t are independent under $\mathcal{P}$ for both $s = c$ and $s = i$. Consequently,

$$E^{\mathcal{P}}\left[\frac{\overline{K}_{c,t+1}}{W_t}Q_{c,t}\right] = E^{\mathcal{P}}\left[\frac{\overline{K}_{c,t+1}}{W_t}\right] E^{\mathcal{P}}\left[Q_{c,t}\right], \qquad E^{\mathcal{P}}\left[\frac{\overline{K}_{i,t+1}}{W_t}Q_{i,t}\right] = E^{\mathcal{P}}\left[\frac{\overline{K}_{i,t+1}}{W_t}\right] E^{\mathcal{P}}\left[Q_{i,t}\right]$$

---

[88]This is a convenient sufficient condition on beliefs about $\overline{K}_{c,t+1}$ and $\overline{K}_{i,t+1}$ for the transversality condition to hold. A functional relationship between Z^t and $\left(\overline{K}_{c,t+1}, \overline{K}_{i,t+1}\right)$ clearly exists in equilibrium, so this condition holds for example, if households have rational expectations about capital bounds. We have not discussed beliefs of households with respect to capital bound processes, because they only affect subjectively expected utility (both its value and whether or not it is well-defined), but are inessential for decision functions and thus any positive prediction of our model.

Substituting these equations into (61) and using that $E^{\mathcal{P}}\left[\frac{\overline{K}_{c,t+1}}{W_t}\right]$, $E^{\mathcal{P}}\left[\frac{\overline{K}_{i,t+1}}{W_t}\right]$ are bounded above by some constant $B < \infty$ (assumption 2), we obtain the inequality

$$\beta^t E^{\mathcal{P}}\left[\frac{1}{W_t}\left(K_{c,t+1}Q_{c,t} + K_{i,t+1}Q_{i,t}\right)\right] \leq B \cdot \beta^t \left(E^{\mathcal{P}}[Q_{c,t}] + E^{\mathcal{P}}[Q_{i,t}]\right).$$

Consequently, if $\beta^t \left(E^{\mathcal{P}}[Q_{c,t}] + E^{\mathcal{P}}[Q_{i,t}]\right) \to 0$ as $t \to \infty$, then the transversality condition (20) must hold. To conclude the proof, it is thus only left to show $\lim_{t\to\infty} \beta^t E^{\mathcal{P}}[Q_{s,t}]$ for $s \in \{c, i\}$, which we do in the following.

Iterating the subjective law of motion for Q_s (equation (59)) backwards yields

$$\ln Q_{s,t} = \ln \beta_{s,t} + \ln \varepsilon_t + (1 - \eta_Q) \ln \bar{Q}_s + \eta_Q \ln Q_{s,t-1}$$
$$= \ln \beta_{s,t} + \eta_Q \ln \beta_{s,t-1} + \ln \varepsilon_{s,t} + \eta_Q \ln \varepsilon_{s,t-1} + (1 + \eta_Q)(1 - \eta_Q) \ln \bar{Q}_s + \eta_Q^2 \ln Q_{s,t-2}$$
$$= \cdots$$
$$= \eta_Q^{t+1} \ln Q_{s,-1} + \sum_{\tau=0}^{t} \eta_Q^{t-\tau} \ln \beta_{s,\tau} + \sum_{\tau=0}^{t} \eta_Q^{t-\tau} \ln \varepsilon_{s,\tau} + \underbrace{\sum_{\tau=0}^{t} \eta_Q^{t-\tau}(1 - \eta_Q) \ln \bar{Q}_s}_{1 - \eta_Q^{t+1}}.$$

Similarly, we obtain for β_s (iterating equation (60) backwards)

$$\ln \beta_{s,t} = \eta_\beta \ln \beta_{s,t-1} + \ln \nu_{s,t}$$
$$= \eta_\beta^2 \ln \beta_{s,t-2} + \ln \nu_{s,t} + \eta_\beta \ln \nu_{s,t-1}$$
$$= \cdots$$
$$= \eta_\beta^{t+1} \ln \beta_{s,-1} + \sum_{\tau=0}^{t} \eta_\beta^{t-\tau} \ln \nu_{s,\tau}$$

Combining the two equations implies

$$
\ln Q_{s,t} = \eta_Q^{t+1} \ln Q_{s,-1} + \sum_{\tau=0}^{t} \eta_Q^{t-\tau} \eta_\beta^{\tau+1} \ln \beta_{s,-1} + \sum_{\tau=0}^{t} \eta_Q^{t-\tau} \sum_{r=0}^{\tau} \eta_\beta^{\tau-r} \ln \nu_{s,r}
$$

$$
+ \sum_{\tau=0}^{t} \eta_Q^{t-\tau} \ln \varepsilon_{s,\tau} + \left(1 - \eta_Q^{t+1}\right) \ln \bar{Q}_s
$$

$$
= \eta_Q^{t+1} \ln Q_{s,-1} + \left(\eta_Q^t \eta_\beta \sum_{\tau=0}^{t} \left(\frac{\eta_\beta}{\eta_Q}\right)^\tau \right) \ln \beta_{s,-1} + \eta_Q^t \sum_{r=0}^{t} \left(\eta_\beta^{-r} \sum_{\tau=r}^{t} \left(\frac{\eta_\beta}{\eta_Q}\right)^\tau \right) \ln \nu_{s,r}
$$

$$
+ \sum_{\tau=0}^{t} \eta_Q^{t-\tau} \ln \varepsilon_{s,\tau} + \left(1 - \eta_Q^{t+1}\right) \ln \bar{Q}_s
$$

$$
= \eta_Q^{t+1} \ln Q_{s,-1} + \frac{\eta_Q^{t+1} - \eta_\beta^{t+1}}{\eta_Q - \eta_\beta} \eta_\beta \ln \beta_{s,-1} + \sum_{\tau=0}^{t} \frac{\eta_Q^{t-\tau} - \eta_\beta^{t-\tau}}{\eta_Q - \eta_\beta} \eta_\beta \ln \nu_{s,\tau}
$$

$$
+ \sum_{\tau=0}^{t} \eta_Q^{t-\tau} \ln \varepsilon_{s,\tau} + \left(1 - \eta_Q^{t+1}\right) \ln \bar{Q}_s
$$

$Q_{s,-1}$ is a constant (part of the initial state) and the prior for $\ln \beta_{s,-1}$ as well as all (subjective) shocks $\ln \nu_{s,\tau}$, $\ln \varepsilon_{s,\tau}$ are normally distributed, so also $Q_{s,t}$ must be lognormally distributed under $\mathcal{P}$ and it is thus fully characterized by the mean and variance of $\ln Q_{s,t}$. The mean is given by

$$
E^{\mathcal{P}}\left[\ln Q_{s,t}\right] = \eta_Q^{t+1} \ln Q_{s,-1} + \frac{\eta_Q^{t+1} - \eta_\beta^{t+1}}{\eta_Q - \eta_\beta} \eta_\beta E^{\mathcal{P}}\left[\ln \beta_{s,-1}\right]
$$

$$
+ \sum_{\tau=0}^{t} \frac{\eta_Q^{t-\tau} - \eta_\beta^{t-\tau}}{\eta_Q - \eta_\beta} \eta_\beta \left(-\frac{\sigma_\nu^2}{2}\right) + \sum_{\tau=0}^{t} \eta_Q^{t-\tau} \left(-\frac{\sigma_\varepsilon^2}{2}\right) + \left(1 - \eta_Q^{t+1}\right) \ln \bar{Q}_s
$$

$$
= \eta_Q^{t+1} \ln Q_{s,-1} + \frac{\eta_Q^{t+1} - \eta_\beta^{t+1}}{\eta_Q - \eta_\beta} \eta_\beta E^{\mathcal{P}}\left[\ln \beta_{s,-1}\right]
$$

$$
- \frac{\sigma_\nu^2}{2} \frac{\eta_\beta}{\eta_Q - \eta_\beta} \left(\frac{1 - \eta_Q^{t+1}}{1 - \eta_Q} - \frac{1 - \eta_\beta^{t+1}}{1 - \eta_\beta}\right) - \frac{\sigma_\varepsilon^2}{2} \frac{1 - \eta_Q^{t+1}}{1 - \eta_Q} + \left(1 - \eta_Q^{t+1}\right) \ln \bar{Q}_s
$$

and taking the limit $t \to \infty$ yields (due to $\eta_Q, \eta_\beta < 1$ by assumption 3)

$$
\lim_{t \to \infty} E^{\mathcal{P}}\left[\ln Q_{s,t}\right] = -\frac{\sigma_\nu^2}{2} \frac{\eta_\beta}{\eta_Q - \eta_\beta} \left(\frac{1}{1 - \eta_Q} - \frac{1}{1 - \eta_\beta}\right) - \frac{\sigma_\varepsilon^2}{2} \frac{1}{1 - \eta_Q} + \ln \bar{Q}_s
$$

$$
= -\frac{\sigma_\nu^2}{2} \frac{\eta_\beta}{\left(1 - \eta_Q\right)\left(1 - \eta_\beta\right)} - \frac{\sigma_\varepsilon^2}{2} \frac{1}{1 - \eta_Q} + \ln \bar{Q}_s < \infty
$$

Next, the (subjective) variance of $\ln Q_{s,t}$ is

$$\text{var}^{\mathcal{P}}(\ln Q_t) = \left(\frac{\eta_Q^{t+1} - \eta_\beta^{t+1}}{\eta_Q - \eta_\beta}\eta_\beta\right)^2 \text{var}^{\mathcal{P}}(\ln \beta_{s,-1}) + \sum_{\tau=0}^{t}\left(\frac{\eta_Q^{t-\tau} - \eta_\beta^{t-\tau}}{\eta_Q - \eta_\beta}\eta_\beta\right)^2 \text{var}^{\mathcal{P}}(\ln \nu_{s,\tau})$$

$$+ \sum_{\tau=0}^{t}\eta_Q^{2(t-s)}\text{var}^{\mathcal{P}}(\ln \varepsilon_{s,\tau})$$

$$= \left(\frac{\eta_Q^{t+1} - \eta_\beta^{t+1}}{\eta_Q - \eta_\beta}\eta_\beta\right)^2 \text{var}^{\mathcal{P}}(\ln \beta_0) + \sigma_\nu^2 \frac{\eta_\beta^2}{(\eta_Q - \eta_\beta)^2}\sum_{\tau=0}^{t}\left(\eta_Q^{2\tau} - 2\eta_Q^\tau\eta_\beta^\tau + \eta_\beta^{2\tau}\right)$$

$$+ \sigma_\varepsilon^2 \sum_{\tau=0}^{t}\eta_Q^{2\tau}$$

$$= \left(\frac{\eta_Q^{t+1} - \eta_\beta^{t+1}}{\eta_Q - \eta_\beta}\eta_\beta\right)^2 \text{var}^{\mathcal{P}}(\ln \beta_0)$$

$$+ \sigma_\nu^2 \frac{\eta_\beta^2}{(\eta_Q - \eta_\beta)^2}\left(\frac{1 - \eta_Q^{2t+2}}{1 - \eta_Q^2} + \frac{1 - \eta_\beta^{2t+2}}{1 - \eta_\beta^2} - 2\frac{1 - \eta_Q^{t+1}\eta_\beta^{t+1}}{1 - \eta_Q\eta_\beta}\right) + \sigma_\varepsilon^2 \frac{1 - \eta_Q^{2t+1}}{1 - \eta_Q^2}.$$

This has the limit

$$\lim_{t\to\infty}\text{var}^{\mathcal{P}}(\ln Q_t) = \sigma_\nu^2\left(\frac{\eta_\beta}{(1 - \eta_Q)(1 - \eta_\beta)}\right)^2 \frac{\frac{1}{1-\eta_Q^2} + \frac{1}{1-\eta_\beta^2} - \frac{2}{1-\eta_Q\eta_\beta}}{\frac{1}{(1-\eta_Q)^2} + \frac{1}{(1-\eta_\beta)^2} - \frac{2}{(1-\eta_Q)(1-\eta_\beta)}} + \sigma_\varepsilon^2\frac{1}{1-\eta_Q^2} < \infty$$

Because both $\lim_{t\to\infty}E^{\mathcal{P}}[\ln Q_{s,t}]$ and $\lim_{t\to\infty}\text{var}^{\mathcal{P}}(\ln Q_t)$ exist and are finite we obtain

$$\lim_{t\to\infty}\beta^t E^{\mathcal{P}}[Q_{s,t}] = \underbrace{\left(\lim_{t\to\infty}\beta^t\right)}_{=0}\exp\underbrace{\left(\lim_{t\to\infty}E^{\mathcal{P}}[\ln Q_{s,t}] + \frac{1}{2}\lim_{t\to\infty}\text{var}^{\mathcal{P}}(\ln Q_t)\right)}_{<\infty} = 0.$$
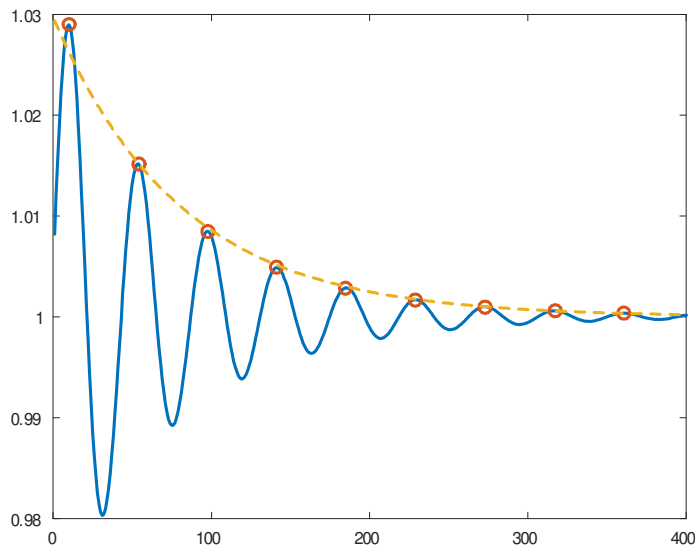
This completes the proof. $\qquad\square$

D.4   Details of the Phase Diagram in Figure 4

Shifting equation (15) one period forward and setting the information shock $\ln \varepsilon_{s,t+1}^1$ to zero, we have from (34)

$$\ln m_{c,t+1} = \ln m_{c,t} + g\left(\ln Q_{c,t} - \ln Q_{c,t-1} - \ln m_{c,t}\right)$$
$$= \ln m_{c,t} + g\left(-\ln(1 - B\cdot m_{c,t}) + \ln(1 - B\cdot m_{c,t-1}) - \ln m_{c,t}\right)$$
$$= (1 - g)\ln m_{c,t} - g\ln(1 - B\cdot m_{c,t}) + g\ln(1 - B\cdot m_{c,t-1})$$
$$m_{c,t+1} = (m_{c,t})^{1-g}\frac{(1 - B\cdot m_{c,t-1})^g}{(1 - B\cdot m_{c,t})^g}$$

Figure 12. Illustration of the decay rate restriction

This nonlinear second-order difference equation can be written as a first order two dimensional difference equation system

$$\begin{pmatrix} m_{c,t+1} \\ m_{c,t} \end{pmatrix} = \begin{pmatrix} (m_{c,t})^{1-g} \frac{(1-B \cdot m_{c,t-1})^g}{(1-B \cdot m_{c,t})^g} \\ m_{c,t} \end{pmatrix},$$

which maps the pair $(m_{c,t}, m_{c,t-1})$ into a new pair $(m_{c,t+1}, m_{c,t})$. Clearly, the second coordinate is stable over time for $m_{c,t-1} = m_{c,t}$, as indicated in figure 4. The first coordinate is stable $(m_{c,t+1} = m_{c,t})$ if

$$m_{c,t} = (m_{c,t})^{1-g} \frac{(1-B \cdot m_{c,t-1})^g}{(1-B \cdot m_{c,t})^g} \Leftrightarrow m_{c,t-1} = \frac{1}{B} - \left(\frac{1}{B} - m_{c,t} \right) m_{c,t},$$

which is the other solid line in Figure 4.

D.5 Details on the IRF Decay Restrictions in Estimation

Capital prices in the subjective belief model show a cyclical pattern in response to a technology shock: as prices fall from a boom to steady state, agents may be already slightly pessimistic and prices undershoot, as prices recover from a bust back to steady state, agents may already be slightly optimistic, triggering another boom. If these dynamics are too strong, deterministic cycles can exist or the cyclical dynamics can even be

self-amplifying (small shocks lead to a sequence of cycles of increasing magnitude). We impose a decay restriction in the estimation to rule out such dynamics. The IRF decay restriction does not impose a certain speed of decay within one cycle, but rather requires the decay of subsequent cycle peaks over time to be sufficiently fast.

Specifically, for each sector $s \in \{c, i\}$ we consider a long deterministic impulse response path (400 quarters) for the capital price Q_s to a one-standard-deviation technology shock starting in the steady state. We identify all peaks (local maxima) of the resulting path $\{Q_{s,t}\}_{t=0}^{400}$, where at t there is a "peak", if $Q_{s,t} > Q_{s,t-1}, Q_{s,t+1}$. Let T_p be the set of all peak times of the impulse response path. If $|T_p| \leq 1$, we set the peak decay rate to ∞, thereby always admitting such a parameter combination. If $|T_p| \geq 2$, we fit an exponential function through the points $\{Q_{s,t}\}_{t \in T_p}$, specifically we estimate the least-squares regression

$$\ln\left(\frac{Q_{s,t}}{Q_s^{ss}} - 1\right) = a + bt + \varepsilon_t, \qquad t \in T_p,$$

where Q_s^{ss} is the steady-state value of Q_s. We call $-b$ the peak decay rate of the impulse response. Figure 12 illustrates the procedure graphically: the blue solid line is the impulse response path, the red circles mark the peaks, i.e. the points $(t, Q_{s,t}/Q_s^{ss})$ for $t \in T_p$, and the yellow dashed line represents the fitted exponential.

On the b parameter we impose the restriction $-b \geq 1.16\%$. This implies a half-life of at most 60 quarters or – given a typical distance of approximately 40 quarters between two peaks – a size reduction of at least one third from one peak to the next.

Parameter combinations are admitted in the estimation, if they satisfy this condition for both Q_c and Q_i.

D.6 Comparison with Boldrin, Christiano, and Fisher (2001) and Adam, Marcet, and Beutel (2017)

In this section we compare the quantitative implications of our subjective belief model with Boldrin, Christiano, and Fisher (2001) and Adam, Marcet, and Beutel (2017). We choose to compare our model to Boldrin, Christiano, and Fisher (2001), because it is one of the leading joint explanations of stock prices and business cycles under rational expectations in the literature and the one closest to ours. We compare our model to Adam, Marcet, and Beutel (2017), because they study the same belief specification as we do, but in an endowment economy.

Boldrin, Christiano, and Fisher (2001) calibrate their model to a different data sample than we consider. In order to make the comparison as fair as possible, we use their reported data moments as a benchmark,[89] not the ones reported in Section 3, but re-estimate

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[89] For financial moments, they consider U.S. data covering the period 1892–1987, for business cycle moments U.S. data covering 1964:Q1–1988:Q2.

our model to fit those moments using the procedure outlined in Section 7. As estimation targets we use all moments reported in Table 8 with the exception of the standard deviation of the risk-free rate.[90] Boldrin, Christiano, and Fisher (2001) report annual instead of quarterly return moments despite their model being quarterly, which necessitates another change to make the two models comparable. As return autocorrelations are not fully in line with the data in either model,[91] we transform the annual return moments reported in Boldrin, Christiano, and Fisher (2001) both for the data and the model to quarterly frequency under the assumption of no return autocorrelation at that frequency.[92]

Table 8
Parameters for subjective belief model and model of Boldrin, Christiano, and Fisher (2001)

Parameter	Subjective Belief Model	BCF
β	0.997	0.999999
α_c	0.36	0.36
α_i	0.6	0.36
δ_c	0.15	0.021
δ_i	0.01	0.021
γ	1.004	1.004
σ	0.015	0.018
g	0.025	–
p	0.4	(0.248)

Note: BCF refers to Boldrin, Christiano, and Fisher (2001).

Table 8 shows the estimated parameters in comparison with the parameters used by Boldrin, Christiano, and Fisher (2001). Due to the similar structure of the two models, the parameters in their model have an identical interpretation as in ours. They are also

---

[90] The standard deviation of the risk-free rate is clearly lower in our model than in the data, particularly relative to the sample starting in 1892 used by Boldrin, Christiano, and Fisher (2001). We do not consider this as a serious shortcoming, since the data moment is likely overstated as argued in Section 3. For this reason, we do not attempt to match this number perfectly.

[91] In our model, returns at the quarterly frequency are positively autocorrelated. This is a known weakness of subjective price belief models of the kind studied here in the absence of transitory shocks. See Adam, Marcet, and Beutel (2017) Section VIII.A for a discussion and solution of this issue. Instead, in Boldrin, Christiano, and Fisher (2001) quarterly returns are strongly negatively correlated.

[92] This assumption is (approximately) correct for the data and it transforms – counterfactually – the good fit of the Boldrin, Christiano, and Fisher (2001) model to quarterly returns, so as to not bias the model comparison towards our model. It implies that means are divided by 4 and standard deviations are divided by 2. Similarly, we divide standard errors of means by 2 and standard errors of standard deviations by $\sqrt{2}$.

quantitatively not too different. Table 9 reports a standard set of business cycle and financial return moments in the data and in both models. The second column reports moments for our model, the third column moments for Boldrin, Christiano, and Fisher (2001). Overall, both models match the set of business cycle moments well. Where there are differences between the two, our model tends to do slightly better. Along the financial dimension, the model by Boldrin, Christiano, and Fisher (2001) is able to match the average levels of the risk-free rate and stock returns perfectly and generates stock return volatility close to the one observed in the data. Our model is less successful with the former two moments, but its prediction lie within a one-standard-error interval around the point estimates in the data. As Boldrin, Christiano, and Fisher (2001), we are able to generate high stock return volatility, but unlike in their paper, this is not achieved by generating counterfactually high volatility in the risk-free rate.

Table 9
Model comparison with Boldrin, Christiano, and
Fisher (2001): targeted moments

	Data (std.dev.)	Subj. Belief Model	BCF
Business Cycle Moments			
$\sigma(Y)$	1.89 (0.21)	1.83	1.97
$\sigma(C)/\sigma(Y)$	0.40 (0.04)	0.67	0.69
$\sigma(I)/\sigma(Y)$	2.39 (0.06)	2.46	1.67
$\sigma(H)/\sigma(Y)$	0.80 (0.05)	0.80	0.51
$\rho(Y,C)$	0.76 (0.05)	0.91	0.95
$\rho(Y,I)$	0.96 (0.01)	0.93	0.69
$\rho(Y,H)$	0.78 (0.05)	0.74	0.86
Financial Moments			
$E[r^f]$	0.30 (0.41)	0.68	0.30
$E[r^e - r^f]$	1.66 (0.89)	0.79	1.66
$\sigma(r^f)$	2.64 (0.52)	0.07	12.30
$\sigma(r^e)$	9.70 (1.10)	9.21	9.20

Notes: BCF refers to Boldrin, Christiano, and Fisher (2001); data and standard errors are taken from BCF and refer to their sample; moments and standard errors for financial moments are transformed to quarterly frequency by the procedure described in the main text.

Table 10 reports additional evidence for the two models. The first four moments are statistics reported in Tables 1 and 2 of Boldrin, Christiano, and Fisher (2001) which have not been discussed in Section 3 of the present paper. $\rho(\Delta Y_t)$, $\rho(\Delta C_t)$ denote the

autocorrelation of log output growth and log consumption growth, respectively, $\sigma(P_{hp})$ is the standard deviation of logged and HP-filtered quarterly stock prices and $\rho(Y, P_{hp})$ is the correlation of output and stock prices, both logged and HP-filtered. The two models perform similarly for the first three moments, both matching the persistence of output growth, but underpredicting the persistence of consumption growth and overstating the volatility of HP-filtered stock prices. Our model generates somewhat more comovement of stock prices with output than Boldrin, Christiano, and Fisher (2001), in line with the data.

Table 10 also shows statistics that relate to the behavior of dividends and the PD ratio. These statistics are not reported by Boldrin, Christiano, and Fisher (2001). We therefore report our own estimates discussed in Section 3 in the data column and compute the respective moments in the Boldrin, Christiano, and Fisher (2001) model ourselves.[93] For the definition of dividends and prices we use the same convention as for our own model. Namely, firms pay out a fixed fraction p of capital rental income each period as dividends and reinvest the remaining fraction $1 - p$ into new capital. Then, dividend growth equals the growth rate of the (sector-weighted) capital rental rates and the PD ratio is an affine linear function of the capital-price-to-rental-rate ratio, as in our model. For this reason, only the moments $E[P/D]$ and $\sigma(P/D)$ depend on the value of the parameter p, which is not present in Boldrin, Christiano, and Fisher (2001). We choose this parameter so as to minimize the sum of squared t statistics for two moments.[94] The resulting value is $p = 0.248$. In our model, dividends and the PD ratio behave qualitatively as discussed in Section 7, although the overall quantitative fit is somewhat worse, because the PD ratio was not targeted by the estimation. Yet, for any of the reported moments, our model clearly outperforms Boldrin, Christiano, and Fisher (2001). Dividend growth is far too volatile in their model and the PD ratio is neither as volatile nor as persistent as in the data and in our model. In addition, the PD ratio in Boldrin, Christiano, and Fisher (2001) displays negative comovement with hours worked and the investment-to-output ratio. We have encountered such negative correlations also in the rational expectations version of our model above. The reason for this negative correlation is, that capital prices are much less persistent than H and I/Y in Boldrin, Christiano, and Fisher (2001) and thus only mildly procyclical, but rental rates are similarly persistent as macro aggregates. This leads to a negative correlation of H (I/Y) and the PD ratio, despite the fact that they both move in the same direction on impact in response to a technology shock.

Next, we compare the financial moments of our model to the baseline specification of the endowment-economy model of Adam, Marcet, and Beutel (2017) reported in their

---

[93]We solve the Boldrin, Christiano, and Fisher (2001) model by solving the associated planner problem using standard Bellman-equation-based iteration methods. We evaluate the accuracy of our solution by computing Euler errors and replicating the model moments reported in Boldrin, Christiano, and Fisher (2001). Additional moments reported by us are computed using the same method as described in footnote a in all of their tables (based on 500 simulations of sample size 200).

[94]This mirrors our estimation procedure for the subjective belief model.

Table 10

Model comparison with Boldrin, Christiano, and
Fisher (2001): additional moments

	Data (std. dev.)	Subj. Belief Model	BCF
Additional Moments reported by BCF			
$\rho(\Delta \log Y)$	0.34 (0.07)	0.39	0.36
$\rho(\Delta \log C)$	0.24 (0.09)	-0.02	-0.05
$\sigma(P_{hp})$	8.56 (0.85)	19.7	12.1
$\rho(Y, P_{hp})$	0.30 (0.08)	0.32	0.16
Dividend and P/D Moments			
$\sigma(D_{t+1}/D_t)$	1.75 (0.38)	1.49	6.87
$E[P/D]$	152.3 (25.3)	110.5	162.4
$\sigma(P/D)$	63.39 (12.39)	32.95	13.20
$\rho(P/D)$	0.98 (0.003)	0.91	0.18
$\rho(H, P/D)$	0.51 (0.17)	0.29	-0.60
$\rho(I/Y, P/D)$	0.58 (0.19)	0.20	-0.60

Notes: BCF refers to Boldrin, Christiano, and Fisher (2001);
p estimated to minimize standard-error-weighted squared dis-
tance of mean and std of P/D from data, estimate is $p =
0.24782$; data and standard errors for first four moments are
taken from BCF and refer to their sample; data and standard
errors for remaining moments are not reported in BCF, we
therefore take the ones from Section 3.

Table 3 (last column, labeled "diagonal Σ matrix"). As their data sample is almost identical to ours, we consider again the estimated model from Section 7. Table 11 reports the results. Not unexpected, our production economy matches the moments less well than the endowment economy studied in Adam, Marcet, and Beutel (2017). Yet, overall the performance of our model is relatively close to that of the Adam, Marcet, and Beutel (2017) model. Given that our model has at the same time realistic business cycle implications, we consider this a substantial achievement.

Table 11
Model comparison with Adam, Marcet, and Beutel (2017)

Moment	Data	Model	AMB
$E[P/D]$	152.3 (25.3)	149.95	115.2
$\sigma(P/D)$	63.39 (12.39)	44.96	88.20
$\rho(P/D)$	0.98 (0.003)	0.97	0.98
$E[r^e]$	1.87 (0.45)	1.25	1.82
$\sigma(r^e)$	7.98 (0.35)	7.07	7.74
$E[r^f]$	0.25 (0.13)	0.78	0.99
$\sigma(r^f)$	0.82 (0.12)	0.06	0.83
$\sigma(D_{t+1}/D_t)$	1.75 (0.38)	2.46	1.92
$\rho(E^{\mathcal{P}}[r^e], P/D)$	0.79 (0.07)	0.52	0.79

Note: Column "Model" refers to our model, column "AMB" refers to Adam, Marcet, and Beutel (2017).

D.7 PD Distribution of the Estimated RE Models

Figure 13 plots the PD distributions implied by the two estimated RE models section 7 against that in the data. It shows that the RE models fail in generating a skewed distribution.
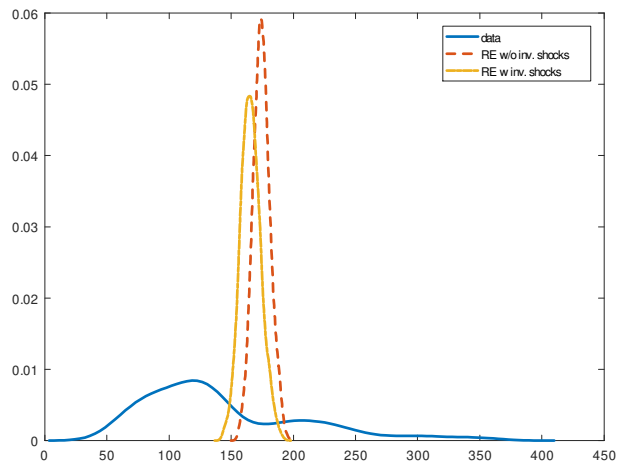
Figure 13. Unconditional density of PD ratio: RE models vs data (not targeted in estimation, kernel estimates)