Discussion Paper Series – CRC TR 224

Discussion Paper No. 120
Project B 01, B 04

The Balance Condition in Search-and-Matching Models

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August 2019

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Funding by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through CRC TR 224 is gratefully acknowledged.

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August 15, 2019

Abstract: Most of the literature that studies frictional search-and-matching models with heterogeneous agents and random search investigates steady-state equilibria. Steady-state equilibrium requires, in particular, that the flows of agents into and out of the population of unmatched agents balance. We investigate the structure of this balance condition, taking agents' matching behavior as given. Building on the "fundamental matching lemma" for quadratic search technologies in Shimer and Smith (2000), we establish existence, uniqueness, and comparative-static properties of the solution to the balance condition for *any* search technology satisfying minimal regularity conditions. Implications for the existence and structure of steady-state equilibria are noted.

Keywords: Search, Matching, Steady State.

*This paper subsumes Nöldeke and Tröger (2009). We are grateful to Daron Acemoglu and Lones Smith for prodding us to pursue the more general results presented here and thank Alfred Galichon, Samuel Häfner, Philipp Kircher, Mihai Manea, and Larry Samuelson for helpful comments and discussions. Financial support from the German Research Foundation (DFG) through SFB-TR 224, Project B01 (Tröger), SFB-TR 224, Project B04 (Lauermann), and Germany's Excellence Strategy—EXC 2126/1390838866 (Lauermann) is gratefully acknowledged.

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1 Introduction

Search-and-matching models with heterogeneous agents are the cornerstone of the literature investigating the economic causes and consequences of sorting and mismatch in labor markets (Marimon and Zilibotti, 1999; Teulings and Gautier, 2004; Eeckhout and Kircher, 2011; Hagedorn, Law, and Manovskii, 2017). More generally, such models, surveyed by Burdett and Coles (1999), Smith (2011) and Chade, Eeckhout, and Smith (2017, Section 4), have provided key insights into the functioning of decentralized markets with trading frictions. Examples include the conditions for assortative matching with transferable utility (Shimer and Smith, 2000), block segregation with nontransferable utility (Burdett and Coles, 1997; Smith, 2006), the emergence of the law of one price (Gale, 1987; Lauermann, 2013), and the convergence of equilibria to stable matches as frictions vanish in marriage markets with nontransferable utility (Lauermann and Nöldeke, 2014). With few exceptions (e.g., Smith, 2009), the literature considers steady-state equilibria.

As emphasized in Burdett and Coles (1999), identifying a steady-state equilibrium in a search-and-matching model involves solving two distinct problems. First, taking a time-invariant population of unmatched agents as given, determine stationary equilibrium strategies in the induced strategic interaction between the agents. Such a "partial equilibrium" (Burdett and Coles, 1997) specifies, in particular, who would like to match with whom. Second, taking such stationary matching behavior as given, determine a time-invariant population of unmatched agents by the "balance condition" that the outflow from the population of unmatched agents (due to the formation of matches or exit) is equal to the corresponding inflow (due to match dissolution or entry) for every type of agent. A steady-state equilibrium obtains if the partial-equilibrium condition and the balance condition both hold.

Much can be learned about the structure of steady-state equilibria from considering the partial-equilibrium condition in isolation.[1] For instance, all of the descriptive theory developed in Shimer and Smith (2000) and Smith (2006) is implied by the partial-equilibrium condition and holds independently of whether or not the balance condition is satisfied. For other purposes, the interaction between the partial-equilibrium condition and the balance condition takes center stage. For instance, multiple steady-state equilibria may arise in the model of Burdett and Coles (1997) even though (as they show) partial equilibrium is unique for a given unmatched population and (as we show) their balance condition has a unique solution for any given specification of matching behavior. Lauermann (2013) demonstrates that the balance condition plays a key role in ensuring convergence of steady-state equilibrium outcomes to competitive outcomes for vanishing search frictions, both because it is

[1] Indeed, there are many papers (e.g., McNamara and Collins, 1990; Morgan, 1998; Bloch and Ryder, 2000; Chade, 2001; Adachi, 2003; Chen, 2005; Coles and Francesconi, 2019), which do exactly this by assuming that agents who leave the pool of unmatched agents are immediately replaced by identical clones. In such cloning models, the balance condition is trivially satisfied for *any* population of unmatched agents. Thus, the conditions for a steady-state equilibrium reduce to the partial-equilibrium condition for an exogenously given population of unmatched agents.

the counterpart to a market clearing condition, ensuring the feasibility of the resulting outcomes given any specification of the agents' matching behavior, and because it ensures that individuals who are less likely to match than others are overrepresented in the unmatched population.

In this paper, we focus on the balance condition, thereby complementing the extensive literature that has considered the behavioral implications of the partial-equilibrium condition. Our main motivation for focussing on the balance condition comes from the elegant, modular proof of existence of steady-state equilibrium in Shimer and Smith (2000).[2] In particular, provided that the conditions determining partial equilibrium (which are independent of the balance condition) are sufficiently well-behaved (as they are not only in Shimer and Smith (2000) but also in Smith (2006)), existence of steady-state equilibrium is assured as long as the so-called "fundamental matching lemma" (Shimer and Smith, 2000, Lemma 4) holds. This lemma asserts that there is a unique unmatched population solving the balance condition for any given specification of matching behavior and, further, that this solution is continuous in the matching behavior. Unfortunately, establishing the fundamental matching lemma is "the hardest step" (Smith, 2011, p. 355) in the existence proof from Shimer and Smith (2000) and the authors only succeed in doing so for the special case of a quadratic search technology, which "seems a poor approximation" unless one deals with "an economy with a low density of potential partners" (Diamond and Maskin, 1979, p.283). In the years since the publication of Shimer and Smith (2000) almost no progress has been made in extending the fundamental matching lemma to other search technologies.[3]

Our main results show that the fundamental matching lemma does not require the assumption that the search technology is quadratic. Rather, the fundamental matching lemma holds whenever the search technology induces an aggregate meeting rate that is continuous and non-decreasing in the mass of unmatched agents and satisfies the boundary condition that the aggregate meeting rate is zero if there are no unmatched agents. These regularity conditions on the behavior of the aggregate meeting rate are not only natural; we also show (cf. Remark 2) that they are minimal in the following sense: if any one of these three conditions (continuity, monotonicity, boundary behavior) fails, examples featuring either no or multiple solutions to the balance condition can be constructed.

Section 2 introduces the balance condition that we study in this paper. This condition coincides with the balance condition from the search-and-matching models with random

[2]For search-and-matching models with a finite number of types, Lauermann and Nöldeke (2015) and Manea (2017) present alternative proofs for the existence of steady-state equilibrium that dispense with many of the structural assumptions on the search technology and the production function maintained in Shimer and Smith (2000) and Smith (2006). Manea (2017, p. 219) discusses the formidable technical difficulties of generalizing his arguments to a setting with a continuum of types and concludes that "extending our results to settings with a continuum of types in order to obtain an exact generalization of Shimer and Smith's (2000) existence theorem" is an "open problem." (Manea, 2017, p. 220). Similarly, the combination of continuity and compactness properties required for the proof from Lauermann and Nöldeke (2015) is difficult to obtain in a setting with a continuum of types, stymieing any straightforward extension of their arguments.

[3]In a precursor to the work reported here, Nöldeke and Tröger (2009) show that the fundamental matching lemma holds for linear search technologies.

search, a one-dimensional continuum of types, and exogenous separation of matches considered in Shimer and Smith (2000) and Smith (2006), except that we allow for general search technologies rather than restricting attention to quadratic search technologies. We thus refer to our balance condition as the "general balance condition" and to the one from Shimer and Smith (2000) and Smith (2006) as the "quadratic balance condition." These two balance conditions are closely linked. Indeed, every solution to the general balance condition is also a solution to the quadratic balance condition for a suitable choice of a parameter that scales the velocity of the underlying quadratic search technology (Lemma 1). Together with the fundamental matching lemma for quadratic search technologies (that we restate as Lemma 2) this simple observation provides the starting point for our subsequent analysis.

Section 3 provides three comparative-statics results that describe how the solution to the quadratic balance condition behaves as a function of the velocity parameter. The most important of these results are Lemmas 3 and 5. Lemma 3 shows that for any given matching pattern an increase in the velocity of a quadratic search technology cannot cause the number (formally: the density) of unmatched agents of any type to increase. In particular, the mass of unmatched agents does not increase when the velocity of a quadratic search technology is increased. On the other hand, Lemma 5 shows that the unmatched mass cannot decrease so quickly as to offset the positive effect of an increase in the velocity on the aggregate meeting rate.

Section 4 leverages the comparative-statics properties for the quadratic search technology established in Section 3 to show our main result: the fundamental matching lemma holds for the general class of search technologies described above (Propositions 1 and 2). In addition, we find that all the comparative-statics properties we have established for solutions of the quadratic balance condition carry over to the solutions of the general balance condition (Proposition 3).

While our analysis in Sections 2 - 4 considers a general class of search technologies, it otherwise retains the structure of the search-and-matching process in Shimer and Smith (2000). Section 5 shows that this isn't required by considering three extensions. The most important and difficult of these is to the case in which—as in most applied work—there are two distinct groups of agents (workers and firms, or women and men) and agents in one group only meet agents from the other group.

Section 6 discusses the implications of our analysis for steady-state equilibria. We explain in detail how the arguments proving the existence of steady-state equilibrium in Shimer and Smith (2000) and Smith (2006) carry over from the special case of the quadratic search technology considered in those papers to general search technologies. We also note implications for the existence of steady-state equilibria in more general models (e.g., with imperfectly transferable utility or multi-dimensional types), as well as for the structure of the set of steady-state equilibria, and their uniqueness. Section 7 concludes.

2 The Balance Condition

We consider a search-and-matching process akin to the one considered in Shimer and Smith (2000) and Smith (2006). Time is continuous and there is a continuum of infinitely lived agents. Agents are characterized by their types $x \in \mathcal{X} = [0,1]$. For any measurable $X \subset \mathcal{X}$, the mass of agents with types in this set is given by $\int_X \ell(x)dx$, where the (exogenous) population density $\ell : \mathcal{X} \to (0,\infty)$ is measurable, bounded, and bounded away from zero. Let $\bar{\ell} = \int \ell(x)dx > 0$ denote the total mass of agents.[4]

Agents are either unmatched or are matched with a single partner. The mass of unmatched agents with types in X is given by $\int_X u(x)dx$, where the (endogenous) unmatched density $u : \mathcal{X} \to (0,\infty)$ is measurable and satisfies $u(x) \leq \ell(x)$ for all $x \in \mathcal{X}$. The mass of matched agents with types in X is $\int_X (\ell(x) - u(x))dx$.

Unmatched agents search for partners and meet other unmatched agents, drawn at random from the population of all unmatched agents. The mass of unmatched agents involved in meetings per unit of time—the aggregate meeting rate—is determined by the search technology as a function of the mass of unmatched agents $\bar{u} = \int u(x)dx$ and a velocity parameter $\sigma \geq 0$. Let $\mathcal{D}$ denote the set of real-valued functions on $\mathcal{X}$ that are positive, measurable, and bounded.[5]

Assumption 1 (General Search Technology)**.** *For any unmatched density $u \in \mathcal{D}$, the aggregate meeting rate is given by $\sigma \cdot m(\bar{u})$, where $\sigma \geq 0$ is the velocity parameter of the search technology and the contact function $m : (0,\infty) \to (0,\infty)$ is continuous, non-decreasing, and satisfies $\lim_{\bar{u}\to 0} m(\bar{u}) = 0$.*

As meetings are random, every unmatched agent meets other unmatched agents with types in X at the rate $\sigma \cdot r(\bar{u}) \int_X u(x)dx$, where

$$r(\bar{u}) = \frac{m(\bar{u})}{\bar{u}^2}. \tag{1}$$

We refer to $r : (0,\infty) \to (0,\infty)$ as the rate function and to $\sigma \cdot r(\bar{u}) \cdot \bar{u}$, i.e., the rate at which an unmatched agent meets some other unmatched agent, as the individual meeting rate.

Matches form whenever two unmatched agents meet and agree to match with each other. We describe the proportion of meetings between agents with types x and y that lead to a match by a symmetric measurable function $\alpha : \mathcal{X} \times \mathcal{X} \to [0,1]$ and refer to α as the matching affinity.[6] Let $\mathcal{A}$ denote the set of all such functions. We treat $\alpha \in \mathcal{A}$ as exogenous, thereby taking the behavior of agents as given.

[4] When no confusion can arise, we use the unadorned integral $\int$ to denote integration over the entire type space $\mathcal{X}$.

[5] We follow the convention of using terms such as positive, negative, increasing, and decreasing in the strict sense, with a prefix "non-" indicating the opposite weak sense.

[6] The symmetry condition $\alpha(x,y) = \alpha(y,x)$ is an accounting identity: every instance in which an agent of type x meets an agent of type y is also an instance in which an agent of type y meets an agent of type x with $\alpha(x,y)$ and $\alpha(y,x)$ both denoting the fraction of such meetings leading to a match.

All matches dissolve at an exogenous rate $\delta > 0$. At the moment a match is dissolved, both agents return to the pool of unmatched agents.

In a steady state, the flow creation and flow destruction of matches for every type of agent must balance. The flow of matches that are created and involve type x is the product of the unmatched density $u(x)$ of type x and the individual matching rate $\sigma \cdot r(\bar{u}) \int \alpha(x,y)u(y)dy$ for agents of type x. The flow of matches that are destroyed and involve type x is the product of the dissolution rate δ and the matched density $\ell(x) - u(x)$ of type x. Therefore, steady state requires

$$\delta \left[\ell(x) - u(x)\right] = u(x) \cdot \sigma \cdot r(\bar{u}) \int \alpha(x,y)u(y)dy \quad \forall x \in \mathcal{X}. \tag{2}$$

From equation (2), it is obvious that there is no loss of generality in normalizing the dissolution rate δ to 1. We do so throughout the following and rewrite (2) as

$$\ell(x) = u(x) \left[1 + \sigma \cdot r(\bar{u}) \int \alpha(x,y)u(y)dy\right] \quad \forall x \in \mathcal{X}. \tag{3}$$

We refer to (3) as the general balance condition.

Remark 1 (Finite Number of Types)**.** The counterpart to (3) for a model with a finite number of types $x = 1, \ldots, n$ is

$$\ell(x) = u(x) \left[1 + \sigma \cdot r(\bar{u}) \sum_{y=1}^{n} \alpha(x,y)u(y)\right] \quad \forall x = 1, \ldots, n, \tag{4}$$

where $\ell(x) > 0$ is now the mass of agents of type x in the population and $u(x)$ is the corresponding unmatched mass. It is easily verified that all of our subsequent analysis and results carry over to (4). ■

A broad class of search technologies is compatible with Assumption 1, including the quadratic search technology considered in Shimer and Smith (2000) and Smith (2006).

Assumption 2 (Quadratic Search Technologies)**.** *For any unmatched density $u \in \mathcal{D}$, the aggregate meeting rate is given by*

$$m(\bar{u}) = \rho \cdot \bar{u}^2 \tag{5}$$

with $\rho \geq 0$.

In Assumption 2, we have embedded a velocity parameter $\rho \geq 0$ in the description of the contact function m, while setting the velocity parameter premultiplying the contact function m in Assumption 1 to $\sigma = 1$. This proves convenient in formulating Lemma 1 below.

From equations (1) and (5), the rate function associated with a quadratic search technology is simply given by its velocity (i.e., $r(\bar{u}) = \rho$ holds for all $\bar{u} > 0$). Thus, under

Assumption 2 the general balance condition (3) simplifies to

$$\ell(x) = u(x)\left[1 + \rho \int \alpha(x,y)u(y)dy\right] \quad \forall x \in \mathcal{X}. \tag{6}$$

We refer to (6) as the quadratic balance condition.

Comparing (3) and (6), the following lemma is immediate:

Lemma 1. *An unmatched density $u \in \mathcal{D}$ solves the general balance condition (3) if and only if u solves the quadratic balance condition (6) for ρ satisfying*

$$\rho = \sigma \cdot r(\bar{u}). \tag{7}$$

Lemma 1 provides a tight link between the solutions to the general balance condition and the solutions to the quadratic balance condition. In the next two sections, we show how this link can be exploited to leverage results for quadratic search technologies (Section 3) to obtain corresponding results for general search technologies (Section 4). Before turning to these tasks, we restate Lemma 4 from Shimer and Smith (2000), the so-called "fundamental matching lemma" (Smith, 2006) for quadratic search technologies, in a form suitable for our subsequent analysis:[7]

Lemma 2 (Fundamental Matching Lemma for Quadratic Search Technologies)**.** *There exists a unique unmatched density $u \in \mathcal{D}$ solving the quadratic balance condition (6) and this solution is a jointly continuous function of ρ and α.*

3 Quadratic Search Technologies

In this section, we investigate the comparative statics of the solution to the quadratic balance condition as a function of the velocity parameter ρ (taking α as given). To simplify notation, we thus write u_ρ for the solution of (6). By Lemma 2, u_ρ is uniquely defined for all $\rho \geq 0$.

Our first result asserts that the solution to the quadratic balance condition is non-increasing in the velocity of the search technology. The proof is in Appendix A.1.

Lemma 3. *The map $\rho \to u_\rho(x)$ is non-increasing for all $x \in \mathcal{X}$.*

To see the intuition behind Lemma 3, it is instructive to consider an example with a finite number of types (cf. Remark 1). Suppose, for instance, that the velocity ρ increases

[7] The continuity claim in the statement of Lemma 2 means that $\lim_{n\to\infty} \int\int |\alpha_n(x,y) - \alpha(x,y)|dxdy + |\rho_n - \rho| = 0$ implies $\lim_{n\to\infty} \int |u_{\rho_n,\alpha_n}(x) - u_{\rho,\alpha}(x)|dx = 0$, where $(\rho,\alpha) \to u_{\rho,\alpha}$ is the map from $[0,\infty)\times\mathcal{A}$ to the unmatched density implied by (6). For fixed ρ, this is equivalent to the continuity claim in Shimer and Smith (2000, Lemma 4), who show that the map $\alpha \to u_{\rho,\alpha}$ is continuous with respect to the pseudometrics on $\mathcal{A}$ and $\mathcal{D}$ induced by the L^1-norms. While Shimer and Smith (2000) do not consider the (joint) continuity in ρ, this is immediate upon noting that (i) their arguments do not require α to be bounded above by 1 and, therefore, show that the solution to (6) is continuous in $\hat{\alpha} = \rho \cdot \alpha$, and that (ii) the map $(\rho,\alpha) \to \hat{\alpha}$ is continuous, too.

by 5 percent. As there is a finite number of types, there must be some type x' for whom the associated percentage change in $u_\rho(x')$ is maximal. Now suppose that, contrary to what is asserted in Lemma 3, the increase in velocity leads to an increase in $u_\rho(x')$ by, say, 1 percent. To maintain the balance condition, such an increase in the unmatched mass of type x' by 1 percent requires that the matching rate for type x' decreases by more than 1 percent (as, otherwise, the right side of (6) increases). Given that the velocity has increased by 5 percent, this, in turn, requires the unmatched mass $u_\rho(y)$ of some other type y to decrease by more than 6 percent. Similar reasoning to the one just given shows that this is only possible if there is some further type x'' whose unmatched mass $u_\rho(x'')$ has increased by more than 1 percent. But the existence of such a type x'' contradicts the hypothesis that the percentage change in the unmatched mass is maximal for type x'. Thus, $u_\rho(x)$ must be non-increasing in ρ for all types x.[8]

For a given unmatched density u, it is trivial that an increase in ρ causes an increase in $\rho \cdot u(x)$ for all types x. The point of the following lemma is that the countervailing effect of an increase in the velocity ρ on the unmatched density u_ρ identified in Lemma 3 cannot overturn this direct effect:

Lemma 4. *The map $\rho \to \rho \cdot u_\rho(x)$ is increasing for all $x \in \mathcal{X}$.*

The proof of Lemma 4 is in Appendix A.2. It is straightforward from the arguments proving Lemma 3.

Obviously, Lemma 4 implies that the individual meeting rate $\rho \cdot \bar{u}_\rho$ is increasing in ρ. For our subsequent analysis we require more, namely that the aggregate meeting rate $\rho \cdot \bar{u}_\rho^2$ is also increasing in ρ.

Lemma 5. *The map $\rho \to \rho \cdot \bar{u}_\rho^2$ is increasing in ρ.*

The proof of Lemma 5 is in Appendix A.3. It proceeds in two steps. The first step uses a transformation of the unmatched density to replace u_ρ by the unique solution $z_\gamma \in \mathcal{D}$ of the condition

$$\ell(x) = z(x) \left[\gamma + \int \alpha(x,y) z(y) dy \right] \quad \forall x \in \mathcal{X} \tag{8}$$

and shows that the claim in Lemma 5 is equivalent to the claim that the mass $\bar{z}_\gamma$ is decreasing in γ. The second step adapts an argument that Decker, Lieb, McCann, and Stephens (2013) have developed in a related context.[9] It shows that (8) corresponds to the first-order condition of a convex minimization problem to infer that $\bar{z}_\gamma$ is indeed decreasing in γ.[10]

[8] Analogous reasoning can be used to infer the uniqueness claim in the fundamental matching lemma for the quadratic search technology. See the remark at the end of the proof of Lemma 3 in Appendix A.1.

[9] Decker, Lieb, McCann, and Stephens (2013) establish the existence of a unique equilibrium in the model of a marriage market from Choo and Siow (2006) and derive comparative-statics properties of this equilibrium. These results are generalized in Galichon and Salanié (2015).

[10] In light of the fact that Lemmas 3 and 4 provide monotone comparative statics results that hold pointwise (i.e., for all types), it is natural to consider the question whether a pointwise counterpart to Lemma 5, asserting that $\rho \cdot u_\rho(x) \cdot u_\rho(y)$ is increasing in ρ for all $(x,y) \in \mathcal{X}^2$, holds. As we show in an earlier version of this paper (Lauermann, Nöldeke, and Tröger, 2018), this is not true in general.

4 General Search Technologies

Consider a general search technology satisfying Assumption 1 and, as before, denote the unique solution to the quadratic balance condition as a function of ρ by $u_\rho \in \mathcal{D}$. The function $F : [0, \infty) \to [0, \infty)$ given by

$$F(\rho) = \frac{\rho}{r(\bar{u}_\rho)} \tag{9}$$

is then well-defined because $\bar{u}_\rho > 0$ holds for all $\rho \geq 0$ and $\bar{u}_\rho > 0$ implies $r(\bar{u}_\rho) > 0$. Now, suppose the equality

$$F(\rho) = \sigma \tag{10}$$

holds. It is then immediate from Lemma 1 that u_ρ solves the general balance condition (3). Hence, to establish existence of a solution to the general balance condition, it suffices to show that equation (10) has a solution. In the same vein, Lemma 1 implies that the general balance condition has a unique solution if (10) has a unique solution. Consequently, to establish existence and uniqueness of the solution to the general balance condition, which is infinite dimensional, it suffices to show that for every $\sigma \geq 0$ there exists a unique $\rho \geq 0$ solving the one-dimensional equation (10), i.e., to show that the function F is a bijection. Lemmas 2 – 5 provide us with all the tools to do so:

Proposition 1. *Let Assumption 1 hold. Then there exists a unique unmatched density $u \in \mathcal{D}$ solving the general balance condition (3).*

Proof. As we have already argued, it suffices to show that $F : F : [0, \infty) \to [0, \infty)$ is a bijection. We clearly have $F(0) = 0$. Hence, the desired result follows if F is continuous, increasing, and satisfies $\lim_{\rho \to \infty} F(\rho) = \infty$. To show these properties, it is convenient to use (1) to rewrite (9) as

$$F(\rho) = \frac{\rho \cdot \bar{u}_\rho^2}{m(\bar{u}_\rho)}. \tag{11}$$

By Assumption 1, m is continuous. From Lemma 2, $\bar{u}_\rho$ is continuous in ρ. Hence, both the numerator and the denominator of the fraction on the right side of (11) are continuous in ρ. Thus, F is continuous.

Lemma 3 implies that $\bar{u}_\rho$ is non-increasing in ρ. Assumption 1 then implies that $m(\bar{u}_\rho)$ is a non-increasing function of ρ. From Lemma 5, $\rho \cdot \bar{u}_\rho^2$ is increasing in ρ. It follows from (11) that F is increasing.

We have already noted that $m(\bar{u}_\rho)$ is non-increasing in ρ, so that $\lim_{\rho \to \infty} m(\bar{u}_\rho)$ is finite. Hence, $\lim_{\rho \to \infty} F(\rho) = \infty$ follows from (11) if $\lim_{\rho \to \infty} \rho \cdot \bar{u}_\rho^2 = \infty$ holds. Suppose, then, that $\rho \cdot \bar{u}_\rho^2$ does not diverge to infinity. Because $\rho \cdot \bar{u}_\rho^2$ is increasing in ρ, it then converges to a finite limit $a^* > 0$. Observing that $\lim_{\rho \to \infty} \rho \cdot \bar{u}_\rho^2 = a^*$ implies $\lim_{\rho \to \infty} \bar{u}_\rho = 0$, Assumption 1 yields $\lim_{\rho \to \infty} m(\bar{u}_\rho) = 0$. Hence, (11) implies $\lim_{\rho \to \infty} F(\rho) = \infty$ in this case, too. □

Remark 2 (Necessity of Assumption 1)**.** The regularity conditions in Assumption 1 are the minimal conditions under which Proposition 1 can be obtained. Specifically, the condition that the aggregate meeting rate is non-decreasing is necessary to exclude the possibility of multiple solutions to the general balance condition. Given that the aggregate meeting rate is non-decreasing, the other two conditions (continuity and boundary behavior) are necessary to exclude the possibility that there is no solution to the general balance condition; Appendix A.4 exhibits simple examples that validate these claims. ■

Having established the existence and uniqueness of a solution to the general balance condition, we next generalize the continuity claim in Lemma 2, thereby showing that the fundamental matching lemma from Shimer and Smith (2000) holds for all search technologies satisfying Assumption 1.[11] The idea underlying the proof of the following proposition is straightforward, namely to use Lemma 1 (and the uniqueness result from Proposition 1) to transfer the continuity result for quadratic search technologies to general search technologies. Some care is required in implementing this idea because we cannot rely on the monotonicity results from Section 3 as we are now considering the effect of changes in the matching affinity α.

Proposition 2. *Let Assumption 1 hold. The unique solution $u \in \mathcal{D}$ to the general balance condition (3) is a jointly continuous function of σ and α.*

Proof. Consider any sequence $\{\sigma_n, \alpha_n, u_n\}$ in $[0,\infty) \times \mathcal{A} \times \mathcal{D}$ satisfying the general balance condition (3) for all $n \in \mathbb{N}$. Let $\rho_n = \sigma_n \cdot r(\bar{u}_n)$ for all n. By Lemma 1, the sequence $\{\rho_n, \alpha_n, u_n\}$ satisfies the quadratic balance condition (6) for all n. As we show in Appendix A.5, we have

Lemma 6. *$\rho_n \to \infty$ holds if and only if $\sigma_n \to \infty$ holds.*

Now suppose that $\{\sigma_n, \alpha_n\}$ converges to $(\sigma_0, \alpha_0) \in [0,\infty) \times \mathcal{A}$. From Lemma 6, this implies that the sequence $\{\rho_n\}$ is bounded. Consider any converging subsequence $\{\rho_{n_k}\}$ of $\{\rho_n\}$ and denote its limit by $\rho_0 \in [0,\infty)$. Lemma 2 then implies that $\{\rho_{n_k}, \alpha_{n_k}, u_{n_k}\}$ converges to (ρ_0, α_0, u_0), where u_0 is the unique unmatched density such that (ρ_0, α_0, u_0) satisfies the quadratic balance condition. As m is continuous (Assumption 1), so is the rate function r defined by (1). Hence, from the convergence of $\{\rho_{n_k}, u_{n_k}\}$ to $\{\rho_0, u_0\}$ and the equality $\rho_n = \sigma_n \cdot r(\bar{u}_n)$, we obtain that σ_0 satisfies

$$\rho_0 = \sigma_0 \cdot r(\bar{u}_0). \tag{12}$$

From Lemma 1, (12) implies that the limit $(\sigma_0, \alpha_0, u_0)$ of the sequence $\{\sigma_{n_k}, \alpha_{n_k}, u_{n_k}\}$ satisfies the general balance condition. From Proposition 1, u_0 is uniquely determined, so that (12) implies that all converging subsequences of $\{\rho_n\}$ converge to the same limit ρ_0. Consequently, the bounded sequence $\{\rho_n\}$ itself converges to ρ_0. Hence, we have shown that the convergence of $\{\sigma_n, \alpha_n\}$ to (σ_0, α_0) implies that the associated sequence $\{u_n\}$ of

[11] The continuity notion in Proposition 2 is the same as in Lemma 2; see footnote 7.

solutions to the general balance condition converges to the unique solution u_0 of the general balance condition for (σ_0, α_0), which is the desired result. □

Finally, we consider the counterparts of the comparative static results in Lemmas 3 to 5 for general search technologies. Using v_σ to denote the solution to the general balance condition (3) as a function of the parameter σ, the counterpart to Lemma 3 is the claim that the density $v_\sigma(x)$ is non-increasing in σ, the counterpart to Lemma 4 is that $\sigma \cdot r(\bar{v}_\sigma) \cdot v_\sigma(x)$ is increasing in σ, and the counterpart to Lemma 5 is that the aggregate meeting rate $\sigma \cdot m(\bar{v}_\sigma)$ is increasing in σ. All of these statements are true if the search technology satisfies Assumption 1:

Proposition 3. *Let Assumption 1 hold and let $v_\sigma \in \mathcal{D}$ denote the unique solution to the general balance condition (3) for given values of the other parameters. Then*

1. *The map $\sigma \to v_\sigma(x)$ is non-increasing for all $x \in \mathcal{X}$.*
2. *The map $\sigma \to \sigma \cdot r(\bar{v}_\sigma) \cdot v_\sigma(x)$ is increasing for all $x \in \mathcal{X}$.*
3. *The map $\sigma \to \sigma \cdot m(\bar{v}_\sigma)$ is increasing.*

Proof. Let $\sigma_1 > \sigma_2 \geq 0$. From Lemma 1, there exists ρ_1 and ρ_2 such that $v_{\sigma_1} = u_{\rho_1}$ and $v_{\sigma_2} = u_{\rho_2}$ holds, where (as before) u_ρ denotes the solution of the quadratic balance condition as a function of ρ. Further, $\rho_i = \sigma_i \cdot r(\bar{u}_{\rho_i})$ holds. As shown in the proof of Proposition 1, the function $F(\rho)$ is increasing, so that $\sigma_1 > \sigma_2$ implies $\rho_1 > \rho_2$. The first claim in the statement of the proposition is then immediate from Lemma 3 and the identity $v_{\sigma_i} = u_{\rho_i}$. Using this identity and $\rho_i = \sigma_i \cdot r(\bar{u}_{\rho_i})$, we have $\rho_i \cdot u_{\rho_i}(x) = \sigma_i \cdot r(\bar{v}_{\sigma_i}) \cdot v_{\sigma_i}(x)$ for all $x \in \mathcal{X}$, so that the second claim in the statement of the proposition follows from Lemma 4. By analogous reasoning, we obtain $\rho_i \cdot \bar{u}^2_{\rho_i} = \sigma_i \cdot r(\bar{v}_{\sigma_i}) \cdot \bar{v}^2_{\sigma_i}$. Upon recalling (1), the third claim in the statement of the Proposition is thus implied by Lemma 5. □

5 Extensions

We have framed our analysis of the balance condition in the context of a search model that differs from the one considered in Shimer and Smith (2000) only in that it replaces their quadratic search technology with general search technologies satisfying our Assumption 1. Our arguments apply more generally, though. To illustrate, we consider three extensions.

Multidimensional type spaces. Our proofs do not use the assumption that the type space $\mathcal{X}$ is the closed unit interval $[0, 1]$ rather than any (measurable) subset of $\mathbb{R}^n$ with positive and finite Lebesgue measure. The same is true for the arguments proving the fundamental matching lemma in Shimer and Smith (2000). Hence, all of our results are applicable when types are multidimensional as in Goussé, Jacquemet, and Robin (2017) or Coles and Francesconi (2019).

Exogenous entry and exit. We have followed Shimer and Smith (2000) in supposing that there is an exogenous population of infinitely lived agents who exit the unmatched pool only if they have found a partner and return to this pool when their partnership dissolves due to exogenous separation. Alternatively, we could have followed Burdett and Coles (1997) and others in supposing that (i) there is a constant exogenous inflow of "newborn" agents into the pool of unmatched agents given by a density $\hat{\ell}$, (ii) matches are permanent, so that matched partners never return to the pool of unmatched agents, and (iii) all unmatched agents abandon the search for a partner with an exogenous exit rate $\delta > 0$. Balancing in- and outflows for such a model yields the condition

$$\hat{\ell}(x) = u(x) \left[\delta + \sigma \cdot r(\bar{u}) \int \alpha(x, y) u(y) dy \right], \quad \forall x \in \mathcal{X}. \tag{13}$$

Upon defining $\ell(x) = \hat{\ell}(x)/\delta$, it is immediate that (13) is equivalent to (2). Thus, all of our results carry over to this alternative specification of the search-and-matching process.

Matching with two groups. We have also followed Shimer and Smith (2000) in assuming that the search technology generates random meetings between *all* agents. This is in contrast to much of the theoretical and applied literature on search-and-matching that considers scenarios in which there are two distinct groups of agents (workers and firms, or men and women); the only meetings that are accounted for in these model are those that feature a pair of agents from distinct groups. All of our results can be extended to this case. We explain how this can be done in Appendix A.6. In particular, the fundamental matching lemma holds for all continuous aggregate meeting rates that are non-decreasing in the unmatched masses of both groups and satisfy the boundary condition that there are no meetings if either of the two groups has no unmatched agents. With the singular exception of the two-group version of the linear search technology (discussed in detail and dismissed as a reasonable model in Stevens, 2007), these conditions are satisfied for all specifications of the aggregate meeting rate considered in the extensive literature studying random search models in labor markets (Petrongolo and Pissarides, 2001; Rogerson, Shimer, and Wright, 2005), including search technologies with constant returns to scale and the Cobb-Douglas specification with arbitrary positive exponents.

6 Implications for Steady-State Equilibrium

So far we have considered the balance condition in isolation. To develop implications of our analysis for steady-state equilibrium, requires us to spell out the partial-equilibrium conditions. These are, of course, model-specific. To fix ideas, we begin by discussing the transferable utility model from Shimer and Smith (2000), while noting extensions to other settings thereafter.

6.1 General Search Technologies in the Shimer-Smith Model

In Shimer and Smith (2000), partial equilibrium requires two conditions: First, the optimality condition,

$$\alpha(x,y) = \begin{cases} 0 & \text{if } w(x) + w(y) > f(x,y), \\ 1 & \text{if } w(x) + w(y) < f(x,y), \end{cases} \tag{14}$$

where $w(x)$ is the equilibrium flow value of an unmatched agent of type x and $f(x,y)$ is the flow output generated when agents with types x and y are matched. Second, the value condition,

$$w(x) = \frac{\rho}{2\tau} \cdot \int \max\{f(x,y) - w(x) - w(y), 0\} u(y) dy, \tag{15}$$

where ρ is the velocity of the quadratic search technology and $\tau > 0$ is the sum of the interest rate and the match dissolution rate. The optimality condition reflects the efficiency of the matching decision in every meeting and the value condition reflects an equal split of the available surplus (Nash bargaining). A steady-state equilibrium in the Shimer-Smith model is a tuple (u, α, w) that satisfies the quadratic balance condition (6) and the two partial equilibrium conditions, (14) and (15).

Throughout their analysis, Shimer and Smith (2000) assume that the type space $\mathcal{X}$ is the unit interval and that the symmetric production function $f(x,y)$ is continuously differentiable, strictly increasing, and either strictly supermodular or strictly submodular.

Replacing the assumption of a quadratic search technology with Assumption 1 has no effect on the optimality condition (14). The only effect on the value condition arises because individual meeting rates are no longer governed by ρ but by $\sigma \cdot r(\bar{u})$, so that the value condition becomes

$$w(x) = \frac{\sigma \cdot r(\bar{u})}{2\tau} \cdot \int \max\{f(x,y) - w(x) - w(y), 0\} u(y) dy. \tag{16}$$

Hence, a steady-state equilibrium in the Shimer-Smith model with a general search technology is given by a tuple (u, α, w) that satisfies the general balance condition (3), the optimality condition (14), and the value condition (16).

Existence of Steady-State Equilibrium. Shimer and Smith (2000) obtain existence in their model from three results: The first, Lemma 3 in Shimer and Smith (2000), shows that the map from value functions into matching affinities defined by the optimality condition (14) is continuous. In the following we will refer to this result as the optimality lemma. The second result is the fundamental matching lemma, Lemma 4 in Shimer and Smith (2000), which shows that the quadratic balance condition (6) defines the unmatched density as a continuous function of the matching affinity. These two results together provide continuity of the map $w \to u_w$ that accounts for the endogenous determination of the unmatched density in steady-state equilibrium. The third result, which follows from a fairly standard

application of Schauder's fixed point theorem, is that there exists a value function w solving the value condition when the unmatched density u on the right side of (15) is replaced by u_w whenever $w \to u_w$ is continuous. In the following we refer to this result, shown in the proof of Proposition 1 in Shimer and Smith (2000), as the value proposition.

Now, consider the Shimer-Smith model with a general search technology satisfying Assumption 1. Because the optimality lemma only refers to the optimality condition (14), it continues to hold. We have shown in Propositions 1 and 2 that the fundamental matching lemma holds for the general balance condition (3). Further, the value proposition continues to hold for the value condition (16).[12] Hence, existence of steady-state equilibrium in the Shimer-Smith model with a general search technology is assured, with the generalization of the fundamental matching lemma being the key (and hardest) step in obtaining this result.

Structure of Steady-State Equilibria. We have observed (cf. Lemma 1) that an unmatched density u solves the general balance condition if and only if it solves the quadratic balance condition with $\rho = \sigma \cdot r(\bar{u})$. As the same equality ensures that the value condition (16) holds if and only if (15) is satisfied, steady-state equilibria inherit this property: a tuple (u, α, w) is a steady-state equilibrium in the Shimer-Smith model with a general search technology if and only if it is a steady-state equilibrium in the Shimer-Smith model for a quadratic search technology with $\rho = \sigma \cdot r(\bar{u})$. This is useful because the quadratic search technology is often easier to analyze as it rules out certain crowding externalities (the rate at which an unmatched agent meets agents with types in X is independent of the mass of unmatched agents with types not in X). We note two implications:

First, for any contact function m satisfying the conditions in Assumption 1, the set of tuples (u, α, w) that are steady-state equilibria for some velocity $\sigma \geq 0$ coincides with the set of steady-state equilibria that arise for the quadratic search technology for some velocity $\rho \geq 0$. Assuming the quadratic search technology is thus without loss of generality if the aim is, say, to characterize the set of matching patterns that can possibly arise in steady-state equilibrium for a given production function f.

Second, using the equivalence result in Lemma 6, we can infer that a tuple (u, α, w) is the limit of steady-state equilibria for the quadratic search technology as $\rho \to \infty$ if and only if it is the limit of steady-state equilibria as $\sigma \to \infty$ for any search technology with contact function m satisfying Assumption 1. Consequently, for the purpose of studying whether steady-state equilibria approximate the outcomes arising in a frictionless market when search frictions disappear (in the sense that the velocity of the search technology goes to infinity), it is without loss of generality to assume that the search technology is quadratic.

Uniqueness of Steady-State Equilibrium Very little is known about the conditions under which steady-state equilibrium is unique, not only in the Shimer-Smith model but

[12] It is straightforward to verify that the proof of Proposition 1 in Shimer and Smith (2000) carries over provided that the rate function r is continuous. Continuity of r is immediate from Assumption 1 and the definition of the rate function in (1).

in general.[13] We will not resolve this issue here. Rather, we point out that the same logic that allowed us to infer uniqueness of the solution to the general balance condition from the uniqueness of the solution to the quadratic balance condition is applicable for steady-state equilibria. Specifically, suppose that for any $\rho \geq 0$ there is a unique steady-state equilibrium in the Shimer-Smith model with associated unmatched mass $\bar{u}^*_\rho$. Consider a general search technology with contact function m (satisfying Assumption 1) and let $G(\rho) = \rho / r(\bar{u}^*_\rho)$. It is then immediate from the observations in the first paragraph in our discussion of the structure of steady-state equilibria that steady-state equilibrium is unique for every choice of the velocity parameter $\sigma \geq 0$ of the general search technology if and only if the function G is increasing. Further, the same arguments as in the proof of Proposition 1 are applicable to provide simple sufficient conditions for G to be increasing. Namely, it suffices that $\bar{u}^*_\rho$ is non-increasing and $\rho \cdot (\bar{u}^*_\rho)^2$ is increasing.

6.2 Going Beyond the Shimer-Smith Model

Our observations about the structure of the set of steady-state equilibria and the conditions for their uniqueness in Section 6.1 do not hinge on the particular structure of the Shimer-Smith model. Rather, all that is required is that the partial-equilibrium conditions can be expressed in terms of an optimality condition that is independent of the search technology and that the relationship $\rho = \sigma \cdot r(\bar{u})$ ensures the equivalence between the value condition for a quadratic search technology and a general search technology.[14] These requirements are satisfied in all of the models we consider in the following, so that we focus our discussion on the existence of steady-state equilibrium.

Smith (2006) shows the existence of steady-state equilibrium in the case of nontransferable utility (NTU) by utilizing the fundamental matching lemma for quadratic search technologies and proving counterparts to the optimality lemma (under the assumption that the production function is strictly increasing) and the valuation proposition. For reasons analogous to the ones explained above, our generalization of the fundamental matching lemma implies that his existence result extends to all search technologies satisfying Assumption 1. In a similar vein, the validity of the fundamental matching lemma for general search technologies in models with two groups and exogenous exit and entry (cf. Section 5) yields the existence of steady-state equilibrium in the NTU model considered in Burdett and Coles (1997).[15]

The valuation proposition (either for quadratic or general search technologies) in the Shimer-Smith model does not require their particular assumptions on type spaces and the

[13] For notable exceptions see the uniqueness results in Burdett and Coles (1997), Burdett and Wright (1998), and Lauermann and Nöldeke (2014).

[14] In models with two distinct groups of agents the corresponding relationship is $\rho = \sigma \cdot r(\bar{u}_A, \bar{u}_B)$; see Appendix A.6.

[15] Burdett and Coles (1997) establish the existence of steady-state equilibrium under restrictive assumptions, including the requirement that both groups have identical masses, that the search technology has constant returns to scale, and that the type distributions are log-concave. In related work, Eeckhout (1999) establishes uniqueness of partial equilibrium for any given unmatched density.

production function. For instance, it suffices to assume that $\mathcal{X} \subset \mathbb{R}^n$ is compact and f is Lipschitz continuous on $\mathcal{X}^2$. As the validity of the fundamental matching lemma is independent of any assumptions on the production function and holds (as discussed in Section 5) for multidimensional type spaces, the only difficulty in generalizing the existence result from Shimer and Smith (2000) in this direction lies in establishing the validity of the optimality lemma. This, to the best of our knowledge, is an unsolved problem. We note, however, that the optimality lemma will hold quite generally if there is a (continuously distributed) match-specific "bliss shock" as in the random-search models of Goussé, Jacquemet, and Robin (2017) and Coles and Francesconi (2019) or the empirical matching literature more generally (Chiappori and Salanié, 2016). Hence, we conjecture that existence of a steady-state equilibrium for general search technologies can be established for such models by following the scheme of the existence proof in Shimer and Smith (2000) and using our generalized fundamental matching lemma.

Finally, we note that the counterparts to the optimality and value conditions from the Shimer-Smith model with a general search technology when utility is imperfectly transferable rather than (perfectly) transferable are

$$\alpha(x,y) = \begin{cases} 0 & \text{if } w(x) > \phi(x,y,w(y)), \\ 1 & \text{if } w(x) < \phi(x,y,w(y)), \end{cases} \tag{17}$$

and,

$$w(x) = \frac{\sigma \cdot \rho(\bar{u})}{2\tau} \cdot \int \max\{\phi(x,y,w(y)) - w(x), 0\} u(y) dy. \tag{18}$$

Here, the function ϕ describes, as in Legros and Newman (2007), the utility frontier by specifying the maximal utility an agent of type x can obtain when matched with an agent of type y receiving utility $w(y)$. For such an ITU model, the optimality lemma and the value proposition follow from arguments analogous to the ones in Shimer and Smith (2000) if the appropriate counterpart to their assumptions are made, i.e, types are one-dimensional ($\mathcal{X} = [0,1]$), the function ϕ is continuously differentiable, strictly increasing in x and y, strictly decreasing in in its third argument, and satisfies a strict single-crossing property that generalizes the strict super- or submodularity condition from the TU case (cf. Nöldeke and Samuelson, 2018, Section 7). As the fundamental matching lemma holds, too, existence of a steady-state equilibrium in such a model is ensured.

7 Concluding Remarks

We have investigated the balance condition in a search-and-matching model with heterogenous agents, treating agents' matching behavior as arbitrary but given. We have obtained novel comparative-static results for the quadratic search technology and, using these results, have shown that both the fundamental matching lemma and these comparative-statics results carry over from the quadratic search technology to the broad class of search tech-

nologies satisfying our Assumption 1. Combined with the observation that the conditions determining partial equilibrium are largely independent of the search technology, our generalization of the fundamental matching lemma can be used as a lever to extend arguments proving the existence of steady-state equilibrium for the quadratic search technology to general search technologies.

Our analysis builds on a simple but essential insight, namely that any solution to the general balance condition is a solution to the quadratic balance condition for a suitable choice of the velocity parameter ρ. As we have noted in Section 6, this insight extends immediately to steady-state equilibria, that is, every steady-state equilibrium in a model for a general search technology is, with a suitable choice of the velocity parameter, a steady-state equilibrium in a model with a quadratic search technology. There is a sense, then, in which the study of steady-state equilibria for general search technologies can be reduced to the study of the comparative statics of steady-state equilibria for the quadratic search technology in ρ. While much work remains to be done, we believe that this observation and our comparative-statics results for the quadratic search technology will prove useful in investigating the uniqueness and comparative statics of steady-state equilibrium.

Appendix

A.1 Proof of Lemma 3

We proceed in three steps. The first step establishes that an unmatched density solving the quadratic balance condition (6) is not only bounded above but also bounded away from zero. This ensures that all expressions considered in the following two steps are finite. The second step shows that for any two velocities ρ_1 and ρ_2 the corresponding solutions to (6), denoted by u_1 and u_2 for simplicity, are ordered, that is, either $u_1(x) \geq u_2(x)$ holds for all $x \in \mathcal{X}$ or the reverse inequality holds for all $x \in \mathcal{X}$. The third step excludes the possibility that a higher velocity leads to an increase in the density solving the quadratic balance condition, thereby finishing the proof that $u_\rho(x)$ is non-increasing in ρ.

Throughout the proof we eschew making use of the uniqueness result from Lemma 2, thereby clarifying that this property plays no role in our argument. Rather, as we explain in a remark at the end of proof, uniqueness of the solution to the quadratic balance condition (6) can be inferred from our argument.

Step 1: Consider any $(\rho, \alpha) \in [0, \infty) \times \mathcal{A}$ and let $u \in \mathcal{D}$ be an unmatched density solving (6). As the population density ℓ is bounded above and bounded away from zero, there exist l and L such that $0 < l \leq \ell(x) \leq L < \infty$ holds for all $x \in \mathcal{X}$. As u satisfies $0 < u(x) \leq \ell(x)$ for all $x \in \mathcal{X}$, it is immediate that $u(x) \leq L$ holds for all $x \in \mathcal{X}$. Further, because $0 \leq \alpha(x, y) \leq 1$ holds for all (x, y), we also have

$$\int \alpha(x, y) u(y) dy \leq \bar{\ell} \quad \forall x \in \mathcal{X}. \tag{19}$$

Hence, (6) implies $\ell(x) \leq u(x)\left[1+\rho\overline{\ell}\right]$ for all x. Thus,

$$0 < \frac{l}{1+\rho\overline{\ell}} \leq u(x) \leq L < \infty \quad \forall x \in \mathcal{X}. \tag{20}$$

Step 2: Let $u_1 \in \mathcal{D}$ and $u_2 \in \mathcal{D}$ be solutions to the quadratic balance conditions (6) for velocities $\rho_1 \geq 0$ and $\rho_2 \geq 0$ respectively:

$$\ell(x) = u_1(x)\left[1+\rho_1 \int \alpha(x,y)u_1(y)dy\right] \quad \forall x \in \mathcal{X}, \tag{21}$$

$$\ell(x) = u_2(x)\left[1+\rho_2 \int \alpha(x,y)u_2(y)dy\right] \quad \forall x \in \mathcal{X}. \tag{22}$$

From (20) in Step 1 we have that

$$\lambda_1 = \sup_{x\in\mathcal{X}} \frac{u_1(x)}{u_2(x)}, \tag{23}$$

$$\lambda_2 = \sup_{x\in\mathcal{X}} \frac{u_2(x)}{u_1(x)} \tag{24}$$

are both finite and positive. We now argue that at most one of these two numbers can be strictly greater than 1, which implies that u_1 and u_2 are ordered, i.e., either $u_1(x) \leq u_2(x)$ holds for all $x \in \mathcal{X}$ or $u_2(x) \leq u_1(x)$ holds for all $x \in \mathcal{X}$.

Suppose $\lambda_2 > 1$ holds. The following shows that this implies $\lambda_1\rho_1 > \lambda_2\rho_2$. Suppose not, so that we have $\lambda_2\rho_2 \geq \lambda_1\rho_1$. Then

$$\begin{aligned} u_2(x)\left[1+\rho_2 \int \alpha(x,y)u_2(y)dy\right] &= u_1(x)\left[1+\rho_1 \int \alpha(x,y)u_1(y)dy\right] \\ &\leq u_1(x)\left[1+\lambda_1\rho_1 \int \alpha(x,y)u_2(y)dy\right] \\ &\leq u_1(x)\left[1+\lambda_2\rho_2 \int \alpha(x,y)u_2(y)dy\right], \end{aligned}$$

for all $x \in \mathcal{X}$, where the equality in the first line is from (21) and (22), the first inequality holds because (23) implies $\lambda_1 u_2(y) \geq u_1(y)$, and the second inequality is from the hypothesis $\lambda_2\rho_2 \geq \lambda_1\rho_1$. Consequently, we obtain

$$\frac{u_2(x)}{u_1(x)} \leq \frac{1+\lambda_2\rho_2 \int \alpha(x,y)u_2(y)dy}{1+\rho_2 \int \alpha(x,y)u_2(y)dy}$$

for all $x \in \mathcal{X}$ and thus

$$\lambda_2 \leq \sup_{x\in\mathcal{X}} \left[\frac{1+\lambda_2\rho_2 \int \alpha(x,y)u_2(y)dy}{1+\rho_2 \int \alpha(x,y)u_2(y)dy} \right]. \tag{25}$$

On the other hand, using the hypothesis $\lambda_2 > 1$ and (19), we have

$$\frac{1+\lambda_2\rho_2 \int \alpha(x,y)u_2(y)dy}{1+\rho_2 \int \alpha(x,y)u_2(y)dy} \leq \frac{1+\lambda_2\rho_2\overline{\ell}}{1+\rho_2\overline{\ell}} < \lambda_2 \quad \forall x \in \mathcal{X}. \tag{26}$$

From (26) we obtain

$$\lambda_2 > \sup_{x\in\mathcal{X}} \left[\frac{1+\lambda_2\rho_2 \int \alpha(x,y)u_2(y)dy}{1+\rho_2 \int \alpha(x,y)u_2(y)dy} \right].$$

and thereby a contradiction to (25). Consequently, $\lambda_2 > 1$ implies $\lambda_1\rho_1 > \lambda_2\rho_2$.

Exchanging the roles of λ_1 and λ_2 in the above argument yields that $\lambda_1 > 1$ implies $\lambda_2\rho_2 > \lambda_1\rho_1$. As at most one of the inequalities $\lambda_1\rho_1 > \lambda_2\rho_2$ and $\lambda_2\rho_2 > \lambda_1\rho_1$ can hold, it follows that $\lambda_1 \leq 1$ or $\lambda_2 \leq 1$ (or both) must hold.

STEP 3: Suppose $\rho_2 \geq \rho_1$. If $\lambda_1 > 1$ holds, then $\lambda_2 \leq 1$ is immediate from the conclusion of Step 2. By the definition of λ_2 in (24) this implies $u_2(x) \leq u_1(x)$ for all $x \in \mathcal{X}$, which is the desired result.

It remains to consider the case $\lambda_1 \leq 1$. From the definition of λ_1 in (23) this implies $u_2(y) \geq u_1(y)$ for all $y \in \mathcal{X}$. Together with the inequality $\rho_2 \geq \rho_1$ this yields

$$\rho_2 \int \alpha(x,y)u_2(y)dy \geq \rho_1 \int \alpha(x,y)u_1(y)dy \quad \forall x \in \mathcal{X}.$$

It is then immediate from (21) and (22) that $u_2(x) \leq u_1(x)$ holds for all $x \in \mathcal{X}$, finishing the proof.

REMARK: Applying the argument from Step 3 to the case $\rho_1 \geq \rho_2$ yields $u_1(x) \leq u_2(x)$ for all $x \in \mathcal{X}$. Consequently, for $\rho_1 = \rho_2$ we have $u_1(x) = u_2(x)$ for all $x \in \mathcal{X}$, showing that (6) cannot have more than one solution for given ρ (and α).

A.2 Proof of Lemma 4

We use the same notation as in the proof of Lemma 3.

Suppose $\rho \cdot u_\rho(x)$ is not increasing in ρ for all $x \in \mathcal{X}$. There then exists $\rho_2 > \rho_1 > 0$ and $x' \in \mathcal{X}$ such that $\rho_2 \cdot u_2(x') \leq \rho_1 \cdot u_1(x')$ holds. We then have $u_1(x') > u_2(x')$ and therefore $\lambda_1 > 1$. From Step 2 in the proof of Lemma 3 this implies $\lambda_2 \leq 1$ and $\lambda_2\rho_2 > \lambda_1\rho_1$. In particular, we have $\rho_2 > \lambda_1\rho_1$. From the definition of λ_1 in (23) this yields $\rho_2 \cdot u_2(x) > \rho_1 \cdot u_1(x)$ for all $x \in \mathcal{X}$, contradicting the hypothesis that the inequality $\rho_2 \cdot u_2(x') \leq \rho_1 \cdot u_1(x')$ holds for some $x' \in \mathcal{X}$.

A.3 Proof of Lemma 5

As $\rho \cdot \bar{u}_\rho^2$ is equal to zero for $\rho = 0$ and positive otherwise, it suffices to consider $\rho > 0$. We proceed in two steps.

STEP 1: For any $\rho > 0$ define $s_\rho \in \mathcal{D}$ by

$$s_\rho(x) = \sqrt{\rho} \cdot u_\rho(x), \quad \forall x \in \mathcal{X}. \tag{27}$$

Using that u_ρ is the unique positive solution to (6), it is immediate from (27) that s_ρ is the unique positive solution to (8) for $\gamma = 1/\sqrt{\rho} > 0$. From (27) we also have

$$\bar{s}_\rho = \sqrt{\rho \cdot \bar{u}_\rho^2}.$$

Therefore, using z_γ to denote the unique positive solution to (8) as a function of γ, it suffices to show that the map $\gamma \to \bar{z}_\gamma$ from $(0, \infty)$ to $(0, \infty)$ is decreasing in γ to establish the lemma.

STEP 2: Let $\mathcal{V}$ be the subspace of essentially bounded functions in $L^2(\mathcal{X})$. For all $v \in \mathcal{V}$ and $\gamma > 0$ define

$$H(v, \gamma) = \gamma \int e^{v(x)} dx + \frac{1}{2} \int \int \alpha(x, y) e^{v(x)+v(y)} dx dy - \int v(x) \ell(x) dx. \tag{28}$$

The function H is convex in v. It is also continuous in v. Its derivative (with respect to v) is the bounded linear operator H_v on $L^2(\mathcal{X})$ defined by

$$\begin{aligned} H_v(v, \gamma)(h) &= \lim_{t \to 0} \frac{H(v + th, \gamma) - H(v, \gamma)}{t} \\ &= \gamma \int e^{v(x)} h(x) dx + \frac{1}{2} \int \int \alpha(x, y) e^{v(x)+v(y)} \left[h(x) + h(y) \right] dx dy \\ &\quad - \int \ell(x) h(x) dx. \end{aligned}$$

Using the symmetry condition $\alpha(x, y) = \alpha(y, x)$, we obtain

$$H_v(v, \gamma)(h) = \gamma \int e^{v(x)} h(x) dx + \int \int \alpha(x, y) e^{v(x)+v(y)} h(x) dx dy - \int \ell(x) h(x) dx. \tag{29}$$

Because H is convex, it follows from (29) that v minimizes $H(v, \gamma)$ over $\mathcal{V}$ if and only if the first order condition

$$\gamma e^{v(x)} + e^{v(x)} \int \alpha(x, y) e^{v(y)} dy - \ell(x) = 0 \tag{30}$$

holds for almost all $x \in \mathcal{X}$.

Applying the transformation $z = e^v$ to (28) and (30) and comparing the resulting first order condition with (8) we obtain that z_γ minimizes

$$G(z,\gamma) = \gamma \int z(x)dx + \frac{1}{2}\int\int \alpha(x,y)z(x)z(y)dxdy - \int \ell(x)\ln(z(x))dx \tag{31}$$

over the set of all positive, essentially bounded functions in $L^2(\mathcal{X})$. Further, as z_γ is uniquely determined, any other such minimizer of $G(z,\gamma)$ must agree with z_γ for almost all $x \in \mathcal{X}$.

Consider now $\gamma_2 > \gamma_1 > 0$. Because (i) z_{γ_1} minimizes $G(z,\gamma_1)$ and z_{γ_2} minimizes $G(z,\gamma_2)$, (ii) these minimizers are essentially unique, and (iii) equation (8) precludes the possibility that $z_{\gamma_1} = z_{\gamma_2}$ holds for almost all $x \in \mathcal{X}$, we have

$$[G(z_{\gamma_1},\gamma_1) - G(z_{\gamma_2},\gamma_1)] + [G(z_{\gamma_2},\gamma_2) - G(z_{\gamma_1},\gamma_2)] < 0. \tag{32}$$

Substituting from (31) into (32) yields

$$[\gamma_2 - \gamma_1]\,[\bar{z}_{\gamma_2} - \bar{z}_{\gamma_1}] < 0.$$

Hence, as was to be shown, $\bar{z}_\gamma$ is decreasing in γ.

A.4 Proof for the Claims in Remark 2

Suppose that the population density is equal to $l > 0$ for all $x \in \mathcal{X} = [0,1]$. Suppose, in addition that all meetings lead to matches, so that the matching affinities are given by $\alpha(x,y) = 1$ for all (x,y). The general balance condition (3) then simplifies to

$$l = u(x)\,[1 + \sigma \cdot r(\bar{u}) \cdot \bar{u}]\,, \quad \forall x \in \mathcal{X}.$$

This condition is satisfied if and only if $u(x) = \bar{u}$ holds for all $x \in \mathcal{X}$ and, using (1), the mass $\bar{u}$ of unmatched agents solves

$$l = \bar{u} + \sigma \cdot m(\bar{u}). \tag{33}$$

In particular, a solution to (3) exists (is unique) if and only if a solution to (33) exists (is unique). Therefore, to validate the claims in the main body of the paper, it suffices to show that (i) for suitable choices of the parameters $\sigma > 0$ and $l > 0$, condition (33) has multiple solutions if the aggregate meeting rate fails to be non-decreasing, and, if the aggregate meeting rate is non-decreasing, the parameters σ and l can be chosen such that no solution to (33) exists if (ii) m fails the boundary condition or (iii) m fails to be continuous.

i) Suppose that m fails to be non-decreasing. Then, there exist $\hat{u}$ and $u^\dagger$ satisfying $\hat{u} > u^\dagger > 0$ and $m(u^\dagger) > m(\hat{u})$. Setting $\sigma = [\hat{u} - u^\dagger]/[m(u^\dagger) - m(\hat{u})] > 0$ and $l = \hat{u} + \sigma \cdot m(\hat{u})$, it is immediate that $\hat{u}$ satisfies (33) and easily verified that $u^\dagger$ does so, too.

ii) Suppose that m is non-decreasing and $\lim_{\bar{u}\to 0} m(\bar{u}) > 0$ holds. Then, as the right side of (33) is increasing in $\bar{u}$, equation (33) has no solution for $\sigma > 0$ and $l > 0$ satisfying

$0 < l/\sigma < \lim_{\bar{u}\to 0} m(\bar{u})$.

iii) Suppose that m is non-decreasing but not continuous. There then exists $\hat{u} > 0$ and $M \neq m(\hat{u})$ such that $\bar{u} < \hat{u}$ implies $m(\bar{u}) < M$ and $\bar{u} > \hat{u}$ implies $m(\bar{u}) > M$. Fix such $\hat{u}$ and M and let $\sigma > 0$ and $l > 0$ be such that $l = \hat{u} + \sigma M$. By construction, we then have that $\hat{u}$ does not solve (33). Further, as the right side of (33) is increasing in $\bar{u}$ and $\bar{u} > \hat{u}$ implies $m(\bar{u}) > M$, also no $\bar{u} > \hat{u}$ solves (33). An analogous argument shows that (33) also has no solution with $\bar{u} < \hat{u}$.

A.5 Proof of Lemma 6

Suppose $\rho_n \to \infty$, whereas $\{\sigma_n\}$ is bounded. From the relationship $\rho_n = \sigma_n \cdot r(\bar{u}_n)$, we then have that $r(\bar{u}_n) \to \infty$ must hold. Due to the monotonicity condition in Assumption 1, $m(\bar{u}_n)$ is bounded above by $m(\bar{\ell})$. From (1), we thus obtain that $r(\bar{u}_n) \to \infty$ implies $\bar{u}_n \to 0$. Using Assumption 1 again, $\bar{u}_n \to 0$ implies $m(\bar{u}_n) \to 0$. Hence, we have $\bar{u}_n + \sigma_n \cdot m(\bar{u}_n) \to 0$. But this is impossible because, by hypothesis, $(\alpha_n, \sigma_n, u_n)$ satisfies the general balance condition for all n, which in turn implies $\bar{u}_n + \sigma_n \cdot m(\bar{u}_n) \geq \bar{\ell} > 0$ for all n:

$$\begin{aligned} 0 < \bar{\ell} = \bar{u}_n + \sigma_n \cdot r(\bar{u}_n) \int\int \alpha_n(x,y) u_n(y) v_n(x) dy dx \\ \leq \bar{u}_n + \sigma_n \cdot r(\bar{u}_n) \cdot \bar{u}_n^2 \\ = \bar{u}_n + \sigma_n \cdot m(\bar{u}_n), \end{aligned}$$

where the equality in the first line is from integrating (3) with respect to x, the inequality in the second line uses $0 \leq \alpha_n(x,y) \leq 1$, and the equality in the third line is from (1). Hence, if $\rho_n \to \infty$, then the sequence $\{\sigma_n\}$ cannot be bounded. As the same argument is applicable to any subsequence, it follows that $\rho_n \to \infty$ implies $\sigma_n \to \infty$.

Suppose $\sigma_n \to \infty$, whereas $\{\rho_n\}$ is bounded. From the relationship $\rho_n = \sigma_n \cdot r(\bar{u}_n)$, we then have that $r(\bar{u}_n) \to 0$ must hold. Because $\bar{u}_n$ is bounded above by $\bar{\ell}$, (1) then implies $m(\bar{u}_n) \to 0$. From Assumption 1, this in turn implies $\bar{u}_n \to 0$. Hence, we have $\bar{u}_n + \rho_n \bar{u}_n^2 \to 0$. On the other hand, by an argument analogous to the one given in the preceding paragraph, we have $\bar{u}_n + \sigma_n \bar{u}_n^2 \geq \bar{\ell} > 0$ for all n because (α_n, ρ_n, u_n) satisfies the quadratic balance condition (6) for all n. Hence, if $\sigma_n \to \infty$, then the sequence $\{\rho_n\}$ cannot be bounded and it follows that $\sigma_n \to \infty$ implies $\rho_n \to \infty$.

A.6 Random Meetings between Two Groups

We modify the search-and-matching process from Section 2 to incorporate two groups as follows: First, we suppose that the type space $\mathcal{X}$ is partitioned into two measurable sets A and B such that $\bar{\ell}^A = \int_A \ell(x)dx > 0$ and $\bar{\ell}^B = \int_B \ell(x)dx > 0$ holds. Agents with types in A are the members of group A and agents with types in B are the members of group B. Second, the aggregate meeting rate only accounts for meetings featuring unmatched

agents from distinct groups and depends on the unmatched masses $\bar{u}^A = \int_A u(x)dx$ and $\bar{u}^B = \int_B u(x)dx$ rather than just on $\bar{u}$. The counterpart to Assumption 1 is

Assumption 3 (General Search Technology with Two Groups)**.** *For any unmatched density $u \in \mathcal{D}$, the aggregate meeting rate is given by $\sigma \cdot m(\bar{u}^A, \bar{u}^B)$, where $\sigma \geq 0$ and $m : (0,\infty)^2 \to (0,\infty)$ is continuous, non-decreasing in both arguments, and satisfies $\lim_{\bar{u}^B \to 0} m(\bar{u}^A, \bar{u}^B) = \lim_{\bar{u}^A \to 0} m(\bar{u}^A, \bar{u}^B) = 0$.*

Let $r : (0,\infty)^2 \to (0,\infty)$ be given by

$$r(\bar{u}^A, \bar{u}^B) = \frac{m(\bar{u}^A, \bar{u}^B)}{\bar{u}^A \cdot \bar{u}^B}. \tag{34}$$

The assumption of random meetings between the unmatched members of the two groups implies that each unmatched agent from group A meets some unmatched agent from group B at the individual meeting rate $\sigma \cdot r(\bar{u}_A, \bar{u}_B) \cdot \bar{u}_B$, whereas each unmatched agent from group B meets some unmatched agent from group A at the individual meeting rate $\sigma \cdot r(\bar{u}_A, \bar{u}_B) \cdot \bar{u}_A$.

With these modifications, the counterpart to the general balance condition (3) is

$$\ell(x) = u(x)\left[1 + \sigma \cdot r(\bar{u}^A, \bar{u}^B) \int_B \alpha(x,y)u(y)dy\right] \quad \forall x \in A, \tag{35a}$$

$$\ell(x) = u(x)\left[1 + \sigma \cdot r(\bar{u}^A, \bar{u}^B) \int_A \alpha(x,y)u(y)dy\right] \quad \forall x \in B, \tag{35b}$$

where $0 \leq \alpha(x,y) = \alpha(y,x) \leq 1$ is the probability that a meeting between a pair of agents with types (x,y) leads to a match, for $(x,y) \in (A \times B) \cup (B \times A)$.[16] Extending the definition of the matching affinities α to $\mathcal{X}^2$ by setting $\alpha(x,y) = \alpha(y,x) = 0$ for all $(x,y) \in (A \times A) \cup (B \times B)$, the balance conditions (35) can also be written as

$$\ell(x) = u(x)\left[1 + \sigma \cdot r(\bar{u}^A, \bar{u}^B) \int_{\mathcal{X}} \alpha(x,y)u(y)dy\right] \quad \forall x \in \mathcal{X}, \tag{36}$$

which provides us with a more natural counterpart to (3).

Quadratic search technologies. With a quadratic search technology, the aggregate meeting rate is equal to the contact rate (i.e., we again set $\sigma = 1$ for quadratic search technologies) and given by

$$m(\bar{u}^A, \bar{u}^B) = \rho \cdot \bar{u}^A \cdot \bar{u}^B. \tag{37}$$

Using (34) and $\sigma = 1$, this implies that $\sigma \cdot r(\bar{u}^A, \bar{u}^B)$ is constant and equal to ρ for such technologies. Hence, for quadratic search technologies (36) is identical to (6). Consequently, all results from Section 3 hold as stated for the model with two groups. This, however, does

[16] Recall that A and B partition $\mathcal{X}$. Hence, $(A \times B) \cap (B \times A) = \emptyset$ and $(x,y) \in A \times B \Leftrightarrow (y,x) \in B \times A$ hold. As in Section 2, the symmetry condition $\alpha(x,y) = \alpha(y,x)$ thus does not impose any restrictions on behavior but is an accounting identity (cf., footnote 6).

not suffice to obtain counterparts to Propositions 1 – 3 for the model with two groups. Rather, we require the following variant of Lemma 5.

Lemma 7. *In the model with two groups, the map $\rho \to \rho \cdot \bar{u}_\rho^A \cdot \bar{u}_\rho^B$ is increasing.*

Proof. Because matches only occur between agents of distinct groups the quadratic balance condition can be written as (cf. (35))

$$\ell(x) = u(x)\left[1 + \rho \int_B \alpha(x,y)u(y)dy\right] \quad \forall x \in A, \tag{38a}$$

$$\ell(x) = u(x)\left[1 + \rho \int_A \alpha(x,y)u(y)dy\right] \quad \forall x \in B. \tag{38b}$$

For $\rho > 0$ let

$$\gamma^A(\rho) = \frac{\sqrt{\bar{u}_\rho^A}}{\sqrt{\rho \cdot \bar{u}_\rho^B}} > 0 \text{ and } \gamma^B(\rho) = \frac{\sqrt{\bar{u}_\rho^B}}{\sqrt{\rho \cdot \bar{u}_\rho^A}} > 0, \tag{39}$$

and define $s_\rho \in \mathcal{D}$ by

$$s_\rho(x) = \begin{cases} u_\rho(x)/\gamma^A(\rho) & \text{if } x \in A, \\ u_\rho(x)/\gamma^B(\rho) & \text{if } x \in B. \end{cases} \tag{40}$$

Arguments analogous to those in Step 1 of the proof of Lemma 5 in Appendix A.3 then show that s_ρ is the unique solution to

$$\ell(x) = z(x)\left[\gamma^A(\rho) + \int_B \alpha(x,y)z(y)dy\right] \quad \forall x \in A, \tag{41a}$$

$$\ell(x) = z(x)\left[\gamma^B(\rho) + \int_A \alpha(x,y)z(y)dy\right] \quad \forall x \in B. \tag{41b}$$

In addition, considering the function

$$H(v,\gamma^A,\gamma^B) = \gamma^A \int_A e^{v(x)}dx + \gamma^B \int_B e^{v(y))}dy + \int_{\mathcal{X}}\int_{\mathcal{X}} \alpha(x,y)e^{v(x)+v(y)}dxdy - \int_{\mathcal{X}} v(x)\ell(x)dx$$

as a starting point, arguments analogous to the ones in Step 2 of the proof of Lemma 5 in Appendix A.3 show that, for $\rho_2 > \rho_1 > 0$, we have

$$\left[\gamma^A(\rho_2) - \gamma^A(\rho_1)\right]\left[\bar{s}_{\rho_2}^A - \bar{s}_{\rho_1}^A\right] + \left[\gamma^B(\rho_2) - \gamma^B(\rho_1)\right]\left[\bar{s}_{\rho_2}^B - \bar{s}_{\rho_1}^B\right] < 0. \tag{42}$$

Observing that Lemmas 3 and 4 imply that the expressions defining $\gamma^A(\rho)$ and $\gamma^B(\rho)$ in (39) are decreasing in ρ and that (40) implies

$$\bar{s}^A_\rho(x) = \bar{s}^B_\rho(x) = \sqrt{\rho \cdot \bar{u}^A_\rho \cdot \bar{u}^B_\rho}, \tag{43}$$

the inequality in (42) yields that $\rho \cdot \bar{u}^A_\rho \cdot \bar{u}^B_\rho$ is increasing in ρ. □

General search technologies. Using (36) in lieu of (3), the counterpart to (7) for the two-group model is

$$\rho = \sigma \cdot r(\bar{u}^A, \bar{u}^B). \tag{44}$$

Lemma 1 holds for the model with two groups if condition (7) is replaced by (44).

Because the map $\rho \to \rho \cdot \bar{u}^A_\rho \cdot \bar{u}^B_\rho$ is increasing by Lemma 7, the proofs of Propositions 1 and 2 then go through with minor modifications to show that these results hold when Assumption 3 and the balance condition (36) rather than Assumption 1 and the balance condition (3) are used.[17] Similarly, the natural counterpart to Proposition 3, asserting monotonicity of the maps $\sigma \to v_\sigma$, $\sigma \to \sigma \cdot r(\bar{v}^A_\sigma, \bar{v}^B_\sigma) \cdot v_\sigma$, and $\sigma \to \sigma \cdot m(\bar{v}^A_\sigma, \bar{v}^B_\sigma)$ for the solution v_σ to the balance condition (36), follows from Assumption 3.

References

Adachi, H. (2003): "A Search Model of Two-Sided Matching under Nontransferable Utility," *Journal of Economic Theory*, 113(2), 182–198.

Bloch, F., and H. Ryder (2000): "Two-Sided Search, Marriages, and Matchmakers," *International Economic Review*, 41(1), 93–116.

Burdett, K., and M. G. Coles (1997): "Marriage and Class," *Quarterly Journal of Economics*, 112, 141–168.

——— (1999): "Long-Term Partnership Formation: Marriage and Employment," *Economic Journal*, 109, F307–F334.

Burdett, K., and R. Wright (1998): "Two-Sided Search with Nontransferable Utility," *Review of Economic Dynamics*, 1(1), 220–245.

Chade, H. (2001): "Two-Sided Search and Perfect Segregation with Fixed Search Costs," *Mathematical Social Sciences*, 42(1), 31–51.

[17] The only somewhat substantial modifications required are in the proof of Lemma 6 that we have relegated to Appendix A.5. Consider the first part of the proof, showing that $\rho_n \to \infty$ implies $\sigma_n \to \infty$. (The difficulty in the second part of the proof and its resolution are analogous.) Here the hypothesis that $\{\sigma_n\}$ is bounded while $\rho_n \to \infty$ no longer implies $\bar{u}_n + \sigma_n \cdot m(\bar{u}_n) \to 0$, but only that either $\bar{u}^A_n + \sigma_n \cdot m(\bar{u}^A_n, \bar{u}^B_n) \to 0$ or $\bar{u}^B_n + \sigma_n \cdot m(\bar{u}^A, \bar{u}^B) \to 0$ must hold. This still yields a contradiction, because integrating (35a) over the types in A and (35b) over the types in B yields $\bar{\ell}^A \leq \bar{u}^A_n + \sigma_n \cdot m(\bar{u}^A_n, \bar{u}^B_n)$ and $\bar{\ell}^B \leq \bar{u}^B_n + \sigma_n \cdot m(\bar{u}^A_n, \bar{u}^B_n)$ respectively, and we have assumed $\bar{\ell}^A > 0$ and $\bar{\ell}^B > 0$.

Chade, H., J. Eeckhout, and L. Smith (2017): "Sorting through Search and Matching Models in Economics," *Journal of Economic Literature*, 55(2), 493–544.

Chen, F. H. (2005): "Monotonic Matching in Search Equilibrium," *Journal of Mathematical Economics*, 41(6), 705–721.

Chiappori, P.-A., and B. Salanié (2016): "The Econometrics of Matching Models," *Journal of Economic Literature*, 54(3), 832–61.

Choo, E., and A. Siow (2006): "Who Marries Whom and Why," *Journal of Political Economy*, 114(1), 175–201.

Coles, M. G., and M. Francesconi (2019): "Equilibrium Search with Multiple Attributes and the Impact of Equal Opportunities for Women," *Journal of Political Economy*, 127(1), 138–162.

Decker, C., E. H. Lieb, R. J. McCann, and B. K. Stephens (2013): "Unique Equilibria and Substitution Effects in a Stochastic Model of the Marriage Market," *Journal of Economic Theory*, 148(2), 778–792.

Diamond, P. A., and E. Maskin (1979): "An Equilibrium Analysis of Search and Breach of Contract I: Steady States," *Bell Journal of Economics*, 10, 282–316.

Eeckhout, J. (1999): "Bilateral Search and Vertical Heterogeneity," *International Economic Review*, 40(4), 869–887.

Eeckhout, J., and P. Kircher (2011): "Identifying Sorting in Theory," *The Review of Economic Studies*, 78(3), 872–906.

Gale, D. (1987): "Limit Theorems for Markets with Sequential Bargaining," *Journal of Economic Theory*, 43, 20–54.

Galichon, A., and B. Salanié (2015): "Cupids Invisible Hand: Social Surplus and Identification in Matching Models," Discussion paper, SSRN.

Goussé, M., N. Jacquemet, and J.-M. Robin (2017): "Marriage, Labor Supply, and Home Production," *Econometrica*, 85(6), 1873–1919.

Hagedorn, M., T. H. Law, and I. Manovskii (2017): "Identifying Equilibrium Models of Labor Market Sorting," *Econometrica*, 85(1), 29–65.

Lauermann, S. (2013): "Dynamic Matching and Bargaining Games: A General Approach," *American Economic Review*, 103(2), 663–689.

Lauermann, S., and G. Nöldeke (2014): "Stable Marriages and Search Frictions," *Journal of Economic Theory*, 151, 163–195.

Lauermann, S., and G. Nöldeke (2015): "Existence of Steady-State Equilibria in Matching Models with Search Frictions," *Economics Letters*, 131, 1 – 4.

Lauermann, S., G. Nöldeke, and T. Tröger (2018): "Steady States in Search-and-Matching Models," CRC TR 224 Discussion Paper 055, University of Bonn and University of Mannheim.

Legros, P., and A. Newman (2007): "Beauty is a Beast, Frog is a Prince: Assortative Matching with Nontransferabilities," *Econometrica*, 75(4), 1073–1102.

Manea, M. (2017): "Steady States in Matching and Bargaining," *Journal of Economic Theory*, 167, 206 – 228.

Marimon, R., and F. Zilibotti (1999): "Unemployment vs. Mismatch of Talents: Reconsidering Unemployment Benefits," *The Economic Journal*, 109(455), 266–291.

McNamara, J. M., and E. J. Collins (1990): "The Job Search Problem as an Employer-Candidate Game," *Journal of Applied Probability*, 27(4), 815–827.

Morgan, P. B. (1998): "A Model of Search, Coordination and Market Segmentation," Mimeo, Department of Economics, University of Buffalo.

Nöldeke, G., and L. Samuelson (2018): "The Implementation Duality," *Econometrica*, 86(4), 1283–1324.

Nöldeke, G., and T. Tröger (2009): "Matching Heterogeneous Agents with a Linear Search Technology," Bonn Econ Discussion Paper 1-2009, Bonnn Graduate School of Economics.

Petrongolo, B., and C. A. Pissarides (2001): "Looking into the Black Box: A Survey of the Matching Function," *Journal of Economic Literature*, 39(2), 390–431.

Rogerson, R., R. Shimer, and R. Wright (2005): "Search-Theoretic Models of the Labor Market: A Survey," *Journal of Economic Literature*, 43(4), 959–988.

Shimer, R., and L. Smith (2000): "Assortative Matching and Search," *Econometrica*, 68(2), 343–369.

Smith, L. (2006): "The Marriage Model with Search Frictions," *Journal of Political Economy*, 114(6), 1124–1144.

——— (2009): "Cross-sectional Dynamics in a Two-Sided Matching Model," Unpublished Manuscript, University of Michigan.

——— (2011): "Frictional Matching Models," *Annual Review of Economics*, 3, 319–38.

Stevens, M. (2007): "New Microfoundations for the Aggregate Matching Function," *International Economic Review*, 48(3), 847–868.

Teulings, C. N., and P. A. Gautier (2004): "The Right Man for the Job," *The Review of Economic Studies*, 71(2), 553–580.