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Persuasion and Information Aggregation in Large Elections

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Abstract

This paper studies the Bayes correlated equilibria of large majority elections in a general environment with heterogeneous, private preferences. Voters have exogeneous private signals and a version of the Condorcet Jury Theorem holds when voters do not receive additional information (Feddersen & Pesendorfer, 1997). We show that any state-contingent outcome can be implemented in some Bayes-Nash equilibrium by an expansion of the exogenous private signal structure. We interpret the result in terms of the possibility of persuasion by a biased sender who provides additional information to voters who also have noisy private information from other sources. The additional information can be an almost public signal that almost reveals the state truthfully. The same additional information is shown to be effective uniformly across environments so that persuasion does not require detailed knowledge of the distribution of the voters' private information and preferences. In a numerical example with uniform voter types, we show the effects of persuasion with already 17 or more voters.

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1 Introduction

In most elections, a voter's ranking of outcomes depends on her information. For example, a shareholder's view of a proposed merger depends on her belief about its profitability, a legislators support of a proposed legislation depends on her belief of its effectiveness. An interested party that has private information may utilize this fact by strategically releasing information to affect voters behavior. Examples of interested parties holding and strategically releasing relevant information for voters are numerous: in a shareholder vote, the management may strategically provide information about the merger through presentations and conversations; similarly, lobbyists provide selected information to legislators to influence their vote.

We are interested in the scope of such "persuasion" (Kamenica & Gentzkow, 2011) in elections. We study this question in the canonical voting setting by Feddersen & Pesendorfer (1997): there are two possible policies (outcomes), A and B. Voters' preferences over policies are heterogenous and depend on an unknown state, α or β, in a general way (some voters may prefer A in state α, some prefer A in state β, and some "partisans" may prefer one of the policies independently of the state). The preferences are drawn independently across voters and are each voters' private information. In addition, all voters privately receive information in the form of a noisy signal. The election determines the outcome by a simple majority rule.

In this setting, Feddersen & Pesendorfer (1997) have shown that, within a broad class of "monotone" preferences and conditionally i.i.d. private signals, all equilibrium outcomes of large elections are equivalent to the outcome with publicly known states ("information aggregation"). We restate their result as a benchmark in Theorem 1. [1]

We ask: can a manipulator ensure that a majority supports his favorite policy—potentially state-dependent—in a large election merely by providing *additional* information to the voters? Formally, the manipulator can choose

[1] Really, they consider a continuum of states, for binary states the result can be found in Bhattacharya (2013).

and commit to any joint distribution over states and signal realizations that are then privately observed by the voters. In particular, the manipulator's additional signal is independent of the voters' exogenous private signals and their individual preferences (it is an "independent expansion"). The previous result by Feddersen & Pesendorfer (1997) suggests that the scope for persuasion may be limited because, if voters simply ignored the additional information, the outcome would be "as if" if the state were known, and, hence, the information provided by the manipulator would be worthless.

Somewhat surprisingly maybe, our main result (Theorem 5) shows that, nevertheless, within the same class of monotone preferences and for any state-contingent policy, there exists an independent expansion of the conditionally i.i.d. signal structure and a natural equilibrium that ensures that the targeted policy is supported by a majority with probability close to one. So, just by providing additional information, a manipulator can implement, for example, a targeted policy that is the opposite of the outcome with publicly known states, for every state.

The additional information affects the voters' behavior *directly*, by changing their beliefs about the state, and *indirectly*, by affecting their inference from being "pivotal" for the election outcome. While the direct effect is limited by the well-known "Bayesian-consistency" requirement of beliefs, the pivotal inference turns out to have no such constraint, and so the indirect effect is critical for the persuasion possibility.

To explain the effectiveness of persuasion, we first consider the case in which all information of voters comes from a manipulator ("monopolistic persuasion") in Section 5. Specifically, the main result for this baseline model is the following, stated as Theorem 2: the manipulator can persuade a large electorate to elect any state-contingent policy with probability close to 1 for any distribution of preferences for which there is one belief about the likelihood that the state is α such that a voter with randomly drawn preferences prefers A with probability larger than $1/2$ given this belief and another belief such that the probability of preferring A given this belief is smaller than $1/2$. Denote these beliefs by p_A and p_B, respectively. In particular this condition guarantees that there is a belief $\bar{r}$ at which a random voter prefers A with probability of exactly 50% since we assume prefereces to be continuous in beliefs. Clearly, some such condition is necessary for persuasion to be effective: For example, if for all beliefs, each voter prefers A with probability less than 1/2, then, whatever the induced beliefs, in a large election the expected share of voters supporting A will be less than 1/2.

We show that the condition is sufficient. For example, when the manipulator's goal is to get A elected in both states we construct a signal structure as follows. Roughly speaking, with high probability, $1-\varepsilon$, the voters receive conditionally independent draws of a binary signal, a or b, with a being relatively more likely in state α and b relatively more likely in state β. With monotone preferences and $\varepsilon = 0$, this would generally ensure information aggregation in equilibrium as in Feddersen & Pesendorfer (1997). However, with probability $\varepsilon > 0$, the manipulator induces an additional *state-of-confusion*: In this additional state, almost all voters will receive a common signal z while only few voters receive signals a or b. Thus, conditional on observing z, a voter knows that most other voters have also observed z. The consequence is that, in contrast to the usual calculus of strategic voting, there is essentially no further information about others' signals contained in the event of being pivotal. This is the critical observation, and it implies that voters behave essentially sincerely conditional on z. By choosing the relative probability of z in the two states appropriately, the posterior conditional on z will be $\bar{r}$, meaning, each voter prefers A with probability 1/2 and, hence, the election is close to being tied in the state-of-confusion. We show that, even from the viewpoint of the few voters observing signals a or b, conditional on the election being tied, it is likely that the other voters received the common signal z. By appropriately choosing the probabilities of a and b in the state-of-confusion, the posterior conditional on the state-of-confusion and conditional on a or b is is the belief p_A for which more than 1/2 of the voters support A. Hence, in the standard state, when there are only signals a and b, a large majority supports A. The main idea of the construction is that one can first characterize equilibrium for voters receiving a z signal and then use that characterization to extend the construction to voters receiving other signals.

We argue that the manipulated equilibrium is robust in various dimensions. First, the played equilibrium is simple and insures voters against errors. Specifically, the equilibrium profile is almost identical to voting sincerely given one's signal, conditional on the state-of-confusion. One may argue that this behavior is simple. In particular, voters just need interpret their own signal conditional on that state; they do not need to make any further inference about other voters' signals using the equilibrium strategy profile or have to know the preference distribution of the electorate. Furthermore, as will be explained in detail later, sincere behavior is 'safe' in the sense of being an ε best response conditional on being pivotal, for a neighborhood around the actual environment. Thus, even if a voter's belief about the environment and the equilibrium is slightly wrong, the cost of this error is small (even conditional on being pivotal). Second, the equilibrium is 'attracting'. In particular, its "basin of attraction" for the best response dynamic is essentially the full set of strategy profiles, except for the one (essentially unique) strategy profile that corresponds to the one type of other equilibrium:[2] If we start with any strategy profile that is close to but not exactly equal to that type of equilibrium and if we consider the voters best response to it and the voters best response to this best response, then the resulting strategy profile is arbitrarily close to the manipulated equilibrium when the number of voters is large (Theorem 3).

We show that the same information structure can be used uniformly across many environments. This implies that the sender does not need to know the exact details of the game. By way of contrast, as discussed momentarily, existing work assumes that the manipulator knows the exact preference of each individual voter and this knowledge is indeed used. First, we show that the mo-

[2]For common values, it follows from a result by McLennan (1998) that the symmetric strategy that maximizes the voters' welfare is an equilibrium. In this equilibrium, information aggregates except in the added state. By continuity, it is impossible that the manipulator implements a prefered outcome in all equilibria with probability 1 when voters have almost common values. Our model nests almost common values. Hence, the robustness result presented here is the strongest possible.

nopolistic manipulator can use the same information structure to implement a preferred policy for a large set of prior beliefs and preferences, including all "monotone preferences and priors.

In the second part of the paper, we consider the setting in which voters already have access to exogeneous information of the form studied in Feddersen & Pesendorfer (1997). We show that, by adding the information structure as before, the manipulator can still persuade the voters effectively to elect any state-contingent policy (Theorem 5). This is surprising since the manipulator cannot block information aggregation in a small added state as in the monopolistic scenario. The voters can infer a lot about the state from the pivotal event when they hold exogenous i.i.d. private information. In fact, if the manipulator adds no further information, the voters would aggregate the exogenous information perfectly.

By releasing the additional information to the voters, the manipulator uses the pivotal inference of the voters and its power in a judoe-esque manner. First, when the manipulator sends the additional signal z to almost all voters, the voters hold almost no further information beyond their private signals. Moreover, from the perspective of almost all voters (those with the z-signal) it is almost common knowledge that the game is close to the game without further information. We show that, after z, the equilibrium played by a large electorate is arbitrarily close to equilibrium without further information. This pins down the margins of victory in the added state and allows to extend the construction to the other signals in the same way as before: we show that there is an equilibrium in which all voters believe that conditional on the election being tied, it is most likely that almost all voters received z, i.e. that they are in the added state. By appropriately choosing the probabilities of a and b in the added state, the posterior conditional on the added state and conditional on a or b is close to 1. Since preferences are 'monotone in beliefs, a belief of 1 implies the maximal support for A. Hence, when voters hold such beliefs, in the standard state, when there are only signals a or b from the manipulator, A receives close to maximal support. We show that, then, from the viewpoint of the voters receiving an a- or b signal, conditional on the election being tied,

it is likely that all other voters received the signal z. This supports the equilibrium belief that they are in the added state.
In the last part of the paper, we show that, even a manipulator with very limited knowledge about the state can persuade a large electorate (Theorem 6 and Theorem 7). Most strikingly, even when all voters hold private signals about the state, typically, already when the manipulator has more information than only two random voters together, he can persuade the voters to elect any policy (which might depend on his private signal).

The rest of the paper is organized as follows: In Section 2 we present the model. In Section 3 contains the preliminary analysis. In Section 4, we discuss a binary-state version of Feddersen & Pesendorfer (1997) as in Bhattacharya (2013). We restate the Condorcet Jury Theorem observed here (Theorem 1). In Section 5, we show that persuasion is essentially limitless when the information designer is monopolistic (Theorem 2), give a numerical example with 17 voters and illustrate the robustness of the 'manipulated equilibrium'; in particular, we discuss other equilibria (Proposition 4) and their non-robustness. In Section 6, we prove the main result of this paper by showing that persuasion is essentially limitless even when a manipulator can only add information to arbitrarily precise exogeneous private signals (Theorem 5). In particular, any state-contingent policy can be an equilibrium outcome. In Section 7, we show that the main result extends to the situation where the manipulator holds incomplete information about the state, as long as his information is not too imprecise. In Section 8, we provide additional results and give another interpretation of the main result: the equilibria of the game with a manipulator are the Bayes correlated equilibria a voting game as in Feddersen & Pesendorfer (1997). In Section 9, we discuss the paper's contribution to the existing literature and compare our results especially to other results on voter persuasion and other reported failures of information aggregation. The conclusion discusses the relation to the literature on auctions with general information structures.

2 Model

There are $2n+1$ voters (or citizens), two policies A and B, and two states of the world $\omega \in \{\alpha, \beta\} = \Omega$. The prior probability of α is $\Pr(\alpha) \in (0,1)$.

Voters have heterogeneous preferences. A voter's preference is described by a type $t = (t_\alpha, t_\beta) \in [-1,1]^2$, with t_ω the utility of A in ω. The utility of B is normalized to 0, so that t_ω is the difference of the utilities of A and B in ω. The types are independently and identically distributed across voters according to a cumulative distribution function $G : [-1,1]^2 \to [0,1]$ with a strictly positive, continuous density g. The own type is private information of the voter.

An *information structure* π is a finite set of signals S and a joint distribution of signal profiles and states that is independent of G. The conditional distribution is exchangeable with respect to the voters. In particular, there is a finite number of substates $\{\alpha_j\}_{j=1,\dots,N_\alpha}$ and $\{\beta_j\}_{j=1,\dots,N_\beta}$ such that the signals are independently and identically distributed conditional on the substates.[3] Abusing notation slightly, we denote by $\Pr(\omega_j|\omega)$ and $\Pr(s_i|\omega_j)$ the corresponding probabilities of the substates and the individual signal s_i, conditional on a substate. So, for a signal profile $\mathbf{s} = (s_i)_{i=1,\dots,2n+1} \in S^{2n+1}$, we have

$$\Pr(\mathbf{s}|\omega) = \sum_j \Pr(\omega_j|\omega) \prod_{i=1,\dots,2n+1} \Pr(s_i|\omega_j). \tag{1}$$

The observed signal is the private information of the voter as well. It will be sufficient to use a class of information structures with two substates, $\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$, and three conditionally independent signals in each substate, $s \in \{a, b, z\}$ that is illustrated in Figure 1.

The voting game is as follows. First, nature draws the state, the profile of preferences types $\mathbf{t}$ and the profile of signals $\mathbf{s}$ according to G and π. Second, after observing her type and signal, each voter simultaneously submits a vote

[3] Note that the Hewitt-Savage-de Finetti theorem (De Finetti (1931), Hewitt & Savage (1955)) states that for any exchangeable infinite sequence of random variables $(X_i)_{i=1,\dots,\infty}$ with values in some set X there exists a random variable Y such that the random variables X_i are independently and identically distributed conditional on Y.

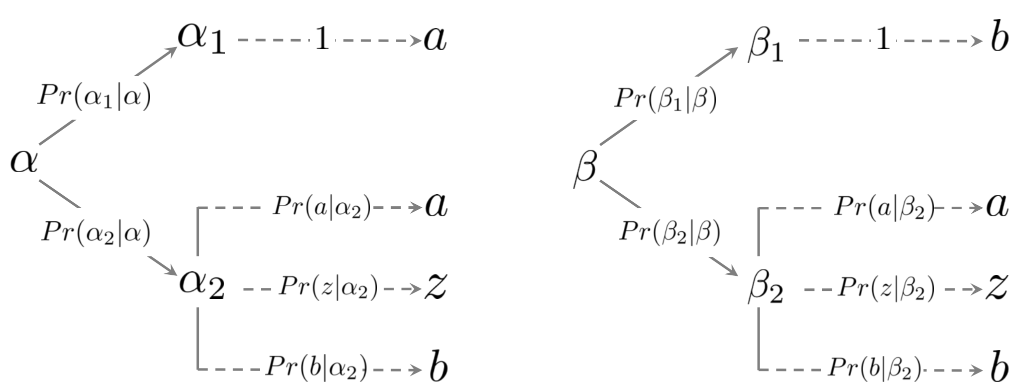

Figure 1: The main class of information structures considered in this paper. Each state ω has two substates $\{\omega_1, \omega_2\}$, occuring with conditional probabilities $\Pr(\omega_j|\omega)$. Conditional on the substate, the distribution of the signals $s_i \in \{a, z, b\}$ is independent and identical with the marginal probabilities denoted by $\Pr(s|\omega_j)$ (the marginals are degenerate in α_1 and β_1) .

for A or B. Finally, the submitted votes are counted and the majority outcome is chosen. This defines a Bayesian game.

A strategy of a voter is a function $\sigma : S \times [-1,1]^2 \to [0,1]$, where $\sigma(s,t)$ is the probability that a voter of type t with signal s votes for A.

We consider only weakly undominated strategies. In particular, we require that

$$\begin{aligned} \sigma(s,t) &= 0 \quad \text{for all} \quad t = (t_\alpha, t_\beta) < (0,0), \\ \sigma(s,t) &= 1 \quad \text{for all} \quad t = (t_\alpha, t_\beta) > (0,0), \end{aligned} \tag{2}$$

where $t > (0,0)$ and $t < (0,0)$ are *partisans* who prefer A and B, respectively, independently of the state. Given our full support assumption on G, this rules out degenerate strategies for which either $\sigma(s,t) = 1$ for all (s,t) or $\sigma(s,t) = 0$ for all (s,t). Here, and in the following, we ignore zero measure sets when writing "*for all*".

From the viewpoint of a given voter and given any strategy σ' used by the other voters, the pivotal event piv is the event in which the realized types and signals of the other $2n$ voters are such that exactly n of them vote for A and n for B. In this event, if she votes A, the outcome is A; if she votes B, the outcome is B. In any other event, the outcome is independent of her vote. Thus, a strategy is optimal if and only if it is optimal conditional on the pivotal event.

Let $\Pr(\alpha|s, \text{piv}; \sigma')$ denote the posterior probability of α conditional on s and conditional on *being pivotal* for the nondegenerate strategy σ'. The strategy σ is a best response to σ' if and only if

$$\Pr(\alpha|s, \text{piv}; \sigma') \cdot t_\alpha + (1 - \Pr(\alpha|s, \text{piv}; \sigma')) \cdot t_\beta > 0 \Rightarrow \sigma(s,t) = 1, \tag{3}$$

and

$$\Pr(\alpha|s, \text{piv}; \sigma') \cdot t_\alpha + (1 - \Pr(\alpha|s, \text{piv}; \sigma')) \cdot t_\beta < 0 \Rightarrow \sigma(s,t) = 0, \tag{4}$$

that is, a voter supports A if the expected value of A conditional on being pivotal is strictly positive and supports B otherwise. Note that indifference holds only for a set of types that has zero measure. For all other types, the best response is pure. It follows that there is no loss of generality to consider pure strategies with $\sigma(s,t) \in \{0,1\}$ for all (s,t).

So, a symmetric, undominated, and pure Bayes-Nash equilibrium of $\Gamma(\pi)$ is a strategy $\sigma : S \times [-1,1]^2 \to \{0,1\}$ that satisfies (2), (3), and (4), with $\sigma' = \sigma$. We refer to such a strategy simply as an *equilibrium*.

3 Preliminary Observations

3.1 Inference from the Pivotal Event

When making an inference from being pivotal, voters ask which state is more likely conditional on a tie, with exactly n voters supporting A and n supporting B. It is intuitive that a tie is evidence in favor of the substate in which the election is closer to being tied in expectation. Thus, conditional on being pivotal, a voter updates toward the substate in which the expected vote share is closer to $\frac{1}{2}$. We now verify this simple intuition and introduce some notation along the way.

Fix some strategy σ for the other voters. Then, the probability that any given voter supports A in substate ω_j is

$$q(\omega_j;\sigma) = \sum_{s \in S} \pi(s|\omega_j) \Pr_G \{t : \sigma(s,t) = 1\}; \tag{5}$$

we refer to $q(\omega_j;\sigma)$ also as the *expected vote share* of A.

Given that the signals and the types of the voters are independent conditional on the substate, the probability of a tie in the vote count is

$$\Pr(\text{piv}|\omega_j;\sigma) = \binom{2n}{n} (q(\omega_j;\sigma))^n (1 - q(\omega_j;\sigma))^n . \tag{6}$$

For any two substates ω_j and $\hat{\omega}_l$, the likelihood ratio of being pivotal is

$$\frac{\Pr\left(\text{piv}|\omega_j;\sigma\right)}{\Pr\left(\text{piv}|\hat{\omega}_l;\sigma\right)} = \left(\frac{q\left(\omega_j;\sigma\right)\left(1-q\left(\omega_j;\sigma\right)\right)}{q\left(\hat{\omega}_l;\sigma\right)\left(1-q\left(\hat{\omega}_l;\sigma\right)\right)}\right)^n. \tag{7}$$

Using the conditional independence, the posterior likelihood ratio of any two substates conditional on a signal s and the event that the voter is pivotal is

$$\frac{\Pr\left(\omega_j|\text{piv},s;\sigma\right)}{\Pr\left(\hat{\omega}_l|\text{piv},s;\sigma\right)} = \frac{\Pr(\omega_j)}{\Pr(\hat{\omega}_l)}\frac{\Pr(s|\omega_j)}{\Pr(s|\hat{\omega}_l)}\frac{\Pr\left(\text{piv}|\omega_j;\sigma\right)}{\Pr\left(\text{piv}|\hat{\omega}_l;\sigma\right)}. \tag{8}$$

Now, we record the intuitive fact that voters update toward the substate in which the vote share is closer to 1/2, that is, in which the election is closer to being tied in expectation.

Claim 1 *Take any two substates ω_j and $\hat{\omega}_l$, and any strategy σ for which* $\Pr\left(\text{piv}|\hat{\omega}_l;\sigma\right) \in (0,1)$*; if*

$$\left|q\left(\omega_j;\sigma\right)-\frac{1}{2}\right| < \left|q\left(\hat{\omega}_l;\sigma\right)-\frac{1}{2}\right|, \tag{9}$$

then

$$\frac{\Pr\left(\text{piv}|\omega_j;\sigma\right)}{\Pr\left(\text{piv}|\hat{\omega}_l;\sigma\right)} > 1. \tag{10}$$

Proof. The function $q(1-q)$ has an inverse u-shape on $[0,1]$ and is symmetric around its peak at $q=\frac{1}{2}$, as is illustrated in Figure (2). So, $\left|q-\frac{1}{2}\right| < \left|q'-\frac{1}{2}\right|$ implies that $q(1-q) > q'(1-q')$. Thus, it follows from (7) that (9) implies (10). ■

3.2 Pivotal Voting

Given any strategy profile σ' used by the others, the vector of posteriors conditional on piv and s is denoted as

$$\boldsymbol{\rho}\left(\sigma'\right) = \left(\Pr(\alpha|s,\text{piv};\sigma')\right)_{s\in S}. \tag{11}$$

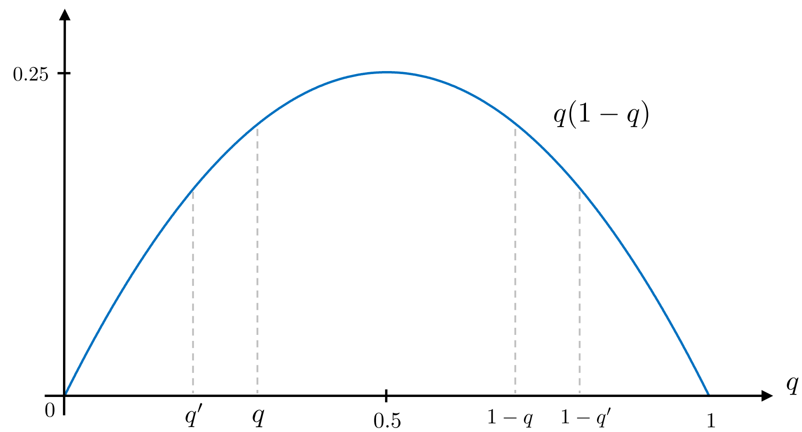

Figure 2: The function $q(1-q)$ for $q \in [0,1]$. If $|q - \frac{1}{2}| < |q' - \frac{1}{2}|$, then $q(1-q) > q(1-q')$.

This vector of posteriors is a sufficient statistic for the unique best response to σ' for all nonpartisan voter types; see (3) and (4).

In particular, given some vector of beliefs $\mathbf{p} = (p_s)_{s\in S}$, let $\sigma^{\mathbf{p}}$ be the unique undominated strategy that is optimal if a voter with a signal s believes the probability of α to be p_s; that is,

$$\forall(s,t) : \sigma^{\mathbf{p}}(s,t) = 1 \Leftrightarrow p_s \cdot t_\alpha + (1-p_s)\cdot t_\beta > 0, \tag{12}$$

and (2) holds for the partisans. Then, the strategy σ is a best response to σ' if and only if $\sigma = \sigma^{\mathbf{p}}$ for $\mathbf{p} = \boldsymbol{\rho}(\sigma')$.

So, σ^* is an equilibrium if and only if $\sigma^* = \sigma^{\boldsymbol{\rho}(\sigma^*)}$. Conversely, an equilibrium can be described by a vector of beliefs $\mathbf{p}^*$ that is a fixed point of $\boldsymbol{\rho}(\sigma^{\mathbf{p}})$, that is

$$\mathbf{p}^* = \boldsymbol{\rho}(\sigma^{\mathbf{p}*}); \tag{13}$$

meaning, the belief $\mathbf{p}^*$ corresponds to an equilibrium if, when voters behave optimally given $\mathbf{p}^*$ (i.e., vote according to $\sigma^{\mathbf{p}^*}$), the posterior conditional on being pivotal is again $\mathbf{p}^*$.

Equation (13) provides an equilibrium existence argument: the expression $\boldsymbol{\rho}\left(\sigma^{\mathbf{p}}\right)$ defines a finite-dimensional mapping $[0,1]^{|S|} \rightarrow [0,1]^{|S|}$ from beliefs $\mathbf{p}$ into posterior beliefs $\boldsymbol{\rho}\left(\sigma^{\mathbf{p}}\right)$, and this mapping is continuous.[4] Thus, an application of Kakutani's theorem implies the existence of a fixed point $\mathbf{p}^*$ that solves (13).[5] The strategy $\sigma^{\mathbf{p}^*}$ is an equilibrium.[6]

The possibility of writing equilibria in terms of posteriors enables us to connect our model and our results to the Bayesian persuasion literature.

3.3 Aggregate Preferences

A central object of the analysis is the *aggregate preference function*

$$\Phi(p) := \Pr_G(\{t : p \cdot t_\alpha + (1-p) \cdot t_\beta > 0\}), \tag{14}$$

which maps a belief $p \in [0,1]$ to the probability that a random type t prefers A under p. The function Φ proves useful to express expected vote shares. If a strategy σ is optimal given beliefs $\mathbf{p}$, i.e. $\sigma = \sigma^{\mathbf{p}}$, then the expected vote share of outcome A in substate ω_j is simply given by

$$q\left(\omega_j; \sigma\right) = \sum_{s \in S} \Pr(s|\omega_j) \Phi\left(p_s\right). \tag{15}$$

Figure 3 illustrates Φ. Given p, the dashed (blue) line corresponds to the plane of indifferent types $t = (t_\alpha, t_\beta)$ with $p \cdot t_\alpha + (1-p) \cdot t_\beta = 0$. Voters having types to the north-east prefer A given p, and Φ is the measure of such types under G. The indifference plane has a slope $-\frac{p}{1-p}$, and a change in p corresponds to a rotation of it. Given that G has a continuous density, it follows that the function Φ is continuous in p. Given that G has a strictly positive density on

[4] To see why $\boldsymbol{\rho}\left(\sigma^{\mathbf{p}}\right)$ is continuous in $\mathbf{p}$, first, note that (12) implies that $\Pr_G\left\{t : \sigma^{\mathbf{p}}\left(s,t\right) = 1\right\}$ is continuous in $\mathbf{p}$ since G has a continuous density. Second, $q(\omega_j; \sigma^{\mathbf{p}})$ are continuous in $\Pr_G\left\{t : \sigma^{\mathbf{p}}\left(s,t\right) = 1\right\}$, given (5). Third, $\rho(\sigma^{\mathbf{p}})$ is continuous in $q(\omega_j; \sigma^{\mathbf{p}})$, given (6) and (8).

[5] The ability to write an equilibrium as a finite-dimensional fixed point via (13) is a significant advantage. This reduction to finite dimensional equilibrium beliefs has been useful in other settings as well; see Bhattacharya (2013) and Ahn & Oliveros (2012).

[6] Note that, because of the partisans, $\sigma^{\mathbf{p}^*}$ is non-degenerate.

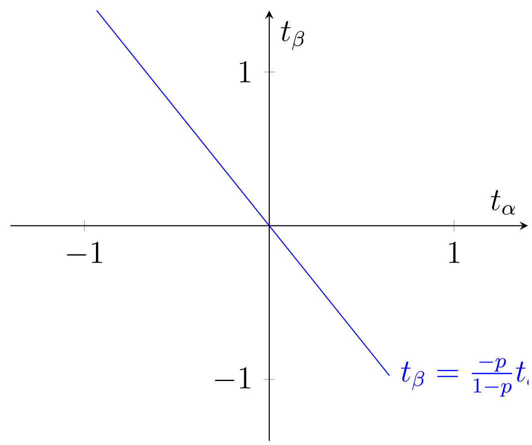

Figure 3: The curve of indifferent types is $t_\beta = \frac{-p}{1-p}t_\alpha$ for any given belief $p = \Pr(\alpha) \in (0,1)$.

$[-1,1]^2$, we also have that

$$0 < \Phi(p) < 1 \quad \text{for all} \quad p \in [0,1]. \tag{16}$$

As observed before, voters having types t in the north-east quadrant prefer A for all beliefs and voters having types t in the south-west quadrant always prefer B (*partisans*). Voters having types t in the south-east quadrant prefer A in state α and B in β (*aligned voters*) and voters having types t in the north-west quadrant prefer B in state α and A in β (*contrarian voters*).

We assume that the distribution of types is rich enough so that there is a belief p for which a majority prefers A and a belief p' for which a majority prefers B,[7] i.e.,

$$\Phi(p') < \frac{1}{2} < \Phi(p). \tag{17}$$

[7] Otherwise, the analysis is trivial: if, for all beliefs $p \in [0,1]$, in expectation a majority prefers A, then, for any information structure, it follows from the weak law of large numbers that in any equilibrium sequence A is elected with probability converging to 1.

4 Large Elections: Basic Results

We consider a sequence of elections along which the electorate's size n grows. For each $2n+1$, we fix some strategy profile σ_n and calculate the probability that a policy $x \in \{A, B\}$ wins the support of the majority of the voters in state ω, denoted $\Pr(x|\omega; \sigma_n, n)$. We will be interested in the limit of $\Pr(x|\omega; \sigma_n^*, n)$ as $n \to \infty$ for equilibrium sequences $(\sigma_n^*)_{n\in\mathbb{N}}$.[8] We first state a central observation regarding the inference from being pivotal in large elections, and then we show how this observation implies the Condorcet Jury Theorem, which we restate as a benchmark.

4.1 Inference in Large Elections

As a first step, we study the properties of the inference from being pivotal in a large election. We show that Claim 1 extends in an extreme form as the electorate grows large ($n \to \infty$): the event that the election is tied is infinitely more likely in the (sub-)state in which the election is closer to being tied in expectation. In fact, the likelihood ratio of the pivotal event diverges exponentially fast.

Since we want to allow the information structure to depend on n, we also include π_n now in the argument. The set of substates is kept fixed.

Claim 2 *Consider any sequence of strategies $(\sigma_n)_{n\in\mathbb{N}}$ and any sequence of information structures $(\pi_n)_{n\in\mathbb{N}}$ and any two substates ω_j and $\hat{\omega}_l$ for which* $\Pr(\mathrm{piv}|\hat{\omega}_l; \sigma, n, \pi_n) \in (0,1)$ *for all n. If*

$$\lim_{n\to\infty} \left| q(\omega_j; \sigma_n, \pi_n) - \frac{1}{2} \right| < \lim_{n\to\infty} \left| q(\hat{\omega}_l; \sigma_n, \pi_n) - \frac{1}{2} \right|, \tag{18}$$

then, for any $d \geq 0$,

$$\lim_{n\to\infty} \frac{\Pr(\mathrm{piv}|\omega_j; \sigma_n, \pi_n)}{\Pr(\mathrm{piv}|\hat{\omega}_l; \sigma_n, \pi_n)} n^{-d} = \infty. \tag{19}$$

[8] Recall that an equilibrium exists, given the representation as a finite-dimensional fixed point of a continuous mapping; see (13).

Proof. Let

$$k_n = \frac{q(\omega_j; \sigma_n, \pi_n)}{q(\hat{\omega}_j; \sigma_n, \pi_n)} \frac{(1 - q(\omega_j; \sigma_n, \pi_n))}{(1 - q(\hat{\omega}_j; \sigma_n, \pi_n))}.$$

From (7), the left-hand side of (19) is $\frac{(k_n)^n}{n^d}$. If (18) holds, then $\lim_{n\to\infty} k_n > 1$, because of the properties of $q(1-q)$ illustrated in Figure 2. So, $\lim_{n\to\infty} (k_n)^n = \infty$. Moreover, $(k_n)^n$ diverges exponentially fast and, hence, dominates the denominator n^d, which is polynomial. ∎

4.2 Benchmark: Condorcet Jury Theorem

The model embeds a special case of the canonical voting game by Feddersen & Pesendorfer (1997) with a binary state. In the following, we restate their full-information equivalence result, assuming, at first, that signals are binary, $S = \{u, d\}$.

As in Feddersen & Pesendorfer (1997), we assume that the signals are independently and identically distributed across voters conditional on the state $\omega \in \{\alpha, \beta\}$.[9] This corresponds to the case of an information structure π^c with a single substate in each state; in the following, we identify the substate with this state. The probabilities $\Pr(s|\omega; \pi^c)$ for $s \in \{u, d\}$ and $\omega \in \{\alpha, \beta\}$ satisfy

$$1 > \Pr(u|\alpha; \pi^c) > \Pr_{\pi^c}(u|\beta; \pi^c) > 0\,; \tag{20}$$

that is, signal u is indicative of α, and signal d is indicative of β. We further assume that

$$\Phi(p) \text{ is strictly increasing in } p. \tag{21}$$

We say that the aggregate preference function is *monotone*.[10] Monotonicity (21) and (17) together imply that $\Phi(0) < \frac{1}{2} < \Phi(1)$; so, the *full information outcome* is A in α and B in β.

[9] Feddersen & Pesendorfer (1997) assume the existence of subpopulations and allow the signal distributions to vary across those. This is not critical. Moreover, they assume a continuum of states ω. Bhattacharya (2013) nests a binary-state version of their model. The binary state version here is a special case of Bhattacharya (2013).

[10] Bhattacharya (2013) says that the distribution of preferences satisfies 'Strong Preference Monotonicity' if (21) holds except possibly for a countable number of beliefs p.

Theorem 1 *Feddersen & Pesendorfer (1997), Bhattacharya (2013)*
Suppose Φ is monotone (i.e., satisfies equation (21)). Then, for every sequence of equilibria $(\sigma_n^*)_{n\in\mathbb{N}}$ given π^c,

$$\begin{aligned}\lim_{n\to\infty} \Pr\left(A|\alpha;\sigma_n^*,\pi^c,n\right) &= 1,\\ \lim_{n\to\infty} \Pr\left(B|\beta;\sigma_n^*,\pi^c,n\right) &= 1.\end{aligned}$$

4.3 Proof of the Condorcet Jury Theorem

The following sketches the proof of Theorem 1. The proof is standard. We state it here to introduce some of the basic arguments that we use for the later analysis as well.

Step 1 *For all n and every equilibrium σ_n^*, the vote share of A is larger in α than in β,*

$$0 < q(\beta;\sigma_n^*,n) < q(\alpha;\sigma_n^*,n) < 1. \tag{22}$$

This ordering of the vote shares follows from the likelihood ratio ordering of the signals. In particular, recall the expression (8) for the posterior likelihood ratio of two states conditional on a given voter's signal s and the event that the voter is pivotal,

$$\frac{\Pr\left(\alpha|s,\text{piv};\sigma_n^*,n\right)}{1-\Pr\left(\alpha|s,\text{piv};\sigma_n^*,n\right)} = \frac{\Pr\left(\alpha\right)}{\Pr\left(\beta\right)}\frac{\Pr\left(\text{piv}|\alpha;\sigma_n^*,n\right)}{\Pr\left(\text{piv}|\beta;\sigma_n^*,n\right)}\frac{\Pr(s|\alpha;\pi^c)}{\Pr(s|\beta;\pi^c)}, \tag{23}$$

where $\Pr\left(\text{piv}|\beta;\sigma_n^*,n\right) > 0$ because σ_n^* is nondegenerate by (2). Therefore, $\frac{\Pr(u|\alpha;\pi^c)}{\Pr(u|\beta;\pi^c)} > \frac{\Pr(d|\alpha;\pi^c)}{\Pr(d|\beta;\pi^c)}$ implies that $\Pr\left(\alpha|u,\text{piv};\sigma_n^*,n\right) > \Pr\left(\alpha|d,\text{piv};\sigma_n^*,n\right)$. Now, (22) follows from (15) and the monotonicity of Φ. Intuitively, the expected posterior in state α is higher and this translates into a larger set of types preferring A given the monotonicity of Φ.

Step 2 *Voters cannot become certain of the state conditional on being pivotal, that is, the inference from the pivotal event must remain bounded,*

$$\lim_{n\to\infty}\frac{\Pr\left(\text{piv}|\alpha;\sigma_n^*,n\right)}{\Pr\left(\text{piv}|\beta;\sigma_n^*,n\right)} \in (0,\infty)\,, \tag{24}$$

for every convergent subsequence in the extended reals.

Suppose not and suppose instead, for example, that conditional on being pivotal voters become convinced that the state is β (i.e., $\eta = \lim_{n\to\infty} \frac{\Pr(\text{piv}|\alpha;\sigma_n^*,n)}{\Pr(\text{piv}|\beta;\sigma_n^*,n)} = 0$). This would imply $\lim_{n\to\infty} \Pr(\alpha|s, \text{piv}; \sigma_n^*, n) = 0$ for $s \in \{u, d\}$. Then, given $\Phi(0) < \frac{1}{2}$, a strict majority would support B in both states. However, the election is then closer to being tied in state α and voters would update towards state α conditional on being pivotal, in contradiction to $\eta = 0$.

Formally, if $\eta = 0$ for some converging subsequence, then $\lim_{n\to\infty} q(\omega; \sigma_n^*) = \Phi(0) < \frac{1}{2}$ for $\omega \in \{\alpha, \beta\}$. Therefore, for large enough n, (22) implies that $q(\beta; \sigma_n^*) < q(\alpha; \sigma_n^*) < 1/2$. Now, Claim 1 implies that voters update towards state α, that is, $\frac{\Pr(\text{piv}|\alpha;\sigma_n^*,n)}{\Pr(\text{piv}|\beta;\sigma_n^*,n)} \geq 1$, in contradiction to $\eta = 0$.

Step 3 *In every equilibrium sequence $(\sigma_n^*)_{n\in\mathbb{N}}$, the limit of the vote share of A is larger in α than in β,*

$$\lim_{n\to\infty} q(\alpha; \sigma_n^*) > \lim_{n\to\infty} q(\beta; \sigma_n^*). \tag{25}$$

From (24) and (125), we have that the limits of the posteriors conditional on being pivotal and $s \in \{u, d\}$ are interior and hence ordered,

$$0 < \lim_{n\to\infty} \Pr(\alpha|d, \text{piv}; \sigma_n^*, n) < \lim_{n\to\infty} \Pr(\alpha|u, \text{piv}; \sigma_n^*, n) < 1.$$

Now, (25) follows from (15) since Φ is strictly increasing.

Step 4 *The election is equally close to being tied in expectation, that is,*

$$\lim_{n\to\infty} q(\alpha; \sigma_n^*) - \frac{1}{2} = \lim_{n\to\infty} \frac{1}{2} - q(\beta; \sigma_n^*). \tag{26}$$

Since voters must not become certain conditional on being pivotal by (24), Claim 2 requires that

$$\lim_{n\to\infty} \left| q(\alpha; \sigma_n^*) - \frac{1}{2} \right| = \lim_{n\to\infty} \left| q(\beta; \sigma_n^*) - \frac{1}{2} \right|. \tag{27}$$

Given the ordering of the limits of the vote shares from (25), the equation (27) implies (26).

It follows from Step 4 and (25) that

$$\lim_{n\to\infty} q(\alpha;\sigma_n^*) > \frac{1}{2} > \lim_{n\to\infty} q(\beta;\sigma_n^*).$$

Therefore, by the weak law of large numbers, A wins in state α with probability converging to 1 as $n \to \infty$ and B wins in state β with probability converging to 1 as $n \to \infty$. This proves the result. ■

Theorem 1 holds more generally for *any* information structure π for which the signals are independent and identically distributed conditional on the state $\omega \in \{\alpha, \beta\}$ (i.e., there is a single substate) and for which signals are not completely uninformative. To see why this is true, note that, given the binary state, the signals can be taken to be ordered by the monotone likelihood ratio, without loss of generality. For any information structure π and any equilibrium σ_n^*, it then follows from (125) that the distribution of posteriors $\Pr(\alpha|\text{piv}, s; \sigma_n^*, \pi, n)$ in the state α (as implied by the distribution over s) first order stochastically dominates the distribution of posteriors $\Pr(\alpha|\text{piv}, s; \sigma_n^*, \pi, n)$ in the state β. Then, given that Φ is monotone, it follows from (15) that the vote shares satisfy the ordering (22). From (22) onward none of the arguments use that the signals are binary.
By the same line of argument, Theorem 1 holds even when we allow the information structure π with a single substate to vary with n (keeping the signal set S fixed), as long as the limit information structure is not completely uninformative, i.e.

$$\exists s \in S : \lim_{n\to\infty} \Pr(s|\pi_n) > 0 \;\text{ and }\; \lim_{n\to\infty} \frac{\Pr(s|\alpha;\pi_n)}{\Pr(s|\beta;\pi_n)} \neq 1. \tag{28}$$

We conclude,

Theorem 1' *Suppose Φ is monotone (i.e. satisfies equation (21)). Then, for every sequence of information structures $(\pi_n)_{n\in\mathbb{N}}$ with a single substate and satisfying (28) and for every sequence of equilibria $(\sigma^*)_{n\in\mathbb{N}}$ given $(\pi_n)_{n\in\mathbb{N}}$,*

$$\begin{aligned}\lim_{n\to\infty}\Pr\left(A|\alpha;\sigma_n^*,\pi_n,n\right) &= 1,\\ \lim_{n\to\infty}\Pr\left(B|\beta;\sigma_n^*,\pi_n,n\right) &= 1.\end{aligned}$$

5 Monopolistic Persuasion

We now consider the case of a sender who aims to affect the election outcome by providing information to voters, and voters have no other source of information on their own. Thus, the sender is the monopolist for information, which is the case studied in much of the literature on persuasion.

When the sender provides no information, the election outcome is trivially the outcome that is preferred by the majority at the prior, as determined by $\Phi(\Pr(\alpha))$. In addition, the sender can implement the full information outcome with public signals by revealing the state. What else can the sender implement?

For example, could the sender implement a constant policy that is the opposite of what the voters prefer at the prior? Or could the sender even implement the inverse of the full information outcome? Clearly, in order to implement these policies, the sender must provide some information to the voters, and, in fact, to implement the inverse of the full information outcome, the sender must provide sufficient information for the voters to be able to collectively distinguish the two states. On the other hand, the Condorcet jury theorem suggests that providing information to voters may easily lead to the full information outcome, suggesting that the possibility of persuasion is limited.

5.1 Result: Full Persuasion

Formally, we study what policies can be implemented in an equilibrium of a large election for some choice of π. This determines the set of feasible policies for a strategic sender.

The choice of the information structure π affects voters by affecting the posteriors $(\Pr(\alpha|s, \text{piv}; \sigma', \pi))_{s\in S}$. There are two effects of π. First, there is a *direct effect* of π on how voters learn from their signal. This effect is known from the work on persuasion. Second, there is an *indirect effect* of π because it affects the inference of the voters from being pivotal.

We show that there is no limit to the set of feasible policies. For any state-dependent policy and for large n, there is an information structure π_n and an equilibrium σ_n for which the targeted policy wins with probability close to 1 in the respective state.[11]

Theorem 2 *Take any* Φ *that satisfies (17) and any prior* $\Pr(\alpha) \in (0,1)$*: for every state-dependent policy* $(x(\alpha), x(\beta)) \in \{A, B\}^2$ *there exists a sequence of signal structures* $(\pi_n)_{n\in\mathbb{N}}$ *and equilibria* $(\sigma_n^*)_{n\in\mathbb{N}}$ *given* $(\pi_n)_{n\in\mathbb{N}}$ *such that*

$$\lim_{n\to\infty} \Pr(x(\alpha)|\alpha; \sigma_n^*, \pi_n, n) = 1,$$
$$\lim_{n\to\infty} \Pr(x(\beta)|\beta; \sigma_n^*, \pi_n, n) = 1.$$

From the previous analysis, with a single substate in each state, the sender can only implement either the policy that gets a majority at the prior or the full-information policy, by choosing uninformative signals or by choosing any other information structure, respectively.[12]

Thus, there need to be at least two substates for any nontrivial result. And, indeed, the class of information structures from the model section with two substates (as illustrated in Figure 1) turns out to be sufficient for the result.

In the following, first, we provide a proof for a special case of the theorem

[11] The sender can also implement any stochastic policy by "mixing" over information structures in the appropriate way.

[12] See Theorem 1 and the remarks after.

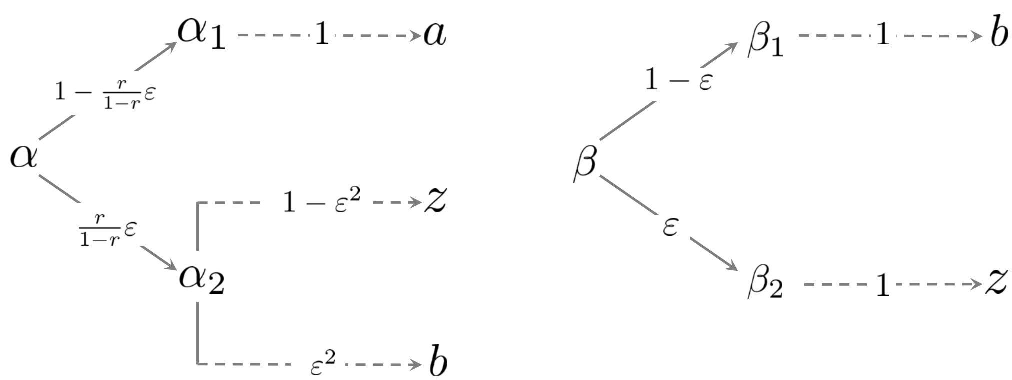

Figure 4: The information structure π_n^r with $\varepsilon = \frac{1}{n}$ and $r \in (0, 1)$.

in Section 5.2. Then, we provide the proof for the general case in Section 5.4.

5.2 Proof: Constant Policy

This section proves Theorem 2 for the case in which Φ is monotonically increasing and the targeted policy is A in both states (i.e., Φ satisfies (21) and $(x(\alpha), x(\beta)) = (A, A)$). We further assume a uniform prior to simplify the algebra, setting $\Pr(\alpha) = 1/2$.

5.2.1 The Information Structure

We specialize the general information structure introduced in the model section to the one defined in Figure 4. Setting $\varepsilon = \frac{1}{n}$, the information structure has a single parameter, $r \in (0, 1)$, and we denote it by π_n^r.

As ε vanishes for large n, the signals are almost public in the following sense: conditional on observing any signal s, a voter believes that any other voter has received the same signal with a probability close or equal to 1. Furthermore, the signals a and b reveal the state (almost) perfectly. In particular, this way the proof implies that even when constraining the sender to (almost) perfectly revealing information structures, persuasion is not constrained. In other words, the sender could be constrained to not 'lie' too often. The signal z contains only limited information since $r \in (0,1)$. When observing the signal z, a voter knows that the substate must be either α_2 or β_2. Moreover, given that a voter receives z with a probability close to 1 in either substate, we have (recall the uniform prior),

$$\lim_{n\to\infty} \Pr(\alpha|z;\pi_n^r) = \lim_{n\to\infty} \Pr(\alpha|\{\alpha_2,\beta_2\},\pi_n^r) = r. \tag{29}$$

5.2.2 Voter Inference

Clearly, for signal a,

$$\Pr(\alpha|a,\mathrm{piv};\sigma_n,\pi_n^r) = 1. \tag{30}$$

Hence, in state α_1, when all voters receive a, the probability that a random citizen votes A is $\Phi(1) > \frac{1}{2}$. It follows from the weak law of large numbers that in any equilibrium A is elected with probability converging to 1 in state α_1. In state β_1 all voters receive b. Conditional on the signal b alone, state β is more likely.

The remaining part of this section shows that the indirect effect from the inference of the voters from being pivotal can dominate and turn around this direct effect of the inference from the signal b alone such that there is an equilibrium sequence $(\sigma_n^*)_{n\in\mathbb{N}}$ for which

$$\lim_{n\to\infty} \Pr(\alpha|b,\mathrm{piv};\sigma_n^*,\pi_n^r) = 1. \tag{31}$$

Consider the signal z and the inference about the relative likelihood of α_2 and β_2. We show that, for *any* strategy used by the other voters, the pivotal event contains no information about the relative probability of α_2 and β_2 as the electorate grows large.

Claim 3 *Given any $r \in (0,1)$ and any sequence of strategies $(\sigma_n)_{n\in\mathbb{N}}$,*

$$\lim_{n\to\infty} \frac{\Pr(\mathrm{piv}|\alpha_2;\sigma_n,\pi_n^r)}{\Pr(\mathrm{piv}|\beta_2;\sigma_n,\pi_n^r)} = 1. \tag{32}$$

The proof is in the Appendix in Section A. The pivotal event contains no information since the distribution of signals is almost identical in the two substates α_2 and β_2 (and the distribution of preference types is identical by construction). Therefore, for any strategy σ, the distribution of votes must be almost identical in the two substates; in particular, the probability of a tie is also almost the same in the two substates.[13]

Claim 3 and (29) imply, in particular, that for any sequence of strategies $(\sigma_n)_{n\in\mathbb{N}}$,

$$\lim_{n\to\infty} \Pr(\alpha|z, \text{piv}; \sigma_n, \pi_n^r) = r. \tag{33}$$

Therefore, the sender can "steer" the behavior of voters with signal z by choosing r.

Next, we consider signal b and the voters' inference about the relative likelihood of α_2 and β_1. We show that, for this signal, the inference from the signal is dominated by the inference from being pivotal, for a large set of voting strategies. Conditional on the signal alone, state β is much more likely. However, this is turned around if the election is closer to being tied in state α_2 than in state β_1:

Claim 4 *Take any sequence of strategies $(\sigma_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty} |q(\sigma_n; \alpha_2, \pi_n^r) - \frac{1}{2}| < \lim_{n\to\infty} |q(\sigma_n; \beta_1, \pi_n^r) - \frac{1}{2}|; \tag{34}$$

then,

$$\lim_{n\to\infty} \frac{\Pr(\alpha|b, \text{piv}; \sigma_n, \pi_n^r)}{\Pr(\beta|b, \text{piv}; \sigma_n, \pi_n^r)} = \infty. \tag{35}$$

[13]The probability that all voters receive signal z in state α_2 is $(1-\frac{1}{n^2})^{2n}$ and $\lim_{n\to\infty}(1-\frac{1}{n^2})^{2n} = 1$, recalling that $\lim_{n\to\infty}(1-\frac{1}{n}\frac{1}{d})^{2n} = e^{-\frac{2}{d}}$. This observation is the critical step in the proof in the appendix.

Proof. The posterior likelihood ratio is

$$\begin{aligned}
& \frac{\Pr(\alpha|b,\mathrm{piv};\sigma_n,\pi_n^r)}{\Pr(\beta|b,\mathrm{piv};\sigma_n,\pi_n^r)} \\
= & \frac{\Pr(\alpha)}{\Pr(\beta)}\frac{\Pr(\alpha_2|\alpha)}{\Pr(\beta_1|\beta)}\frac{\Pr(b|\alpha_2;\pi_n^r)}{\Pr(b|\beta_1;\pi_n^r)}\frac{\Pr(\mathrm{piv}|\alpha_2;\sigma_n,\pi_n^r)}{\Pr(\mathrm{piv}|\beta_1;\sigma_n,\pi_n^r)} \\
= & \frac{\Pr(\alpha)}{\Pr(\beta)}\frac{r\frac{1}{n}}{1-(1-r)\frac{1}{n}}\frac{\frac{1}{n^2}}{1}\frac{\Pr(\mathrm{piv}|\alpha_2;\sigma_n,\pi_n^r)}{\Pr(\mathrm{piv}|\beta_1;\sigma_n,\pi_n^r)} \\
\approx & \frac{\Pr(\mathrm{piv}|\alpha_2;\sigma_n,\pi_n^r)}{\Pr(\mathrm{piv}|\beta_1;\sigma_n,\pi_n^r)}n^{-3}.
\end{aligned} \tag{36}$$

For the approximation on the last line we used that the prior is uniform. Given (34), equation (35) follows from applying Claim 2 for $d = 3$. ■

Thus, for any sequence of strategies that satisfies (34),

$$\lim_{n\to\infty}\Pr(\alpha|b,\mathrm{piv};\sigma_n,\pi_n^r) = 1. \tag{37}$$

5.2.3 Fixed Point Argument

By the richness assumption on Φ (see (17)), there is some $\hat{r}$ such that $\Phi(\hat{r}) = \frac{1}{2}$. We will show that, for the information structure $\pi_n^{\hat{r}}$ and n large enough, there is an equilibrium in which A receives a strict majority of votes in both states in expectation.

Recall that equilibrium is equivalently characterized by a vector of beliefs, $\mathbf{p}^* = (p_a^*, p_z^*, p_b^*)$ such that $\mathbf{p}^* = \boldsymbol{\rho}\left(\sigma^{\mathbf{p}^*}\right)$; see (13). Now, take any $\delta > 0$ and let

$$\mathrm{B}_\delta = \left\{\mathbf{p}\in[0,1]^3\,|\,|\mathbf{p}-(1,\hat{r},1)|\leq\delta\right\},$$

so that B_δ is the set of beliefs at most δ away from $(1,\hat{r},1)$. Take any $\mathbf{p}\in\mathrm{B}_\delta$ and the corresponding strategy $\sigma^{\mathbf{p}}$. Since $\Phi(1) > \frac{1}{2}$, this means that A receives a strict majority of votes in the states α_1 and β_1 for δ small enough. In the states α_2 and β_2, (almost) all voters observe signal z, so $q(\alpha_2;\sigma^{\mathbf{p}},\pi_n^{\hat{r}}) \approx \Phi(\hat{r})$ and $q(\beta_2;\sigma^{\mathbf{p}},\pi_n^{\hat{r}}) \approx \Phi(\hat{r})$. Since $\Phi(\hat{r}) = \frac{1}{2}$, the vote share for A is approximately $\frac{1}{2}$.

Now, we show that our two previous claims, Claim 3 and 4, imply that, given $\sigma^{\mathbf{p}}$, the posterior conditional on being pivotal is again in B_δ, for any $\mathbf{p} \in \mathrm{B}_\delta$, any sufficiently small δ and any sufficiently large n:

Claim 5 *For any δ sufficiently small, there exists $n(\delta)$ such that for all $n \geq n(\delta)$,*

$$\forall \mathbf{p} \in B_\delta : \boldsymbol{\rho}\left(\sigma^{\mathbf{p}}; \pi_n^{\hat{r}}, n\right) \in B_\delta \quad . \tag{38}$$

Proof. Take any $\mathbf{p} \in \mathrm{B}_\delta$ and its corresponding behavior $\sigma^{\mathbf{p}}$. For the posterior following signal a it is immediate that, for all δ and n,

$$\boldsymbol{\rho}_a\left(\sigma^{\mathbf{p}}; \pi_n^{\hat{r}}, n\right) = 1; \tag{39}$$

see (31). Secondly,

$$\lim_{n\to\infty} \boldsymbol{\rho}_z\left(\sigma^{\mathbf{p}}; \pi_n^{\hat{r}}, n\right) = \hat{r}, \tag{40}$$

follows from Claim 3 for all δ; see (33).

Finally, for δ small enough and n large enough, the election is closer to being tied in α_2 than in β_1,

$$\forall \mathbf{p} \in \mathrm{B}_\delta\text{: } |q(\alpha_2; \sigma^{\mathbf{p}}, \pi_n^{\hat{r}}) - \frac{1}{2}| < |q(\beta_1; \sigma^{\mathbf{p}}, \pi_n^{\hat{r}}) - \frac{1}{2}| \tag{41}$$

To see why, note that for n large enough, $q(\alpha_2; \sigma^{\mathbf{p}}, \pi_n^{\hat{r}}) \approx \Phi(p_z)$ and $q(\beta_1; \sigma^{\mathbf{p}}, \pi_n^{\hat{r}}) = \Phi(p_b)$ since almost all voters receive z in α_2 and all voters receive b in β_1. In addition, by the continuity of Φ, for δ small enough, we have that $\Phi(p_z) \approx \Phi(\hat{r})$ and $\Phi(p_b) \approx \Phi(1)$. Finally, (41) follows then from $\Phi(\hat{r}) = \frac{1}{2}$ and $\Phi(1) > \frac{1}{2}$.

Now, it follows from (41) and from Claim 4 that

$$\lim_{n\to\infty} \boldsymbol{\rho}_b\left(\sigma^{\mathbf{p}}; \pi_n^{\hat{r}}, n\right) = 1. \tag{42}$$

Thus, the claim follows from (39),(40), and (42). ■

Since $\boldsymbol{\rho}(\sigma^{\mathbf{p}})$ is continuous in $\mathbf{p}$ by the arguments after (13), it follows from (38) and Kakutani's fixed point theorem that there exists a fixed point $\mathbf{p}_n^* \in B_\delta$

for all n large enough. By the arguments from the proof of Claim 5,

$$\lim_{n\to\infty} \mathbf{p}_n^* = (1, \hat{r}, 1)\,, \tag{43}$$

see (39), (40), and (42). Finally, for the corresponding sequence of equilibrium strategies, $(\sigma^{\mathbf{p}_n^*})_{n\in\mathbb{N}}$, the policy A wins in both states; this follows from (43), which implies that voters with signals a and b are supporting A with a probability converging to $\Phi(1) > \frac{1}{2}$, and from the weak law of large numbers.

This finishes the proof of the theorem for the special case in which Φ is monotone, the targeted policy is A in both states, and the prior is uniform. When the prior is not uniform, the only piece of the argument that needs to be adjusted is the choice of $\hat{r}$. For a general prior $\Pr(\alpha)$, the value of $\hat{r}$ should be such that

$$\Phi\left(\frac{\Pr(\alpha)\,\hat{r}}{\Pr(\alpha)\,\hat{r} + (1-\Pr(\alpha))\,(1-\hat{r})}\right) = \frac{1}{2}. \tag{44}$$

We provide the proof for the general case (i.e. non-constant policies) in Section 5.4.

5.3 Numerical Example with 17 voters

Let $\Phi(p) = p$ for all $p \in [0,1]$.[14] Further, we set $p_0 = \frac{1}{4}$ and let the information structure be π_n^r with $r = \frac{1}{2}$.

In the Online Supplement, we show that under these primitives, when there are at least $2n+1 = 17$ voters, there is an equilibrium σ_n^* for which A is elected with a probability larger than 99.9% in the states α_1 and β_1. So, the overall probability of A being elected exceeds $0.999(1-\frac{1}{n})$ which is larger than 87% when there are at least $2n+1 = 17$ voters. To do so, we show that under the specified primitives, when $n \geq 8$, the best reponse is a self-map

[14]In the Online Supplement, we give an example of a preference distribution G such that $\Phi(p) = p$ for all p and such that $\Pr(t : t_\alpha > 0, t_\beta < 0) = 1$. Note that this is slightly inconsistent with the assumption that G has a strictly positive density on $[-1,1]^2$, but is done for the simplicity of presentation.

on the set of strategies σ satisfying $q(\omega_1; \sigma, \pi_n^r) \geq 0.95$ for $\omega_1 \in \{\alpha_1, \beta_1\}$, and $q(\omega_2; \sigma; \pi_n^r) \in [0.32, 0.68]$ for $\omega_2 \in \{\alpha_2, \beta_2\}$. This yields an equilibrium in which voters with an a-or b-signal vote A with a probability of at least 95%.

5.4 Proof: How to shape the voter beliefs freely

Now we allow for non-monotone Φ. The way we prove Theorem 2 is by showing that the sender can implement almost any belief μ_α in state α and simultaneously any belief μ_β in state β as $n \to \infty$, in the sense that, with probability close to one, (almost) all voters will have such beliefs conditional on being pivotal.

This implies the theorem as follows: the richness assumption (17) states that there is a belief p for which a majority prefers A in expectation and a belief p' for which a majority prefers B in expectation, i.e. $\Phi(p) < \frac{1}{2} < \Phi(p')$. So, given belief p', it follows from the weak law of large numbers that B is elected with probability converging 1. Given belief p, it follows from the weak law of large numbers that A is elected with probability converging 1. Hence, the sender can implement any state-contingent policy $(x_\alpha, x_\beta) \in \{A, B\}^2$ simply by implementing belief p' in any state ω for which $x_\omega = A$ and by implementing belief p in any state for which $x_\omega = B$.

Formally, a pair of beliefs $(\mu_\alpha, \mu_\beta) \in (0,1)^2$ is *implementable* if there is a sequence of information structures $(\pi_n)_{n\in\mathbb{N}}$ and an equilibrium sequence $(\sigma_n^*)_{n\in\mathbb{N}}$ given $(\pi_n)_{n\in\mathbb{N}}$ and two signals $s_\alpha, s_\beta \in S_2$ such that for all $\omega \in \{\alpha, \beta\}$

$$\lim_{n\to\infty} \Pr(s_\omega|\omega; \pi_n) = 1, \tag{45}$$

and the posterior after s_ω converges to μ_ω, i.e.

$$\lim_{n\to\infty} \hat{\rho}_{s_\omega}(\sigma_n^*; \pi_n, n) = \mu_\omega. \tag{46}$$

Lemma 1 *Take any Φ that satisfies (17) and any prior $\Pr(\alpha) \in (0,1)$: any pair of beliefs $(\mu_\alpha, \mu_\beta) \in [0,1]^2$ with $\Phi(\mu_\alpha) \neq \frac{1}{2}$ and $\Phi(\mu_\beta) \neq \frac{1}{2}$ is imple-*

mentable.

5.4.1 The Information Structure

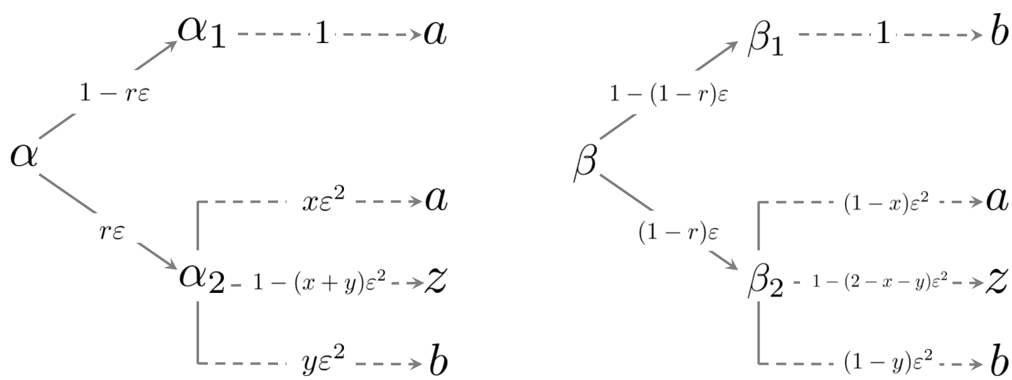

Figure 5: The information structure $\pi_n^{x,r,y}$ with $\varepsilon = \frac{1}{n}$ and $(x, r, y) \in (0, 1)^3$. The parameter r controls the posterior after z and the parameters x and y control the beliefs after a and b, respectively, conditional on being in substate α_2 or β_2.

We consider the information structure depicted in Figure 5. the signals are (almost) public, similar to the information structure in the previous section (see Figure 4), . Also, the signals a and b reveal the state (almost) perfectly. The signal z contains only limited information since $r \in (0, 1)$. When observing the signal z, a voter knows that the substate must be either α_2 or β_2. Moreover, given that a voter receives z with a probability close to 1 in either substate,

$$\lim_{n\to\infty} \frac{\Pr(\alpha|z;\pi_n^{x,r,y})}{\Pr(\beta|z;\pi_n^{x,r,y})} = \lim_{n\to\infty} \frac{\Pr(\alpha|\{\alpha_2,\beta_2\},\pi_n^{x,r,y})}{\Pr(\beta|\{\alpha_2,\beta_2\},\pi_n^{x,r,y})} = \frac{\Pr(\alpha)}{\Pr(\beta)}\frac{r}{1-r}. \tag{47}$$

5.4.2 Voter Inference

The basic arguments of the previous discussion of the voters' inference extend (compare with Section 5.2.2):

Consider the signal z and the inference about the relative likelihood of α_2 and β_2. As in the previous section (compare to Claim 3), for *any* strategy used by the other voters, the pivotal event contains no information about the relative probability of α_2 and β_2 as the electorate grows large.

Claim 6 *Given any parameters* $(x, r, y) \in (0,1)^3$ *and any sequence of strategies* $(\sigma_n)_{n\in\mathbb{N}}$,

$$\lim_{n\to\infty} \frac{\Pr(\text{piv}|\alpha_2; \sigma_n, \pi_n^{x,r,y})}{\Pr(\text{piv}|\beta_2; \sigma_n, \pi_n^{x,r,y})} = 1. \tag{48}$$

The proof follows from previous arguments: the arguments from the proof of Claim 3 hold verbatim with the required changes in notation. Claim 6 and (47) imply, in particular, that

$$\lim_{n\to\infty} \frac{\Pr(\alpha|z, \text{piv}; \sigma_n, \pi_n^{x,r,y})}{\Pr(\beta|z, \text{piv}; \sigma_n, \pi_n^{x,r,y})} = \frac{\Pr(\alpha)}{\Pr(\beta)} \frac{r}{1-r}. \tag{49}$$

Therefore, the sender can "steer" the behavior of voters with signal z by choosing r.

Next, we consider a signal $s \in \{a, b\}$ and the voters' inference about the relative likelihood of α and β. We show that, analogous to Claim 4, for this signal, the inference from the signal is dominated by the inference from being pivotal, for a large set of voting strategies. Conditional on the signal a alone, state α is much more likely. Conditional on the signal b alone, state β is much more likely. However, if the election is closer to being tied in states α_2 and β_2 than in the states α_1 and β_1, this inference after receiving a signal $s \in \{a, b\}$ can be turned around.

Claim 7 *Take any sequence of strategies* $(\sigma_n)_{n\in\mathbb{N}}$ *such that*

$$\begin{aligned} &\lim_{n\to\infty} \max_{\omega_2\in\{\alpha_2,\beta_2\}} |q(\sigma_n; \omega_2, \pi_n^{x,r,y}) - \frac{1}{2}| \\ < \; &\lim_{n\to\infty} \min_{\omega_1\in\{\alpha_1,\beta_1\}} |q(\sigma_n; \omega_1, \pi_n^{x,r,y}) - \frac{1}{2}|; \end{aligned} \tag{50}$$

then, for $s \in \{a, b\}$,

$$\lim_{n\to\infty} \frac{\Pr(\{\alpha_2, \beta_2\}|s, \text{piv}; \sigma_n, \pi_n^{x,r,y})}{\Pr(\{\alpha_1, \beta_1\}|s, \text{piv}; \sigma_n, \pi_n^{x,r,y})} = \infty. \tag{51}$$

The proof is in the Appendix in Section A. The proof is the same as for Claim 4, except for minor modifications.

For any sequence of strategies that satisfies (50), Claim 7 implies that for signal a,

$$\begin{aligned} \lim_{n\to\infty} \frac{\Pr(\alpha|a, \text{piv}; \sigma_n, \pi_n^{x,r,y})}{\Pr(\beta|a, \text{piv}; \sigma_n, \pi_n^{x,r,y})} &= \frac{\Pr(\alpha_2|\{\alpha_2, \beta_2\}, a; \sigma_n, \pi_n^{x,r,y})}{\Pr(\beta_2|\{\alpha_2, \beta_2\}, a; \sigma_n, \pi_n^{x,r,y})} \\ &= \frac{\Pr(\alpha)}{\Pr(\beta)} \frac{r}{1-r} \frac{x}{1-x} \end{aligned} \tag{52}$$

and that for signal b,

$$\begin{aligned} \lim_{n\to\infty} \frac{\Pr(\alpha|b, \text{piv}; \sigma_n, \pi_n^{x,r,y})}{\Pr(\beta|b, \text{piv}; \sigma_n, \pi_n^{x,r,y})} &= \frac{\Pr(\alpha_2|\{\alpha_2, \beta_2\}, b; \sigma_n, \pi_n^{x,r,y})}{\Pr(\beta_2|\{\alpha_2, \beta_2\}, b; \sigma_n, \pi_n^{x,r,y})} \\ &= \frac{\Pr(\alpha)}{\Pr(\beta)} \frac{r}{1-r} \frac{y}{1-y}. \end{aligned} \tag{53}$$

So, for the signals $s \in \{a, b\}$, the limits of the beliefs conditional on being pivotal are pinned down by the inference from the signal probabilities in the states α_2 and β_2 (i.e. the parameters $x \in (0, 1)$ and $y \in (0, 1)$).

5.4.3 Implementable Beliefs

In this section, we prove Lemma 1, using the observations from the preceding section.

Similar to before in section 5.2.3 we use that an equilibrium is equivalently characterized by a vector of beliefs, $\mathbf{p}^* = (p_a^*, p_z^*, p_b^*)$ such that $\mathbf{p}^* = \boldsymbol{\rho}\left(\sigma^{\mathbf{p}^*}\right)$; see (13). Take any $\delta > 0$ and let

$$\mathrm{B}_\delta = \left\{\mathbf{p} \in [0, 1]^3 \,|\, |\mathbf{p} - (\mu_\alpha, r', \mu_\beta)| \leq \delta\right\}, \tag{54}$$

so that B_δ is the set of beliefs at most δ away from $(\mu_\alpha, r', \mu_\beta)$.

We show that our two previous claims, Claim 6 and 7, imply that there is a large set of belief triples $(\mu_\alpha, r', \mu_\beta)$ such that, given $\sigma^{\mathbf{p}}$, the posterior conditional on being pivotal is again in B_δ, for any $\mathbf{p} \in \mathrm{B}_\delta$, any sufficiently small δ and any sufficiently large n.[15]

Claim 8 *Let $(\mu_\alpha, \mu_\beta) \in [0,1]^2$ and $r' \in (0,1)$ with*

$$|\Phi(\mu_\alpha) - \frac{1}{2}| > |\Phi(r') - \frac{1}{2}| \text{ and } |\Phi(\mu_\beta) - \frac{1}{2}| > |\Phi(r') - \frac{1}{2}|. \tag{55}$$

For any $\delta > 0$ small enough, there exists $n(\delta)$ such that for all $n \geq n(\delta)$,

$$\forall \mathbf{p} \in B_\delta : \boldsymbol{\rho}\left(\sigma^{\mathbf{p}}; \pi_n^{x,r,y}, n\right) \in B_\delta \tag{56}$$

with $\frac{\Pr(\alpha)}{\Pr(\beta)} \frac{r}{1-r} \frac{x}{1-x} = \frac{\mu_\alpha}{1-\mu_\alpha}$, $\frac{\Pr(\alpha)}{\Pr(\beta)} \frac{r}{1-r} \frac{y}{1-y} = \frac{\mu_\beta}{\mu_\beta}$, *and* $\frac{\Pr(\alpha)}{\Pr(\beta)} \frac{r}{1-r} = \frac{r'}{1-r'}$.

Proof. Let $\pi_n = \pi_n^{x,r,y}$. Take any $\mathbf{p} \in B_\delta$ and consider the corresponding strategy $\sigma^{\mathbf{p}}$. The condition (55) implies that for δ small enough, the election is closer to being tied in the states α_2 and β_2 than in the states α_1 and β_1 in expectation as $n \to \infty$:

$$\begin{aligned} \forall \mathbf{p} \in \mathrm{B}_\delta: \qquad & \lim_{n\to\infty} \max_{\omega_2 \in \{\alpha_2, \beta_2\}} |q(\omega_2; \sigma^{\mathbf{p}}, \pi_n) - \frac{1}{2}| \\ & < \lim_{n\to\infty} \min_{\omega_1 \in \{\alpha_1, \beta_1\}} |q(\omega_1; \sigma^{\mathbf{p}}, \pi_n) - \frac{1}{2}| \end{aligned} \tag{57}$$

To see why, note that for n large enough, $q(\alpha_2; \sigma^{\mathbf{p}}, \pi_n) \approx \Phi\left(p_z\right)$ and $q(\beta_2; \sigma^{\mathbf{p}}, \pi_n) \approx \Phi\left(p_z\right)$ since almost all voters receive z in α_2 and β_2. Also, $q(\alpha_1; \sigma^{\mathbf{p}}, \pi_n) = \Phi\left(p_a\right)$ since all voters receive a in α_1 and $q(\beta_1; \sigma^{\mathbf{p}}, \pi_n) = \Phi\left(p_b\right)$ since all voters receive b in β_1. In addition, by the continuity of Φ, for δ small enough, we have that $\Phi\left(p_z\right) \approx \Phi\left(\hat{r}\right)$, $\Phi\left(p_a\right) \approx \Phi\left(\mu_\alpha\right)$ and $\Phi\left(p_b\right) \approx \Phi\left(\mu_\beta\right)$. Finally, (57) follows then from $\Phi\left(\hat{r}\right) = \frac{1}{2}$ and $\Phi\left(\mu_\omega\right) \neq \frac{1}{2}$ for $\omega \in \{\alpha, \beta\}$. Now, it follows from (57),

[15] In the following, we use the convention that dividing by zero yields a result of infinity such that formulas like $\frac{\Pr(\alpha)}{\Pr(\beta)} \frac{r}{1-r} \frac{x}{1-x} = \frac{\mu_\alpha}{1-\mu_\alpha}$ make sense for $\mu_\alpha \in \{0,1\}$.

Claim 7, and its implications (52) and (53) that

$$\lim_{n\to\infty} \boldsymbol{\rho}_a\left(\sigma^{\mathbf{p}};\pi_n,n\right) = \mu_\alpha, \tag{58}$$
$$\lim_{n\to\infty} \boldsymbol{\rho}_b\left(\sigma^{\mathbf{p}};\pi_n,n\right) = \mu_\beta \tag{59}$$

for any $\delta > 0$ small enough. Thus, the claim follows from (49), (58) and (59). ■

We finish the proof of Lemma 1. Let $r = \hat{r}$ such that $\Phi(r') = \frac{1}{2}$ with $r' = \frac{\Pr(\alpha)\hat{r}}{\Pr(\alpha)\hat{r}+(1-\Pr(\alpha))(1-\hat{r})}$; see (44). Take any (μ_α, μ_β) with $\Phi(\mu_\alpha) \neq \frac{1}{2}$ and $\Phi(\mu_\beta) \neq \frac{1}{2}$. Then, given Claim 8, $\rho(\sigma^{\mathbf{p}})$ is a self-map on B_δ for δ small enough and $n \geq n(\delta)$ Since $\boldsymbol{\rho}(\sigma^{\mathbf{p}})$ is continuous in $\mathbf{p}$, it follows from Kakutani's fixed point theorem that there exists a fixed point $\mathbf{p}_n^* \in B_\delta$ for all n large enough, i.e. $\mathbf{p}_n^* = \rho(\sigma^{\mathbf{p}_n^*})$ and the corresponding behaviour $\sigma^{\mathbf{p}_n^*}$ forms a sequence of equilibria. Since the probability that a random voter receives a in α converges to 1 and since the probability that a random voter receives b in β converges to 1, Lemma 1 follows from (58) and (59) with $a = s_\alpha$ and $b = s_\beta$.

5.5 Robustness

In this section, we discuss the robustness of Theorem 2. In particular, we ask: can the sender persuade the voters even when he does not know the exact details of the game? Can the sender release information to the voters such that there is a unique manipulated equilibrium that implements the target policy? If not, will the voters play the manipulated equilibrium?

5.5.1 Robustness: Detail-Freeness

In this section, we show that, to persuade the voters, the signal structure does not need to be finely tuned to the details of the environment. In fact, there is a single information structure that implies the manipulated outcome uniformly across a large set of environments. These environment are given by the set of nondegenerate priors about α and preference distributions for which

$$|\Phi(0)-\frac{1}{2}| \quad > \quad |\Phi(\Pr(\alpha))-\frac{1}{2}|, \tag{60}$$

$$|\Phi(1)-\frac{1}{2}| \quad > \quad |\Phi(\Pr(\alpha))-\frac{1}{2}|, \tag{61}$$

hold, where (60) and (61) imply that, when the citizens vote optimally given a common belief, the election is closer to being tied when they hold a degenerate belief about the state relative to when they hold the prior belief (recall here the definition of Φ in (14)).

Proposition 1 *For every state-dependent policy $(x(\alpha), x(\beta)) \in \{A,B\}^2$ there is a sequence of signal structures $(\pi_n)_{n\in\mathbb{N}}$ such that for any $\Pr(\alpha) \in (0,1)$ and any Φ for which 60) and (61) hold, there is a sequence of equilibria $(\sigma_n^*)_{n\in\mathbb{N}}$ given $(\pi_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty} \Pr(x(\alpha)|\alpha;\sigma_n^*,\pi_n,n) = 1,$$
$$\lim_{n\to\infty} \Pr(x(\beta)|\beta;\sigma_n^*,\pi_n,n) = 1.$$

Proof. We provide the proof for the constant target policy A in both states, i.e. $(x(\alpha), x(\beta)) = (A, A)$. The other cases are analogous. Let the sender use the information structures $\pi_n = \pi_n^{x,r,y}$ with $x = y = 1$ and $r = \frac{1}{2}$. It follows from Claim 8 that, for any Φ for which (60) and (61) hold, that there is a δ small enough such that $\rho(\sigma^{\mathbf{p}})$ is a self-map on $B_\delta = \{\mathbf{p} \in [0,1]^3 : |\mathbf{p}-(1,\Pr(\alpha),1)| \leq \delta\}$ for all n large enough.
Since $\boldsymbol{\rho}(\sigma^{\mathbf{p}})$ is continuous in $\mathbf{p}$, it follows from Kakutani's fixed point theorem that there exists a fixed point $\mathbf{p}_n^* \in B_\delta$ for all n large enough, i.e. $\mathbf{p}_n^* = \rho(\sigma^{\mathbf{p}_n^*})$ and the corresponding behaviour $\sigma^{\mathbf{p}_n^*}$ forms a sequence of equilibria that implements the beliefs $(\mu_\alpha, \mu_\beta) = (1,1)$. Given $(\sigma^{\mathbf{p}_n^*})_{n\in\mathbb{N}}$, the policy A wins in both states; this follows since voters with an a and b-signal are supporting A with a probability converging to $\Phi(1) > \frac{1}{2}$ and from the weak law of large numbers. This finishes the proof of the proposition. ■

5.5.2 Robustness: Basin of Attraction

We provide a robustness argument by showing that for a large set of initial strategies, an iterated best response leads quickly to the "manipulated equilibrium" of Theorem 2 described before:

Let (μ_α, μ_β) be any pair of beliefs with $\Phi(\mu_\alpha) \neq \frac{1}{2}$ and $\Phi(\mu_\beta) \neq \frac{1}{2}$. By Lemma 1, there is a sequence of information structures and equilibria $(\sigma_n^*)_{n\in\mathbb{N}}$ that implements the pair of beliefs as $n \to \infty$, in the sense that, with probability close to one, (almost) all voters will have such beliefs conditional on being pivotal; see (45) and (79) for the definition of implementable beliefs. Lemma 1 was instrumental to show Theorem 2, namely, that the sender can implement any target policy $(x(\alpha), x(\beta)) \in \{A, B\}^2$ since he can do so by implementing a belief μ_ω with $\Phi(\mu_\omega) > \frac{1}{2}$ whenever $x(\omega) = B$ and a belief μ'_ω with $\Phi(\mu'_\omega) > \frac{1}{2}$ whenever $x(\omega) = A$.[16]

The next result shows that, for almost any other strategy $\sigma \neq \sigma_n^*$, the twice iterated best response is arbitrarily close to σ_n^* when n is large enough.

First, let us define the twice iterated best response: take any belief $\mathbf{p}$ and the strategy $\sigma^{\mathbf{p}}$ that is optimal given these beliefs. Then, $\sigma^{\boldsymbol{\rho}(\sigma^{\mathbf{p}})}$ is the best response to $\sigma^{\mathbf{p}}$ and is optimal given the beliefs

$$\boldsymbol{\rho}^1(\mathbf{p}) = \boldsymbol{\rho}(\sigma^{\mathbf{p}}) \tag{62}$$

where $\boldsymbol{\rho}(\sigma^{\mathbf{p}})$ is the vector of the posteriors conditional on the pivotal event and the signal s. In the same way, $\sigma^{\boldsymbol{\rho}\left(\sigma^{\boldsymbol{\rho}^1(\mathbf{p})}\right)}$ is the best response to $\sigma^{\boldsymbol{\rho}^1(\mathbf{p})}$ (so it is the twice iterated best reponse to $\sigma^{\mathbf{p}}$) and is optimal given the beliefs

$$\boldsymbol{\rho}^2(\mathbf{p}) = \boldsymbol{\rho}(\sigma^{\boldsymbol{\rho}^1(\mathbf{p})}). \tag{63}$$

Theorem 3 shows that for almost any $\mathbf{p}$, we have $|\boldsymbol{\rho}^2(\mathbf{p}) - (\mu_\alpha, \hat{r}, \mu_\beta)| < \epsilon$ when n is large enough. This means that the twice iterated best response is arbitrarily close to the manipulated equilibrium σ_n^* since the equilibrium is

[16] Recall that such beliefs exist by the richness assumption (17).

consistent with the belief $\boldsymbol{\rho}(\sigma_n^*) \approx (\mu_\alpha, \hat{r}, \mu_\beta)$; see (13). More generally, for almost any strategy σ, which does not need to be optimal given some belief $\mathbf{p}$, the twice iterated best is arbitrarily close to the manipulated equilibrium when n is large enough; we omit the proof of the more general case.

Theorem 3 *Let $\Phi(0) \neq \frac{1}{2}$ and $\Phi(1) \neq \frac{1}{2}$. Take any beliefs $(\mu_\alpha, \mu_\beta) \in [0,1]^2$ with $\Phi(\mu_\alpha) \neq \frac{1}{2}$ and $\Phi(\mu_\beta) \neq \frac{1}{2}$ and the information structures $(\pi_n^{x,\hat{r},y})_{n\in\mathbb{N}}$ with $\frac{\Pr(\alpha)}{\Pr(\beta)}\frac{r}{1-r}\frac{x}{1-x} = \frac{\mu_\alpha}{1-\mu_\alpha}$, $\frac{\Pr(\alpha)}{\Pr(\beta)}\frac{r}{1-r}\frac{y}{1-y} = \frac{\mu_\beta}{\mu_\beta}$, and $\hat{r}$ such that $\Phi(r') = \frac{1}{2}$ with $\frac{\Pr(\alpha)}{\Pr(\beta)}\frac{\hat{r}}{1-\hat{r}} = \frac{r'}{1-r'}$ (see Figure 5). For any $\delta > 0$, there is some $B \subset [0,1]^{|S|}$ with Lebesgue-measure at least $1-\delta$ and $\bar{n} \in \mathbb{N}$ such that, for all $n \geq \bar{n}$,*

$$\forall \mathbf{p} \in B : |\boldsymbol{\rho}^2(\mathbf{p}) - (\mu_\alpha, r', \mu_\beta)| < \delta. \tag{64}$$

Proof. Recall that for any strategy σ, the distance between the margin of victory in α_2 and β_2 is smaller than $\frac{1}{n^2}$ in expectation since the probability that a random voter receives the signal z is at least $1-\frac{1}{n^2}$ in both the substates. Now, consider any belief $\mathbf{p} \in [0,1]^3$ such that under the corresponding strategy $\sigma^{\mathbf{p}}$ the margins of victory differ by at least $\delta > 0$ for any other pair of substates. The theorem follows from the following claim: we show that for any such belief $\mathbf{p}$, the twice iterated response is δ-close to the manipulated equilibrium when n is large enough.

Claim 9 *For any $\delta > 0$, there exists $\bar{n} \in \mathbb{N}$ such that the following holds: take any $\mathbf{p} \in [0,1]^3$ and the information structures $(\pi_n^{x,\hat{r},y})_{n\in\mathbb{N}}$ with $\frac{\Pr(\alpha)}{\Pr(\beta)}\frac{r}{1-r}\frac{x}{1-x} = \frac{\mu_\alpha}{1-\mu_\alpha}$, $\frac{\Pr(\alpha)}{\Pr(\beta)}\frac{r}{1-r}\frac{y}{1-y} = \frac{\mu_\beta}{\mu_\beta}$, and $\hat{r}$ such that $\Phi(r') = \frac{1}{2}$ with $\frac{\Pr(\alpha)}{\Pr(\beta)}\frac{\hat{r}}{1-\hat{r}} = \frac{r'}{1-r'}$ (see Figure 5). If $n \geq \bar{n}$ and*

$$\left| |q(\omega_i, \sigma^{\boldsymbol{p}}, \pi_n) - \frac{1}{2}| - |q(\omega_j', \sigma^{\mathbf{p}}, \pi_n) - \frac{1}{2}| \right| > \delta, \tag{65}$$

for all $\omega_i \in (\alpha_1, \alpha_2, \beta_1, \beta_2\}$ and, $\omega_j' \in \{\alpha_1, \beta_1\}$ with $\omega_i \neq \omega_j'$, then $|\boldsymbol{\rho}^2(\mathbf{p}) - (\mu_\alpha, \hat{r}, \mu_\beta)| < \epsilon$.

Proof.

Take any $\mathbf{p} \in [0,1]^3$ such that (65) holds and consider the corresponding behaviour $\sigma^{\mathbf{p}}$. Denote the best response to $\sigma^{\mathbf{p}}$ by $\tilde{\sigma} = \sigma^{\boldsymbol{\rho}(\sigma^{\mathbf{p}};\pi_n,n)}$ and let

$\pi_n = \pi_n^{x,\hat{r},y}$ with $x = \mu_\alpha$ and $y = \mu_\beta$. The critical step is to show the following assertion: given $\tilde{\sigma}$, that is, given , the expected margin of victory in the states α_1 and β_1 is larger than in the states α_2 and β_2, i.e. $\tilde{\sigma}$ satisfies (50). We show one part of (50), i.e.

$$\lim_{n\to\infty} \max_{\omega_2\in\{\alpha_2,\beta_2\}} |q(\tilde{\sigma};\omega_2,\pi_n) - \frac{1}{2}| < \lim_{n\to\infty} |q(\tilde{\sigma};\alpha_1,\pi_n) - \frac{1}{2}|. \tag{66}$$

The proof for the second part, the analogous statement where we replace α_1 by β_1, is verbatim with the required changes in notation. To prove (66), we distinguish two cases. Note that the two cases are exhaustive since (65) holds.

Case 1 $\lim_{n\to\infty} |q(\sigma^{\mathbf{P}};\omega_2,\pi_n) - \frac{1}{2}| < \lim_{n\to\infty} |q(\sigma^{\mathbf{P}};\alpha_1,\pi_n) - \frac{1}{2}|$

Given (65), the difference is at least δ. Since almost all voters receive signal z in α_2 and β_2, the expected vote shares in α_2 and β_2 differ by much less than $\frac{\delta}{2}$ for n large enough. So, the expected margin of victory in α_1 is larger than the expected margin of victory in both α_2 and β_2 for n large enough. It follows from Claim 2 that for any $\omega_2 \in \{\alpha_2, \beta_2\}$,

$$\lim_{n\to\infty} \frac{\Pr(\omega_2|\text{piv}, a; \sigma^{\mathbf{P}}, \pi_n, n)}{\Pr(\alpha_1|\text{piv}, a; \sigma^{\mathbf{P}}, \pi_n, n)} = \infty. \tag{67}$$

Since all voters receive a in α_1, it holds $q(\alpha_1;\tilde{\sigma},\pi_n) = \Phi(\boldsymbol{\rho}_a(\sigma^{\mathbf{P}}))$. Since almost all voters receive z in α_2 and β_2 (see Figure 5), it holds $q(\alpha_2;\tilde{\sigma},\pi_n) \approx \Phi(\boldsymbol{\rho}_z(\sigma^{\mathbf{P}}))$ and $q(\beta_2;\tilde{\sigma},\pi_n) \approx \Phi(\boldsymbol{\rho}_z(\sigma^{\mathbf{P}}))$. It follows from (67) and Claim 6, which says that conditional on α_2 and β_2, there is nothing to be learnt from the pivotal event, that, when a voter observes signal a, the inference from the signal probabilities in the states α_2 and β_2 pins down the limits of the beliefs conditional on being pivotal,

$$\begin{aligned}\lim_{n\to\infty} \Pr(\alpha|a, \text{piv}; \sigma^{\mathbf{P}}, \pi_n, n) &= \lim_{n\to\infty} \Pr(\alpha|a, \{\alpha_2,\beta_2\}; \sigma^{\mathbf{P}}, \pi_n, n) \\ &= \mu_\alpha; \end{aligned} \tag{68}$$

compare to (52). Finally, (66) follows from (68) and (49) together with

$\Phi(\mu_\alpha) \neq \frac{1}{2}$ and $\Phi(r') = \frac{1}{2}$. This finishes the first case.

Case 2 $\lim_{n\to\infty} |q(\sigma^{\mathbf{P}}; \omega_2, \pi_n) - \frac{1}{2}| > \lim_{n\to\infty} |q(\sigma^{\mathbf{P}}; \alpha_1, \pi_n) - \frac{1}{2}|$

Given (65), the difference is at least δ. Since almost all voters receive signal z in α_2 and β_2 (see Figure 5), the expected vote shares in α_2 and β_2 differ by much less than $\frac{\delta}{2}$ for n large enough. So, the expected margin of victory in α_1 is smaller than the expected margin of victory in both α_2 and β_2 for n large enough. It follows from Claim 2 that for $\omega_2 \in \{\alpha_2, \beta_2\}$,

$$\lim_{n\to\infty} \frac{\Pr(\text{piv}|\alpha_1; \sigma^{\mathbf{P}}, \pi_n, n)}{\Pr(\text{piv}|\omega_2; \sigma^{\mathbf{P}}, \pi_n, n)} = \infty. \tag{69}$$

Therefore,

$$\begin{aligned}
& \lim_{n\to\infty} \frac{\rho_a(\sigma^{\mathbf{P}}; \pi_n, n)}{1 - \rho_a(\sigma^{\mathbf{P}}; \pi_n, n)} \\
\geq \; & \lim_{n\to\infty} \frac{\Pr(\alpha)\Pr(\alpha_1|\alpha)\Pr(a|\alpha_1)\Pr(\text{piv}|\alpha_1; \sigma^{\mathbf{P}}, \pi_n, n)}{\sum_{j=1,2}\Pr(\beta)\Pr(\beta_j|\beta)\Pr(a|\beta_j)\Pr(\text{piv}|\beta_j, a; \sigma^{\mathbf{P}}, \pi_n, n)}, \\
= \; & \frac{\Pr(\alpha)}{\Pr(\beta)} \frac{(1 - \frac{r}{n^2})}{(1-r)\frac{1}{n}} \frac{1}{(1-x)\frac{1}{n^2}} \frac{\Pr(\text{piv}|\alpha_1; \sigma^{\mathbf{P}}, \pi_n, n)}{\Pr(\text{piv}|\beta_2; \sigma^{\mathbf{P}}, \pi_n, n)} \\
= \; & \infty,
\end{aligned} \tag{70}$$

where the equality on the third line follows since the probability of signal a is zero in β_1 and where we used (69) for the equality on the last line.
We will show now that (70) implies (66): to see why, recall that for n large enough, $q(\alpha_2; \tilde{\sigma}, \pi_n) \approx \Phi(\boldsymbol{\rho}_z(\sigma^{\mathbf{P}}; \pi_n, n))$ and $q(\beta_2; \tilde{\sigma}, \pi_n) \approx \Phi(\boldsymbol{\rho}_z(\sigma^{\mathbf{P}}; \pi_n, n))$ since almost all voters receive z in α_2 and β_2. Also, $q(\alpha_1; \tilde{\sigma}, \pi_n) = \Phi(\boldsymbol{\rho}_a(\sigma^{\mathbf{P}}; \pi_n, n))$ since all voters receive a in α_1. In addition, we have that $\rho_z(\sigma^{\mathbf{P}}; \pi_n, n) \approx \hat{r}$ by (49) and $\rho_a(\sigma^{\mathbf{P}}; \pi_n, n) \approx 1$ by (70). Finally, (66) follows since $\Phi(r') = \frac{1}{2}$ and since $\Phi(1) \neq \frac{1}{2}$. This finishes the second case.

Now, we finish the proof of Claim 9. Since we just showed that, given $\tilde{\sigma} = \sigma^{\boldsymbol{\rho}(\sigma^{\mathbf{P}}; \pi_n, n)}$, the expected margin of victory in α_1 and β_1 is larger than in α_2 and β_2, it follows from Claim 7 that

$$\lim_{n\to\infty} \frac{\Pr\left(\{\alpha_2, \beta_2\}|\text{piv}, s; \tilde{\sigma}, \pi_n, n\right)}{\Pr\left(\{\alpha_1, \beta_1\}|\text{piv}, s; \tilde{\sigma}, \pi_n, n\right)} = \infty \tag{71}$$

for any $s \in \{a, b\}$. It follows from (71) and Claim 6, which says that conditional on α_2 and β_2, there is nothing to be learnt from the pivotal event, that, given $\tilde{\sigma}$; when a voter observes signal a, the inference from the signal probabilities in the states α_2 and β_2 pins down the limits of the beliefs conditional on being pivotal, such that (52) and (53) hold for $\sigma_n = \tilde{\sigma}$. This, together with (49) yields Claim 9. So, we are also done with the proof of Theorem 3. ■ ■

Simple Reasoning. Theorem 3 illustrates that a simple reasoning is underlying the manipulated equilibrium σ_n^*. The result loosely relates to the concepts of level k-thinking and level-k-implementability (De Clippel *et al.* (2016)). The theorem implies that for almost any strategy (a 'behavioral anchor'), the strategies that are consistent with level-2-thinking are close to the manipulated equilibrium. In this sense, any state-dependent target policy $(x(\alpha), x(\beta)) \in \{A, B\}^2$ is level-2-implementable.[17]

5.5.3 Other Equilibria

Theorem 3 shows that the basin of attraction of an arbitrarily small neigbourhood of the manipulated equilibria consists of almost all strategies when n is large enough. However, this still leaves open the possibility that there are other equilibria such that, if we start exactly at such a strategy profile, the best reponse dynamic stays there. We show that this is indeed the case. There exists another equilibrium and that equilibrium is not "manipulated", but implements the full information outcome as $n \to \infty$.

Theorem 4 *Let Φ be stricly increasing. Take information structures $(\pi_n)_{n\in\mathbb{N}} = \pi_n = \pi_n^{x,r,y}$ with $(x, r, y) \in (0, 1)^3$ as illustrated in Figure 5. There exists an equilibrium sequence $(\sigma_n^*)_{n\in\mathbb{N}}$ for which the full information outcome is elected*

[17] De Clippel *et al.* (2016) consider different notions of level-2-implementability which demand that there is *some* behavioral anchor such that *any* profile of strategies that are level-1-consistent or level-2-consistent for this anchor implement a given social choice function; see their paper for the exact definitions.

as $n \to \infty$,

$$\lim_{n\to\infty} \Pr(A|\alpha; \sigma_n^*, \pi_n) = 1,$$
$$\lim_{n\to\infty} \Pr(B|\beta; \sigma_n^*, \pi_n) = 1.$$

The proof is in the Appendix in Section B.

Intuition. Note that the signal π_n almost always sends an (almost) perfectly revealing signal when n is large. Hence, there is a sequence of strategies (e.g. given by sincere voting) for which the full-information outcome is elected as $n \to \infty$. The question is if such a sequence of strategies can be an equilibrium sequence. The theorem shows that, whenever Φ is monotone, the answer is yes.
This is easy to see when voters have a common type $t > 0$.[18] A result of McLennan (1998) states that the utility maximizing symmetry strategy is a symmetric equilibrium. Hence, for this special case, the existence of a sequence of strategies that aggregates information implies the existence of an equilibrium sequence that aggregates information.

Non-Robustness. Consider the information structures $\pi_n = \pi_n^{x,\hat{r},y}$ used to construct the manipulated equilibria in the proof of Lemma 1. Recall that, given $\hat{r}$, it holds that $\Phi(r') = \frac{1}{2}$ for $\frac{\Pr(\alpha)}{\Pr(\beta)}\frac{r}{1-r} = \frac{r'}{1-r'}$; see (44). We claim that, given π_n, any equilibrium sequence other than the sequence of manipulated equilibria must be close to being tied in either α_1 or β_1 as $n \to \infty$. To see why, consider any strategy sequence $(\sigma_n)_{n\in\mathbb{N}}$. It follows from (33) and $\Phi(r') = \frac{1}{2}$ that the margin of victory in α_2 and β_2 converges to zero. Hence, unless the margin of victory in α_1 or β_1 converges to zero, Claim 7 and its implications (52) and (53) together with (33) yield that $\lim_{n\to\infty} \boldsymbol{\rho}(\sigma_n; \pi_n^{x,\hat{r},y}) = (\mu_\alpha, r', \mu_\beta)$ with $\frac{\Pr(\alpha)}{\Pr(\beta)}\frac{r}{1-r}\frac{x}{1-x} = \frac{\mu_\alpha}{1-\mu_\alpha}$ and $\frac{\Pr(\alpha)}{\Pr(\beta)}\frac{r}{1-r}\frac{y}{1-y} = \frac{\mu_\beta}{\mu_\beta}$, i.e. the best response to σ_n converges to the manipulated equilibrium. So, for any equilibrium other than

[18] Recall that our model imposes no restriction on the preference distribution G other than that G has a strictly positive density. In particular, the model embeds preferences that are close to being common.

the manipulated equilibrium there is a marginal perturbation such that the best response to the perturbed strategy is close to the manipulated equilibrium, illustrating that the other equilibrium is not stable.

6 Persuasion of Privately Informed Voters

Recall the binary information structure from the CJT, defined by the signal probabilities $\Pr(s|\omega)_{\omega\in\{\alpha,\beta\}}$ for $s \in \{u, d\}$ such that (20) holds. We will think of this as exogenous private information that is held by the voters and denote the information structure by π^c. We say that an information structure π with signal set S is an *independent expansion* of π^c if there exists an information structure π^p with signal set S_2 and substates $\{\alpha_1, \ldots, \alpha_{N_\alpha}\}$ and $\{\beta_1, \ldots, \beta_{N_\beta}\}$ such that

$$S = \{u, d\} \times S_2, \tag{72}$$

and

$$\Pr(\mathbf{s}|\omega_j; \pi) = \Pr(\mathbf{s}_1|\omega; \pi^c)\Pr(\mathbf{s}_2|\omega_j; \pi^p) \tag{73}$$

for all $\omega_j \in \{\alpha_1, \ldots, \alpha_{N_\alpha}\} \cup \{\beta_1, \ldots, \beta_{N_\beta}\}$ and all signal profiles $\mathbf{s} = (\mathbf{s}_1, \mathbf{s}_2) \in (\{u, d\} \times S_2)^{2n+1}$. We write $\pi = \pi^c \times \pi^p$. We think of the expansion as resulting from additional information that is provided by a sender to voters who also receive private signals from π^c, the *exogenous information structure*. In this sense, we call π^p the *additional information structure*. By considering only independent expansions, we do not allow the sender's signal to condition directly on the realization of π^c. As before, we do not allow the sender to elicit the voters' private information (the preference type the signal). As in the setting of the CJT, the preferences of the voters are such that the aggregate preference function Φ is strictly increasing, i.e. (21)) holds.

What outcomes can the sender implement when the voters have exogenous signals and how should he communicate with the voters? Clearly, to implement any policy other than the full information outcome, the sender has to communicate with the voters in some way, since without additional information (i.e. when π_2 is uninformative) it follows from the CJT that the unique equilibrium outcome is the full information outcome as the electorate grows large.

Consider a sender who communicates with *public signals* $s \in S_2$, meaning, that the signals are commonly received by all the voters.[19] When the voters receive a public signal s, this shifts the common belief from the prior $\Pr(\alpha)$ to $\Pr(\alpha|s)$. Since the CJT (Theorem 1) holds for any common prior, it follows that in the subgame following any public signal, the full information outcome is elected with probability converging to 1 as $n \to \infty$.[20]

Therefore, in order to implement any outcome other than the full information outcome, the sender has to communicate privately with the voters. We provide a possibility result; even when the voters hold arbitrarily precise exogenous information, the sender can release additional information π^p to the voters such that the voters elect any arbitrary target policy of the sender with probability close to 1 (Section 6.1).

6.1 Result: Full Persuasion

The following theorem shows that there exists an independent expansion of the private information of the voters that allows to implement any arbitrary state-dependent policy, including, e.g., the policy that inverts the full-information outcome.

Theorem 5 *Take any exogenous private signals π^c of the voters and any strictly increasing Φ satisfying (17). For every state-dependent policy $(x(\alpha), x(\beta)) \in \{A, B\}^2$, there exists a sequence of independent expansions $(\pi_n)_{n \in \mathbb{N}}$ of π_1 and*

[19] Alonso & Câmara (2015) have studied persuasion with public signals when voters do not have exogenous private signals.

[20] To be precise, the CJT only applies to any non-degenerate prior $\Pr(\alpha) \in (0, 1)$. However, if the sender reveals the state publicly such that $\Pr(\alpha|s) \in \{0, 1\}$, trivially, the full-information outcome is elected as $n \to \infty$.

equilibria $(\sigma_n^*)_{n\in\mathbb{N}}$ *given* $(\pi_n)_{n\in\mathbb{N}}$ *such that*

$$\lim_{n\to\infty} \Pr\left(x\left(\alpha\right)|\alpha;\sigma_n^*,\pi_n,n\right) = 1,$$
$$\lim_{n\to\infty} \Pr\left(x\left(\beta\right)|\beta;\sigma_n^*,\pi_n,n\right) = 1.$$

In the next section 6.2, we state Lemma 2, a result more general than Theorem 5, which describes the beliefs that the sender can implement in equilibrium in states α and β by releasing independent additional information. A proof of Lemma 2 is in the Appendix in Section C.

6.2 Implementable Beliefs

We provide a compact representation of equilibrium as a belief vector, which will be used in the following; it simplifies the one given before by (13). Given any strategy σ' used by the others, the vector of posteriors conditional on piv and the additional signal $s_2 \in S_2$ is denoted as

$$\hat{\boldsymbol{\rho}}(\sigma';\pi,n) = (\Pr(\alpha|s_2,\text{piv};\sigma',\pi))_{s_2\in S_2} \tag{74}$$

and called the vector of *induced priors*.[21] It follows from the independence of the additional information and the exogenous information π^c that the vector of induced priors pins down the vector of the beliefs $\Pr(\alpha|s_1,s_2,\text{piv};\sigma',\pi)$: for any $s_2 \in S_2$ and any $s_1 \in \{u,d\}$,

$$\Pr(\alpha|s_1,s_2,\text{piv};\sigma',\pi) = \frac{\hat{\rho}_{s_2}(\sigma';\pi,n)\Pr(s_1|\alpha;\pi^c)}{\hat{\rho}_{s_2}(\sigma';\pi,n)\Pr(s_1|\alpha) + (1-\hat{\rho}_{s_2}(\sigma';\pi,n))\Pr(s_1|\beta)}. \tag{75}$$

Recall that the vector of beliefs $(\Pr(\alpha|s_1,s_2,\text{piv};\sigma',\pi))_{(s_1,s_2)\in\{u,d\}\times S_2}$ is a sufficient statistic for the unique best response to σ' for all types; see (11). Hence, the vector of induced priors pins down the best response for all types. Slightly abusing notation, for any $\mathbf{p} = (p_a,p_z,p_b) \in [0,1]^3$ we let $\sigma^{\mathbf{p}}$ be the unique strategy that is optimal if a voter with exogenous signal s_1 and additional

[21] We adopt the wording from Bhattacharya (2013).

signal $s_2 \in S_2$ believes the probability of α to be given by

$$\frac{p_{s_2} \Pr(s_1|\alpha; \pi^c)}{p_{s_2} \Pr(s_1|\alpha; \pi^c) + (1 - p_{s_2}) \Pr(s_1|\beta; \pi^c)}. \tag{76}$$

Equilibrium can be equivalently characterized by a vector of beliefs $\mathbf{p}^* = (p_a^*, p_z^*, p_b^*)$ such that

$$\mathbf{p}^* = \hat{\boldsymbol{\rho}}(\sigma^{\mathbf{p}^*}; \pi, n); \tag{77}$$

compare to the alternative representation (13).

Lemma 2 shows that, independent of the exogenous information π^c of the voters, the sender can implement an extreme belief $\mu_\omega \approx 0$ as induced prior with probability close to 1 in any state $\omega \in \{\alpha, \beta\}$. Similarly, the sender can implement an extreme belief $\mu_\omega \approx 1$ as induced prior in any state. However, more generally, the set of implementable beliefs depends on π^c: he can implement any belief as an induced prior in state α outside some intermediate interval $[\lambda_\alpha, \lambda]$ and any belief as an induced prior in state β outside some intermediate interval $[\lambda, \lambda_\beta]$.
Formally, a pair of beliefs $(\mu_\alpha, \mu_\beta) \in (0,1)^2$ is implementable[22] if there is a sequence of independent expansions $(\pi_n)_{n\in\mathbb{N}}$ of π^C and an equilibrium sequence $(\sigma_n^*)_{n\in\mathbb{N}}$ given $(\pi_n)_{n\in\mathbb{N}}$ and two signals $s_\alpha, s_\beta \in S_2$ such that for all $\omega \in \{\alpha, \beta\}$

$$\lim_{n\to\infty} \Pr(s_\omega|\omega; \pi_n) = 1, \tag{78}$$

and the induced prior after s_ω converges to μ_ω, i.e.

$$\lim_{n\to\infty} \hat{\rho}_{s_\omega}(\sigma_n^*; \pi_n, n) = \mu_\omega. \tag{79}$$

Lemma 2 *Take any exogenous private signals π^c of the voters and any strictly increasing Φ satisfying (17). There exist $\lambda_\alpha < \lambda < \lambda_\beta$ such that any pair of*

[22] We slightly adapt the definition of implementable belief pairs from Section 5.4 to account for the exogenous signals of the voters.

beliefs $[\mu_\alpha, \mu_\beta] \in [0,1]^2$ *with* $\mu_\alpha \notin [\lambda_\alpha, \lambda]$ *and* $\mu_\beta \notin [\lambda, \lambda_\beta]$ *is implementable.*

The proof is in the Appendix in Section C. The rest of this section describes the boundaries λ_α and λ_β.

For any belief $p \in (0,1)$,

$$\hat{q}(\omega; p, \pi^c) = \sum_{s_1 \in \{u,d\}} \Pr(s_1|\omega; \pi^c)\Phi(\frac{p\Pr(s_1|\alpha)}{p\Pr(s_1|\alpha) + (1-p)\Pr(s_1|\beta)}) \tag{80}$$

is the probability that a random voter with induced prior p votes for the outcome A in state ω. We illustrate the functions $\hat{q}(\omega; p, \pi^c)$ in Figure 6.

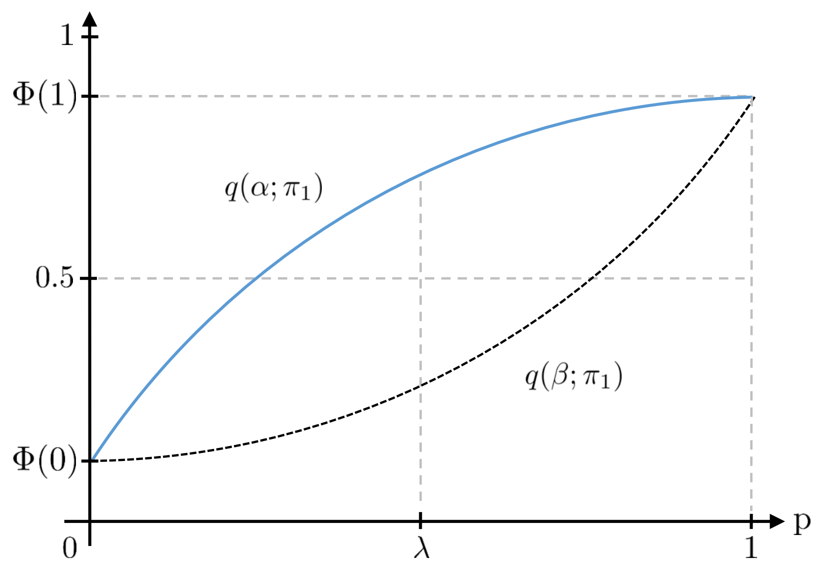

Figure 6: The function $\hat{q}(\alpha; p, \pi^c)$ of the implied vote share in state α and the function $\hat{q}(\beta; p, \pi^c)$ of the implied vote share in state β.

Since Φ is continuous, it follows from (17) and the intermediate value theorem that there exists a belief λ such that the implied vote shares satisfy

$$\hat{q}(\alpha; \lambda, \pi^c) - \frac{1}{2} = \frac{1}{2} - \hat{q}(\beta; \lambda, \pi^c); \tag{81}$$

this is also illustrated in Figure 6. Since Φ is strictly increasing, for any given π^c , $q(\alpha; p, \pi^c)$ and $q(\beta; p, \pi^c)$ are strictly increasing and λ is unique.
The boundaries λ_α and λ_β of the lemma are such that all beliefs outside the intermediate intervals $[\lambda_\alpha, \lambda]$ and $[\lambda, \lambda_\beta]$ imply margins of victory that are larger than the ones implied by λ in *any* state $\omega \in \{\alpha, \beta\}$, i.e. margins of victory that are larger than $q(\alpha; \lambda, \pi^c) - \frac{1}{2}$. Formally, λ_α and λ_β are given by

$$q(\alpha; \lambda_\alpha, \pi^c) = q(\beta; \lambda, \pi^c), \tag{82}$$

$$q(\beta; \lambda_\beta, \pi^c) = q(\alpha; \lambda, \pi^c). \tag{83}$$

Figure 7 illustrates the boundaries λ_α and λ_β of the set of implementable induced priors. Intuitively, when the exogenous information π^c of the voters

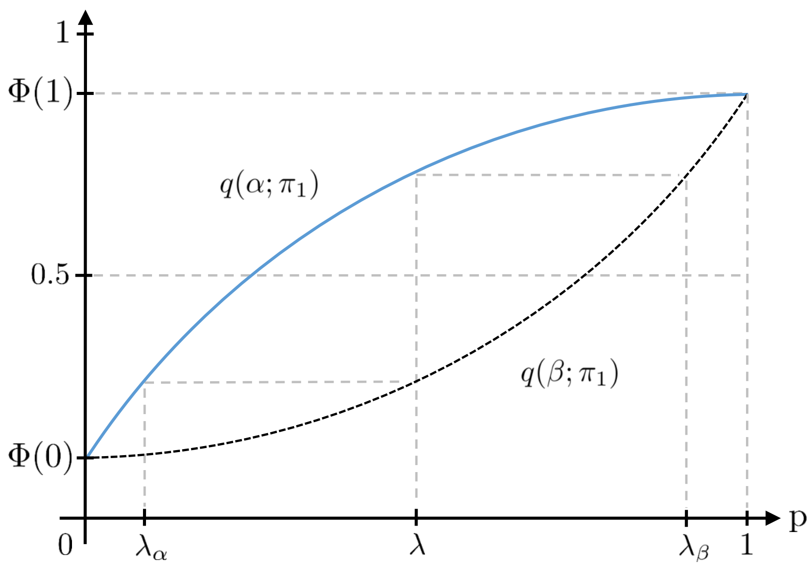

Figure 7: All pairs of induced priors (μ_α, μ_β) with $\mu_\alpha \in [\lambda_\alpha(\pi^c), \lambda(\pi^c)]^c$ and $\mu_\beta \in [\lambda(\pi^c), \lambda_\beta(\pi^c)]^c$ are implementable in a limit equilibrium through some independent expansion of π^c.

becomes arbitrarily precise, the lower boundary λ_α converges to zero and the upper boundary λ_β to 1. In the Appendix, we formally show that, indeed, when $\Pr(u|\alpha;\pi^c) \to 1$ and $\Pr(d|\beta;\pi^c) \to 1$, then

$$\lambda_\alpha \to 0, \tag{84}$$

$$\lambda_\beta \to 1. \tag{85}$$

6.3 Sketch of Proof: Full Persuasion

In the following, first, we make two observations about the induced priors. These two observations parallel previous ones from the analysis of monopolistic persuasion in Section 5. Then, we use these observations to provide intuition for Theorem 5 and sketch the reasoning of the equilibrium where A is the target policy in both states. There, a new subtlety arises, driven by the presence of more information, i.e. the exogenous information of the voters, relative to the monopolistic scenario. We cannot apply the fixed point arguments as before, but need an additional argument to construct equilibria.

We observe that, when the sender provides additional information $(\pi_n^{x,r,y})_{n\in\mathbb{N}}$ as illustrated in Figure 5), the induced prior after z, and thereby the margin of victory in the states α_2 and β_2 is pinned down uniquely by the exogenous information π^c of the voters.[23]

Claim 10 *Suppose that the additional information is given by $\pi_n^{x,r,y}$ for some $(x,r,y) \in (0,1)^3$ (see Figure 5) and consider the corresponding sequence $(\pi_n)_{n\in\mathbb{N}}$ of independent expansions of π^c. Then, for any equilibrium sequence $(\sigma_n^*)_{n\in}$ given $(\pi_n)_{n\in\mathbb{N}}$,*

$$\lim_{n\to\infty} \hat{\rho}_z(\sigma_n^*, \pi_n, n) = \lambda. \tag{86}$$

Sketch of Proof. The key insight why (86) holds is the following: given π_n, in the substates α_2 and β_2, a random voter receives the additional signal z with probability converging to 1 . Voters who received z know that either α_2 or β_2 holds and that almost all other voters got a signal z as well. Hence, from their perspective, it is close to common knowledge that the game is close

[23]This observation is analogous to Claim 6 and its implication (49).

to a game with a binary state and binary signals π^c, as in the original setting of the CJT. Then, using arguments similar to the proof of the Condorcet Jury Theorem, we show that for any equilibrium sequence $(\sigma_n^*)_{n\in\mathbb{N}}$, the election is equally close to being tied in expectation in α_2 and β_2 as $n \to \infty$ (compare with (26)), that is

$$\lim_{n\to\infty} q(\sigma_n^*; \alpha_2, \pi_n) - \frac{1}{2} = \lim_{n\to\infty} \frac{1}{2} - q(\sigma_n^*; \beta_2, \pi_n). \tag{87}$$

Since almost all voters receive z in α_2 and β_2, the expected vote share in any of these states converges to the vote share implied by the induced prior after z; for any $\omega_2 \in \{\alpha_2, \beta_2\}$,

$$\lim_{n\to\infty} q(\sigma_n^*; \omega_2, \pi_n) = \lim_{n\to\infty} q(\omega; \hat{\rho}_z(\sigma_n^*, \pi_n, n)). \tag{88}$$

Recall that λ is the unique induced prior such that the margins of victory are equal given the implied vote shares; see (81). So, (87) and (88) imply

$$\lim_{n\to\infty} \hat{\rho}_z(\sigma_n^*, \pi_n, n) = \lambda. \tag{89}$$

Let $M = q(\alpha, \lambda) - \frac{1}{2}$ be the margin of victory implied by λ. Now, (87) - (89) yield

$$\lim_{n\to\infty} q(\sigma_n^*; \alpha_2, \pi_n) - \frac{1}{2} = \lim_{n\to\infty} \frac{1}{2} - q(\sigma_n^*; \beta_2, \pi_n) = M. \tag{90}$$

Consider a belief vector $\mathbf{p}$ with $p_z = \lambda$ and $p_a > \lambda_\beta$. Then,

$$\hat{q}\left(\beta, p_a, \pi_1\right) - \frac{1}{2} > M \tag{91}$$

Similarly, if $p_b > \lambda$ then

$$\hat{q}\left(\beta, p_b, \pi_1\right) - \frac{1}{2} > M \tag{92}$$

Since $\hat{q}(\alpha, p) > \hat{q}(\alpha, p)$, it follows that the margins of victory implied by p_a and p_b are larger than the margin of victory implied by λ, i.e. $M = \hat{q}(\alpha, \lambda) - \frac{1}{2}$, in any state $\omega \in \{\alpha, \beta\}$.

Given Claim 10, for intuition, let us fix the belief after z to be λ. The following claim takes a sequence of belief vectors $(\mathbf{p}_n)_{n\in\mathbb{N}}$ with $p_{z,n} = \lambda$, $\lim_{n\to\infty} p_{a,n} > \lambda_\beta$ and, $\lim_{n\to\infty} p_{b,n} > \lambda_\beta$ and then characterizes the vector of induced priors of the corresponding strategy $\sigma^{\mathbf{p}_n}$, that is $\hat{\rho}(\sigma^{\mathbf{p}_n}; \pi_n, n)$.
To prove the claim, we use (91) and (92) and give arguments similar to the ones given for Claim 7 and Claim 6 and show that, as $n \to \infty$, the inference from being pivotal, given $\sigma^{\mathbf{p}_n}$, is that the state is in $\{\alpha_2, \beta_2\}$. If the voters believe that the state is in $\{\alpha_2, \beta_2\}$, then after receiving $s \in \{a, z, b\}$, they hold beliefs

$$\begin{aligned}
\Pr\left(\alpha | \left\{\alpha_2, \beta_2\right\}, a; \pi_n\right) &= \frac{xr}{xr + (1-x)(1-r)}, \\
\lim_{n\to\infty} \Pr\left(\alpha | \left\{\alpha_2, \beta_2\right\}, z; \pi_n\right) &= r, \\
\Pr\left(\alpha | \left\{\alpha_2, \beta_2\right\}, b; \pi_n\right) &= \frac{yr}{yr + (1-y)(1-r)}.
\end{aligned}$$

where, for simplicity, we assumed a uniform prior.[24]

Claim 11 *Let* $\Pr(\alpha) = \frac{1}{2}$. *Take any sequence of belief vectors* $(\mathbf{p}_n)_{n\in\mathbb{N}}$ *such that* $p_{z,n} = \lambda$ *for all* $n \in \mathbb{N}$, $\lim_{n\to\infty} p_{a,n} > \lambda_\beta$ *and* $\lim_{n\to\infty} p_{b,n} > \lambda_\beta$. *Then*

$$\lim_{n\to\infty} \hat{\rho}\left(\sigma_n^{\mathbf{p}_n}\right) = \lim_{n\to\infty} \left(\Pr\left(\alpha | \left\{\alpha_2, \beta_2\right\}, s\right)\right)_{s\in\{a,z,b\}},$$

Proof. The result parallels Claim 6 and the implications (52) and (53) of Claim 7.

Step 1 *For any* $\omega_1 \in \{\alpha_1, \beta_1\}$ *and* $\omega_2' \in \{\alpha_2, \beta_2\}$, *it holds* $\lim_{n\to\infty} \frac{\Pr(\omega_1|\text{piv};\sigma^{\mathbf{p}_n},\pi_n)}{\Pr\left(\omega_2'|\text{piv};\sigma^{\mathbf{p}_n},\pi_n\right)} = 0$.

First, if $\mathbf{p}_n$ is such that $p_{a,n} > \lambda_\beta$, then, since all voters receive a in α_1, it follows from (92) and (15) that, given $\sigma^{\mathbf{p}_n}$, we have $\left|q\left(\alpha_1; \sigma^{\mathbf{p}_n}, n\right) - \frac{1}{2}\right| >$

[24]The claim provides intuition, but is not used to prove the Theorem 5.

M. Given that almost all voter receive z in β_2 and α_2 and since $p_{z,n} = \lambda$, $\lim_{n\to\infty} q(\omega_2; \sigma^{\mathbf{P}_n}) = \hat{q}(\omega, \lambda; \pi^c)$ for $\omega_2 \in \{\alpha_2, \beta_2\}$. Recalling $M = q(\alpha, \lambda, \pi^c) - \frac{1}{2} = \frac{1}{2} - q(\beta, \lambda, \pi^c)$, Claim 2 implies that

$$\lim_{n\to\infty} \frac{\Pr(\beta_1|\text{piv})}{\Pr(\beta_2|\text{piv})} = \lim \frac{\Pr(\beta_1|\text{piv})}{\Pr(\alpha_2|\text{piv})} = 0. \tag{93}$$

Step 2 $\lim_{n\to\infty} \frac{\Pr(\alpha_2|\text{piv};\sigma^{\mathbf{P}_n},n)}{\Pr(\beta_2|\text{piv};\sigma^{\mathbf{P}_n},n)} = 1$

As $p_{z,n} = \lambda$ and since $q(\alpha, \lambda, \pi^c) - \frac{1}{2} = \frac{1}{2} - q(\beta, \lambda, \pi^c)$ and since almost all voters receive z in α_2 and β_2, the election is almost equally close to being tied in α_2 and β_2. Intuitively, this is why the voters cannot infer anything about the likelihood of α_2 and β_2 from the pivotal event. To show the claim of the second step formally, note that $q(\alpha, \lambda, \pi^c) - \frac{1}{2} = \frac{1}{2} - q(\beta, \lambda, \pi^c)$ implies

$$\left(\frac{q(\alpha; \lambda,\pi^c)(1 - q(\alpha; \lambda,\pi^c))}{q(\beta; \lambda,\pi^c)(1 - q(\beta; \lambda,\pi^c))}\right)^n = 1 \tag{94}$$

for all n. We claim that

$$\lim_{n\to\infty} \left(\frac{q(\omega; p_z,\pi^c)(1 - q(\omega; p_z,\pi^c))}{q(\omega_2; \sigma_n^{\mathbf{P}_n},n)(1 - q(\omega_2; \sigma_n^{\mathbf{P}_n},n))}\right)^n = 1 \tag{95}$$

The claim follows from the argument as in the proof of Claim 3 since $\frac{q(\omega;p_z,n)}{q\left(\omega_2;\sigma_n^{\mathbf{P}_n},n\right)}$ converges to 1 sufficiently fast given that each voter receives a signal z in ω_2 with probability converging to 1 sufficiently quickly. More precisely, the likelihood of not receiving z is of order $\frac{1}{n^2}$, which implies that $\left(\frac{q(\omega;p_z,\pi^c)(1-q(\omega;p_z,\pi^c))}{q\left(\omega_2;\sigma_n^{\mathbf{P}_n},n\right)\left(1-q\left(\omega_2;\sigma_n^{\mathbf{P}_n},n\right)\right)}\right) \in [1 - \frac{K}{n^2}, 1 + \frac{K}{n^2}]$ for some constant $K \neq 0$; compare to (131). The equality follows since $\lim_{n\to\infty}[1 \pm \frac{K}{n^2}]^n = 1$. The claim finally follows from (94) and (95) and (7).

Step 1 and 2 together imply that the voter's inference from the pivotal event is that the state is in $\{\alpha_2, \beta_2\}$; hence, for all s,

$$\lim_{n\to\infty} \Pr(\alpha|\text{piv}, s; \sigma_n^{\mathbf{P}_n},n) = \Pr(\alpha|\{\alpha_2, \beta_2\}, s). \tag{96}$$

This finishes the proof of Claim 11. ■

Fixed Point. To construct an equilibrium with a majority supporting A in α_1 and β_1, we let $x = y = 1$ and $r = \lambda$ and find an equilibrium belief vector close to $(1, \lambda, 1)$. Let $\delta > 0$ and

$$\mathrm{B}_\delta = \left\{ \mathbf{p} \in [0,1]^3 \mid |\mathbf{p} - (1, \lambda, 1)| \leq \delta \right\},$$

so that B_δ is the set of beliefs at most δ away from $(1, \lambda, 1)$. Take any $\mathbf{p} \in \mathrm{B}_\delta$ and the corresponding strategy $\sigma^{\mathbf{p}}$. For δ small enough, $p_a > \lambda_\beta$ and $p_b > \lambda_\beta$ for all $p = (p_a, p_z, p_z) \in B_\delta$. By Claim 10 and Claim 11, if we start with beliefs $\mathbf{p} \in B_\delta$ such that $p_z = \lambda$ holds exactly, $\hat{\rho}(\sigma^{\mathbf{p}})$ response maps back into the neighborhood, i.e. $\hat{\rho}(\sigma^{\mathbf{p}}) \in B_\delta$. However, for beliefs $p_z \neq \lambda$ that are δ-close to λ, this is not the case for n large enough. This is because, in that case, the margins of victory in α_2 and β_2 are strictly different, i.e. $\lim_{n\to\infty} |q(\alpha_2; \sigma^{\mathbf{p}}) - \frac{1}{2}| \neq \lim_{n\to\infty} |\frac{1}{2} - q(\beta_2; \sigma^{\mathbf{p}})|$. Therefore, being pivotal contains information about the relative likelihood of α_2 and β_2, in contrast to the previous case of a monopolistic sender (see Claim 3). To deal with this problem, we consider a constrained version of $\hat{\rho}(\sigma^{\mathbf{p}})$, namely, the projection of the induced prior mapping $\hat{\rho}(\sigma^{\mathbf{p}})$ onto B_δ. We then show that, when n is large enough, for any fixed point p_n^* of the projection the constraints do not bind, i.e. $\hat{\rho}(\sigma^{\mathbf{p}_n^*}) \in B_\delta$; establishing that a fixed point $\mathbf{p}_n^*$ is an equilibrium belief. The corresponding strategy $\sigma^{\mathbf{p}_n^*}$ is an equilibrium where a majority supports A in α_1 and β_1 since $p_s \approx 1$ for $s \in \{a, b\}$ and $\Phi(1) > \frac{1}{2}$.

6.4 Robustness of Theorem 5

Detail-Freeness. Can the sender persuade the voters even when he does not know the exact details of the game? We argue that Proposition 1 extends in a more general form to the situation when the voters hold exogenous private signals: to be able to persuade the voters, it is sufficient that the sender knows that Φ is monotone, i.e. (21) holds, and that Φ satisfies the richness assumption (17). We claim that he can release information to the voters such that,

uniformly, for any any prior $\Pr(\alpha) \in (0,1)$, any exogenous information π^c of the voters (see Section 6 for the definition of π^c) and any aggregate preference function Φ with (17) and (21), his target policy is implemented.
Consider e.g. the constant policy A. We have already seen in Lemma 2 that, given (17) and (21), the sender can always implement very extreme beliefs $p \approx 1$, meaning that with probability close to 1, all the voters are almost certain that the state is α. When all voters hold an extremely high belief $p \approx 1$, then A is elected as $n \to \infty$ since $\Phi(1) > \frac{1}{2}$. The proof of Lemma 1 in the Appendix shows, that, to implement beliefs such high beliefs, the sender can uniformly choose the same sequence of information structures $(\pi_n)_{n\in\mathbb{N}} = (\pi_n^{x,r,y})_{n\in\mathbb{N}}$ with $x = y = 1$; see in particular Claim 13 and the paragraph thereafter. This is analogous to the situation when the sender is the monopolistic information provider; see Section 5.5.1.

The results from Section 5.5.2 about the basin of attraction of the manipulated equilibria do not extend to the situation when the sender is not a monopolistic information provider.[25]

7 Partially Informed Sender

We now consider a sender who does not know the state $\omega \in \{\alpha, \beta\}$. Instead, the sender receives a private signal m. Conditional on the private signal m, the sender can release signals to the voters. In the following, we consider information structures π of the voters that are consistent with the sender's private information π_0 and call them *coarsenings* of π_0.

The sender's signal is finite, $m \in \{m_1, \dots, m_k\}$ and boundedly informative

[25]Instead, one can show the following: let the sender release the information $(\pi_n^{x,y})_{n\in\mathbb{N}}$ to the voters as in the proof of Theorem 5. Then, when the electorate is large enough, for almost any initial strategy, under the iterated best response, the voter behaviour after signal z jumps back and forth infinitely from voting approximately according to $\sigma^{\mathbf{p}}$ with $\mathbf{p} = \Pr(\alpha|s)_{s\in\{a,z,b\}}$ to voting approximately as if one of the states is known to be the true state. We omit the proof.

about the state, i.e. $0 < \Pr(m_i|\alpha) < 1$ and $0 < \Pr(m_i|\beta) < 1$ for all $i = 1, \ldots, k$. After seeing his private signal, the sender randomizes how he releases information to the voters. More precisely, for any m, there exist probabilities $(\Pr(j|m;\pi))_{j\in\{1,\ldots,N_m\},m\in\{m_1,\ldots,m_k\}}$ and these probabilities define a distribution over substates $\omega_{m,j}$,

$$\Pr(\omega_{m,j}|\omega;\pi) = \Pr(m|\omega;\pi_0)\Pr(j|m;\pi). \tag{97}$$

The sender uses an information structure (as defined in Section 2) such that the substates of each state $\omega \in \{\alpha,\beta\}$ can be partitioned into sets $P_m(\omega) = \{\omega_{m,1}, \omega_{m,2}, \ldots, \omega_{m,N_m}\}$ for $m = m_1, \ldots, m_k$ and the probabilities of the substates are given by (97). We call such an information structure a coarsening of π_0.

7.1 Monopolistic Persuasion

First, we consider the situation when all of the information of the voters comes from a partially informed sender and the sender receives a binary signal $m \in \{h, \ell\}$.[26] The sender is free to release his information to the voters: to do so, he chooses a coarsening π of π_0. We consider preferences of the voters such that Φ is strictly increasing and satisfies (17). Recall that this implies that is exists a unique belief $\hat{r} \in [0,1]$ for which the electorate is split between A and B, i.e. $\Phi(\hat{r}) = \frac{1}{2}$.
Clearly, the voters cannot learn more about the state than the sender since all their information comes from the sender. The private signal $m \in \{h, \ell\}$ of the sender is what the voters learn about and, effectively, h and ℓ take the role of the binary state from before.

Whenever

$$\frac{\Pr(\alpha)}{\Pr(\beta)}\frac{\Pr(\ell|\alpha;\pi_0)}{\Pr(\ell|\beta;\pi_0)} < \frac{\hat{r}}{1-\hat{r}} < \frac{\Pr(\alpha)}{\Pr(\beta)}\frac{\Pr(h|\alpha)}{\Pr(h|\beta)}, \tag{98}$$

[26] The restriction to a binary signal is made for the ease of exposition.

there exists a belief $p = \Pr(h)$ about the sender's binary signal such that a majority of the voters prefers A given p; similarly there exists p' such that a majority of the voters prefers B given p'. So, (98) is the analogue of the richness condition (17).

The sender might have very little information such that even when he releases all his information to the voters, the alternative favored by the majority under the prior is elected. When $\frac{\Pr(\alpha)}{\Pr(\beta)}\frac{\Pr(h|\alpha)}{\Pr(h|\beta)} < \frac{\hat{r}}{1-\hat{r}}$, this is in fact the case: to see why, note that all voters necessarily hold beliefs smaller than $\hat{r}$,[27] so, the alternative B is elected as $n \to \infty$. Similarly, alternative A is elected, as $n \to \infty$, when $\frac{\Pr(\alpha)}{\Pr(\beta)}\frac{\Pr(\ell|\alpha)}{\Pr(\ell|\beta)} > \frac{\hat{r}}{1-\hat{r}}$.

As a corollary of Theorem 2, we show that a monopolistic information provider is able to persuade the voters to elect any policy contingent on his private signal when he possesses sufficiently precise information about the state, i.e. (98) holds.

Theorem 6 *Let Φ be strictly increasing and satisfy (17). Take any binary signal π_0 of the sender. If (98) holds, then, for every signal-dependent policy $(x(h), x(\ell)) \in \{A, B\}^2$ there is a sequence of coarsenings $(\pi_n)_{n\in\mathbb{N}}$ of π_0 and a sequence of equilibria $(\sigma_n^*)_{n\in\mathbb{N}}$ given $(\pi_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty} \Pr(x(h))\,|h; \sigma_n^*, \pi_n, n) = 1, \tag{99}$$

$$\lim_{n\to\infty} \Pr(x(\ell)\,|\ell; \sigma_n^*, \pi_n, n) = 1. \tag{100}$$

To prove the result, we recast the model with the partially informed sender into a model of a sender who is perfectly informed about the states $m \in \{h, \ell\}$. Then, the result follows from Theorem 2. A proof is in the Appendix in Section D.

[27] This holds regardless of the coarsening π and regardless of the signal received by the voter.

7.2 Persuasion of Privately Informed Voters

In this section, we analyse the situation when the voters hold exogenous information from a binary information structure π^c (satisfying (20) as in the setting of the CJT) and the partially informed sender can release additional information to the voters. Formally, we consider the setting of Section 6, but we restrict the analysis to the Bayesian games of voters that are induced by independent expansions π of π_1 that are coarsenings of π_0, reflecting the partial informedness of the sender; see Section 7 for the definition of a coarsening.

The next result provides a weak condition on the informativeness of the sender's private signal m that guarantees that he can release information to the voters such that the voters elect any arbitrary policy contingent on his signal. The condition is that there exist two signals $h, \ell \in \{m_1, \ldots, m_k\}$ of the sender such that

$$\frac{\lambda}{1-\lambda}\frac{\Pr(h|\alpha)}{\Pr(h|\beta)} > \frac{\lambda_\beta}{1-\lambda_\beta} \tag{101}$$

as well as

$$\frac{\lambda}{1-\lambda}\frac{\Pr \ell|\alpha)}{\Pr(\ell|\beta)} < \frac{\lambda_\alpha}{1-\lambda_\alpha}. \tag{102}$$

We show momentarily that these conditions, for example, are usually fulfilled when the sender holds more information than two random voters together; see Remark 1.

Theorem 7 *Let Φ be strictly increasing and satisfy (17). If the information structure π_0 of the sender satisfies (101) and (102), then, for every signal-dependent policy $(x(m_i))_{i=1,\ldots,k} \in \{A, B\}^k$, there exists a sequence of independent expansions $(\pi_n)_{n\in\mathbb{N}}$ of π^c that are coarsenings of π_0 and a sequence of equilibria $(\sigma_n^*)_{n\in\mathbb{N}}$ given $(\pi_n)_{n\in\mathbb{N}}$ such that for all $i = 1, \ldots, k$,*

$$\lim_{n\to\infty} \Pr\left(x(m_i)|m_i; \sigma_n^*, \pi_n, n\right) = 1. \tag{103}$$

The proof of the theorem follow the same ideas as the proof of Theorem 5 for the case of an informed sender who perfectly knows the state $\omega \in \{\alpha, \beta\}$.[28] We delegate the proof to the Appendix Section D.

Intuition. Recall the definitions (82) and (83) of λ_α and λ_β which imply: when voters hold a common belief $p > \lambda_\beta$ and vote accordingly, then in both states, a majority of citizens votes A and it follows from the weak law of large numbers that A is elected. Conversely, when voters hold a common belief $p < \lambda_\alpha$ and vote accordingly, then in both states, a majority of citizens votes B and it follows from the weak law of large numbers that B is elected.
So, to implement any target policy $x(m) \in \{A, B\}$ after receiving signal m , it is sufficient to implement the respective belief $p > \lambda_\beta$ or $p < \lambda_\alpha$. Recall that a sender with perfect information can implement any pair of beliefs (μ_α, μ_β) with $\mu_\alpha \in [\lambda_\alpha, \lambda]^c$ and $\mu_\beta \in [\lambda, \lambda_\beta]^c$ (see Lemma 2). Intuitively, given this result, when the sender has sufficiently good information about the state, he is able to implement sufficiently extreme beliefs. In the Appendix Section D we prove Theorem 7 and show: when the information structure π^0 of the sender satisfies the weak conditions (101) and (102), then he he is able to implement a belief $p > \lambda_\beta$ or a belief $p < \lambda_\alpha$ after any m.

The Information Structure with Binary Signals. In the Appendix, first, we show the result of the theorem for the case when the signals of the sender are binary, i.e. $m \in \{h, \ell\}$, and then explain how the result generalizes. The information structures in the binary signal case parallel the information structures used by the perfectly informed sender in Section 5.4: the sender releases information to the voters given by the sequence of coarsenings $(\pi_{2,n}^{x,y})_{n\in\mathbb{N}}$ in Figure 8.

Remark 1 *Let the sender's signal be Blackwell more informative than the signals of two random voters, i.e.* π_0 *is Blackwell more informative than the independent expansion* $\pi^c \times \pi^c$. *Then, there are signals* $h, \ell \in \{m_1, \ldots, m_k\}$

[28] We conjecture that the sender cannot persuade voters in the sense of Theorem 7 when the signal m of the sender is sufficiently uninformative, i.e. $\frac{\Pr(m|\alpha)}{\Pr(m|\beta)}$ is sufficiently close to 1 for all realizations $m \in \{m_1, \ldots, m_k\}$. We were not able to prove this conjecture.

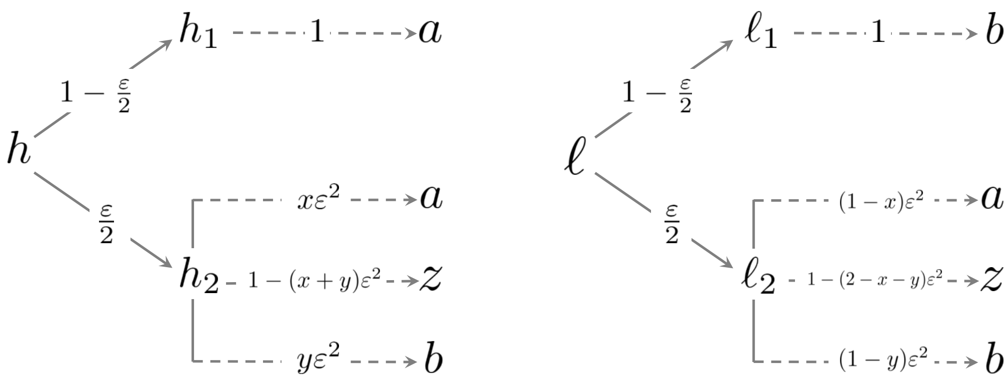

Figure 8: The coarsenings $\pi_{2,n}^{x,y}$ with $\varepsilon = \frac{1}{n}$, and $(x, y) \in \{0, 1\}^2$.

with

$$\frac{\Pr(h|\alpha;\pi_0)}{\Pr(h|\beta;\pi_0)} \geq \left[\frac{\Pr(u|\alpha;\pi^c)}{\Pr(u|\beta;\pi^c)}\right]^2, \tag{104}$$

$$\frac{\Pr(\ell|\alpha;\pi_0)}{\Pr(\ell|\beta;\pi_0)} \geq \left[\frac{\Pr(d|\alpha;\pi^c)}{\Pr(d|\beta;\pi^c)}\right]^2. \tag{105}$$

Recall the definitions of λ, λ_α and λ_β through (81), (82) and (83). So, (101) means that the vote share implied by the posterior p with $\frac{p}{1-p} = \frac{\Pr(h|\alpha;\pi_0)}{\Pr(h|\beta;\pi_0)}\frac{\lambda}{1-\lambda}$ in β is larger than the vote share implied by the belief λ in α. We argue that (101) is implied by (104) when the signals of the voters are symmetric, i.e. $\Pr(u|\alpha;\pi^c) = \Pr(d|\beta;\pi^c)$ *holds. Then, (104) implies that* $\frac{\Pr(h|\alpha;\pi_0)}{\Pr(h|\beta;\pi_0)}\frac{\Pr(d|\alpha;\pi^c)}{\Pr(d|\beta;\pi^c)} \geq \frac{\Pr(u|\alpha;\pi^c)}{\Pr(u|\beta;\pi^c)}$. *So,* $\frac{p}{1-p}\frac{\Pr(d|\alpha;\pi^c)}{\Pr(d|\beta;\pi^c)} > \frac{\lambda}{1-\lambda}\frac{\Pr(u|\alpha;\pi^c)}{\Pr(u|\beta;\pi^c)}$. *Now, this implies (101), given (80). Similarly, (102) means that the vote share implied in α by the posterior p' with $\frac{p'}{1-p'} = \frac{\Pr(\ell|\alpha;\pi_0)}{\Pr(\ell|\beta;\pi_0)}\frac{\lambda}{1-\lambda}$ is smaller than the vote share implied in β by the belief λ. We argue that (102) is implied by (105) when the signals of the voters are symmetric. Then, (105) implies that* $\frac{\Pr(\ell|\alpha;\pi_0)}{\Pr(\ell|\beta;\pi_0)}\frac{\Pr(u|\alpha;\pi^c)}{\Pr(u|\beta;\pi^c)} \leq \frac{\Pr(d|\alpha;\pi^c)}{\Pr(d|\beta;\pi^c)}$. *So,* $\frac{p'}{1-p'}\frac{\Pr(u|\alpha;\pi^c)}{\Pr(u|\beta;\pi^c)} < \frac{\lambda}{1-\lambda}\frac{\Pr(d|\alpha;\pi^c)}{\Pr(d|\beta;\pi^c)}$. *Now, this implies (102), given (80).*

8 Remarks and Extensions

8.1 Bayes Correlated Equilibria

The Bayes correlated equilibria given some exogenous information structure π^c are the Bayes-Nash equilibria that arise from expansions π of π^c (see Bergemann & Morris (2016) for the definition of an expansion and the characterization of Bayes correlated equilibria). In terms of Bayes correlated equilibria, Theorem 5 means that for any state-dependent policy $(x(\alpha), x(\beta)) \in \{A, B\}^2$, there exists a sequence of Bayes correlated equilibria given π^c that implements the policy as $n \to \infty$.

8.2 Non-Implementability of Intermediate Beliefs

Theorem 2 shows that the sender can implement any pair of beliefs μ_α, μ_β that do not come from intermediate intervals $[\lambda_\alpha, \lambda]$ and $[\lambda, \lambda_\beta]$ respectively, which depend on the exogenous information of the voters. In this section, we study whether all pairs of beliefs can be implemented, including the intermediate ones. Recall that this is essentially the case when the sender is the monopolistic information provider (see Theorem 1). However, we show that this is not the case when the voters hold exogenous information. Specifically, we show that when the aggregate preference function Φ is linear, then no belief from the intermediate intervals of Theorem 2 can be implemented (8).

Thus, the presence of exogenous private information implies a true constraint on the implementable beliefs: whatever signals from the broad class of exchangeable information structures with finite substates the sender releases to the voters and whatever equilibrium sequence given these signals we look at, no equilibrium sequence implements intermediate beliefs.

Theorem 8 *Let any exogenous information structure π^c be given that satisfies (20) and (21) (as in the CJT), and Φ be linear. Any pair of beliefs $(\mu_\alpha, \mu_\beta) \in (0,1)^2$ with $\mu_\alpha \in (\lambda_\alpha, \lambda)$ or $\mu_\beta \in (\lambda, \lambda_\beta)$ is not implementable.*

Proof. In the Online Supplement. ■

8.3 Known Preferences: Targeted Persuasion

When the types of the voters are known to a potential sender, voters can be 'targeted' with recommendations; formally, a revelation principle applies saying that any equilibrium is equivalent to a recommendation policy that will be followed by the voters.[29] Below, we show that when the preference types are known, there is a simple way how the sender can persuade the voters to elect a constant policy via private recommendations.[30] We also show that, with known preferences, the possibility of persuasion is unaffected by the presence of a private signal of the voters.

Targeted Persuasion. Suppose that the voters' preference types $t^i = (t^i_\alpha, t^i_\beta)$ are commonly known, and $t^i \neq 0$ for any $i \in \{1, \ldots, 2n+1\}$. The voters receive exogeneous private signals as in the setting of the CJT (Section 4.2) (the following result extends when these exogeneous signals are uninformative). Suppose that the voters $1, \ldots, m$ weakly prefer A in α and B in β, that is $t^i_\alpha \geq 0$ and $t^i_\beta \leq 0$ and without loss let $m > n$. The remaining voters $m+1, \ldots, 2n+1$ weakly prefer B in α and A in β, that is $t^i_\alpha \leq 0$ and $t^i_\beta \geq 0$. The following recommendation policy implements the outcome A with probabability of at least $1-\epsilon$ in an equilibrium, for some arbitrarily small $\epsilon > 0$: in both states, with probability $1-\epsilon$, all voters receive the recommendation 'vote A' (signal a). In state α, with the remaining probability ϵ, a random subset of size $n+1$ of the voters $1, \ldots, m$ receives the recommendation 'vote A' and the remaining n voters receive the recommendation 'vote B' (signal b). In state β, with the remaaining probability $\epsilon > 0$, a random subset of size $n+1$ of the voters $1, \ldots, m$ receives b and the remaining n voters receive a. Voting A after an a-signal and B after a b-signal constitutes an equilibrium: given this strategy, denoted by σ, voters $i \in \{1, \ldots, m\}$ with an a-signal are only pivotal in α, and voters $i \in \{1, \ldots, m\}$ with a b-signal are only pivotal in β, that is

[29] For a formal proof of the general revelation principle see e.g. Bergemann & Morris (2017).

[30] This has been observed by Chan *et al.* (2016) and in Bardhi & Guo (2016a) in similar settings. Therefore, the main part of these papers considers a setting with voting costs ('expressive voting') and unanimity, respectively.

$\Pr(\alpha|\text{piv}, a, i \leq m; \sigma) = 1$ and $\Pr(\alpha|\text{piv}, b, i \leq m; \sigma) = 0$. Hence, voting A after a and B after b is a strict best reponse for any voter $i \in \{1, \dots, m\}$. Voters $i \in \{m+1, \dots, 2n+1\}$ are never pivotal if the other voters follow the recommendations. Hence, following the recommendation is a best response also for them and therefore σ is an equilibrium. Since with probability $1-\epsilon$ all citizens vote A, given σ, the recommendation policy implements the outcome A with probability of at least $1-\epsilon$.

9 Literature

We contribute to several strands of the literature: we contribute to the literature on information design in general (see Bergemann & Morris (2017) for a survey) and especially on persuasion with multiple receivers (e.g., Mathevet *et al.* (2017)), e.g. voter persuasion (Section 9.2). The paper also contributes to the literature on information aggregation in elections (Section 9.1). Section 9.3 discusses relations to further literature; in particular, to the literature on Bayes correlated equilibria in auctions and to the literature on information transmission between informed experts and an uninformed decision maker.

9.1 Information Aggregation Literature

The literature has identified several circumstances in which information may fail to aggregate. We discuss the studies that are most closely related: Feddersen & Pesendorfer (1997) (FP, Section 6) show that an invertibility problem causes a failure when there is aggregate uncertainty with respect to the preference distribution conditional on the state. A specific case of the model in this paper has been studied in Bhattacharya (2013) (BH) who shows that failure can happen when preference monotonicity is violated. However, when the preferences are monotone, i.e. when (21) holds, information is aggregated perfectly; we show that in the setup with monotone preferences, a sender can implement any state-contigent outcome (Theorem 5, Theorem 7) and thereby create a failure of information aggregation simply by providing additional information about the state to the voters.

In a pure common-values setting, Mandler [2012] (MA) shows that failure can happen when there is aggregate signal uncertainty conditional on the state. The paper does not discuss persuasion, but the results can be understood in terms of it. Similar to (MA), in our model, the sender uses signals with aggregate signal uncertainty to persuade the voters and thereby creates a failure of information aggregation. We discuss in Section 8.3 persuasion via private signals when the preference types are known since then the sender can rely on techniques of targeted persuasion.[31] Further related models of elections that perform poorly in aggregating information are Razin (2003), Acharya (2016), Ekmekci & Lauermann (2016), Ali *et al.* (2018).

9.2 Voter Persuasion Literature

Previous papers on voter persuasion have studied persuasion through public signals (Alonso & Câmara (2015)), persuasion with conditionally independent private signals (Wang (2013)) and targeted persuasion with private signals (Bardhi & Guo (2016a), Chan *et al.* (2016)) in situations when the preferences of the voters are commonly known. In contrast to the existing literature, we revisit the general voting setting of Feddersen & Pesendorfer (1997) with private preferences: we discussed how in this setup, as a consequence of the Condorcet Jury Theorem, there is no scope for persuasion with public signals and no scope for persuasion with conditionally independent private signals; see Theorem 1' and the discussion in the introduction of Section 6. We discussed persuasion with private signals if preferences of the voters would be known, see Section 8.3.

More generally, most of the Bayesian persuasion literature assumes that the sender has much knowledge about the the environment; for example, typically, perfect knowledge about the state and the receiver's types is assumed. In this paper, the informational requirements for persuasion are considerably weak; we allow for private preferences and exogeneous private signals of the receivers;

[31] This has been observed by Chan *et al.* (2016) and in Bardhi & Guo (2016a) in similar settings.

we also consider the case when the sender has incomplete information about the state (see Section 7) and the case when the sender has incomplete or misspecified information about the prior probabilities of the state, the distribution of the private preference types of the voters or the distribution of the private signals of the voters (see Section 5.5.1 and Section 6.4). A notable exception in the literature are Bobkova & Fuchs (2018) who study persuasion of voters when voting is by unanimity and voters hold private information.[32] Further are Guo & Shmaya (2017) who study persuasion of a receiver with exogenous private signals and Kolotilin *et al.* (2015) who study persuasion of a privately informed receiver and show that optimal information structures do not need to screen types. Correspondingly, we showed that for large electorates with private preferences, the set of equilibrium outcomes that can be obtained by information design is the same with and without the option to screen types. This results holds when the information designer is monopolistic (Theorem 2) as well as when voters have private signals from an exogeneous source, see (Theorem 5, Theorem 7).

Several other paperps study how groups can be influenced through strategic information transmission, but are less closely related: Kerman *et al.* (2019) study targeted persuasion via private signals when the sender is restricted to use signals that induce the voters to sincerely in some equilibrium; compare to the discussion of targeted persuasion in Section 8.3. Levy *et al.* (2018) study persuasion of voters with correlation neglect. Schipper & Woo (2012) is an early paper and studies persuasion of unaware voters. Schnakenberg (2015) studies a cheap talk setting in which an expert tries to manipulate a voting body. Salcedo (2019) studies persuasion of subgroups of receivers via private messages in a setting where each receiver's payoff only depends on his own action and the state. Bardhi & Guo (2016b) study sequential persuasion of a group of receivers.

[32]This is complementary to our paper. The results presented here are for the simple majority rule. However, they generalize to any any majority rule except unanimity: to see why, recall that we did not impose restrictions on the likelihood of the partisans and note that a shift in the majority threshold is strategically equivalent to a shift in the likelihood of the partisans.

9.3 Further Literature

The paper is related to work on information design in general (see Bergemann & Morris (2017) for a survey) and especially to persuasion with multiple receivers (e.g., Mathevet *et al.* (2017)). The paper naturally relates to the literature on information design in auctions since the analysis of auctions and elections is largely connected.[33] Bergemann *et al.* (2016) and Du (2017) studied Bayes correlated equilibria of common value auctions, and in particular calculated the minimum revenue across all models of information and all Bayesian equilibria for the mechanisms that maximize minimum revenue. Yamashita *et al.* (2016) studies such optimal mechanisms in an auction setting where each bidder may have additional information about the other bidders' valuations, e.g. through information acquisition. In comparison to the auctions literature, we fixed the voting rule and characterized the Bayes correlated equilibrium outcomes both when the information designer is monopolistic and also when a minimum level of private information of the voters is imposed where this minimum level of information can be arbitrarily precise. By correlating the signals of voters, the information designer can implement any state-contingent outcome (see Theorem 5 and Theorem 7).

Gerardi *et al.* (2009) study aggregation of expert information by an uninformed decision maker. By giving each expert a small chance of being a dictator, information can be extracted at a small loss when either the correlation of the experts' information or the number of experts is high, while implementing an adversarial outcome with a high probability otherwise. Relatedly, Feng & Wu (2019) show that, even when there is little or no correlation between the experts' information, information extraction is possible and they provide conditions when this is the case.

[33] For example, the analogue of the swing voter's curse is the winner's curse in auctions.

Appendices

A Large Elections and Monopolistic Persuasion

Proof of Claim 3

Suppose w.l.o.g. that $q(\alpha_2\,; \sigma_n)(1 - q(\alpha_2\,; \sigma_n)) < q(\beta_2; \sigma_n)(1 - q(\beta_2; \sigma_n))$ for all n. It follows directly from (7) that

$$\lim_{n\to\infty} \frac{\Pr(\text{piv}|\alpha_2; \sigma_n, \pi_n)}{\Pr(\text{piv}|\beta_2; \sigma_n, \pi_n)} \leq 1. \tag{106}$$

We now show that the reverse inequality also holds and thereby finish the proof of the lemma. For this, note the following: first, it follows from (15) that the expected vote share for A in α_2 differs from the expected vote share for A in β_2 maximally by the probability that b is observed in α_2, that is by $\varepsilon^2 = \frac{1}{n^2}$,

$$|q(\alpha_2\,; \sigma_n) - q(\beta_2\,; \sigma_n)| \leq \epsilon^2, \tag{107}$$

for all n. Second, recall that $\Phi(0) < q(\omega_j; \sigma) < \Phi(1)$ for any strategy and any substate ω_j and note that the derivative of $q(1-q)$ is bounded by some $L > 0$ on the compact interval $[\Phi(0), \Phi(1)]$. These observations taken together imply that

$$h(q(\beta_2\,; \sigma_n))\Big|\frac{h(q(\alpha_2\,; \sigma_n))}{h(q(\beta_2\,; \sigma_n)} - 1\Big| = |h(q(\alpha_2\,; \sigma_n)) - h(q(\beta_2\,; \sigma_n))| \leq L\epsilon^2. \tag{108}$$

for $h(q) = q(1-q)$ and all n. Since $0 < \Phi(0) < q(\alpha_2\,; \sigma_n) < \Phi(1)$ and h is inverse U-shaped with maximum at $\frac{1}{2}$, this bound implies

$$\frac{h(q(\alpha_2\,; \sigma_n))}{h(q(\beta_2\,; \sigma_n)} \geq 1 - \frac{L}{h(q(\beta_2\,; \sigma_n))n^2} \geq 1 - \frac{L}{Mn^2} \tag{109}$$

for $M = \min\left(h(\Phi(0)), h(\Phi(1))\right)$ and all n. It follows from (7) that $\frac{\Pr(\text{piv}|\alpha_2;\sigma_n,\pi_n)}{\Pr(\text{piv}|\beta_2;\sigma_n,\pi_n)} \geq (1-\frac{L}{Mn^2})^n$. However, $\lim_{n\to\infty}(1-\frac{L}{Mn^2})^n = 1$ such that

$$\lim_{n\to\infty} \frac{\Pr(\text{piv}|\alpha_2;\sigma_n,\pi_n)}{\Pr(\text{piv}|\beta_2;\sigma_n,\pi_n)} \geq 1. \tag{110}$$

(To see in more detail why $\lim_{n\to\infty}(1-\frac{L}{Mn^2})^n = 1$, note that $\lim_{n\to\infty}(1-\frac{L}{Mn^2})^{2n} = (1-\frac{\sqrt{L}}{\sqrt{M}n})^{2n}(1+\frac{\sqrt{L}}{\sqrt{M}n})^{2n} = \lim_{n\to\infty} e^{2\sqrt{\frac{L}{M}}}e^{-2\sqrt{\frac{L}{M}}} = e^0 = 1$ where we used the limit description $\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$ for the exponential function.)

Proof of Claim 7

Let $\pi_n = \pi_n^{x,y}$. Let for example $\omega_1' = \alpha_1$ and $\omega_2 = \beta_2$ and $s = a$. Then,

$$\begin{aligned}
& \frac{\Pr(\beta_2|s,\text{piv};\sigma_n,\pi_n)}{\Pr(\alpha_1|s,\text{piv};\sigma_n,\pi_n)} \\
= & \frac{\Pr(\beta)\Pr(\beta_2|\omega)\Pr(s|\beta_2;\pi_n)\Pr(\text{piv}|\beta_2;\sigma_n,\pi_n)}{\Pr(\alpha)\Pr(\alpha_1|\omega')\Pr(s|\alpha_1;\pi_n)\Pr(\text{piv}|\alpha_1;\sigma_n,\pi_n)} \\
= & \frac{\Pr(\beta)}{\Pr(\alpha)}\frac{(1-r)\frac{1}{n}}{(1-r\frac{1}{n})}\frac{(1-x)\frac{1}{n^2}}{1}\frac{\Pr(\text{piv}|\beta_2;\sigma_n,\pi_n)}{\Pr(\text{piv}|\alpha_1;\sigma_n,\pi_n)} \\
\approx & \frac{\Pr(\beta)}{\Pr(\alpha)}\frac{\Pr(\text{piv}|\beta_2;\sigma_n,\pi_n)}{\Pr(\text{piv}|\alpha_1;\sigma_n,\pi_n)}(1-r)(1-x)n^{-3} \\
\to & \infty.
\end{aligned} \tag{111}$$

where the convergence on the last line follows from applying Claim 2 for $d = 3$, given (50). In the same way, we obtain more generally for any $s \in \{a, b\}$,

$$\forall \omega_1' \in \{\alpha_1, \beta_1\}, \omega_2 \in \{\alpha_2, \beta_2\} : \frac{\Pr(\omega_2|s,\text{piv};\sigma_n,\pi_n)}{\Pr(\omega_1'|s,\text{piv};\sigma_n,\pi_n)} \;\to\; \infty. \tag{112}$$

Equation (51) follows from (112).

B Other Equilibria

Proof of Theorem 4

Lemma 3 *Consider any sequence of strategies $(\sigma_n)_{n\in\mathbb{N}}$ and any sequence of information structures $(\pi_n)_{n\in\mathbb{N}}$ with a common set of substates. Then, for any substate $\omega_i \in \{\alpha_1, \ldots, \alpha_{N_\alpha}\} \cup \{\beta_1, \ldots, \beta_{N_\beta}\}$ let*

$$c = \lim_{n\to\infty} \left(q(\omega_i, \sigma_n) - \frac{1}{2}\right) \cdot \left[\frac{2n+1}{q(\omega_i, \sigma_n)(1 - q(\omega_i, \sigma_n))}\right]^{\frac{1}{2}}$$

be the limit of the expected difference of the vote share to $\frac{1}{2}$ measured in standard deviations of the expected vote share.[34] *The probability that A gets elected in ω_i converges to*

$$\lim_{n\to\infty} \Pr(A|\omega_i; \sigma_n) = \Phi(c),$$

where $\Phi(\cdot)$ is the cumulative distribution of the standard normal distribution.

Proof. Let $q_n = q(\omega_i, \sigma_n)$. By using the normal approximation[35]

$$\mathcal{B}(2n+1, q_n) \simeq \mathcal{N}((2n+1)q_n, (2n+1)q_n(1-q_n)),$$

we see that the probability that A wins the election in ω converges to

$$\Phi(\frac{\frac{1}{2}(2n+1) - (2n+1) \cdot q_n}{((2n+1)q_n(1-q_n))^{\frac{1}{2}}}).$$

[34] We allow for $c = \pm\infty$.

[35] For this normal approximation we cannot rely on the standard central limit theorem, because q_n varies with n. Recall that for any undominated strategy, types t with $t_\alpha > 0, t_\beta > 0$ vote A and types t with $t_\alpha < 0, t_\beta < 0$ vote B. Hence, since the type distribution has a strictly positive density, there exists $\epsilon > 0$ such that $\epsilon < q_n < 1 - \epsilon$ for all $n \in \mathbb{N}$. We claim, that, as a consequence, we can apply the Lindeberg-Feller central limit theorem (see Billingsley (2008), Theorem 27.2). To see why, one checks that a sufficient condition for the the Lindeberg condition is that $(2n+1)q_n(1-q_n) \to \infty$ as $n \to \infty$ since this implies that for n sufficiently large the indicator function in the condition takes the value zero.

Taking limits $n \to \infty$, gives

$$
\begin{aligned}
& \lim_{n\to\infty} \Phi\left(\frac{\frac{1}{2}(2n+1) - (2n+1)\cdot q_n}{(2n+1)q_n(1-q_n))^{\frac{1}{2}}}\right) \\
= & \lim_{n\to\infty} \Phi\left(\frac{(2n+1)\frac{1}{2} - (2n+1)(\frac{1}{2} + (q_n - \frac{1}{2}))}{((2n+1)^{\frac{1}{2}}(q_n(1-q_n))^{\frac{1}{2}}}\right) \\
= & \lim_{n\to\infty} \Phi\left((q_n - \frac{1}{2})\Big[\frac{(2n+1)}{q_n(1-q_n)}\Big]^{\frac{1}{2}}\right) \\
= & \Phi(c),
\end{aligned}
$$

where the equalities on the last two lines hold both when $c \in \{\infty, -\infty\}$ and when $c \in \mathbb{R}$. ■

Lemma 4 *Consider any sequence of strategies $(\sigma_n)_{n\in\mathbb{N}}$ and any sequence of information structures $(\pi_n)_{n\in\mathbb{N}}$ with a common set of substates. Then, for any substates $\omega_j, \hat{\omega}_l \in \{\alpha_1, \dots, \alpha_{N_\alpha}\} \cup \{\beta_1, \dots, \beta_{N_\beta}\}$:*

$$
\begin{aligned}
& \lim_{n\to\infty} |q(\omega_j;\sigma_n) - \frac{1}{2}|n^{\frac{1}{2}} \in \mathbb{R} \\
\Rightarrow \quad & \lim_{n\to\infty} \frac{\Pr(\text{piv}|\omega_j;\sigma_n)}{\Pr(\text{piv}|\hat{\omega}_l;\sigma_n)} \in \mathbb{R} \cup \infty.
\end{aligned} \tag{113}
$$

Proof. Let $x_n = q(\omega_j;\sigma_n) - \frac{1}{2}$. It suffices to show the claim for the case when $q(\hat{\omega}_l;\sigma_n) = \frac{1}{2}$, since in any other case the election is less likely to being tied in $\hat{\omega}_l$. Then,

$$
\begin{aligned}
\frac{q(\omega_j;\sigma_n)(1-q(\omega_j;\sigma_n))}{q(\hat{\omega}_l;\sigma_n)(1-q(\hat{\omega}_l;\sigma_n))} &= \frac{(\frac{1}{2}+x_n)(\frac{1}{2}-x_n)}{\frac{1}{4}} \\
&= \frac{\frac{1}{2}^2 - x_n^2}{\frac{1}{4}} \\
&= \Big[\frac{\frac{1}{4} - x_n^2}{\frac{1}{4}}\Big] \\
&= \Big[1 - \frac{x_n^2}{\frac{1}{4}}\Big].
\end{aligned} \tag{114}
$$

Then, it follows from (7) and (114) that

$$\lim_{n\to\infty}\frac{\Pr(\text{piv}|\omega_i;\sigma_n)}{\Pr(\text{piv}|\hat{\omega}_l;\sigma_n)}=\lim_{n\to\infty}\left[1-\frac{x_n^2}{\frac{1}{4}}\right]^n \tag{115}$$

Now, the assumption of the lemma implies

$$\lim_{n\to\infty}x_n^2 n=\lim_{n\to\infty}(x_n n^{\frac{1}{2}})^2=k \tag{116}$$

for some $k\in\mathbb{R}$. Then (115) and (116) together with $\lim_{n\to\infty}(1+\frac{x}{n})^n=e^x$ imply

$$\lim_{n\to\infty}\frac{\Pr(\text{piv}|\omega_j;\sigma_n)}{\Pr(\text{piv}|\hat{\omega}_l;\sigma_n)}=e^{-4k}\in\mathbb{R}. \tag{117}$$

This finishes the proof of the lemma. ■

Proof of Theorem 4. Let $\pi_n=\pi_n^{x,r,y}$. Recall that equilibrium is equivalently characterized by a belief vector $\mathbf{p}^*=(p_a^*,p_z^*,p_b^*)$ such that $\mathbf{p}^*=\boldsymbol{\rho}(\sigma^{\mathbf{p}^*})$; see (13). Recall that there exists $\hat{r}$ with $\Phi(\hat{r})=\frac{1}{2}$ by the richness assumption (17) and since Φ is continuous. For any strategy σ, we consider a constrained variant $\bar{\boldsymbol{\rho}}$ of the belief vector $\boldsymbol{\rho}(\sigma)$ corresponding to the best response $\sigma^{\boldsymbol{\rho}(\sigma)}$, given by

$$\bar{\rho}_a(\sigma)=\begin{cases}\hat{r} & \text{if} \quad \rho_a(\sigma)<\hat{r},\\ \rho_a(\sigma) & \text{else.}\end{cases}, \tag{118}$$

$$\bar{\rho}_b(\sigma)=\begin{cases}\hat{r} & \text{if} \quad \rho_b(\sigma)>\hat{r},\\ \rho_b(\sigma) & \text{else.}\end{cases}, \tag{119}$$

$$\bar{\rho}_z(\sigma)=\rho_z(\sigma). \tag{120}$$

The function that maps $\mathbf{p}\in[0,1]^3$ to $\bar{\boldsymbol{\rho}}(\sigma^{\mathbf{p}})=(\bar{\rho}_a(\sigma^{\mathbf{p}}),\bar{\rho}_z(\sigma^{\mathbf{p}}),\bar{\rho}_b(\sigma^{\mathbf{p}}))$ is continuous since $\boldsymbol{\rho}(\sigma^{\mathbf{p}})$ is continuous.

Step 1 *For any strategy sequence $(\sigma_n)_{n\in\mathbb{N}}$: if $\lim_{n\to\infty}|q(\alpha_1;\sigma_n)-\frac{1}{2}|n^{\frac{1}{2}}\in\mathbb{R}$,* *then* $\lim_{n\to\infty} \Pr(\alpha|\text{piv}, a; \sigma_n, \pi_n) = 1$. *If* $\lim_{n\to\infty} |q(\beta_1; \sigma_n) - \frac{1}{2}|n^{\frac{1}{2}} \in \mathbb{R}$, *then* $\lim_{n\to\infty} \Pr(\beta|\text{piv}, b; \sigma_n, \pi_n) = 1$.

If $\lim_{n\to\infty} |q(\alpha_1; \sigma_n) - \frac{1}{2}|n^{\frac{1}{2}} \in \mathbb{R}$, then

$$\begin{aligned}
& \lim_{n\to\infty} \frac{\Pr(\beta|\text{piv}, a; \sigma_n, \pi_n)}{\Pr(\alpha_1|\text{piv}, a; \sigma_n, \pi_n)} \\
\geq\ & \lim_{n\to\infty} \frac{\Pr(\beta_2|\text{piv}, a; \sigma_n, \pi_n)}{\Pr(\alpha_1|\text{piv}, a; \sigma_n, \pi_n)} \\
=\ & \lim_{n\to\infty} \frac{\Pr(\beta)}{\Pr(\alpha)} \frac{\Pr(\beta_2|\beta; \pi_n)}{\Pr(\alpha_1|\alpha; \pi_n)} \frac{\Pr(a|\beta_2; \pi_n)}{\Pr(a|\alpha_1; \pi_n)} \frac{\Pr(\text{piv}|\beta_2; \sigma_n, \pi_n)}{\Pr(\text{piv}|\alpha_1; \sigma_n, \pi_n)} \\
=\ & 0. \qquad (121)
\end{aligned}$$

where we used Lemma 4 and that $\lim_{n\to\infty} \frac{\Pr(\beta_2|\beta;\pi_n)}{\Pr(\alpha_1|\alpha;\pi_n)} \frac{\Pr(a|\beta_2;\pi_n)}{\Pr(a|\alpha_1;\pi_n)} = 0$ for the equality on the last line. This implies the first statement. The second statement follows in the same way. This finishes the first step.

Step 2 *For any n large enough, any fixed point $\mathbf{p}_n^*$ of $\bar{\boldsymbol{\rho}}(\sigma^{\mathbf{p}})$ is interior.*

First, recall that $\Pr(\text{piv}|\omega_j; \sigma_n, \pi_n) > 0$ for all ω_j and all n since σ_n is non-degenerate by (2). So, for any fixed point p_n^* of $\bar{\boldsymbol{\rho}}(\sigma^{\mathbf{p}})$, we have $0 < p_n^* < 1$; see (11) and (118) - (120) for the definition of $\bar{\boldsymbol{\rho}}(\sigma^{\mathbf{p}})$. Suppose that there exists a sequence of fixed points $(\mathbf{p}_n^*)_{n\in\mathbb{N}}$ with $(p_n)_a = \hat{r}$ for all n large enough. Then, it follows from (15) and since all voters receive a in α_1 that $q(\alpha_1; \sigma^{p_n^*}) = \phi(\hat{r})$ for n large enough. Recall that $\Phi(\hat{r}) = \frac{1}{2}$. It follows from Step 1 that $\lim_{n\to\infty} \Pr(\alpha|\text{piv}, a; \sigma^{p_n^*}, \pi_n) = 1$. But then, it follows from the definition of $\bar{\rho}_a$ that $\lim_{n\to\infty} \bar{\rho}_a(\sigma^{p_n^*}) = 1$. Since $(\mathbf{p}_n^*)_{n\in\mathbb{N}}$ is a sequence of fixed points of $\bar{\boldsymbol{\rho}}(\sigma^{\mathbf{p}})$, we have $\lim_{n\to\infty}(p_n^*)_a = 1$. This yields a contradiction to the assumption that $(p_n^*)_a = \frac{1}{2}$ for all n large enough. Suppose that there exists a sequence of fixed points $(\mathbf{p}_n^*)_{n\in\mathbb{N}}$ with $(p_n)_b = \hat{r}$ for all n large enough. We obtain a contradiction by the analogous argument. This finishes the second step.

Step 3 *Consider any sequence $\mathbf{p}_n^*$ of fixed points of $\bar{\boldsymbol{\rho}}(\sigma^{\mathbf{p}})$. Then, the corresponding sequence of strategies $\sigma^{\mathbf{p}_n^*}$ aggregates information, i.e.*

$$\begin{aligned}\lim_{n\to\infty} \Pr(A|\alpha; \sigma^{\mathbf{p}_n^*}, \pi_n, n) &= 1,\\ \lim_{n\to\infty} \Pr(B|\beta; \sigma^{\mathbf{p}_n^*}, \pi_n, n) &= 1.\end{aligned}$$

Recall that it follows from the formula for the expected vote share (15) and since all voters receive a in α_1 that $q(\alpha_1; \sigma^{p_n^*}) = \phi((p_n^*)_a)$. It follows from the definition of $\bar{\boldsymbol{\rho}}(\sigma^{\mathbf{p}})$ and since Φ is strictly increasing that $q(\alpha_1, \sigma^{\mathbf{p}_n^*}) \geq \frac{1}{2}$ for all $n \in \mathbb{N}$. Suppose that information is not aggregated in α. Then, it follows from Lemma 3 that

$$\lim_{n\to\infty} (q(\alpha_1, \sigma^{\mathbf{p}_n^*}) - \frac{1}{2})n^{\frac{1}{2}} \in \mathbb{R}. \tag{122}$$

Consequently, it follows from Step 1 that $\lim_{n\to\infty} \Pr(\alpha|\text{piv}, a; \sigma^{\mathbf{p}_n^*}, \pi_n) = 1$. So, given the definition of $\bar{\boldsymbol{\rho}}(\sigma^{\mathbf{p}})$, we obtain $\lim_{n\to\infty} \bar{\boldsymbol{\rho}}_a(\sigma^{\mathbf{p}_n^*}) = 1$. Since $\mathbf{p}_n^*$ is a sequence of fixed points, $\lim_{n\to\infty}(p_n^*)_a = 1$. Then, it follows from (15) and since all voters receive a in α_1 that

$$\lim_{n\to\infty} q(\alpha_1; \sigma^{\mathbf{p}_n^*}) = \Phi(1). \tag{123}$$

However, $\Phi(1) > \frac{1}{2}$ and we see that (122) and (123) contradict each other. Consequently, information is aggregated in α. The analogous argument shows that information is aggregated in β. This finishes the third step.

Step 4 *There exists an equilibrium sequence $(\sigma_n)_{n\in\mathbb{N}}$ that aggregates information.*

Note that Step 2 implies the following: consider any sequence $\mathbf{p}_n^*$ of fixed points of $\bar{\boldsymbol{\rho}}(\sigma^{\mathbf{p}})$. Then, the sequence of the corresponding strategies $(\sigma^{\mathbf{p}_n^*})_{n\in\mathbb{N}}$ is an equilibrium sequence. The claim of the fourth step therefore follows from Step 3. Step 4 finishes the proof of the theorem.

C Persuasion of Privately Informed Voters

Proof of Lemma 2

Fix a sequence of additional information structures $(\pi_n)_{n\in\mathbb{N}} = (\pi_n^{x,r,y})_{n\in\mathbb{N}}$ (as in Figure 5). At first, we make some observations on the voter's inference (Section C.1.1) and then, in Section C.1.2, we use these observations to construct equilibrium sequences and finally prove the theorem.

C.1.1 Voter Inference

Consider the additional signal z. A voter who observes z infers that the state is α_2 or β_2. This is the direct effect of the signal z. Now, we prove Claim 10, showing that after z the joint inference from the pivotal event and the signal z is asymptotically the same across all equilibrium sequences as $n \to \infty$.

Claim 10 *Suppose that the additional information is given by $\pi_n^{x,r,y}$ for some $(x, r, y) \in (0, 1)^3$ (see Figure 5) and consider the corresponding sequence $(\pi_n)_{n\in\mathbb{N}}$ of independent expansions of π^c. Then, for any equilibrium sequence $(\sigma_n^*)_{n\in}$ given $(\pi_n)_{n\in\mathbb{N}}$,*

$$\lim_{n\to\infty} \hat{\rho}_z(\sigma_n^*, \pi_n, n) = \lambda. \tag{86}$$

Proof. It remains to show (87). The sketch of the proof in Section 6.3 then shows how (87) implies (86). To show (87), we use arguments similar to the proof of the Condorcet Jury Theorem; compare to Section 4.3. The arguments are more subtle since, conditional on the substate being α_2 or β_2 the game is close to a game with a binary state and binary signals, as in the setting of the CJT, but not identical; so, we have to show that the arguments of the proof of the CJT extend.

Step 1 *For all n and every equilibrium σ_n^*, voters with z-signal and an u-signal are more likely to vote A than voters with a z-signal and a d-signal when n is large enough, i.e.*

$$\Phi(\rho_{z,u}(\sigma_n^*)) > \Phi(\rho_{z,d}(\sigma_n^*)). \tag{124}$$

This ordering follows from the likelihood ratio ordering of the signals u and d. In particular, if follows from the independence of $\pi_n^{x,r,y}$ and of π^c that the posterior conditional on on a given voter's signals z and $s \in \{u, d\}$ and the event that the voter is pivotal is

$$\begin{aligned} & \Pr(\alpha|z, s, \text{piv}; \sigma_n^*, \pi_n, n) \\ = \; & \frac{\Pr(\alpha|z, \text{piv}; \sigma_n^*, \pi_n, n) \Pr(s|\alpha; \pi^c)}{\Pr(\alpha|z, \text{piv}; \sigma_n^*, \pi_n, n) \Pr(s|\alpha; \pi^c) + \Pr(\beta|z, \text{piv}; \sigma_n^*, \pi_n, n) \Pr(s|\beta; \pi^c)} \end{aligned} \tag{125}$$

Therefore, $\frac{\Pr(u|\alpha;\pi^c)}{\Pr(u|\beta;\pi^c)} > \frac{\Pr(d|\alpha;\pi^c)}{\Pr(d|\beta;\pi^c)}$ implies that $\Pr(\alpha|z, u, \text{piv}; \sigma_n^*, \pi_n, n) > \Pr(\alpha|z, d, \text{piv}; \sigma_n^*, \pi_n, n)$. Now, (124) follows from (20), (80) and the monotonicity of Φ.

Step 2 *For all n and every equilibrium σ_n^*, the vote share of A is at most $\frac{1}{n^2}$ smaller in α_2 than in β_2,*

$$q(\alpha_2; \sigma_n^*) - q(\beta_2; \sigma_n^*) \geq -\frac{1}{n^2} \tag{126}$$

This follows from (15), given (20) and (124) and since in both α_2 and β_2 the likelihood that a voter does not receive signal z is smaller than $\frac{1}{n^2}$.

Step 3 *For every equilibrium sequence $(\sigma_n^*)_{n\in\mathbb{N}}$,*

$$\lim_{n\to\infty} \hat{\rho}_z(\sigma_n^*, \pi_n, n) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \frac{\Pr(\text{piv}|\alpha_2; \sigma_n^*, \pi_n, n)}{\Pr(\text{piv}|\beta_2; \sigma_n^*, \pi_n, n)} \geq 1, \tag{127}$$

$$\lim_{n\to\infty} \hat{\rho}_z(\sigma_n^*, \pi_n, n) = 1 \quad \Rightarrow \quad \lim_{n\to\infty} \frac{\Pr(\text{piv}|\alpha_2; \sigma_n^*, \pi_n, n)}{\Pr(\text{piv}|\beta_2; \sigma_n^*, \pi_n, n)} \leq 1 \tag{128}$$

Suppose that $\lim_{n\to\infty} \hat{\rho}_z(\sigma_n^*, \pi_n, n) = 0$. This implies $\lim_{n\to\infty} \hat{q}(\omega; \rho_z(\sigma_n^*, \pi_n, n)) = \Phi(0)$ for $\omega \in \{\alpha, \beta\}$. Since almost all voters receive z in α_2 and β_2 and since $\Phi(0) < \frac{1}{2}$,

$$\lim_{n\to\infty} q(\alpha_2, \sigma_n^*) < \frac{1}{2}, \text{ and } \lim_{n\to\infty} q(\beta_2, \sigma_n^*) < \frac{1}{2}. \tag{129}$$

Recall that $\Phi(0) < q(\omega_j; \sigma) < \Phi(1)$ for any strategy and any substate ω_j and note that the derivative of $h(q) = q(1-q)$ is bounded below by some

Lipschitz constant $L > 0$ on the compact interval $[\Phi(0), \Phi(1)]$. Hence, Step 2 implies

$$h(q(\beta_2, \sigma_n^*))(\frac{h(q(\alpha_2, \sigma_n^*))}{h(q(\beta_2, \sigma_n^*))} - 1) = h(q(\alpha_2, \sigma_n^*)) - h(q(\beta_2, \sigma_n^*)) \geq -\frac{L}{n^2}. \quad (130)$$

Recall that the function $h(q) = q(1-q)$ is inverse U-shaped with peak at $q = \frac{1}{2}$ and note that it follows from (17) and since Φ is strictly increasing (see (21)) that $0 < \Phi(0) < \frac{1}{2}$ and $\Phi(1) > \frac{1}{2}$. Since $\Phi(0) < q(\beta_2; \sigma_n) < \Phi(1)$,

$$\frac{h(q(\alpha_2, \sigma_n^*))}{h(q(\beta_2, \sigma_n^*))} \geq 1 - \frac{L}{h(q(\beta_2\,; \sigma_n))n^2} \geq 1 - \frac{L}{Mn^2} \quad (131)$$

for $M = \min\left(h(\Phi(0)), h(\Phi(1))\right)$ and all n. It follows from (7) that $\frac{\Pr(\text{piv}|\alpha_2;\sigma_n^*,\pi_n,n)}{\Pr(\text{piv}|\beta_2;\sigma_n^*,\pi_n,n)} \geq (1 - \frac{L}{Mn^2})^n$. Now, (127) follows since $\lim_{n\to\infty}(1 - \frac{L}{Mn^2})^n = 1$; to see why this is true, see the discussion at the end of the proof of Claim 3. The proof of (128) is analogous. This finishes the third step.

Step 4 *For every equilibrium sequence* $(\sigma_n^*)_{n\in\mathbb{N}}$,

$$\lim_{n\to\infty} \hat{\rho}_z(\sigma_n^*, \pi_n, n) \notin \{0, 1\}. \quad (132)$$

Suppose that $\lim_{n\to\infty} \hat{\rho}_z(\sigma_n^*, \pi_n, n) = 0$. We have

$$\frac{\hat{\rho}_z(\sigma_n^*, \pi_n, n)}{1 - \hat{\rho}_z(\sigma_n^*, \pi_n, n)} = \frac{\Pr(\alpha)}{\Pr(\beta)} \frac{\Pr(\alpha_2|\alpha; \pi_n)}{\Pr(\beta_2|\beta; \pi_n)} \frac{\Pr(\text{piv}|\alpha_2, \sigma_n^*, \pi_n, n)}{\Pr(\text{piv}|\beta_2, \sigma_n^*, \pi_n, n)}. \quad (133)$$

If follows from (127) that $\lim_{n\to\infty} \hat{\rho}_z(\sigma_n^*, \pi_n, n) > 0$ which contradicts the assumption $\lim_{n\to\infty} \hat{\rho}_z(\sigma_n^*, \pi_n, n) = 0$. The analogous argument leads the assumption that $\lim_{n\to\infty} \hat{\rho}_z(\sigma_n^*, \pi_n, n) = 1$ to a contradiction. This finishes the fourth step.

Step 5 *In every equilibrium sequence* (σ_n^*), *the limit of the vote share of* A *is larger in* α_2 *than in* β_2,

$$\lim_{n\to\infty} q(\alpha_2; \sigma_n^*) > q(\beta_2; \sigma_n^*). \quad (134)$$

Since almost all voters receive z in α_2 and β_2, we have

$$\lim_{n\to\infty} q(\alpha_2; \sigma_n^*) = \lim_{n\to\infty} \hat{q}(\alpha; \hat{\rho}_z(\sigma_n^*, \pi_n, n)), \tag{135}$$
$$\lim_{n\to\infty} q(\beta_2; \sigma_n^*) = \lim_{n\to\infty} \hat{q}(\beta; \hat{\rho}_z(\sigma_n^*, \pi_n, n)). \tag{136}$$

From (132) and (125), we have that, the limits of the posteriors conditional being pivotal, the signal z and the signals $s \in \{u, d\}$ are interior and hence strictly ordered,

$$0 < \lim_{n\to\infty} \Pr\left(\alpha | z, d, \text{piv}; \sigma_n^*, \pi_n, n\right) < \lim_{n\to\infty} \Pr\left(\alpha | z, u, \text{piv}; \sigma_n^*, \pi_n, n\right) < 1. \tag{137}$$

Now, (134) follows from (135), (136), and (80), given (20), (137), and since Φ is strictly increasing.

Step 6 *The election is equally close to being tied in expectation in α_2 and β_2, that is, (87) holds.*

It follows from (132) that voters must not become certain conditional on being pivotal and the substate being α_2 or β_2, i.e. $\lim_{n\to\infty} \Pr(\alpha|\{\alpha_2, \beta_2\}, \text{piv}; \sigma_n^*, \pi_n) \notin \{0, 1\}$. Hence, Claim 2 requires that

$$\lim_{n\to\infty} \left| q(\alpha_2; \sigma_n^*) - \frac{1}{2} \right| = \lim_{n\to\infty} \left| q(\beta_2; \sigma_n^*) - \frac{1}{2} \right|. \tag{138}$$

Given the ordering of the limits of the vote shares from (134), the equation (138) implies (87). ■

Now, we consider a voter who received an additional signal $s_2 \in \{a, b\}$. The following result shows that, independent of the private signal $s_1 \in \{u, d\}$ received, the inference from the signals is dominated by the inference from the pivotal event, for an open set of voting strategies: if the election is closer to being tied in states α_2 and β_2 than in the states α_1 and β_1, after receiving the additional signal a and any signal $s_1 \in \{u, d\}$, the voter infers that conditional on being pivotal the state is either α_2 or β_2, even though the likelihood of the signal a is infinitely higher in the substate α_1 than in any other substate as $n \to \infty$. In the same way, the voter infers that the state is either α_2 or β_2 after receiving the additional signal b.

Claim 12 *Suppose that the additional information is given by $\pi_n^{x,r,y}$ for some $(x, r, y) \in (0, 1)^3$ (see Figure 5) and consider the corresponding sequence $(\pi_n)_{n\in\mathbb{N}}$ of independent expansions of π^c. Take any sequence of strategies $(\sigma_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty} \min_{\omega_1\in\{\alpha_1,\beta_1\}} |q(\sigma_n; \omega_1, \pi_n) - \frac{1}{2}| > \lim_{n\to\infty} \max_{\omega_2\in\{\alpha_2,\beta_2\}} |q(\sigma_n; \omega_2, \pi_n) - \frac{1}{2}|; \quad (139)$$

then, for any $s \in \{u, d\} \times \{a, b\}$,

$$\lim_{n\to\infty} \frac{\Pr(\{\alpha_2, \beta_2\}|s, \text{piv}; \sigma_n, \pi_n)}{\Pr(\{\alpha_1, \beta_1\}|s, \text{piv}; \sigma_n, \pi_n)} = \infty. \quad (140)$$

Proof. The proof follows from previous arguments: the arguments from the proof of Claim 7 hold verbatim with the required changes in notation. ■

Claim 12 implies that for any sequence of equilibria $(\sigma_n^*)_{n\in\mathbb{N}}$ that satisfies (139),

$$\lim_{n\to\infty} \frac{\Pr(\alpha|a, \text{piv}; \sigma_n^*, \pi_n, n)}{\Pr(\beta|a, \text{piv}; \sigma_n^*, \pi_n, n)} = \lim_{n\to\infty} \frac{\Pr(\alpha_2|a, \text{piv}\sigma_n^*, \pi_n, n)}{\Pr(\beta_2|a, \text{piv}, \sigma_n^*, \pi_n, n)} \quad (141)$$

In the following formula we omit the dependence on σ_n^* and π_n. Using Bayes' rule,

$$\begin{aligned} \lim_{n\to\infty} \frac{\Pr(\alpha_2|a, \text{piv})}{\Pr(\beta_2|a, \text{piv})} &= \lim_{n\to\infty} \frac{\Pr(\alpha)}{\Pr(\beta)} \frac{\Pr(\alpha_2|\alpha)}{\Pr(\beta_2|\beta)} \frac{\Pr(a|\alpha_2)}{\Pr(a|\beta_2)} \frac{\Pr(\text{piv}|\alpha_2)}{\Pr(\text{piv}|\beta_2)} \\ &= \lim_{n\to\infty} \frac{\Pr(\alpha|\{\alpha_2, \beta_2\}, \text{piv})}{\Pr(\beta|\{\alpha_2, \beta_2\}, \text{piv})} \frac{\Pr(a|\alpha_2)}{\Pr(a|\beta_2)} \end{aligned} \quad (142)$$

We see that for a voter who received an additional signal $s_2 \in \{a, b\}$, the inference about the state is asymptotically pinned down by the inference from the pivotal event conditional on the state being α_2 or β_2, and by the ratio of the signal probabilities in the states α_2 and β_2. Note that $\lim_{n\to\infty} \hat{\rho}_z(\sigma_n^*, \pi_n, n) = \lim_{n\to\infty} \Pr(\alpha|\{\alpha_2, \beta_2\}, \text{piv}; \sigma_n^*, \pi_n, n)$ such that Claim 10 implies

$$\lim_{n\to\infty} \Pr(\alpha|\{\alpha_2, \beta_2\}, \text{piv}; \sigma_n^*, \pi_n, n) = \lambda. \tag{143}$$

Using (141), (142), (143) and the definition of the information structure $\pi_n^{x,r,y}$ (see Figure 5), we conclude

$$\lim_{n\to\infty} \frac{\Pr(\alpha|a, \text{piv}; \sigma_n^*, \pi_n)}{\Pr(\beta|a, \text{piv}; \sigma_n^*, \pi_n)} = \frac{x}{1-x}\frac{\lambda}{1-\lambda}, \tag{144}$$

Similarly, for the additional signal b,

$$\lim_{n\to\infty} \frac{\Pr(\alpha|b, \text{piv}; \sigma_n^*, \pi_n, n)}{\Pr(\beta|b, \text{piv}; \sigma_n^*, \pi_n, n)} = \frac{y}{1-y}\frac{\lambda}{1-\lambda}. \tag{145}$$

C.1.2 Fixed Point Argument

In this section, we prove Theorem 2, using the observations from the preceding section. Let us consider some belief $\mu_\alpha \in [\lambda_\alpha, \lambda]^c$ and some belief $\mu_\beta \in [\lambda, \lambda_\beta]^c$ with λ, λ_α and λ_β given by (81), (82) and (83).

Recall from Section 6.2 that equilibrium can be equivalently characterized by a vector of beliefs $\mathbf{p}^* = (p_a^*, p_z^*, p_b^*)$ such that $\mathbf{p}^* = \hat{\boldsymbol{\rho}}(\sigma^{\mathbf{p}^*}; \pi, n)$; see (77). Now, take any $\delta > 0$ and let

$$\mathrm{B}_\delta = \left\{\mathbf{p} \in [0,1]^3 \,|\, |\mathbf{p} - (\mu_\alpha, \lambda, \mu_\beta)| \le \delta\right\},$$

so that B_δ is the set of beliefs at most δ away from $(\mu_\alpha, \lambda, \mu_\beta)$. Take any $\mathbf{p} \in \mathrm{B}_\delta$ and the corresponding strategy $\sigma^{\mathbf{p}}$. We define a constrained best reponse function that ensures that the best reponse to any $\mathbf{p} \in [0,1]^3$ is contained in B_δ:

$$\hat{\rho}_a^{tr}(\sigma^{\mathbf{p}}) = \begin{cases} \mu_\alpha - \delta & \text{if} \quad \hat{\rho}_a(\sigma^{\mathbf{p}}) < \mu_\alpha - \delta, \\ \mu_\alpha + \delta & \text{if} \quad \hat{\rho}_a(\sigma^{\mathbf{p}}) > \mu_\alpha + \delta, \\ \hat{\rho}_a(\sigma^{\mathbf{p}}) & \text{else.} \end{cases} \tag{146}$$

The components $\hat{\rho}_z^{tr}$ and $\hat{\rho}_b^{tr}$ are defined in the analogous way. Note that the vector $\hat{\boldsymbol{\rho}}^{tr}(\sigma^{\mathbf{p}})$ is continuous in $\mathbf{p}$ such that it follows from the Kakutani fixed point theorem that $\hat{\boldsymbol{\rho}}^{tr}(\sigma^{\mathbf{p}})$ has a fixed point $\mathbf{p}^* \in \mathrm{B}_\delta$.
Now, we show that the previous Claim 12 implies that any fixed point $\mathbf{p}^*$ of $\hat{\boldsymbol{\rho}}^{tr}(\sigma^{\mathbf{p}})$ is interior when n is large enough and δ is small enough.

Claim 13 *Consider any $\mu_\alpha \in [\lambda_\alpha, \lambda]^c$ and any $\mu_\beta \in [\lambda, \lambda_\beta]^c$. Consider the sequence of independent expansions $(\pi_n)_{n\in\mathbb{N}}$ of π^c with additional information $\pi_n^{x,r,y}$ where $\mu_\alpha = \frac{x\lambda}{x\lambda+(1-x)(1-\lambda)}$ and $\mu_\beta = \frac{y\lambda}{y\lambda+(1-y)(1-\lambda)}$ and $r \in (0,1)$ (see Figure 5). For any $\delta > 0$ small enough, there exists $n(\delta) \in \mathbb{N}$ such that for all $n \geq n(\delta)$ and any fixed point of $\hat{\boldsymbol{\rho}}^{tr}(\sigma^{\mathbf{p}})$ is interior.*

Proof. Given the assumptions on μ_α, μ_β, we can choose $\delta > 0$ small enough such that for any $\mathbf{p} \in \mathrm{B}_\delta$ and the corresponding behavior $\sigma^{\mathbf{p}}$ the expected margins of victory in the states α_2 and β_2 are strictly smaller than the expected margins of victory in the states α_1 and β_1, i.e. $\sigma^{\mathbf{p}}$ satisfies (50). Therefore, it follows from Claim 12 and its implications (144) and (145) that

$$\lim_{n\to\infty} \rho_a(\sigma^{\mathbf{p}}, n) = \mu_\alpha, \tag{147}$$
$$\lim_{n\to\infty} \rho_b(\sigma^{\mathbf{p}}, n) = \mu_\beta. \tag{148}$$

This means in particular that $\hat{\rho}_a$ and $\hat{\rho}_b$ are interior for n large enough. Consequently, for any fixed point $\mathbf{p}^* = (p_a^*, p_z^*, p_b^*)$ of $\hat{\boldsymbol{\rho}}^{tr}$, the beliefs p_a^* and p_b^* are interior when n is large enough. If $p_z^* = \lambda - \delta$, then, given $\sigma^{\mathbf{p}^*}$ and as $n \to \infty$, the margin of victory in α_2 is strictly smaller than the margin of victory in β_2, given the definition of λ; see (81). Hence, Claim 2 implies that $\lim_{n\to\infty} \frac{\Pr(\text{piv}|\alpha_2;\sigma^{\mathbf{p}^*},\pi_n^{x,r,y},n)}{\Pr(\text{piv}|\beta_2,\sigma^{\mathbf{p}^*},\pi_n^{x,r,y},n)} = \infty$. This implies, $\lim_{n\to\infty} \rho_z(\sigma^{\mathbf{p}^*}; \pi_n^{x,r,y}, n) = 1$. For any n large enough this contradicts with $p_z^* = \lambda - \delta$ since $\mathbf{p}^*$ is a fixed point of $\hat{\boldsymbol{\rho}}^{tr}(\sigma^{\mathbf{p}})$. In the same way we can exclude that $p_z^* = \lambda + \delta$ for any n large enough. We conclude that any fixed point $\mathbf{p}^*$ of $\hat{\boldsymbol{\rho}}^{tr}(\sigma^{\mathbf{p}})$ is interior when δ is small enough and n is large enough. ∎

Now, we finish the proof of Theorem 2. Note that the behavior $\sigma^{\mathbf{p}^*}$ corresponding to any interior fixed point $\mathbf{p}^*$ of $\hat{\boldsymbol{\rho}}^{tr}$ is an equilibrium. Therefore, Claim 13 implies the existence of a sequence of equilibria $(\sigma_n^*)_{n\in\mathbb{N}}$ given additional information $\pi_n^{x,r,y}$ for which (147) and (148) hold. Since in state α the probability that a random voter receives a converges to 1 and since in state β the probability that a random voter receives a converges to 1, (147) and (148) imply that (μ_α, μ_β) is implementable. This finishes the proof of Theorem 2.

D Partially Informed Sender

Proof of Theorem 6

Proof. For each voter type $t = (t_\alpha, t_\beta)$ and any $m \in \{h, \ell\}$, let

$$t_m^{\pi_0} = \Pr(\alpha|m)t_\alpha + \Pr(\beta|m)t_\beta \tag{149}$$

be the expected utility of a voter of type t from the outcome A being elected when she received m. Let G^{π_0} be the distribution of types $t^{\pi_0} = (t_h^{\pi_0}, t_\ell^{\pi_0})$ induced by the preference distribution G. Note that any coarsening π of π_0 can be viewed as an information structure for the state space $\{h, \ell\}$ with substates $(m, j) \in \{h\} \times \{1, \dots, N_h\} \cup \{\ell\} \times \{1, \dots, N_\ell\}$. So, the state space $\{h, \ell\}$ together with the preference distribution G^{π_0} and a coarsening π_2 of π_0 define a game of voters as in Section 2. To apply Theorem 2, we have to show that the distribution G^{π_0} satisfies the richness condition 17: let

$$\Phi^{\pi_0}(\tilde{p}) = \Pr_{G^{\pi_0}}(\{t^{\pi_0} : \tilde{p} \cdot t_h^{\pi_0} + (1 - \tilde{p}) \cdot t_\ell^{\pi_0} > 0\}), \tag{150}$$

be the function that maps a belief $\tilde{p} \in [0, 1]$ about the state h to the probability that a random type t^{π_0} prefers A given belief $\tilde{p}$. We have to show that there exists a belief $\tilde{p}' \in [0, 1]$ for which a majority prefers A and a belief $\tilde{p}$ for which a majority prefers B, i.e.

$$\Phi^{\pi_0}(\tilde{p}) < \frac{1}{2} < \Phi^{\pi_0}(\tilde{p}'). \tag{151}$$

To see, why such beliefs p and p' exist, note that any belief $\tilde{p}$ about h induces a belief $p(\tilde{p}) = \tilde{p}\Pr(\alpha|h) + (1-\tilde{p})\Pr(\alpha|\ell)$ about the state α. The assumption (98) made in the theorem means that

$$p(\tilde{p}) < \hat{r} \quad \text{for} \quad \tilde{p} = 0, \tag{152}$$

$$p(\tilde{p}) > \hat{r} \quad \text{for} \quad \tilde{p} = 1. \tag{153}$$

Now,

$$\begin{aligned} & \Phi^{\pi_0}(\tilde{p}) \\ = \ & \Pr_G(\{t : (\tilde{p}\Pr(\alpha|h) + (1-\tilde{p})\Pr(\alpha|\ell))t_\alpha + (\tilde{p}\Pr(\beta|h) + (1-\tilde{p})\Pr(\beta|\ell))t_\beta > 0) \\ = \ & \Phi(p(\tilde{p})). \end{aligned} \tag{154}$$

Hence, given that Φ is strictly increasing and given that $\Phi(\hat{r}) = \frac{1}{2}$, the assumption (98) made in the theorem together with the richness condition (17) implies the richness condition (151). We conclude that the result of the theorem follows from Theorem 2. ■

Proof of Theorem 7

In Section D.2.1 we provide the proof for the case when the sender receives binary signals $m \in \{h, \ell\}$ and the goal is to persuade the voters to elect B after h and A after ℓ. In Section D.2.2, we explain how the result generalizes.

D.2.1 Proof: Binary Signals of the Sender

Let the sender release information to the voters given by the sequence of coarsenings $(\pi^{x,y}_{2,n})_{n\in\mathbb{N}}$ in Figure 8 (compare to Figure 5). Let $\pi_n = \pi^{x,y}_{2,n} \times \pi^c$ be the arising independent expansions of π^c.

Step 1 *We show that the margin of victory in the states $\alpha_{h,2}$, α_{ℓ_2} and $\beta_{h,2}$, $\beta_{\ell,2}$ is pinned down uniquely by the exogenous information π^c of the voters: there exists $M > 0$ such that for any $(x, y) \in \{0, 1\}^2$, any equilibrium sequence $(\sigma^*_n)_{n\in}$ given $(\pi_n)_{n\in\mathbb{N}}$, and any $m \in \{h, \ell\}$,*

$$M = \lim_{n\to\infty} q(\alpha_{m,2}; \sigma_n^*, \pi_n) - \frac{1}{2} = \lim_{n\to\infty} \frac{1}{2} - q(\beta_{m,2}; \sigma_n^*, \pi_n). \tag{155}$$

The key insight is the following: given the additional information $\pi_n^{x,y}$ provided by the sender, in the substates $\alpha_{h,2}$, $\alpha_{\ell,2}$ and $\beta_{h,2}$, $\beta_{\ell,2}$, a random voter receives the additional signal z with probability converging to 1 and infers that either of these substates holds. Hence, conditional on the substate being $\alpha_{h,2}$, $\alpha_{\ell,2}$, $\beta_{h,2}$ or $\beta_{\ell,2}$, the game of voters converges to a game with binary signals drawn from π^c that are independently and identically distributed conditional on the state $\omega \in \{\alpha, \beta\}$, as in the setting of the CJT. Moreover, the limits of the vote shares in these substates are given by the vote shares implied by the induced prior after z: for any $\omega_{m,2} \in \{\alpha_{h,2}, \alpha_{\ell,2}, \beta_{h,2}, \beta_{\ell,2}\}$,

$$\lim_{n\to\infty} q(\omega_{m,2}; \sigma_n^*, \pi_n) = \lim_{n\to\infty} q(\omega; \hat{\rho}_z(\sigma_n^*, \pi_n, n)). \tag{156}$$

So, in particular, for any $\omega \in \{\alpha, \beta\}$,

$$\lim_{n\to\infty} q(\omega_{h,2}; \sigma_n^*, \pi_n) = \lim_{n\to\infty} q(\omega_{\ell,2}; \sigma_n^*, \pi_n).$$

As in the proof of the CJT, we can show that the election is equally close to being tied in the substates $\alpha_{h,2}$ and $\alpha_{\ell,2}$ compared to the substates $\beta_{h,2}$ and $\beta_{\ell,2}$, that is for any $m \in \{h, \ell\}$,[36]

$$\lim_{n\to\infty} q(\alpha_{m,2}; \sigma_n^*, \pi_n) - \frac{1}{2} = \lim_{n\to\infty} \frac{1}{2} - q(\beta_{m,2}; \sigma_n^*, \pi_n). \tag{157}$$

Since λ is the unique induced prior such that the margins of victory in the states α and β are equal given the implied vote shares (see (81)), we must have

$$\lim_{n\to\infty} \hat{\rho}_z(\sigma_n^*, \pi_n, n) = \lambda. \tag{158}$$

such that (156) - (158) together yield the assertion of the first step.

[36]More precisely, the arguments from the proof of (87) in Section C hold verbatim with the required changes in notation.

Step 2 *We show that for any strategy sequence $(\sigma_n)_{n\in\mathbb{N}}$, the pivotal event does not contain any information about the relative likelihood of the states $\alpha_{h,2}$ and $\alpha_{\ell,2}$ or the relative likelihood of the states $\beta_{h,2}$ and $\beta_{\ell,2}$: for any $\omega \in \{\alpha, \beta\}$, we have*

$$\lim_{n\to\infty} \frac{\Pr(\text{piv}|\omega_{h,2}, \sigma_n, \pi_n)}{\Pr(\text{piv}|\omega_{\ell,2}, \sigma_n, \pi_n)} = 1. \tag{159}$$

To see why, note that almost all voters receive z in $\alpha_{h,2}$ and $\alpha_{\ell,2}$ such that the signal distribution in these two states is almost the same. Similarly, the signal distribution in the states $\beta_{h,2}$ and $\beta_{\ell,2}$ is almost the same. Now, the statement formally follows by the same proof as provided for Claim 3. As a consequence of (159), for any $m \in \{h, \ell\}$

$$\lim_{n\to\infty} \frac{\Pr(\text{piv}|\alpha_{m,2}; \sigma_n, \pi_n)}{\Pr(\text{piv}|\beta_{m,2}; \sigma_n, \pi_n)} = \lim_{n\to\infty} \frac{\Pr(\text{piv}|\{\alpha_{h,2}, \alpha_{\ell,2}\}; \sigma_n, \pi_n)}{\Pr(\text{piv}|\{\beta_{h,2}, \beta_{\ell,2}\}; \sigma_n, \pi_n)}. \tag{160}$$

Fixed Point Argument. We provide a fixed point argument which shows that there exists an equilibrium sequence $(\sigma_n^*)_{n\in\mathbb{N}}$, first, for which the limit of the margin of victory in any state $\omega_j \in \{\alpha_{h,2}, \alpha_{\ell,2}, \beta_{h,2}, \beta_{\ell,2}\}$ is smaller than the limit of the margin of victory in any state $\hat{\omega}_l \in \{\alpha_{h,1}, \alpha_{\ell,1}, \beta_{h,1}, \beta_{\ell,1}\}$, i.e.

$$\lim_{n\to\infty} |q(\omega_j, \sigma_n^*) - \frac{1}{2}| < \lim_{n\to\infty} |q(\hat{\omega}_l, \sigma_n^*) - \frac{1}{2}|. \tag{161}$$

and, second, for which B is elected after h and A after ℓ, as $n \to \infty$. For this, we let $x = 1$ and $y = 0$. The argument for any other signal-dependent policy $(x(h), x(\ell)) \in \{A, B\}^2$ is analogous. Consider a strategy sequence $(\sigma_n)_{n\in\mathbb{N}}$ such that (161) holds. Then, it follows from Claim 2 that any voter, independently of the signal he received, infers that conditional on the pivotal event, it is most likely that the substate is one of $\alpha_{h,2}, \alpha_{\ell,2}, \beta_{h,2}, \beta_{\ell,2}$, i.e. for any $s \in \{u, d\} \times \{a, b, z\}$,[37]

$$\lim_{n\to\infty} \frac{\Pr(\{\alpha_{h,2}, \alpha_{\ell_2}, \beta_{h_2}, \beta_{\ell_2}\}|s, \text{piv}; \sigma_n, \pi_n)}{\Pr(\{\alpha_{h,1}, \alpha_{\ell,1}, \beta_{h,1}, \beta_{\ell,1}\}|s, \text{piv}; \sigma_n, \pi_n)} = \infty. \tag{162}$$

[37] This is analogous to the observation on the voters' inference in section 5.4.2 (see Claim 7).

So, as $n \to \infty$ the voter learns from the pivotal event that $\alpha_{h,2}, \alpha_{\ell,2}, \beta_{h,2}$ or $\beta_{\ell,2}$ holds and something about the relative likelihood of these substates. When the voter receives a, she further learns that the signal of the sender must have been ℓ since $x = 0$ and $y = 1$ (see Figure 8). To the end of this section, we delegate to show that in fact, as $n \to \infty$, the belief $\Pr(\alpha|\text{piv}, a; \sigma_n, \pi_n)$ has a simple description in terms of these two inferences,

$$\lim_{n\to\infty} \frac{\Pr(\alpha|\text{piv}, a; \sigma_n, \pi_n)}{\Pr(\beta|\text{piv}, a; \sigma_n, \pi_n)} = \frac{\lambda}{1-\lambda} \frac{\Pr(\ell|\alpha; \pi_0)}{\Pr(\ell|\beta; \pi_0)}, \tag{163}$$

where λ is the posterior conditional on being pivotal and the substate being $\alpha_{h,2}, \alpha_{\ell,2}, \beta_{h,2}$ or $\beta_{\ell,2}$, as $n \to \infty$, i.e. $\lambda = \lim_{n\to\infty} \Pr(\alpha|\text{piv}, \{\alpha_{h,2}, \alpha_{\ell_2}, \beta_{h_2}, \beta_{\ell_2}\}; \sigma_n, \pi_n)$; compare to (158). Similarly, we show that

$$\lim_{n\to\infty} \frac{\Pr(\alpha|\text{piv}, b; \sigma_n, \pi_n)}{\Pr(\beta|\text{piv}, b; \sigma_n, \pi_n)} = \frac{\lambda}{1-\lambda} \frac{\Pr(h|\alpha; \pi_0)}{\Pr(h|\beta; \pi_0)}. \tag{164}$$

We omit the dependence of beliefs on σ_n and π_n in the following. Now, we claim that the sequence of the best responses to σ_n satisfies (161), i.e. the sequence $(\sigma^{\mathbf{p}_n})_{n\in\mathbb{N}}$ with $\mathbf{p}_n$ the vector $(\Pr(\alpha|\text{piv}, s))_{s\in\{u,d\}\times\{a,z,b\}}$.
To see why, note that in the states $\alpha_{\ell,1}, \beta_{\ell,1}$ all voters receive b such that $q(\alpha_{\ell,1}; \sigma^{\mathbf{p}_n}) = q(\alpha; \Pr(\alpha|\text{piv}, b))$ and $q(\beta_{\ell,1}; \sigma^{\mathbf{p}_n}) = q(\beta; \Pr(\alpha|\text{piv}, b))$. In states $\alpha_{h,2}, \beta_{h,2}$ all voters receive a such that $q(\alpha_{h,2}; \sigma^{\mathbf{p}_n}) = q(\alpha; \Pr(\alpha|\text{piv}, a))$ and $q(\beta_{h,2}; \sigma^{\mathbf{p}_n}) = q(\beta; \Pr(\alpha|\text{piv}, a))$. In any state $\alpha_{m,2} \in \{\alpha_{h,2}, \alpha_{\ell,2}\}$, almost all voters receive z such that $\lim_{n\to\infty} q(\alpha_{m,2}, \sigma^{\mathbf{p}_n}) = \lim_{n\to\infty} q(\alpha; \Pr(\alpha|\text{piv}, z))$. In any state $\beta_{m,2} \in \{\beta_{h,2}, \beta_{\ell,2}\}$, almost all voters receive z such that $\lim_{n\to\infty} q(\beta_{m,2}, \sigma^{\mathbf{p}_n}) = \lim_{n\to\infty} q(\beta; \Pr(\alpha|\text{piv}, z))$. Now, (163) and (101) imply that $\lim_{n\to\infty} \Pr(\alpha|\text{piv}, a) < \lambda_\alpha$ and the monotonicity of the implied vote share $q(-, p)$ in p implies that $\lim_{n\to\infty} q(\alpha; \Pr(\alpha|\text{piv}, a)) < q(\alpha, \lambda_\alpha)$. Similarly, (164) and (102) imply that $\lim_{n\to\infty} \Pr(\alpha|\text{piv}, b) > \lambda_\beta)$ and then the monotonicity of $q(-, p)$ in p implies that $\lim_{n\to\infty} q(\beta; \Pr(\alpha|\text{piv}, b)) > q(\beta, \lambda_\beta)$. Also, recall (158) which states that $\lim_{n\to\infty} \Pr(\alpha|\text{piv}, z)) = \lambda$. Now, finally, the claim about $(\sigma^{\mathbf{p}_n})_{n\in\mathbb{N}}$ follows since $q(\alpha, \lambda_\alpha) = q(\beta, \lambda) < \frac{1}{2}$ and $\frac{1}{2} < q(\alpha, \lambda) = q(\beta, \lambda_\beta)$ by the definitions of λ, λ_α and λ_β given by (81), (82) and (83).

It follows from the Kakutani fixed point theorem that there exists an equilibrium sequence $(\sigma_n^*)_{n\in\mathbb{N}}$ such that (161) holds. Therefore, as just shown, for any $\omega \in \{\alpha, \beta\}$, we have $\lim_{n\to\infty} q(\omega_{\ell,1}; \sigma_n^*) > \frac{1}{2}$ and since $\lim_{n\to\infty} \Pr(\{\alpha_{\ell,1}, \beta_{\ell,1}\}|\ell) = 1$, it follows from the weak law of large numbers that the probability that A gets elected converges to 1 conditional on ℓ. Similarly, the probability that B gets elected converges to 1 conditional on h. This finishes the proof of the Theorem 7 for the case when the sender has binary signals and the goal is to implement B after h and A after ℓ. We provide the proof for the general case in Section D.2.2.

Proof of (163) and (164).

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\Pr(\alpha|\text{piv}, a)}{\Pr(\beta|\text{piv}, a)} \\
=\ &\lim_{n\to\infty} \frac{\Pr(\alpha|\text{piv}, \{\alpha_{h,2}, \alpha_{\ell,2}, \beta_{h,2}, \beta_{\ell,2}\}, a)}{\Pr(\beta|\text{piv}, \{\alpha_{h,2}, \alpha_{\ell,2}, \beta_{h,2}, \beta_{\ell,2}\}, a)} \\
=\ &\lim_{n\to\infty} \frac{\Pr(\alpha)}{\Pr(\beta)} \frac{\Pr(\ell|\alpha)}{\Pr(\ell|\beta)} \frac{\Pr(\ell_2|\ell)}{\Pr(\ell_2|\ell)} \frac{\Pr(\text{piv}|\alpha_{\ell,2})}{\Pr(\text{piv}|\beta_{\ell,2})} \\
=\ &\lim_{n\to\infty} \frac{\Pr(\alpha)}{\Pr(\beta)} \frac{\Pr(\ell|\alpha)}{\Pr(\ell|\beta)} \frac{\Pr(\text{piv}|\{\alpha_{h,2}, \alpha_{\ell,2}\})}{\Pr(\text{piv}|\{\beta_{h,2}, \beta_{\ell,2}\})} \\
=\ &\lim_{n\to\infty} \frac{\Pr(\alpha)}{\Pr(\beta)} \frac{\Pr(\{\alpha_{\ell,2}, \alpha_{h,2}\}|\alpha)}{\Pr(\{\beta_{\ell,2}, \beta_{h,2}\}|\beta)} \frac{\Pr(z|\{\alpha_{\ell,2}, \alpha_{h,2}\})}{\Pr(z|\{\beta_{\ell,2}, \beta_{h,2}\})} \frac{\Pr(\text{piv}|\{\alpha_{h,2}, \alpha_{\ell,2}\})}{\Pr(\text{piv}|\{\beta_{h,2}, \beta_{\ell,2}\})} \frac{\Pr(\ell|\alpha)}{\Pr(\ell|\beta)} \\
=\ &\lim_{n\to\infty} \frac{\Pr(\alpha|\text{piv}, \{\alpha_{h,2}, \alpha_{\ell,2}, \beta_{h,2}, \beta_{\ell,2}\}, z)}{\Pr(\beta|\text{piv}, \{\alpha_{h,2}, \alpha_{\ell,2}, \beta_{h,2}, \beta_{\ell,2}\}, z)} \frac{\Pr(\ell|\alpha)}{\Pr(\ell|\beta)} \\
=\ &\frac{\lambda}{1-\lambda} \frac{\Pr(\ell|\alpha)}{\Pr(\ell|\beta)},
\end{aligned} \tag{165}$$

where we omitted the dependence on σ_n and π_n in the notation. For the equality on the second line we used (162). For the equality on the third line, we used Bayes' rule and that a is send with probability 0 in the substates $\alpha_{h,2}$ and $\beta_{h,2}$ and with probability $\frac{1}{n^2}$ in the substates $\alpha_{\ell,2}$ and β_{ℓ_2}. For the equality on the fourth line we used (160). For the equality on the fifth line, I used that $\frac{\Pr(\{\alpha_{\ell,2},\alpha_{h,2}\}|\alpha)}{\Pr(\{\beta_{\ell,2},\beta_{h,2}\}|\beta)} = 1$ and I used that z is send with the same probability in all substates $\alpha_{h,2}, \alpha_{\ell,2}, \beta_{h,2}, \beta_{\ell,2}$ (see Figure 8). For the equality on the sixth line, I used Bayes' rule. For the equality on the seventh line, we used that $\lim_{n\to\infty} \hat{\rho}_z(\sigma_n^*, \pi_n, n) = \lim_{n\to\infty} \Pr(\alpha|\text{piv}, \{\alpha_{h,2}, \alpha_{\ell,2}, \beta_{h,2}, \beta_{\ell,2}\}, z)$ and 158. This finishes the proof of (163). The proof of (164) is analogous.

D.2.2 Proof: the General Case

Now, we study the general case when there exist two signals $h, \ell \in \{m_1, \ldots, m_k\}$ of the sender such that (101) and (102) hold. Consider any target policy vector $(x(m_i))_{i=1,\ldots,k}$.

The Information Structure. Consider $m \in \{h, \ell\}$. After receiving m, the sender releases signals according to the coarsenings $\pi_{2,n}^{x,y}$ in Figure 8 with the following slight modification of the signal probabilities in the substate m_1: if the target policy is $x(m) = A$, the sender releases signal b to all voters conditional on m_1. Conversely, if $x(m) = B$, the sender releases signal a to all voters conditional on m_1. Consider any other signal $m \notin \{h, \ell\}$. If $x(m) = B$, the sender releases signal a to all voters. If $x(m) = A$, the sender releases signal b to all voters.

Consider for example the case when $x(h) = B$ and $x(\ell) = A$ such that all voters receive a conditional on h_1 and all voters receive b conditional on ℓ_1. Now, abusing notation, we identify all substates of $\omega \in \{\alpha, \beta\}$ in which all voters receive a with the substate $\omega_{h,1}$ and all substates of ω in which all voters receive b with the substate $\omega_{\ell,1}$. With this notation, the proof of Theorem 7 for this case is the same as the proof for the case when the signals of the sender are binary (see Section D.2.1). If the target policies for the signals h and ℓ are different, i.e. $x(h) \neq B$ or $x(\ell) \neq A$, we proceed analogously. This finishes the proof of Theorem (7).

E Online Supplement

Computational Example

Note that one example of a distribution G on $[0,1]\times[-1,0]$ that induces a uniform distribution of 'thresholds of doubt', i.e. Φ with $\Phi(p)=p$ for all $p\in[0,1]$ is given by the density

$$g(t_\alpha,t_\beta)=\begin{cases}\sqrt{1+(\frac{t_\beta}{t_\alpha})^2}\cdot(2\cdot\int_{|t_\alpha|>|t_\beta|}\sqrt{1+(\frac{t_\beta}{t_\alpha})^2}dt)^{-1} & \text{if } \frac{-t_\beta}{t_\alpha-t_\beta}\le\frac{1}{2},\\ \sqrt{1+(\frac{t_\alpha}{t_\beta})^2}\cdot(2\cdot\int_{|t_\alpha|>|t_\beta|}\sqrt{1+(\frac{t_\beta}{t_\alpha})^2}dt)^{-1} & \text{if } \frac{-t_\beta}{t_\alpha-t_\beta}\ge\frac{1}{2}.\end{cases}$$

Lemma 5 *Consider any sequence of strategies $(\sigma_n)_{n\in\mathbb{N}}$ and any sequence of information structures $(\pi_n)_{n\in\mathbb{N}}$ with a common set of substates across n. Then, for any substates $\omega_i,\omega'_j\in\{\alpha_1,\ldots,\alpha_{N_\alpha}\}\cup\{\beta_1,\ldots,\beta_{N_\beta}\}$ and any $n\in\mathbb{N}$,*

$$\frac{\Pr(\text{piv}|\omega_i;\sigma_n,\pi_n)}{\Pr(\text{piv}|\omega'_j;\sigma_n,\pi_n)}=\left[1+\frac{(q(\omega'_j;\sigma^{\mathbf{P}})-\frac{1}{2})^2-(q(\omega_i;\sigma^{\mathbf{P}})-\frac{1}{2})^2}{\frac{1}{4}-(q(\omega'_j;\sigma^{\mathbf{P}})-\frac{1}{2})^2}\right]^n \tag{166}$$

Proof. Let $x_n=q(\omega_i;\sigma_n)-\frac{1}{2}$ and $y_n=q(\omega'_j;\sigma_n)-\frac{1}{2}$. Then,

$$\begin{aligned}\frac{q(\omega_i;\sigma_n)(1-q(\omega_i;\sigma_n))}{q(\omega'_j;\sigma_n)(1-q(\omega'_j;\sigma_n))} &= \frac{(\frac{1}{2}+x_n)(\frac{1}{2}-x_n)}{(\frac{1}{2}+y_n)(\frac{1}{2}-y_n)}\\ &= \frac{\frac{1}{4}-y_n^2+y_n^2-x_n^2}{\frac{1}{4}-y_n^2}\\ &= 1+\frac{y_n^2-x_n^2}{\frac{1}{4}-y_n^2}\end{aligned}$$

The claim follows from (8). ■

Fixed Point Argument.

Consider a belief $\mathbf{p}=(p_a,p_z,p_b)$ with

$$p_a\ge 0.95, \tag{167}$$

$$p_b\ge 0.95, \tag{168}$$

$$p_z\in[0.32,0.68]. \tag{169}$$

Given $(\pi_n)_{n\in\mathbb{N}} = (\pi_n^r)_{n\in\mathbb{N}}$ with $r = \frac{1}{2}$, we have the following bounds for $n \geq 8$:

$$\begin{aligned}
q(\omega_1; \sigma^{\mathbf{P}}, n) &\geq 0.95 \quad \text{for} \quad \omega_1 \in \{\alpha_1, \beta_1\}, && (170)\\
q(\alpha_2; \sigma^{\mathbf{P}}, n) &> 0.3 && (171)\\
q(\beta_2; \sigma^{\mathbf{P}}, n) &\leq 0.7. && (172)
\end{aligned}$$

In the following, we omit the dependence on $\sigma^{\mathbf{P}}$ and on π_n most of the time.

Step 1 *For any $n \in \mathbb{N}$ and any $\omega_1 \in \{\alpha_1, \beta_1\}, \omega_2' \in \{\alpha_2, \beta_2\}$,*

$$\frac{\Pr(\text{piv}|\omega_2')}{\Pr(\text{piv}|\omega_1;)} \geq (3.2)^n \tag{173}$$

Indeed,

$$\begin{aligned}
&\frac{\Pr(\text{piv}|\omega_2')}{\Pr(\text{piv}|\omega_1)}\\
\geq\ & [1 + \min_{\omega_1, \omega_2'} \frac{(q(\omega_1; \sigma^{\mathbf{P}}) - \frac{1}{2})^2 - (q(\omega_2'; \sigma^{\mathbf{P}}) - \frac{1}{2})^2}{\frac{1}{4} - (q(\omega_1; \sigma^{\mathbf{P}}) - \frac{1}{2})^2}]^n\\
\geq\ & (1 + (\frac{(\frac{9}{20})^2 - (\frac{4}{20})^2}{\frac{1}{4} - (\frac{9}{20})^2}))\\
\geq\ & (1 + \frac{65}{19})^n\\
\geq\ & (3.4)^n. && (174)
\end{aligned}$$

where we used Lemma 5 for the inequality on the second line.

Step 2 *For $n \geq 8$: $\rho_a(\sigma^{\mathbf{P}}) \geq 0.95, \rho_b(\sigma^{\mathbf{P}}) \geq 0.95$ and $\rho_z(\sigma^{\mathbf{P}}) \in [0.32, 0.68]$.*

First,

$$\rho_a(\sigma^{\mathbf{P}}) = 1 \tag{175}$$

since a is only sent in α. Second,

$$\begin{aligned}
\frac{\rho_b(\sigma^{\mathbf{P}})}{1-\rho_b(\sigma^{\mathbf{P}})} &= \frac{p_0}{1-p_0}\frac{\Pr(\alpha_2|\alpha)}{\Pr(\beta_1|\beta)}\frac{\Pr(b|\alpha_2)}{\Pr(b|\beta_1)}\frac{\Pr(\text{piv}|\alpha_2)}{\Pr(\text{piv}|\beta_1)} \\
&\geq \frac{1}{3}\frac{\frac{3}{n}\frac{1}{n^2}}{(1-\frac{1}{n})}(3.4)^n \\
&\geq 30 \quad \text{for} \quad n \geq 8.
\end{aligned}$$

where we used (174) for the inequality on the second line. Hence, for $n \geq 8$,

$$\rho(\sigma^{\mathbf{P}})_b \geq \frac{30}{1+30} > 0.95. \tag{176}$$

Third,

$$\begin{aligned}
\frac{\Pr(\text{piv}|\alpha_2)}{\Pr(\text{piv}|\beta_2)} &\leq [1+\frac{|(q(\beta_2;\sigma^{\mathbf{P}})-\frac{1}{2})^2-(q(\alpha_2;\sigma^{\mathbf{P}})-\frac{1}{2})^2|}{\frac{1}{4}-(q(\beta_2;\sigma^{\mathbf{P}})-\frac{1}{2})^2}]^n \\
&\leq (1+\frac{\frac{1}{n^4}+\frac{1}{n^2}}{\frac{1}{4}-\frac{16}{400}})^n \\
&\leq 2. \quad \text{for} \quad n \geq 8.
\end{aligned}$$

where we used Lemma 5 for the inequality on the first line. For the inequality on the second line, we used that z is sent with probability $1-\frac{1}{n^2}$ in both α_2 and β_2 such that the difference in the squared margins of victory cannot exceed $(x+\frac{1}{n^2})^2-x^2 \leq \frac{2x}{n^2}+\frac{1}{n^4}$ where x is the minimum margin of victory in the states α_2, β_2. Finally, the inequality follows since the margin of victory in both α_2 and β_2 is bounded by 0.2. So,

$$\begin{aligned}
\frac{\rho_z(\sigma^{\mathbf{P}})_z}{1-\rho_z(\sigma^{\mathbf{P}})} &= \frac{\Pr(\alpha)}{\Pr(\beta)}\frac{\Pr(\alpha_2|\alpha)}{\Pr(\beta_2|\beta)}\frac{\Pr(z|\alpha_2)}{\Pr(z|\beta_2)}\frac{\Pr(\text{piv}|\alpha_2)}{\Pr(\text{piv}|\beta_2)} \\
&= (1-\frac{1}{n^2})\frac{\Pr(\text{piv}|\alpha_2;\sigma^{\mathbf{P}})}{\Pr(\text{piv}|\beta_2;\sigma^{\mathbf{P}})} \\
&\leq 2 \quad \text{for} \quad n \geq 8.
\end{aligned}$$

Consequently, for all $n \geq 8$,

$$\rho(\sigma^{\mathbf{p}})_z \quad \leq \quad \frac{2}{3}. \tag{177}$$

Fourth,

$$\begin{aligned} \frac{\Pr(\text{piv}|\alpha_2)}{\Pr(\text{piv}|\beta_2)} &\geq (1 - \frac{|(q(\beta_2;\sigma^{\mathbf{p}}) - \frac{1}{2})^2 - (q(\alpha_2;\sigma^{\mathbf{p}}) - \frac{1}{2})^2|}{\frac{1}{4} - (q(\beta_2;\sigma^{\mathbf{p}}) - \frac{1}{2})^2} \\ &\geq (1 - \frac{\frac{1}{n^4} + \frac{1}{n^2}}{\frac{1}{4} - \frac{16}{400}})^n \\ &\geq 0.53 \quad \text{for} \quad n \geq 8. \end{aligned} \tag{178}$$

So, for all $n \geq 8$,

$$\begin{aligned} \frac{\rho(\sigma^{\mathbf{p}})_z}{1 - \rho(\sigma^{\mathbf{p}})_z} &= (1 - \frac{1}{n^2})\frac{\Pr(\text{piv}|\alpha_2;\sigma^{\mathbf{p}})}{\Pr(\text{piv}|\beta_2;\sigma^{\mathbf{p}})} \\ &\geq 0.5. \end{aligned}$$

This gives for all $n \geq 8$,

$$\rho(\sigma^{\mathbf{p}})_z \quad \geq \quad \frac{0.5}{1 + 0.5} \geq 0.32. \tag{179}$$

The claim follows from (175) - (179).

Step 3 *For $n \geq 8$, there is an equilibrium σ_n^* which satisfies (170) - (172).*

It follows from Step 2 that, for any $n \geq 8$, the continuous map that sends $\mathbf{p}$ to $\boldsymbol{\rho}(\sigma^{\mathbf{p}})$ is a self-map on the set of beliefs that satisfy (167) - (169). It follows from the Kakutani fixed point theorem that there exists fixed points $\mathbf{p}_n^*$ that satisfy (167) - (169). The corresponding strategies $\sigma^{\mathbf{p}_n^*}$ are equilibria (compare to (13)) and they satisfy (170) - (172).

Step 4 *Given the equilibrium σ_n^* for $n \geq 8$, the probability that A is elected is larger than $99.9\% \cdot (1 - \frac{3}{n})$.*

Evaluation of the binomial distribution shows that $\Pr(\mathcal{B}(2n+1, x)) > n) \geq 0.999999$ if $n \geq 8$ and $x \geq 0.95$. Hence, given σ_n^*, A is elected with probability larger than 99.9% in the states α_1 and β_1. Finally, the claim follows since these states occur with probability larger than $(1 - \frac{3}{n})$. The fourth step finishes the calculations for the example.

Proof of Theorem 8

To prove the theorem we use that Bayesian updating exhibits a supermartingale property: for any binary random variable $x \in \{\underline{x}, \overline{x}\}$ given by a prior probability $\Pr(\overline{x})$ and any finite random variable $y \in Y$,

$$\mathrm{E}_y(\Pr(\overline{x}|y)|\overline{x}) \quad \geq \quad \Pr(\overline{x}). \tag{180}$$

This means that when the state is $\overline{x}$, the average posterior conditional on y exceeds the prior. Similarly,

$$\mathrm{E}_y(\Pr(\overline{x}|y)|\underline{x}) \leq \Pr(\overline{x}). \tag{181}$$

For a given strategy σ, we apply (180) and (181) to the binary event where a given voter receives a given private signal $s_1 \in \{u, d\}$, the voter is pivotal and the state is α or where the given voter receives a given private signal $s_1 \in S_1$, the voter is pivotal and the state is β:

Lemma 6 *For any independent expansion π of π^c with finite signal set $\{u, d\} \times S_2$, any equilibrium σ given π and any $s_1 \in \{u, d\}$,*

$$\begin{aligned} & \mathrm{E}_{s_2}(\Pr(\alpha|\mathrm{piv}, s_1, s_2; \sigma)|\mathrm{piv}, s_1, \alpha) \\ \geq \; & \Pr(\alpha|\mathrm{piv}, s_1; \sigma) \\ \geq \; & \mathrm{E}_{s_2}(\Pr(\alpha|\mathrm{piv}, s_1, s_2; \sigma)|\mathrm{piv}, s_1, \beta). \end{aligned}$$

where the expectation is taken over the realizations of the additional signal s_2.

Consider any sequence of independent expansions $(\pi_n)_{n \in \mathbb{N}}$ of π^c and any equilibrium sequence $(\sigma_n^*)_{n \in \mathbb{N}}$ given π. We provide the proof for the case when

$\lim_{n\to\infty} \Pr(\alpha|\text{piv}; \sigma_n, \pi_n) \geq \lambda$.[38] Then, there must be a substate α_j of α for which the posterior likelihood conditional on being pivotal does not vanish as $n \to \infty$,

$$\alpha_j \in \Omega_\alpha = \{\alpha_i : \lim_{n\to\infty} \Pr(\alpha_i|\text{piv}; \sigma_n^*, \pi_n) \neq 0\}. \tag{182}$$

In the following, we sometimes omit the dependence on the information structures and the strategies from the notation. Now, we use Lemma 6 and the linearity of the aggregate preference function Φ to establish the first step.

Step 1 *The average vote share for A across the substates of α, weighted by their likelihood conditional on the pivotal event, exceeds the vote share implied by λ as $n \to \infty$, i.e.*

$$\lim_{n\to\infty} \mathrm{E}_{\alpha_j}(q(\alpha_j; \sigma_n)|\alpha, \text{piv}) \geq q(\alpha; \lambda). \tag{183}$$

We see that

[38] If $\lim_{n\to\infty} \Pr(\text{piv}|\sigma_n^*) \leq \lambda$, the proof is analogous and follows the same arguments where one just needs to replace α with β and larger equal signs with smaller equal signs.

$$
\begin{aligned}
& \mathrm{E}_{\alpha_j}\left(q\left(\alpha_j\right)|\alpha,\mathrm{piv}\right) \\
= & \sum_j \Pr\left(\alpha_j|\mathrm{piv},\alpha\right) q\left(\alpha_j\right) \\
= & \sum_j \Pr\left(\alpha_j|\mathrm{piv},\alpha\right) \sum_{s_1} \Pr\left(s_1|\alpha_j\right) \sum_{s_2} \Pr\left(s_2|\alpha_j\right)\left(\Phi\left(\Pr\left(\alpha|s_1,s_2,\mathrm{piv}\right)\right)\right) \\
= & \sum_{s_1} \Pr\left(s_1|\alpha\right)\left[\sum_j \Pr\left(\alpha_j|\mathrm{piv},\alpha\right)\left(\sum_{s_2} \Pr\left(s_2|\alpha_j\right)\Phi\left(\Pr\left(\alpha|s_1,s_2,\mathrm{piv}\right)\right)\right)\right] \\
= & \sum_{s_1} \Pr\left(s_1|\alpha\right) \mathrm{E}_{s_2}\left(\Phi\left(\Pr\left(\alpha|s_1,s_2,\mathrm{piv}\right)\right)|\mathrm{piv},\alpha\right) \\
= & \sum_{s_1} \Pr\left(s_1|\alpha\right) \mathrm{E}_{s_2}\left(\Phi\left(\Pr\left(\alpha|s_1,s_2,\mathrm{piv}\right)\right)|\mathrm{piv},s_1,\alpha\right) \\
= & \sum_{s_1} \Pr\left(s_1|\alpha\right) \Phi\left(\mathrm{E}_{s_2}\left[\Pr\left(\alpha|s_1,s_2,\mathrm{piv}\right)|\mathrm{piv},s_1,\alpha\right]\right) \\
\geq & \sum_{s_1} \Pr\left(s_1|\alpha\right)\left[\Phi(\Pr\left(\alpha|s_1,\mathrm{piv}\right))\right] \\
= & q(\alpha;\Pr(\alpha|\mathrm{piv}))
\end{aligned} \tag{184}
$$

where we used the formula (15) for the vote share of outcome A in a given substate for the equality on the third line. For the equality on the fourth line, we used that the probability of receiving a private signal $s_1 \in \{u,d\}$ only depends on the state $\omega \in \{\alpha,\beta\}$. For the equality on the sixth line we used that the private signals s_1 are independently and identically distributed conditional on ω. For the equality on the seventh line, we used the linearity of Φ. For the equality on the eigth line, we used Lemma 6. For the equality on the ninth line we use the definition of the vote share $q(\alpha;p)$ of A in α implied by a belief $p \in [0,1]$ (see (80)).

Since the function $q(\alpha,p)$ of the vote share implied by a belief $p \in [0,1]$ is strictly increasing in p, the assumption that $\lim_{n\to\infty}\Pr(\alpha|\mathrm{piv};\sigma_n^*,\pi_n) \geq \lambda$ together with (184) implies the assertion (183).

Step 2 *For any substate ω_i, the limit of the margin of victory weakly exceeds* *the margin of victory implied by λ in α as $n \to \infty$, i.e.*

$$\forall \omega_i : \lim_{n\to\infty} |q(\omega_i; \sigma_n^*, \pi_n, n) - \frac{1}{2}| \geq q(\alpha; \lambda, \pi^c) - \frac{1}{2} \tag{185}$$

For this, at first, we argue that for any $\alpha_j \in \Omega_\alpha$ and any other substate ω_i, the limit of the expected margin of victory in ω_i is weakly larger than in α_j, as $n \to \infty$, i.e.

$$\forall \alpha_j \in \Omega_\alpha, \forall \omega_i : \lim_{n\to\infty} |q(\alpha_j; \sigma_n^*) - \frac{1}{2}| \leq \lim_{n\to\infty} |q(\omega_i; \sigma_n^*) - \frac{1}{2}| \tag{186}$$

Suppose that the inequality in (186) does not hold for some $\alpha_j \in \Omega_\alpha$ and some ω_i. Then, Claim 2 implies that $\lim_{n\to\infty} \Pr(\alpha_j|\text{piv}; \sigma_n^*) = 0$, contradicting the assumption that $\alpha_j \in \Omega_\alpha$.
Now, note that (186) implies that the limit of the margin of victory is the same across all states $\alpha_j \in \Omega_\alpha$. Then, it follows from Step E that the limit margin of victory for the states in Ω_α weakly exceeds $q(\alpha; \lambda(\pi^c), \pi^c) - \frac{1}{2}$. The assertion follows from (186) for all other substates.

Step 3 *The equilibrium sequence $(\sigma_n)_{n\in\mathbb{N}}$ does not implement any pair of beliefs $(\mu_\alpha, \mu_\beta) \in (0,1)^2$ with $\mu_\alpha \in (\lambda_\alpha, \lambda)$ or $\mu_\beta \in (\lambda, \lambda_\beta)$.*

Step 2 implies that the equilibrium sequence can possibly only implement beliefs (μ_α, μ_β) for which the implied margins of victory $|q(\alpha; \mu_\alpha, \pi^c) - \frac{1}{2}|$ and $|q(\beta; \mu_\beta, \pi^c) - \frac{1}{2}|$ exceed the lower bound $q(\alpha; \lambda, \pi^c) - \frac{1}{2}$. Note that

$$|q(\alpha; \mu_\alpha, \pi^c) - \frac{1}{2}| \Rightarrow \mu_\alpha \in (\lambda_\alpha, \lambda)^c, \tag{187}$$

$$|q(\beta; \mu_\beta, \pi^c) - \frac{1}{2}| \Rightarrow \mu_\beta \in (\lambda, \lambda_\beta)^c, \tag{188}$$

given the definitions of λ, λ_α and λ_β through (81), (82) and (83). This finishes the proof of the theorem.

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