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Bargaining and Delay in Thin Markets

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Abstract

This paper presents a model of a decentralized thin market with endogenous and stochastic arrival of traders. We obtain that trade delay and price dispersion persist even when sellers and buyers are homogeneous and bargaining frictions are small. We characterize the time-dependent transaction prices as a function of the dynamics of the number of traders in the market, and provide some comparative statics results. Our analysis illustrates the necessity to properly incorporating the submarket structure into the study of job and housing markets, where the trade opportunities are typically constrained both by each trader's geographical location and individual characteristics.

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1 Introduction

Many markets are thin: at any given moment in time, they have few active traders. For example, there is increasing evidence that job seekers are typically locked into thin job markets determined by their commuting areas and their specific skills.[1] Similarly, people looking for renting or buying housing units typically focus on geographically reduced areas and a narrow range of characteristics. In these markets, the endogenous and stochastic evolution of the composition of the market (number of active sellers and buyers) and the market conditions (economic activity, legislation, market's "hotness," etc.) are crucial to determine the current and future trade opportunities, and hence the timing and the pricing of the transactions.

This paper studies how the endogenous evolution of a thin market shapes the timing and pricing of transactions. We document that the trade outcome of a thin market substantially differs from that predicted by models featuring a big decentralized market (see the literature review below). Most notably, we obtain that thin markets exhibit trade delay and price dispersion even when traders are homogeneous and bargaining frictions are small. We argue that prices are mostly determined by the stochastic dynamics of the market composition, and not the particular bargaining protocol used to set the prices. Some of the features of the predicted outcome of a big market, such as the absence of both delay and price dispersion, are recovered when the thin market is let grow by replication. Our analysis illustrates that adequately modeling some economic sectors as a sum of dynamic thin markets instead of as a static big market has a significant impact on the resulting predictions.

We construct a thin-market version of the Gale (1987) model. At any given moment in time, the market is composed by a finite number of sellers who own one unit of a homogeneous indivisible good, and a finite number of homogeneous buyers with a unit demand. There is also a multi-dimensional state capturing the market conditions, with exogenously evolving components (economic cycle, legislation) and endogenously evolving components (market hotness, local government policies). Once in the market, each trader keeps being randomly matched with traders from the other side of the market until she/he trades. Within each match, one of the traders is randomly chosen to make a take-it-or-leave-it offer. The other trader either accepts the offer, and both traders leave the market, or rejects it, and both traders continue in the market. The evolution of the market conditions depends on the composition of the market, the transactions that occur in the market, and the arrivals of sellers and buyers into the market. The composition and conditions of the market determine the arrival and matching rates. We study Markov perfect equilibria using the composition and conditions of the market as the state variable.

Our first result establishes that equilibrium offers are sometimes rejected. That is, a buyer and a seller sometimes fail to agree to trade, despite the fact that such disagreement does

[1] See, for example, Artuç, Chaudhuri, and McLaren (2010), Dix-Carneiro (2014), Artuç and McLaren (2015),Marinescu and Rathelot (2018), Manning and Petrongolo (2017), and Azar, Marinescu, Steinbaum, and Taska (2018).

not increase the gains from trade and delays their realization. The result follows from the observation that the rate at which the state of the market changes (either through a transaction or an arrival) when a trader decides not to trade depends on the side of the market the trader belongs to. Such rate is larger when the trader belongs to the long side of the market, since matches involving traders other than her/him are more likely to occur. We show that, when traders on the long side of the market expect their position to improve after a transaction takes place, their reservation value from not trading may be higher than the gains from trade that a trader short side of the market is willing to offer. In practice, this may correspond to the anticipation that real-state purchases by celebrities may initiate gentrification of some residential areas, and so some sellers may decide to wait to trade as a result. Also, as local authorities may incentivize entry in markets for specialized labor (teachers, nurses, etc) when supply is insufficient, some employers to wait to hire until such policies take place, so the supply increases (and wages go down). We prove that, nonetheless, there is never a "market breakdown;" that is, there is a strictly positive probability of trade in every match.

The second result establishes that price dispersion remains sizable when bargaining frictions are small. To prove this, we first show that when traders get matched increasingly often, the transaction price depends on the market composition and conditions, but not on whether the offer is made by a seller or a buyer. We call the equilibrium state-dependent price the "market price." We then argue that, at times when the market is imbalanced, traders on the long side of the market Bertrand compete, and the market price makes them indifferent between trading or not. When, instead, the market is balanced, we demonstrate that each trader gets a share of the trade surplus equal to the equilibrium share she/he would get in a Rubinstein bargaining game with stochastic outside options given by the possibility that there is an arrival or a change in the market conditions. These findings bring us to define a "risk-neutral measure" for each equilibrium, under which the market evolves *as if* a trader on the long side of the market deviated to not trading. We show that the market price at a given point it time is proportional to the discounted future time the market exhibits excess demand under the risk-neutral measure, adjusted by the discounted future time the market is balanced multiplied by the seller's Rubinstein payoff.

We obtain some properties of the market price process when bargaining frictions are small. First, even when the market composition does not drift toward being balanced, the market price increases in expectation when there is excess supply, and it decreases in expectation when there is excess demand. Second, akin to the standard results for big decentralized markets, trade delay disappears in the limit where the arrival rate of traders into the market increases. In this case, only the future expected excess demand and the market conditions are relevant to determine the current market price, but not the current market composition. Finally, we provide conditions that ensure immediate trade, which take the form of bounds on the effect of individual transactions on the arrival rates of traders to the market. We show that, under some conditions, raising the interest rate results in a mean-preserving spread of the equilibrium distribution of market prices.

We argue that our results are robust to many extensions of our model. For example, our characterization of the evolution of the market price (as a function of the dynamics of the number of traders) holds when the entry and exit of traders is endogenous. Generalizing the bargaining protocol to a general Nash bargaining one only affects the price proportionally to the fraction of time the market is balanced, so the bargaining protocol does not affect prices significantly if the market is rarely balanced. Finally, we argue that trade delay may be more prevalent when gains from trade depend on the market's conditions and, in this case, there may be a "market breakdown" in some states of the market.

1.1 Literature review

Our paper contributes to the literature on thin markets with arrival of traders. The paper closest to ours, Taylor (1995), analyzes a centralized market where sellers and buyers arrive over time. In every period, traders on the short side of the market make price offers, and when the market is balanced one side the market is chosen at random. Coles and Muthoo (1998) consider a similar model where sellers and buyers arrive in pairs, and they allow for heterogeneity in both buyers and goods. Similarly, Said (2011) studies a dynamic market in which buyers compete in a sequence of private-value second-price auctions. These papers analyze price dynamics under different price mechanisms in centralized markets with either constant arrival rates or immediate replacement of traders. Our focus is, instead, on analyzing decentralized bargaining with an endogenous arrival process. We characterize how the arrival process and bargaining asymmetries affect price dynamics and trade delay. This allows us to compare our results with some of the literature on big markets (see below).

Our paper is also related to the extensive literature on bargaining and matching in large markets, reviewed in Osborne and Rubinstein (1990) and Gale (2000).[2] Models in this literature feature a continuum of traders and non-stochastic population dynamics, which is often assumed to be in a stationary state.[3] By contrast, we focus, on how the endogenous stochastic process determining the number of traders on each side of the market affects and is affected by the trade outcome, and how both of these are determined by the bargaining protocol.

Finally, there has been some recent interest in thin markets in a network of traders. For example, Condorelli, Galeotti, and Renou (2016), Talamàs (2016), and Elliott and Nava (2019) look at bargaining in networks with immediate replacement of traders, and allow for differences in the valuation of the good by sellers and buyers. By contrast, we focus on how the dynamics

[2]Important contributions are Rubinstein and Wolinsky (1985), Gale (1987), Burdett and Coles (1997), Shimer and Smith (2000), Atakan (2006), Satterthwaite and Shneyerov (2007), Manea (2011) and Lauermann (2012). Section 4.3 considers the limit where traders arrive increasingly frequently, which is interpreted as the market growing by replication, and we compare the results on convergence to the competitive outcome of this literature.

[3]An exception is Manea (2017a), who studies a non-stationary market with a continuum of agents.

of the population determines the price process and bargaining outcomes in an endogenously growing complete network.

The rest of the paper is organized as follows. Section 2 introduces our model, and Section 3 provides the equilibrium analysis. Section 4 analyzes the equilibrium outcome when the bargaining frictions are small. Section 5 discusses time-varying gains from trade processes, bargaining protocols, and entry/exit of traders, and concludes. The Appendix provides proofs of the results.

2 The model

In this section, we introduce a model similar to Rubinstein and Wolinsky (1985) and Gale (1987). We keep the assumption that sellers and buyers are homogeneous to make the model tractable, and we focus the analysis on the endogenous evolution of the market.

Our setting has two key distinguishing features. The first is that the market is assumed to be "thin;" that is, the number of traders in the market at any given moment in time is a non-negative integer (instead of a mass) that stochastically changes over time. The second is that we allow the arrival and the matching processes to depend on the composition of the market and an endogenously evolving variable that encodes the "market conditions." Thus, we do not impose any restrictions on the matching technology, and account for the case where entry decisions are endogenous (see Section 5.1 below).

State of the market. Time is continuous with an infinite horizon, $t \in \mathbb{R}_+$. There are an infinite number of potential sellers and buyers. At any given moment in time t, there are $S_t \in \{0, ..., \bar{S}\}$ sellers and $B_t \in \{0, ..., \bar{B}\}$ buyers in the market, for some large $\bar{S}, \bar{B} > 0$. Also, we let ω_t denote the *market conditions*, which belongs to a finite set $\Omega \subset \mathbb{R}^n$. The *state of the market* at time t is defined to be (S_t, B_t, ω_t).[4]

Arrival process. Sellers arrive to the market at a Poisson rate $\gamma_{\mathrm{s}}(S_t, B_t, \omega_t) \in \mathbb{R}_+$, and buyers arrive to the market at a Poisson rate $\gamma_{\mathrm{b}}(S_t, B_t, \omega_t) \in \mathbb{R}_+$.

Bargaining. For ease of exposition, our base model uses a simplistic (yet canonical) bargaining protocol. As noted in Section 5.1, our results can be straightforwardly generalized to allow for general Nash bargaining.

If, at time t, there are sellers and buyers in the market (i.e., $S_t, B_t > 0$), a match occurs at a Poisson rate $\lambda(S_t, B_t, \omega_t) > 0$. When a match occurs, nature selects one of the sellers and one of the buyers in the market uniformly randomly, and also chooses the trader who makes a price offer. The probability of the seller being chosen is $\xi \in (0, 1)$. The trading counter-party

[4]The assumption that the number of traders in the market is bounded is technical and simplifies the intuitions and the proofs. Standard arguments—that is, taking sequences of models where $\bar{B}$ and $\bar{S}$ tend to $+\infty$—permit showing that our results apply when $\bar{S} = \bar{B} = \infty$ and the arrival rates atre bounded.

decides whether to accept the offer or not. If the offer is accepted, the good is transacted and the traders leave the market, whereas if the offer is rejected they continue in the market.

Dynamics of the market conditions. We allow the market conditions ω_t to stochastically evolve in three different ways. First, an exogenous change of the market conditions occurs at Poisson rate $\gamma_c(S_t, B_t, \omega_t) \geq 0$. Second, the market conditions change when there is a transaction at time t. Finally, market conditions change when a trader arrives into the market. In all three cases, the new market conditions are drawn from a distribution that depends on the time-t's state of the market and the kind of event triggering the change (exogenous change, transaction, arrival of a seller, or arrival of a buyer).[5]

Payoffs. All sellers value the good at 0 and all buyers value it at 1. Both sellers and buyers discount the future at rate $r > 0$. If a seller and a buyer trade at time t at price p they obtain, respectively, $e^{-r\,t}\,p$ and $e^{-r\,t}\,(1-p)$. If they never trade they both obtain 0. Both sellers and buyers are risk-neutral and expected-utility maximizers. Even though the formal expressions for the payoffs (and the conditions for the optimality of a strategy profile) are obtained using a standard recursive analysis, their length makes it convenient to defer them to Appendix A.1.

Strategies. To simplify the model setting, we focus directly on Markov strategies using the state of the market as the state variable. Thus, the strategy of a trader (seller or buyer) maps each state (S, B, ω) with $S, B > 0$ both to a price offer distribution in $\Delta(\mathbb{R}_+)$ and to a probability of acceptance for each possible offer received. These shall interpreted to be her/his strategy in the bargaining game if she/he is matched and the market state is (S, B, ω).[6]

Equilibrium concept. We focus on *symmetric Markov perfect equilibria*, where all traders on each side of the market use the same strategy (see Appendix A.1 for the formal definition). We will refer to symmetric Markov perfect equilibria as just "equilibria."

3 Equilibrium analysis

3.1 Equilibrium continuation values and existence

We begin this section by presenting the equations that the continuation value of each type of trader satisfies in an equilibrium, and then stating the existence of an equilibrium.

Fix an equilibrium. We use $V_s(S, B, \omega)$ to denote a seller's continuation value at some

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[5]The market conditions capture a broad set of economic variables relevant for determining the entry of traders into the market. Some of the components of ω may evolve independently of the rest, like for example shocks to the country's or world's economy. The evolution of some other components may be endogenous, and depend on the number of traders, the transactions or the arrival of traders in the market.

[6]We implicitly assume that traders observe the state of the market. Markov perfect equilibria (see the definition below) remain equilibria independently of the information structure as long as the current state of the market is known to the traders in the market.

state (S, B, ω) with $S > 0$, and $V_{\mathrm{b}}(S, B, \omega)$ to denote a buyer's continuation value at some state (S, B, ω) with $B > 0$, both defined formally in Appendix A.1. To ease notation, we will sometimes use N_{s} and N_{b} to denote S and B, respectively. We will also refer to sellers and buyers as, respectively, s-traders and b-traders.

For $\theta \in \{\mathrm{s}, \mathrm{b}\}$, the continuation value of a θ-trader at some state (S, B, ω) can be written as

$$V_\theta = \overbrace{\frac{\frac{1}{N_\theta}\lambda}{\lambda+\gamma+r} V_\theta^{\mathrm{m}}}^{\text{own match}} + \overbrace{\frac{\frac{N_\theta-1}{N_\theta}\lambda}{\lambda+\gamma+r} V_\theta^{\mathrm{o}}}^{\text{others match}} + \overbrace{\frac{\gamma}{\lambda+\gamma+r} V_\theta^{\mathrm{e}}}^{\text{exog. change}} , \tag{3.1}$$

where here, and in the rest of the paper, we omit the dependence of all V_θ's, λ, and $\gamma \equiv \gamma_{\mathrm{s}} + \gamma_{\mathrm{b}} + \gamma_{\mathrm{c}}$, on the state of the market when there is no risk of confusion, and where V_θ^{m}, V_θ^{o}, and V_θ^{e} are defined below. The continuation value is divided into the following three components:

1. **Own match:** Consider a seller who is matched with a buyer. If she is chosen to make the offer, she can offer an unacceptable price (say above 1), which provides her with a continuation value of V_{s}. She can alternatively make an offer intended to be acceptable to the buyer. Since the continuation value of a buyer from rejecting the offer is V_{b}, he accepts for sure price offers strictly lower than $1 - V_{\mathrm{b}}$, and rejects offers strictly above $1 - V_{\mathrm{b}}$. Using the standard argument for take-it-or-leave-it offers, equilibrium offers by the seller that are accepted with positive probability are equal to $1 - V_{\mathrm{b}}$. If, instead, the buyer is chosen to make the offer, the seller receives a payoff of V_{s}: in equilibrium, if the offer is acceptable, the buyer makes her indifferent whether to accept it or not. Hence, we have

$$V_{\mathrm{s}}^{\mathrm{m}} = \xi \max\{V_{\mathrm{s}}, 1-V_{\mathrm{b}}\} + (1-\xi)\, V_{\mathrm{s}} . \tag{3.2}$$

The analogous equation for a buyer is given by

$$V_{\mathrm{b}}^{\mathrm{m}} = \xi\, V_{\mathrm{b}} + (1-\xi) \max\{V_{\mathrm{b}}, 1-V_{\mathrm{s}}\} . \tag{3.3}$$

2. **Others match:** The continuation value of a θ-trader if other traders match depends on the acceptance probability of the equilibrium offers. This value can be written as

$$V_\theta^{\mathrm{o}} = \alpha\, \mathbb{E}^{\mathrm{t}}[V_\theta(S-1, B-1, \tilde{\omega})] + (1-\alpha)\, V_\theta , \tag{3.4}$$

where $\alpha \equiv \alpha(S, B, \omega)$ is the equilibrium probability that a seller and a buyer trade when they are matched in state (S, B, ω), and where $\mathbb{E}^{\mathrm{t}}$ is the expectation conditional on trade (as before, we omit the dependence of this expectation on the state of the market). It is important to notice that, if the net surplus from trade is positive (i.e., $1 - V_{\mathrm{s}} - V_{\mathrm{b}} > 0$), the equilibrium offer is accepted for sure in any match in state (S, B, ω) (hence $\alpha = 1$), whereas if it is negative (i.e., $1 - V_{\mathrm{s}} - V_{\mathrm{b}} < 0$), the equilibrium offer is rejected for sure (hence $\alpha = 0$).

3. **Exogenous change:** The state of the market changes exogenously at rate $\gamma = \gamma_{\mathrm{s}} + \gamma_{\mathrm{b}} + \gamma_{\mathrm{c}}$. Conditional on an exogenous change occurring, it consists on the arrival of a seller with probability $\frac{\gamma_{\mathrm{s}}}{\gamma}$, the arrival of a buyer with probability $\frac{\gamma_{\mathrm{b}}}{\gamma}$, and a change in the conditions of the market with probability $\frac{\gamma_{\mathrm{c}}}{\gamma}$. This implies that the continuation value of a θ-trader conditional on an exogenous change in the state of the market arrival can be written as

$$V_\theta^{\mathrm{e}} = \tfrac{\gamma_{\mathrm{s}}}{\gamma}\,\mathbb{E}^{\mathrm{s}}[V_\theta(S{+}1,B,\tilde{\omega})] + \tfrac{\gamma_{\mathrm{b}}}{\gamma}\,\mathbb{E}^{\mathrm{b}}[V_\theta(S,B{+}1,\tilde{\omega})] + \tfrac{\gamma_{\mathrm{c}}}{\gamma}\,\mathbb{E}^{\mathrm{c}}[V_\theta(S,B,\tilde{\omega})] \tag{3.5}$$

for both $\theta \in \{\mathrm{s},\mathrm{b}\}$, where $\mathbb{E}^{\mathrm{s}}$, $\mathbb{E}^{\mathrm{b}}$ and $\mathbb{E}^{\mathrm{c}}$ are, respectively, the expectations conditional on the arrival of a seller, the arrival of a buyer, or a change in the market's conditions.

It is important to notice that α is the only equilibrium object that appears in the equations for the continuation payoffs (3.1)–(3.5). It is not difficult to see that for any choice of α (as a map from the state space to $[0,1]$) determines the continuation values uniquely. Also, as we argued before, there is an equilibrium generating some α only if $\alpha = 1$ when $V_{\mathrm{s}}+V_{\mathrm{b}} < 1$ and $\alpha = 0$ when $V_{\mathrm{s}}+V_{\mathrm{b}} > 1$, for all states (S,B,ω). The proof of the following result uses these observations to prove that, in fact, there is some α satisfying these necessary conditions, and that an equilibrium exists as a result. (Note that, given our focus on symmetric MPE, the existence of an equilibrium is not guaranteed by standard fixed-point arguments.)

Proposition 3.1. *An equilibrium exists. The continuation values in an equilibrium are uniquely determined by the probability of agreement α, and satisfy equations* (3.1)–(3.5).

3.2 Equilibrium properties

We continue our analysis with a result that characterizes some important features of equilibrium behavior. In the following result, and in the rest of the paper, we use $\bar{\theta}$ to denote the complementary type of a trader's type $\theta \in \{\mathrm{s},\mathrm{b}\}$, and so $\{\theta,\bar{\theta}\} = \{\mathrm{s},\mathrm{b}\}$.

Proposition 3.2. *In any equilibrium, for any state (S,B,ω) with $S,B > 0$,*

1. *there is strictly positive probability of trade in every match; that is, $\alpha > 0$;*
2. *there is trade for sure when the market is balanced; that is, $\alpha = 1$ when $S = B$; and*
3. *if $\alpha < 1$ then the traders who are on the long (short) side of the market gain (lose) from other trades; that is, for $\theta \in \{\mathrm{s},\mathrm{b}\}$ such that $N_\theta > N_{\bar{\theta}}$, we have $V_\theta^{\mathrm{o}} > V_\theta$ and $V_{\bar{\theta}}^{\mathrm{o}} < V_{\bar{\theta}}$.*

The first part of Proposition 3.2 establishes that there is no equilibrium and state of the market where equilibrium offers are rejected for sure. Hence, even though equilibrium offers may be rejected with strictly positive probability, there is never a "market breakdown." In other words, in equilibrium, there are no periods of time where there are both sellers and buyers in the market and trade happens with zero probability.

The proof proceeds by contradiction. Assume that there is an equilibrium and a state (S, B, ω) where equilibrium offers are rejected for sure; that is, $\alpha = 0$. This implies that the joint continuation value of a seller and a buyer in this state, which is denoted

$$V(S, B, \omega) \equiv V \equiv V_{\mathrm{b}} + V_{\mathrm{s}} \, ,$$

is weakly higher than the trade surplus; that is, $V \geq 1$. There thus exists a state (S', B', ω') (possibly equal to (S, B, ω)) in which $V(S', B', \omega')$ is maximal across all states and $\alpha(S', B', \omega') = 0$. Nevertheless, in this case, we have a contradiction:

$$V(S', B', \omega') = \tfrac{\gamma(S', B', \omega')}{\gamma(S', B', \omega') + r} \, V^{\mathrm{e}}(S', B', \omega') \leq \tfrac{\gamma(B', S', \omega')}{\gamma(S', B', \omega') + r} V(S', B', \omega') < V(S', B', \omega') \, .$$

The second part of Proposition 3.2 establishes that, when there is a match and the market is balanced (i.e., $S = B$), there is trade with probability one. Intuitively, when the market is balanced, a seller and a buyer "agree" on the relative likelihood of the three events that potentially change the state (matching, others matching, and exogenous changes). Since their joint surplus is never higher than 1 (by part 1), we have that

$$V = \tfrac{\frac{1}{S} \lambda}{\lambda + \gamma + r} \underbrace{V^{\mathrm{m}}}_{=1} + \tfrac{\frac{S-1}{S} \lambda}{\lambda + \gamma + r} \underbrace{V^{\mathrm{o}}}_{\leq 1} + \tfrac{\gamma}{\lambda + \gamma + r} \underbrace{V^{\mathrm{e}}}_{\leq 1} \leq \tfrac{\lambda + \gamma}{\lambda + \gamma + r} < 1 \, .$$

As we see, their joint surplus of a seller and a buyer from not agreeing is strictly lower than 1 since they discount the time when the next event occurs.

The last part of Proposition 3.2 establishes that, if equilibrium offers are rejected with a positive probability at some state (S, B, ω), then a trader on the long side of the market benefits from other traders' transactions in such a state.

To shed some light on this result, consider the case where sellers are on the long side of the market (i.e., $S > B$). As equation (3.1) shows, the rate at which there is a match involving other traders is, from a seller's perspective, $\frac{S-1}{S} \lambda$. This rate is lower from a buyer's perspective, for whom it equals $\frac{B-1}{B} \lambda$. Thus, the weight of the event "others match" is larger in determining the sellers' continuation value than in determining the buyers' (see equation (3.1)). The joint continuation value of a seller and a buyer can then be written as

$$V = \tfrac{\lambda}{\lambda + \gamma + r} \Big(\overbrace{\tfrac{1}{S} V_{\mathrm{s}}^{\mathrm{m}} + \tfrac{S-1}{S} V_{\mathrm{s}}^{\mathrm{o}} + \tfrac{1}{B} V_{\mathrm{b}}^{\mathrm{m}} + \tfrac{B-1}{B} V_{\mathrm{b}}^{\mathrm{o}}}^{(*)} \Big) + \tfrac{\gamma}{\lambda + \gamma + r} V^{\mathrm{e}} \, . \tag{3.6}$$

Assume that offers are rejected with a positive probability, and so $V = V^{\mathrm{m}} = 1$. Since, by the first part of Proposition 3.2, V^{o} and V^{e} are weakly lower than 1, the previous equation holds only if $V_{\mathrm{s}}^{\mathrm{o}} > V_{\mathrm{s}}$ and $V_{\mathrm{b}}^{\mathrm{o}} < V_{\mathrm{b}}$. In this case, the greater weight that a seller assigns to the event that two other traders match makes the term $(*)$ in the previous expression strictly greater than 1 (which is necessary for V to be equal to 1). In fact, it can be written as

$$1 < (*) = \underbrace{\tfrac{S-B}{S\,B} \left(V_{\mathrm{s}}^{\mathrm{o}} - V_{\mathrm{s}} \right)}_{>0} + \underbrace{\tfrac{1}{B} V + \tfrac{B-1}{B} V^{\mathrm{o}}}_{\leq 1} = \underbrace{\tfrac{S-B}{S\,B} \left(V_{\mathrm{b}} - V_{\mathrm{b}}^{\mathrm{o}} \right)}_{>0} + \underbrace{\tfrac{1}{S} V + \tfrac{S-1}{S} V^{\mathrm{o}}}_{\leq 1} \, .$$

The next section provides an illustrative example of a thin market where all equilibria exhibit trade delay.

3.3 An example with trade delay

This section sheds light on how trade delay may arise in equilibrium. The example illustrates that, even though all sellers and all buyers are homogeneous and do not have private information, equilibria with immediate trade may fail to exist.

We focus on one state of the world $(S, B, \omega) = (2, 1, \omega)$, that is, a state with two sellers and one buyer in the market, and where the market conditions are equal to some $\omega \in \Omega$. The sellers are two fading stars looking for selling their mansions in a neighborhood in Los Angeles, both next to the ocean. The buyer is a rising Hollywood star who wants to buy a mansion next to the ocean in the same neighborhood. We assume there is another rising star who is initially out of the market. In the begining, the neighborhood is unfashionable, and so it is more likely that another fading star goes bankrupt than that the other rising actor becomes interested in buying a mansion. We then set, for simplicity, $\gamma_{\mathrm{s}} > \gamma_{\mathrm{b}} = \gamma_{\mathrm{c}} = 0$ (all rates are evaluated at $(2, 1, \omega)$). If a seller arrives, the strong competition between the three sellers gives the buyer a high continuation payoff, which is assumed to be 1, and the sellers obtain 0. If, instead, a transaction occurs before the arrival of a seller, the sale is publicized by tabloids, and the neighborhood becomes "trendy." This attracts the other rising star and the remaining seller obtains a high continuation payoff, which is assumed to be equal to 1.[7]

We first compute the continuation values of the sellers and the buyer under the assumption that, in each match, the price offer is equal to the continuation value of the trader receiving the offer, and that such an offer is accepted for sure (i.e., equations (3.1)–(3.5) hold with $\alpha = 1$). The continuation values in state $(2, 1, \omega)$ solve the following system of equations:

$$
\begin{aligned}
V_{\mathrm{s}} &= \overbrace{\tfrac{\lambda/2}{\lambda+\gamma+r}\left(\xi(1-V_{\mathrm{b}})+(1-\xi)V_{\mathrm{s}}\right)}^{\text{own match}} + \overbrace{\tfrac{\lambda/2}{\lambda+\gamma+r}\,1}^{\text{others match}} + \overbrace{\tfrac{\gamma}{\lambda+\gamma+r}\,0}^{\text{exog. change}}\ , \\
V_{\mathrm{b}} &= \underbrace{\tfrac{\lambda}{\lambda+\gamma+r}\left(\xi V_{\mathrm{b}}+(1-\xi)(1-V_{\mathrm{s}})\right)}_{\text{own match}} + \underbrace{\tfrac{0}{\lambda+\gamma+r}\,0}_{\text{others match}} + \underbrace{\tfrac{\gamma}{\lambda+\gamma+r}\,1}_{\text{exog. change}}\ .
\end{aligned}
$$

Solving for the previous system of equations we obtain

$$
V_{\mathrm{s}} + V_{\mathrm{b}} = 1 + \tfrac{\gamma\,(\lambda-2\,r)-2\,r^2}{(\gamma+\lambda+r)\,(2\,\gamma+(1-\xi)\,\lambda+2\,r)}\ ,
$$

[7]The continuation values after arrivals are assumed to be 0 or 1 for simplicity. These can be approximated, in a full version of our model, while preserving the important features of the example. For instance, a high continuation payoff for the remaining seller after a transaction is obtained if the other rising star arrives fast after the transaction and ξ is close to 1. Analogously, a high continuation value of the rising star when a seller arrives follows if each of the three sellers anticipates that, even if she lets another seller trade with the rising star, there will be a fierce competition when the other rising star enters the market.

which is strictly higher than 1 if λ is large or r is small. As we see, a seller's value if she decides not to trade at state $(2,1,\omega)$ is $\frac{\lambda/2}{\lambda/2+\gamma+r}$, so she is not willing to trade at a price lower than this value. Also, the buyer obtains a continuation value equal to $\frac{\gamma}{\gamma+r}$ from not trading, so he is not willing to trade at a price higher than $\frac{r}{\gamma+r} = 1 - \frac{\gamma}{\gamma+r}$. As a result, if either λ is large or r is small, any equilibrium in this reduced version our model has the property that offers are rejected with positive probability. Using α to denote the probability of agreement in a match in state $(2,1,\omega)$ then, in any equilibrium of the game, we have

$$\alpha = \min\left\{1, \tfrac{2\,r\,(\gamma+r)}{\gamma\,\lambda}\right\} .$$

Notice that the rate at which an agreement occurs in state $(2,1,\omega)$ (which equals $\alpha\,\lambda$) converges to $\frac{2\,r\,(\gamma+r)}{\gamma}$ as λ becomes big; that is, a significant trade delay remains even in the limit where bargaining frictions disappear.

To obtain further intuition, observe that the weight of a (own or other's) match in both the sellers' and the buyer's payoffs is the same, equal to $\frac{\lambda}{\lambda+\gamma+r}$. Nevertheless, conditional on a match occurring, each seller is chosen with probability $\frac{1}{2}$ and therefore she obtains $\frac{1}{2}\,V_{\text{s}}^{\text{m}} + \frac{1}{2}\,V_{\text{s}}^{\text{o}}$; while the buyer is chosen for sure and he obtains V_{b}^{m}. Hence, even if $V^{\text{m}} \leq 1$, the sum of the payoffs of a seller and a buyer conditional on a match may be bigger than 1 if sellers benefit from other's transactions (note that $\frac{1}{2}\,V_{\text{s}}^{\text{o}} + \frac{1}{2}\,V_{\text{b}}^{\text{m}}$ may be bigger than $\frac{1}{2}$). Equilibrium trade delay occurs then when agents on the long side of the market benefit when other traders trade, as they assign a larger probability that other traders' trade than the agents on the short side of the market.

The equilibrium behavior of the sellers in the market resembles a war of attrition: each of them trades at the rate that makes the other seller (and the buyer) indifferent whether to trade at price $\frac{r}{\gamma+r}$ or not. From each seller's perspective, such delay lowers the value of making unacceptable offers, since doing so comes with the risk of another seller arriving. As time passes, either one of the sellers trades (and the remaining seller obtains a high payoff), or another seller arrives (and all sellers obtain a low continuation payoff).

Our example illustrates that trade delay may occur when transactions affect the future arrival rates of traders into the market. In the example, a house sold to a celebrity attracts other celebrities in the same neighborhood. Similarly, in a local labor market, the absence of some specialized workers (teachers, doctors, ...) may lead the local government to initiate a subsidy program (free housing, lower taxes, ...) to increase the arrival of workers into the market. More generally, some transactions may make the market more visible to otherwise inattentive traders or generate externalities over the value of the goods being sold in the market (see the discussion on time-varying gains from trade in Section 5.1), or may trigger actions by third parties (regulators, market makers, ...) that change the dynamics of some thin markets.

Remark 3.1. Inefficient delay can also be found in other bargaining models with complete information. For example, Cai (2000) analyzes a model of one-to-many bargaining between farmers and a railroad company, where the gains from trade are realized only if all the farmers agree. Similar to our model, the farmers want other farmers to trade, to gain monopsony power. Also, Abreu and Manea (2012b,a) and Elliott and Nava (2019) analyze bargaining models in networks in which delay may happen only when traders are heterogeneous, in terms of their value from trade or their position in the network. Our example illustrates that the endogenous evolution of a thin market may generate equilibrium trade delay, as traders may benefit from transactions by other traders.

4 Small bargaining frictions

We now turn to the case where the bargaining frictions are small, that is, where traders in the market are matched frequently. This may be a plausible assumption in some thin markets such as localized housing markets or job markets for specific occupations, where the rate at which traders (can) meet once they are in the market is much higher than the arrival rate to the market. As in the large markets literature, studying the case where frictions are small will allow us to provide a sharper characterization of the equilibrium outcome.

4.1 Notation and preliminary result

In order to analyze the case where bargaining frictions are small, we separate each state's matching rate $\lambda = \lambda(S, B, \omega)$ into two parts. The first is a state-independent common factor $k > 0$, which will taken to be big. The second is a function $\ell(S, B, \omega)$, measuring the relative rate at which traders match in each state. Thus, from now on, we use $\lambda(S, B, \omega)$ and $k\,\ell(S, B, \omega)$ interchangeably.

The difficulty of characterizing how some properties of equilibrium outcomes change "when bargaining frictions are small" is that our model may have multiple equilibria. The following notation is then convenient in comparing the properties of equilibrium outcomes as k increases. The notation "$\simeq$" indicates that terms on each of the sides are equal in any equilibrium, except for terms that go to 0 as k goes to $+\infty$ (see footnote 8). The next result establishes that when bargaining frictions are small, the joint continuation value of a seller and a buyer is close to the joint surplus they obtain from trade.

Lemma 4.1. *$V \simeq 1$ for all states (S, B, ω) with $S, B > 0$.*[8]

To get an intuition for Lemma 4.1 note that, for a fixed equilibrium, there are three kinds of states. The first kind comprises all states where $S, B > 0$ and equilibrium offers are rejected with a positive probability. For these states, $V = 1$. The second kind comprises all states where

[8]The notation "$V \simeq 1$" should be read as "For any sequence $(k_n)_n$ and corresponding sequence of equilibria, and for any state (S, B, ω), we have that $\lim_{n\to\infty}(V_n - 1) = 0$."

either $S = 1$ or $B = 1$ (or both). If for example there is only one buyer, so $S \geq B = 1$, his continuation value can be approximated as follows:

$$V_{\rm b} \simeq \xi\, V_{\rm b} + (1-\xi)\,(1-V_{\rm s}) \quad \Rightarrow \quad V_{\rm b} \simeq 1-V_{\rm s}\ ,$$

and so $V \simeq 1$. Intuitively, given that matches occur very frequently, the buyer can almost costlessly wait until he makes the offer and obtain $1-V_{\rm s} \geq V_{\rm b}$. Finally, the third kind comprises states where $S, B > 1$ and there is immediate trade. In a state in this set, a buyer has the option to wait for a transaction to occur, and so $V_{\rm b} \succeq \mathbb{E}^{\rm t}[V_{\rm b}(S-1, B-1, \tilde{\omega})]$, where "$\succeq$" means that the terms on left side are bigger than those on the right hand side except for terms that vanish as $k \to \infty$. By the same argument we have $V_{\rm s} \succeq \mathbb{E}^{\rm t}[V_{\rm s}(S-1, B-1, \tilde{\omega})]$. This implies that

$$1 \geq V \succeq \mathbb{E}^{\rm t}[V(S-1, B-1, \tilde{\omega})].$$

For each realization of $\tilde{\omega}$, the state $(S-1, B-1, \tilde{\omega})$ belongs to one of the previous three kinds of states. If it belongs to the first or second kind, we have $V(S-1, B-1, \tilde{\omega}) \simeq 1$. If it belongs to the third kind, we can use the same argument again. It is clear that after at most $\min\{S, B\} - 1$ times, a state of one of the first two kinds is going to be reached. Since the expected time for this to happen will shrink to 0 as $k \to \infty$, we have $V \simeq 1$ also for states of the third kind.

An immediate and important consequence of Lemma 4.1 is that, when bargaining frictions are small, a seller is approximately indifferent between whether to trade or not in all states (S, B, ω) where $S > B \geq 1$. This is obviously true if $\alpha < 1$ (the first kind of states defined above). When, instead, $\alpha = 1$ and $S > 1$, the payoff of a seller is

$$V_{\rm s} \simeq \tfrac{1}{S}\, V_{\rm s} + \tfrac{S-1}{S}\, \mathbb{E}^{\rm t}[V_{\rm s}(S-1, B-1, \tilde{\omega})]\ . \tag{4.1}$$

Thus, from the previous equation, it follows that $V_{\rm s} \simeq \mathbb{E}^{\rm t}[V_{\rm s}(S-1, B-1, \tilde{\omega})]$; that is, the seller obtains approximately the same payoff if she follows the equilibrium strategy and if she decides not to trade at state (S, B, ω).

Another implication of Lemma 4.1 is that there within-state price dispersion vanishes when the bargaining frictions are small. Indeed, the transaction price in state (S, B, ω) is either $V_{\rm s}$ (if the buyer makes the offer) or $1 - V_{\rm b}$ (if the seller makes the offer). Since $V \simeq 1$, we have that $V_{\rm s} \simeq 1-V_{\rm b}$. We therefore call $V_{\rm s} = V_{\rm s}(S, B, \omega)$ the *market price*, which is the approximate price at which transactions take place in each state (S, B, ω). The following section shows that there is market price dispersion across states.

4.2 Characterization of the market price

We now provide a characterization of the market price. We do so by changing the probability measure that determines the evolution of the state of the market.

Each equilibrium induces a measure for the evolution of the state of the market. Under such a measure, transactions occur at rate $\alpha(S_t, B_t, \omega_t)\, \lambda(S_t, B_t, \omega_t)$ and exogenous changes in the state of the market occur at rate $\gamma(S_t, B_t, \omega_t)$.

With the aim of characterizing the market price, we now fix an equilibrium, and define a measure different from the equilibrium measure. Under the new measure, exogenous changes of the state of the market occur indentically as under the equilibrium measure. Now, under the new measure, transactions occur at rate

$$\tilde{\delta}(S_t, B_t, \omega_t) \equiv \begin{cases} \frac{B_t-1}{B_t}\,\alpha(S_t, B_t, \omega_t)\,\lambda(S_t, B_t, \omega_t) & \text{if } S_t \leq B_t, \\ \frac{S_t-1}{S_t}\,\alpha(S_t, B_t, \omega_t)\,\lambda(S_t, B_t, \omega_t) & \text{if } S_t > B_t, \end{cases} \tag{4.2}$$

instead of at rate $\alpha(S_t, B_t, \omega_t)\,\lambda(S_t, B_t, \omega_t)$. Such measure will prove to be very useful to compute the market price.

Changing the measure to compute equilibrium prices is in the same spirit as the use of risk-neutral measures in the study of financial markets. The risk-neutral measure is often defined as the evolution that prices would follow if sellers were competitive and risk-neutral. So, typically, the current value of a financial asset is equal to its payoffs in the future discounted at the risk-free rate, averaged using the risk-neutral measure. A risk-neutral agent would then be indifferent between trading the asset at any moment under the risk-neutral measure.

By a slight abuse of language, we call our modified measure measure the *risk-neutral measure* (of the fixed equilibrium). Since, in our model, the side of the market with more traders stochastically changes over time, the risk-neutral measure corresponds to the evolution of the state of the market "when, at each time, one trader on the long side of the market deviates to not trading." Note that the dynamics of the state of the market under the risk-neutral measure are determined from the equilibrium dynamics of the state of the market (and can therefore be uniquely pinned down by an external observer who only observes the evolution of the state). The following proposition uses the risk-neutral measure to characterize the market price.

Proposition 4.1. *For any state (S_0, B_0, ω_0) we have that*

$$V_{\mathrm{s}}(S_0, B_0, \omega_0) \simeq \tilde{\mathbb{E}}\left[\int_0^{\infty} e^{-r\,t}\left(\mathbb{I}_{S_t<B_t} + \xi\,\mathbb{I}_{S_t=B_t}\right) r\,\mathrm{d}t\right], \tag{4.3}$$

where $\tilde{\mathbb{E}}$ is the expectation using the risk-neutral measure.

Proposition 4.1 gives an approximation of the transaction (and market) price at each state (S_0, B_0, ω_0) in terms of the equilibrium dynamics of the state of the market and the probability that a seller makes an offer. It is a discounted average (under the risk-neutral measure) of the future time the market exhibits excess demand, adjusted by the times it is balanced.

To obtain some intuition for Proposition 4.1, fix a state (S, B, ω) where the market is imbalanced. If there are more sellers than buyers (i.e., $S > B$), sellers are approximately indifferent whether to trade or not, and this implies that

$$V_{\mathrm{s}} \simeq \frac{\frac{S-1}{S}\,\alpha\,\lambda}{\frac{S-1}{S}\,\alpha\,\lambda+\gamma+r}\,V_{\mathrm{s}}^{\mathrm{t}} + \frac{\gamma}{\frac{S-1}{S}\,\alpha\,\lambda+\gamma+r}\,V_{\mathrm{s}}^{\mathrm{e}}\,, \tag{4.4}$$

where $V_s^t \equiv \mathbb{E}^t[V_s(S-1,B-1,\tilde{\omega})]$. A similar equation can be obtained when there are more buyers than sellers in the market (replacing s by b and S by B). Using Lemma 4.1 we can write, when $S < B$,

$$\overbrace{1-V_b}^{\simeq V_s} \simeq \frac{r}{\frac{B-1}{B}\,\alpha\,\lambda+\gamma+r} + \frac{\frac{B-1}{B}\,\alpha\,\lambda}{\frac{B-1}{B}\,\alpha\,\lambda+\gamma+r}\,(\overbrace{1-V_b^t}^{\simeq V_s^t}) + \frac{\gamma}{\frac{B-1}{B}\,\alpha\,\lambda+\gamma+r}\,(\overbrace{1-V_b^e}^{\simeq V_s^e})\;. \tag{4.5}$$

Hence, when the market is imbalanced, the outcome of the market resembles the outcome typically obtained in models of Bertrand competition. Indeed, in a match, the payoff of a trader on the long side of the market if she/he trades is very close to her/his continuation value if she/he does not trade and, instead, waits until the state of the market changes. Importantly, in a dynamic thin market, the continuation value is endogenous, and driven by the expectation about future trade opportunities.[9]

Consider now a state (S,B,ω) where the market is balanced (i.e., $S = B$). Proposition 3.2 establishes that there is trade in every match. Consequently, when $S > 1$, equation (4.1) holds, and so $V_s \simeq \mathbb{E}^t[V_s(S-1,B-1,\tilde{\omega})]$. Each seller is close to indifferent between trading or letting other traders trade until she is alone in the market with a single buyer. Assume then that there are only one seller and one buyer in the market (i.e., $S = B = 1$). The reservation value of the seller (i.e., her value from not trading) is $\frac{\gamma}{\gamma+r}\,V_s^e$. Similarly, the reservation value of the buyer is $\frac{\gamma}{\gamma+r}\,V_b^e$. As the frequency with which offers are made increases, the transaction price is determined by the limit outcome of a two-player bargaining game à la Rubinstein (1982) with randomly arriving outside options (given by the potential exogenous changes in the state of the market). The "size of the pie" over which they bargain is not 1, but the trade surplus net of the sum of the outside options, which is

$$1 - \frac{\gamma}{\gamma+r}\,(V_s^e + V_b^e) \simeq \frac{r}{\gamma+r}\;.$$

As in the standard Rubinstein bargaining game, the seller obtains, on top of her reservation value, a fraction of the pie equal to the probability with which she makes offers, ξ. Hence, the Rubinstein payoff of the seller is given by

$$V_s \simeq \frac{r}{\gamma+r}\,\xi + \frac{\gamma}{\gamma+r}\,V_s^e\;. \tag{4.6}$$

Equations (4.4)–(4.6) indicate that, under the risk-neutral measure, V_s approximately follows the same equations as the continuation payoff of a fictitious agent who receives a flow payoff equal to 1 when there is excess demand (i.e., $S_t < B_t$), a flow payoff equal to 0 when there is excess supply (i.e., $S_t > B_t$), and a flow payoff equal to ξ when the market is balanced (i.e., $S_t = B_t$). The right-hand side of equation (4.3) gives an expression of such value.

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[9] This result can be interpreted as micro-founding, using a decentralized approach, the assumption in Taylor (1995) that, at any given time when the market is imbalanced, the transaction price is equal to the one obtained in a static market with Bertrand competition between the traders on its long side.

An implication of Proposition 4.1 is that only the evolution of the sign of the net supply in the market (i.e., the number of sellers minus the number of buyers) is relevant for determining the market price. The reason is that the intensity of the competition between traders on the long side of the market is irrelevant for determining the price when the market is unbalanced: the price equals their reservation value independently of their number. The price only depends on the details of the bargaining protocol proportionally to the future discounted time the market is balanced.

Remark 4.1 (Diamond's paradox). Corollary 4.1 shows that, in the limit where bargaining frictions disappear, the payoff of each trader in each state is strictly positive as long as there is a positive probability that her or his side of the will become the short side of the market in the future. This may be surprising since, in bargaining models with one-sided offers (which in our model would correspond to $\xi = 0$ or $\xi = 1$), the side of the market making the offers obtains all the surplus from trade, independently of how big the imbalance is. This is usually known as Diamond's paradox (see Diamond, 1971). In our model, the order of setting these limits matters: our claim implicitly sets the limit of small bargaining frictions first, and the limit of one-sided offers second. The claim would not hold if we first assumed that ξ is equal to either 0 or 1, and then we set the limit where the bargaining frictions disappear. In such a case, the type of traders making all the offers would obtain all gains from trade.

Changes in continuation values

As we argued before, when the bargaining frictions are small and $S_0 \geq B_0$, a close-to-optimal strategy for a seller at time 0 is not to trade until the market is balanced. If, instead, $S_0 < B_0$, the same applies to a buyer. It then follows that

$$V_{\mathrm{s}}(S_0, B_0, \omega_0) \simeq \begin{cases} \tilde{\mathbb{E}}[e^{-r\,\tau_0}\, V_{\mathrm{s}}(1,1,\tilde{\omega}_{\tau_0})] & \text{if } S_0 \geq B_0\ , \\ 1 - \tilde{\mathbb{E}}[e^{-r\,\tau_0}\,(1 - V_{\mathrm{s}}(1,1,\tilde{\omega}_{\tau_0}))] & \text{if } S_0 < B_0\ , \end{cases} \tag{4.7}$$

where τ_0 is the (stochastic) time it takes for the market to have only one seller and one buyer. This observation can be used to prove the following result:

Corollary 4.1. *For any imbalanced state (S_t, B_t, ω_t) where θ-traders are on the long side of the market,*[10]

$$r\,V_{\theta,t} \simeq \frac{\tilde{\mathbb{E}}_t[V_{\theta,t+\Delta}] - V_{\theta,t}}{\Delta} + o(\Delta) \preceq \frac{\mathbb{E}_t[V_{\theta,t+\Delta}] - V_{\theta,t}}{\Delta} + o(\Delta) \tag{4.8}$$

as $\Delta \to 0$, where $V_{\theta,t} \equiv V_\theta(S_t, B_t, \omega_t)$.

[10]Consistently with footnote 8, the equality "$\simeq$" in equation (4.8) should be read as "For any sequence $(k_n)_n$ and corresponding sequence of equilibria, and for any state (S_t, B_t, ω_t), the function $\Delta \mapsto R(\Delta) \equiv \limsup_{n\to\infty} \big(\frac{\tilde{\mathbb{E}}_t[V_{\theta,n,t+\Delta}] - V_{\theta,n,t}}{\Delta} - r\,V_{\theta,n,t}\big)$ is such that $\lim_{\Delta \searrow 0} R(\Delta) = 0$."

To obtain some intuition for Corollary 4.1, assume that at time t there is excess supply, i.e., $S_t > B_t$. Since sellers are approximately indifferent between trading or not, the expected increase in the price under the risk neutral measure must approximately grow at rate r to compensate for the cost of waiting to trade. Such expected increase is typically larger under the equilibrium measure. To see this, fix a state (S_t, B_t, ω_t) with trade delay. The rate at which transactions happen in this state when all sellers follow the equilibrium strategy (equal to $\alpha\,\lambda$) is higher than the transaction rate when one the sellers deviates and decides not to trade (equal to $\frac{S_t-1}{S_t}\,\alpha\,\lambda$). Given that, by Proposition 3.2, the seller continuation payoff increases when other traders trade, the expected increase in the continuation payoff of the sellers is larger under the equilibrium measure than under the risk-neutral measure.

An important implication of Corollary 4.1 is that when the market is imbalanced, the market price increases in expectation when there is excess supply, and decreases when there is excess demand. Remarkably, this result is independent of whether the state of the market tends towards being balanced or not.

4.3 No delay

We now study the case where trade delay vanishes when the bargaining frictions get small. To this end, we first present a condition on the primitives, which will turn out to be sufficient for our analysis.

Condition 1. (a) $(\omega_t)_t$ is an autonomous process.

(b) $\frac{\gamma_x(S-1,B-1,\omega)}{\gamma(S-1,B-1,\omega)+r} \leq \frac{\gamma_x}{\gamma+r} + \frac{r}{\gamma+r}\,\frac{1}{3}$ for both $x \in \{\mathrm{s}, \mathrm{b}, \mathrm{c}\}$ and each state (S, B, ω) with $S, B > 0$.

Condition 1(a) requires market conditions to evolve independently from what occurs in the market. Condition 1(b) requires that transactions do not accelerate arrivals much. Together, Conditions 1(a) and 1(b) limit the possibility that traders on the long side of the market benefit significantly from transactions of other traders, and therefore prevent trade delay from occurring in equilibrium. Formally:

Proposition 4.2. *Assume Condition 1 holds. Then, there is no equilibrium with trade delay if bargaining frictions are small enough; that is, there is some $\bar{k}$ such that if $k > \bar{k}$ then $\alpha(S, B, \omega) = 1$ whenever $S, B > 0$. Furthermore, there exists a function $p : \mathbb{Z} \times \Omega \to [0, 1]$ such that $V_{\mathrm{s}}(S, B, \omega) \simeq p(S - B, \omega)$ for all states (S, B, ω).*

Proposition 4.2 establishes that Condition 1 is sufficient to ensure that trade delay disappears when bargaining frictions are small. Intuitively, by Proposition 3.2, trade delay occurs in a given state (S, B, ω) with $S > B$ only if sellers gain from other traders' transactions, that is, if $\mathbb{E}^{\mathrm{t}}[V_{\mathrm{s}}(S-1, B-1, \tilde{\omega})] > V_{\mathrm{s}}$. The proof of Proposition 4.2 shows that since by Condition 1 the arrival of buyers cannot increase by much after a transaction, this gain is small, and so there is no equilibrium with trade delay.

Under Condition 1, the time it takes for the short side of the market to clear is increasingly small as the bargaining frictions disappear. Hence, when bargaining frictions are small, there is an empty side of the market most of the time; that is, at any time t, the state is very likely to be of the form either $(S_t, 0, \omega_t)$ or $(0, B_t, \omega_t)$. It is then clear that the limit dynamics of the state of the market (under either the equilibrium or the risk-neutral measures) can be described by the evolution of the net supply, $N_t \equiv S_t - B_t$, and the market conditions, ω_t.

The limit equilibrium and risk-neutral dynamics of the net supply coincide when the market imbalanced. Hence, for example, equation (4.7) holds also for the equilibrium measure. If, for example, $N_t > 0$, the net supply N_t increases by 1 at rate $\gamma_{\rm s}(N_t, 0, \omega_t)$ and decreases by 1 at rate $\gamma_{\rm b}(0, N_t, \omega_t)$, while the market conditions ω_t change at rate $\gamma_{\rm c}(\omega_t)$ (recall that, by Condition 1, ω_t is autonomous). When the market is balanced, instead, they may differ: under the risk-neutral measure, a seller and a buyer remain in the market (the market state changes at rate $\gamma(1, 1, \omega_t)$), but they immediately agree under the equilibrium measure (and so the market state changes at rate $\gamma(0, 0, \omega_t)$).

Note that Condition 1 holds trivially in the big markets studied by Rubinstein and Wolinsky (1985) and Gale (1987), which exhibit no trade delay. Indeed, the equilibrium arrival rate of traders—which, in their models, is a discrete-time flow—is independent of the state of the market. Hence, in these models, delaying trade does not change the joint continuation value of the traders. This argument cannot be applied to a thin market when Condition 1 does not hold: since transactions affect the state of the market, traders may have the incentive to let other traders trade, and to trade only when their bargaining power is higher.

Changes in r

In this section, we consider the effect that changing r has on the distribution of market prices. We start by presenting a condition that will ensure that the state of the market has an ergodic distribution. It requires the arrival rate of agents on the short side of the market be higher than the arrival rate of agents on the long side of the market.

Condition 2. (a) $\Omega = \{\omega\}$.

(b) For any state (S, B, ω), $\gamma_{\rm s} - \gamma_{\rm b} > 0$ if $S < B$, and $\gamma_{\rm b} - \gamma_{\rm s} > 0$ if $S > B$.

It is not difficult to see that if Conditions 1 and 2 hold, then each equilibrium generates an ergodic distribution of the state of the market. In particular, since by Proposition 4.2 there is no equilibrium with trade delay when k is large, the limit process for N_t has also an ergodic distribution. In other words, under Conditions 1 and 2, there is a unique F such that, for each value of the net supply N, $\lim_{t\to\infty} \Pr(N_t = N | N_0) \simeq F(\{N\})$ independently of N_0. Such limit ergodic distribution does not depend on the discount rate, but the ergodic distribution of market prices does. This observation can be used to prove the following result:

Corollary 4.2. *Assume Conditions 1 and 2 hold. Then, $p(\cdot,\omega) : \{-\bar{B},...,\bar{S}\}$ is increasing. Furthermore, increases in r generate spreads of the ergodic distribution of market prices. If, additionally, $\gamma_\theta(0,0,\omega)=\gamma_\theta(1,1,\omega)$ for both $\theta\in\{\text{s,b}\}$, such spreads are mean-preserving.*

Corollary 4.2 establishes three important features of the market price. Its first claim establishes an intuitive result: the market price depends negatively on the net market supply, independently of the process governing the evolution of the market composition (satisfying Conditions 1 and 2). It follows from equation (4.7) (recall that, under Condition 1, equation (4.7) holds also for the equilibrium measure). Indeed, when the state of the market is unidimensional (by Condition 2(a) the market conditions do not change), the time it takes for the market to become balanced is increasing in the degree of unbalancedness of the market. Hence, for positive net supply values, adding an extra trader on the long side of the market makes the discount factor assigned to the time the market becomes balanced smaller.

The second result in Corollary 4.2 establishes that when market conditions do not change and there is no trade delay, an increase in r raises the ergodic market price dispersion.[11] This follows because, again by equation (4.7), an increase in the discount rate r lowers the discount factor of the time it takes the market to become balanced. Hence, the market price for a given N tends to become more extreme when r increases: it tends to decrease when $N > 0$, and it tends to increase when $N < 0$.[12] For instance, in the limit where $r \to \infty$, we have that $p(N,\omega) \to 0$ for all $N > 0$, and $p(N,\omega) \to 1$ for all $N < 0$.

Finally, the third result in Corollary 4.2 establishes that, under some conditions, the spread generated by an increase in r is mean-preserving. It can be understood as follows. Notice first that, since trade delay when the bargaining fractions disappear, the limit equilibrium and risk-neutral measures coincide when $\gamma_\theta(0,0,\omega)=\gamma_\theta(1,1,\omega)$ for both $\theta \in \{\text{s,b}\}$. Also, from equation (4.3), the ergodic mean of the market price can be approximated by the ergodic probability that the market exhibits excess demand plus the ergodic probability that the market is balanced multiplied by ξ. Since under Condition 1 there is no trade delay when bargaining frictions are small, the ergodic probability that the market has excess demand is independent of r, and so is the long-run expected market price. Hence, while changes in r change the ergodic distribution of market prices, they do not change its mean. In general, if the arrival rates in states $(0,0,\omega)$ and $(1,1,\omega)$ are close (but not equal), or if the ergodic likelihood that the market is balanced is low, Corollary 4.2 establishes that increases in the interest rate will increase the spread of prices, and will keep its mean approximately unchanged.

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[11] Recall that a CDF F_1 in $\mathbb{R}$ is said to be a *spread* of another CDF F_2 if they satisfy that there exists a value $\bar{x} \in \mathbb{R}$ such that $F_1(x) \geq F_2(x)$ for all $x < \bar{x}$ and $F_1(x) \leq F_s(x)$ for all $x > \bar{x}$.

[12] The negative direct effect of an increase in r in $p(N,\omega)$ when $N > 0$ (on the right hand side of equation (4.7)) may sometimes be compensated by an increase in $p(0,\omega)$. Nevertheless, as the proof of Corollary 4.2 shows, the direct effect dominates if N is large enough. The restriction that $|\Omega| = 1$ (i.e., the arrival rates only depend on the composition of the market) is necessary to make the state space one-dimensional, since then N_t summarizes the state of the market at time t when the bargaining frictions are small.

4.4 Large market limit

In this section we analyze the trade outcome when the arrival rate of traders in the market is large. The analysis sheds light on the role of the friction that remains in the market when the matching frequency is high, that is, the time that a trader has to wait for trading when she or he is on the long side of the market. We then provide an answer, from the thin-markets perspective, to one of the salient questions in the literature on decentralized bargaining in large markets: whether lowering frictions leads to a competitive outcome. As in the previous models, our answer to this question will shed light on whether and how frictions may be magnified or mitigated by the equilibrium behavior of the traders in the market.[13]

Increasing the arrival rates can be interpreted as unifying similar markets into bigger ones. In practice, such market unification may correspond to the launch of websites providing information on job offerings, rental prices, or housing prices in close locations. Such websites may make it easier for sellers and buyers in different markets to meet each other, which may de facto unify the different markets into a single market. Market unification may also be the result of improvements in the transportation infrastructure that reduce the commuting time, such as new metro stations or new roads.

We begin with some notation. Fix some sequence $(k_n)_n$. Fix also some functions $\tilde{\gamma}_{\mathrm{s}}$ and $\tilde{\gamma}_{\mathrm{b}}$, and a non-negative sequence $(M_n)_n$ tending to $+\infty$. For each n, we consider the model with arrival rates $\gamma_{\mathrm{s}} = M_n \tilde{\gamma}_{\mathrm{s}}$, $\gamma_{\mathrm{b}} = M_n \tilde{\gamma}_{\mathrm{b}}$, and, as before, $\lambda = k_n \ell$. In this section, the notation "$\simeq$" indicates that, for any sequences $(k_n)_n$ and $(M_n)_n$ tending to $+\infty$ and sequence of corresponding equilibria, the terms on each of the sides are equal in any sequence of equilibria (except for terms that go to 0 as n goes to $+\infty$ (see footnote 8). The following characterizes the limit market outcome characterizes as the market grows and bargaining frictions disappear.

Proposition 4.3. *For all states (S, B, ω) with $S, B > 0$ we have $\frac{\alpha \lambda}{\alpha \lambda + r} \simeq 1$. Also, if Conditions 1 and 2 hold, there is some $p^* \in (0, 1)$ such that $V_{\mathrm{s}} \simeq p^*$ for all states (S, B, ω).*

Proposition 4.3 provides two results regarding the trade limit outcome as the market frictions disappear and the market grows. The first establishes that the discount factor until a transaction happens, which is equal to $\frac{\alpha \lambda}{\alpha \lambda + r}$, tends to one; that is, trade delay disappears. The second result prescribes that the dispersion of the market prices vanishes; that is, the market price in all states of the market tends to the same "competitive price." Hence, as it is the case

[13] For example, Gale (1987) characterizes the trade outcome in the large-market version of our model in the limit where the discount rate tends to 0, and obtains that the trade outcome converges to that of a competitive market. In this limit, the price is either 0 (if there are more sellers than buyers) or 1 (if there are more buyers than sellers). Other papers have identified settings where the trade outcome fails to convergence to the competitive one. Such failure of convergence may be due, among other reasons, to symmetric information between traders (Satterthwaite and Shneyerov, 2007; Lauermann and Wolinsky, 2016), the heterogeneity on each side of the market (Lauermann, 2012), or lack of knowledge about the state of the market (Lauermann, Merzyn, and Virág, 2017). See also Lauermann (2013) for an analysis of other causes of trade delay.

for big decentralized markets, thin markets with a high entry of traders and low bargaining frictions feature both a low trade delay and a single "competitive" price.

An intuition for Proposition 4.3 is obtained as follows. As the arrival of traders increases, the current state of the market becomes progressively less relevant in determining the current price, since each trader in the market can wait for the state of the market to change without incurring a big delay cost. In particular, the delay cost of not trading until the net state reaches some given state in the support of the ergodic distribution of the state of the market tends to 0 as $n \to \infty$. It may then seem that the option waiting to trade becomes increasingly attractive to all traders in the market. This is not possible when bargaining frictions are small: the sum of the continuation values of a seller and a buyer in the market is always close to 1, independently of the value of the arrival rates. Hence, even though waiting is increasingly cheap, it is also increasingly worthless, in the sense that the price variation across states becomes increasingly small.

To provide further intuition for the previous results, let S_t^Σ and B_t^Σ denote, respectively, the number of sellers and buyers who arrived to the market between time 0 and time t, including the ones who "arrived" (or were present) at time 0. Then, $N_t = S_t - B_t = S_t^\Sigma - B_t^\Sigma$; thus equation (4.3) holds replacing S_t and B_t by S_t^Σ and B_t^Σ, respectively. As traders arrive more frequently, the price (at time 0, for example) approximates the (ergodic) probability that more sellers than buyers arrive in the future. Hence, the effective market that a trader (at time 0) has access to grows intertemporally. Given that the endogenous arrival process may tend to keep the thin market balanced, the competitive price is not necessary either 0 (when there is excess supply) or 1 (when there is excess demand). Instead, in contrast to the big market case, the competitive price of a thin market is a convex combination of the two extremes, each of them weighted according to the probability that the market features excess supply and demand.

The proof of Proposition 4.3 shows how its statement can be generalized to the case where Condition 2(a) fails to hold. When the market conditions vary over time (in an autonomous manner, by Condition 1(a)), there is a unique limit price for each value of the market conditions, $p^* \equiv p^*(\omega)$. Such price can be obtained by solving the system of equations (one equation for each value $\omega \in \Omega$)

$$p^*(\omega) = \tfrac{r}{\gamma_c + r}\, \bar{p}(\omega) + \tfrac{\gamma_c}{\gamma_c + r}\, \mathbb{E}^c[p^*(\tilde{\omega})] \,, \tag{4.9}$$

where $\bar{p}(\omega) \equiv \mathbb{E}[\mathbb{I}_{S<B} + \xi\, \mathbb{I}_{S=B} | \omega]$ is the expected market price under the ergodic distribution for the market conditions ω. Hence, current and future market conditions affect the current price in the limit where the market grows by replication, but the price is independent of the current composition of the market.

5 Extensions and conclusions

5.1 Extensions

This section discusses different generalizations and extensions of our model. These generalizations and extensions illustrate how our results can be extended beyond some of the assumptions we made. They also indicate that different specifications of a thin market give rise to the same set of results, and hence give a sense of the robustness of our findings.

Endogenous entry. In some cases, entering a market requires a sunk investment. For example, house sellers may have to condition their housing units and design the advertisement before posting it, and workers may have to update their submarket-specific knowledge and to prepare some documentation (CV, cover/reference letters, etc.) before entering a job market. Hence, the decision to enter a market may be the result of a cost-benefit analysis, where potential traders compare the cost of entering the market with the expected gains from trade. To accommodate such a possibility, we could extend our model in the following way.[14] Consider an extended model where sellers and buyers become active at some respective (state-independent) rates $\bar{\gamma}_{\mathrm{s}}$ and $\bar{\gamma}_{\mathrm{b}}$ instead of directly entering the market. Once a seller or a buyer becomes active, she/he draws a cost c from some distribution F_{s} or F_{b}, respectively. In an equilibrium of this model, if for example a seller becomes active and the state is (S, B, ω), she enters the market if the net payoff from doing so, $\mathbb{E}^{\mathrm{s}}[V_{\mathrm{s}}(S+1, B, \tilde{\omega})] - c$, is above some fixed outside option normalized to be 0 (which may come from the option to sell the good in another market or keeping it for herself). This implies that the arrival of sellers into the market is $\gamma_{\mathrm{s}}(S, B, \omega) = \bar{\gamma}_{\mathrm{s}}\, F_{\mathrm{s}}(\mathbb{E}^{\mathrm{s}}[V_{\mathrm{s}}(S+1, B, \tilde{\omega})])$. Any equilibrium outcome in such a model corresponds to an equilibrium outcome of some specification of our model, and so our results hold. In the described model with endogenous arrival, the arrival of agents on the short side of the market will tend to be higher than the arrival of agents on the long side, and hence the market will tend to remain approximately balanced.

Exogenous and endogenous exit. A common assumption in the large-market literature is that θ-traders leave the market at some (typically state-independent) Poisson rate $\rho_\theta > 0$, for each $\theta \in \{\mathrm{s}, \mathrm{b}\}$. This assumption is often made to keep the size of the market stationary when the arrival rates are constant, and incorporates the observation that traders some times exit the market for exogenous reasons. Making such an assumption in our model adds an extra term equal to $S\,\rho_{\mathrm{s}} + B\,\rho_{\mathrm{b}}$ to each denominator in equation (3.1), as well as the term

$$\tfrac{1}{\lambda+\gamma+r+S\,\rho_{\mathrm{s}}+B\,\rho_{\mathrm{b}}}\left((S-1)\,\rho_{\mathrm{s}}\,V_{\mathrm{s}}(S-1, B, \omega) + B\,\rho_{\mathrm{b}}\,V_{\mathrm{s}}(S, B-1, \omega)\right)$$

on the right-hand side of the equation when $\theta = \mathrm{s}$ (and a similar term when $\theta = \mathrm{b}$). The additional term plays a role similar to that of the term corresponding to the exogenous change of

[14] In the large markets literature, Manea (2017b) shows the existence of steady states in a similar specification.

the state of the market in equation (3.1): it also corresponds to an exogenous (i.e., equilibrium-independent) change in the state of the market.

More generally, our model can be adapted to accommodate endogenous exit. In a job market, endogenous exit may correspond to workers moving to other commuting areas when wages are low in their current commuting areas. Similarly, house-seekers may end up looking for houses with different characteristics when the prices of houses in their desired sub-market are high (families may end up buying/renting an apartment instead of a house with a garden, for example). To incorporate endogenous exit, we can add to our model a stochastic process determining the decision times (arriving at some Poisson rate) where a trader can decide whether to leave the market or not. Leaving the market gives a θ-trader an exogenous continuation value equal to $\underline{V}_\theta \geq 0$ (possibly depending on ω and satisfying $\underline{V}_{\rm s}+\underline{V}_{\rm b} < 1$). Equilibria of this extended model would feature some states (typically highly imbalanced) with exit of the traders on the long side of the market, and other states with no exit. Additionally, there would be some highly unbalanced states such that, after the arrival of a trader on the long side of the market, another trader on the long side of the market would almost immediately leave. As in the previous case with endogenous entry, the state of the market would tend to stay approximately balanced in a model with either exogenous or endogenous exit.

Time-varying gains from trade. The gains from trade are constant and normalized to 1 in our model. This assumption permits focussing our analysis on how the arrival process affects the outcome of a thin market. In practice, the gains from trade may change over time depending on the market's conditions. In the example used in Section 3.3, the purchase of a house by a celebrity increases the social status that having a house in the same neighborhood gives. Similarly, subsidies may increase the joint gains that an employer and an employee obtain when, for example, a local government offers free housing to newly-hired workers in response to a shortage of specialized labor.

Our analysis can be easily extended to the case where the gains from trade in state (S, B, ω) are $g(\omega)$. It is easy to see that the same arguments we use to show Proposition 3.2 continue to hold when

$$\min_{\omega'} g(\omega') > \frac{\gamma_{\rm s} + \gamma_{\rm b} + \gamma_{\rm c}}{\gamma_{\rm s} + \gamma_{\rm b} + \gamma_{\rm c} + r} \max_{\omega'} g(\omega')$$

for all states (S, B, ω) (note that the equation holds if $g(\cdot) \equiv 1$). This condition imposes a limit on the size that the increase of the gains from trade can have. If the condition does not hold there may be states exhibiting market breakdown, that is, where trade occurs with probability zero.

Attention frictions. The matching rate of our model can be interpreted as an attention friction faced by traders. Consider, for example, a model like the one presented in Section 2 where now traders draw "attention times" instead of "meeting" other traders. In this model, a θ-trader in the market draws attention times at a (possibly state-independent) Poisson rate $\lambda_\theta > 0$, for

$\theta \in \{\mathrm{s}, \mathrm{b}\}$. When a trader draws an attention time, she/he chooses a trader on the other side of the market (if any) and makes him/her an offer. This model with attention frictions would generate the same (symmetric Markov perfect) equilibria as our model with[15]

$$\lambda \equiv S\,\lambda_{\mathrm{s}} + B\,\lambda_{\mathrm{b}} \;\text{ and }\; \xi \equiv \tfrac{S\,\lambda_{\mathrm{s}}}{S\,\lambda_{\mathrm{s}}+B\,\lambda_{\mathrm{b}}} \;.$$

The limit "where bargaining frictions vanish" considered in Section 4 corresponds, in the model with attention frictions, to the limit "where attention frictions vanish."

Nash bargaining. In our base model, the bargaining protocol in each match consists of a take-it-or-leave-it offer by a randomly chosen trader. In a more general bargaining protocol, such as Nash bargaining, a match results in some (potentially stochastic) transfers, and a probability of agreement. Thus, we can write the payoffs for traders when they are matched as

$$\begin{aligned} V_{\mathrm{s}}^{\mathrm{m}} &= \alpha\,\mathbb{E}[p|\text{agree}] + (1-\alpha)\,(V_{\mathrm{s}} + \mathbb{E}[p|\text{disagree}])\;, \\ V_{\mathrm{b}}^{\mathrm{m}} &= \alpha\,\mathbb{E}[1-p|\text{agree}] + (1-\alpha)\,(V_{\mathrm{b}} - \mathbb{E}[p|\text{disagree}])\;, \end{aligned}$$

where $\alpha \equiv \alpha(S, B, \omega)$ is an endogenous probability of agreement in state (S, B, ω). Under the assumption of individual rationality by sellers and buyers (that is, the assumption that they can opt out from bargaining and obtain their continuation value instead), we have that $V_\theta^{\mathrm{m}} \geq V_\theta$ for both $\theta \in \{\mathrm{s}, \mathrm{b}\}$ (i.e., $V_\theta^{\mathrm{m}} \in [V_\theta, 1 - V_{\bar{\theta}}]$). Consequently, $\alpha = 0$ and $\mathbb{E}[p|\text{disagree}] = 0$ whenever $V_{\mathrm{s}} + V_{\mathrm{b}} > 1$. Then, as long as the bargaining protocol has the property that the probability of agreement increases in the bargaining surplus, our results apply. In particular, if there is a cap $\bar{\alpha}$ to the probability of agreement (or a probability $1 - \bar{\alpha}$ of exogenous breakdown), the match frequency λ can be renormalized to recover our model (where $\bar{\alpha} = 1$).

The arguments behind the results Sections 3 and 4 for a generalized bargaining protocol. Indeed, the particular structure of the bargaining protocol is not used to show Proposition 3.2, Lemma 4.1 and Corollary 4.1, only the individual rationality of the traders. In Proposition 4.1, ξ has to be replaced by the expected fraction of the net surplus captured by a seller when there are only one seller and one buyer in the market. (As footnote 15 clarifies, our model easily accommodates to the case where ξ depends on the state of the market.) Finally, the arguments showing that "size of the pie" over which traders bargain in every match (equal to $1 - V_{\mathrm{s}} - V_{\mathrm{b}}$) shrinks when bargaining frictions disappear still apply.

5.2 Conclusions

We have studied decentralized bargaining in dynamic thin markets. In these markets, the endogenous arrival of traders in the market is crucial to determine the market outcome.

[15]Our results can be straightforwardly generalized to the case where ξ depends on the state of the market. In this case, equation (4.3) should be changed replacing ξ by $\xi(1, 1, \omega_t)$. Our results also apply when r, interpreted as the interest rate, depends on the market conditions.

Our results stress that modeling big decentralized markets as the sum of small thin markets has important implications for the predicted trade outcome. For example, in a thin market, trade delay and price dispersion arise even when bargaining frictions are small and traders are not significantly heterogeneous. Delay occurs because some transactions may make the remaining traders on one side of the market better off, and then induce them to wait for other traders to trade. Price dispersion arises from the fact that prices depend on the current and future trade opportunities that each trader has, which themselves depend on the stochastic evolution of the market. The slow arrival of traders makes their reservation value depend on the current market composition in a nontrivial way even when the bargaining frictions are small. Still, even though the market is thin, the particular bargaining protocol used to set prices in the market has a small effect on the equilibrium price. Our characterization of the price in terms of the evolution of the market may serve as a guide for future empirical work.

We obtain some novel implications for the price process that have the potential of being tested using disaggregated data from individual buyers (employers/house seekers) and sellers (workers/house owners). First, market prices drift toward the price of a balanced market. Second, increases in the interest rate result in increases in the spread of the distribution of market prices. Finally, if different markets are unified, current transaction prices do not depend on the current number traders in the market, but rather on the future evolution of both the balancedness of the market and the market conditions.

Our model can be generalized in multiple directions. Of particular interest is the possibility of allowing sellers and buyers to be heterogeneous in terms of the quality of their goods or their valuations for them. This would make the analysis much less tractable, as it would enlarge the dimensionality of the state of the market.[16] Another possible extension consists on explicitly modeling costly relocation of traders between different sub-markets. This would allow providing new insights on endogenous gentrification (see Guerrieri, Hartley, and Hurst (2013) for a centralized large-market approach) or sectorial mobility of workers (see Artuç and McLaren (2015) for evidence), as well as analyzing the effects that idiosyncratic and common shocks have on mobility across markets. The analysis of these and other extensions is left to future research.

[16] Abreu and Manea (2012b,a) and Elliott and Nava (2019) show that, in a model of bargaining in networks, the outcome of bargaining is stochastic even in the limit where bargaining frictions vanish, as sometimes transactions with low gains from trade are realized in the presence of more beneficial trade opportunities.

A Omitted expressions and proofs of the results

A.1 Payoffs and equilibria

In the Appendix we use $\mathcal{S} \equiv \{0, ..., \bar{S}\}$, $\mathcal{B} \equiv \{0, ..., \bar{B}\}$, $\mathcal{S}^* \equiv \mathcal{S} \setminus \{0\}$, and $\mathcal{B}^* \equiv \mathcal{B} \setminus \{0\}$. Fix a strategy for the sellers $(\pi_{\mathrm{s}}, \alpha_{\mathrm{s}})$ and a strategy for the buyers $(\pi_{\mathrm{b}}, \alpha_{\mathrm{b}})$. For each type $\theta \in \{\mathrm{s}, \mathrm{b}\}$ and state $(S, B, \omega) \in \mathcal{S}^* \times \mathcal{B}^* \times \Omega$, $\pi_\theta(S, B, \omega) \in \Delta(\mathbb{R})$ is the distribution of price offers that θ-traders make if they are matched and chosen to make the offer in state (S, B, ω), and $\alpha_\theta(\,\cdot\,; S, B, \omega) : \mathbb{R} \to [0, 1]$ maps each price offer received to a probability of acceptance.

Fix a strategy profile $\{(\pi_\theta, \alpha_\theta)\}_{\theta \in \{\mathrm{s}, \mathrm{b}\}}$ and state (S, B, ω). We compute the continuation values the strategy gives to a seller (denoted $V_{\mathrm{s}}(S, B, \omega)$) and to a buyer (denoted $V_{\mathrm{b}}(S, B, \omega)$) using standard recursive analysis. They satisfy equation (3.1) (for both $\theta \in \{\mathrm{s}, \mathrm{b}\}$), where now the expected continuation values conditional on being selected in the match given by

$$
\begin{aligned}
V_{\mathrm{s}}^{\mathrm{m}}(S, B, \omega) \equiv\ & \xi\, \mathbb{E}_{\tilde{p}}\big[\, \alpha_{\mathrm{b}}(\tilde{p})\, \tilde{p} + (1 - \alpha_{\mathrm{b}}(\tilde{p}))\, V_{\mathrm{s}}(S, B, \omega) \,\big|\, \pi_{\mathrm{s}} \,\big] \\
& + (1 - \xi)\, \mathbb{E}_{\tilde{p}}\big[\, \alpha_{\mathrm{s}}(\tilde{p})\, \tilde{p} + (1 - \alpha_{\mathrm{s}}(\tilde{p}))\, V_{\mathrm{s}}(S, B, \omega) \,\big|\, \pi_{\mathrm{b}} \,\big] \quad \text{and} \qquad \text{(A.1)} \\
V_{\mathrm{b}}^{\mathrm{m}}(S, B, \omega) \equiv\ & \xi\, \mathbb{E}_{\tilde{p}}\big[\, \alpha_{\mathrm{b}}(\tilde{p})\, (1 - \tilde{p}) + (1 - \alpha_{\mathrm{b}}(\tilde{p}))\, V_{\mathrm{b}}(S, B, \omega) \,\big|\, \pi_{\mathrm{s}} \,\big] \\
& + (1 - \xi)\, \mathbb{E}_{\tilde{p}}\big[\, \alpha_{\mathrm{s}}(\tilde{p})\, (1 - \tilde{p}) + (1 - \alpha_{\mathrm{s}}(\tilde{p}))\, V_{\mathrm{b}}(S, B, \omega) \,\big|\, \pi_{\mathrm{b}} \,\big] \qquad \text{(A.2)}
\end{aligned}
$$

instead of equations (3.2) and (3.3), where the continuation value of the type-θ trader conditional on some other traders being selected in the match is given by

$$
\begin{aligned}
V_\theta^{\mathrm{o}}(S, B, \omega) \equiv\ & \xi\, \mathbb{E}_{\tilde{p}}\big[\, \alpha_{\mathrm{b}}(\tilde{p})\, \mathbb{E}^{\mathrm{t}}[V_\theta(S-1, B-1, \tilde{\omega})] + (1 - \alpha_{\mathrm{b}}(\tilde{p}))\, V_\theta(S, B, \omega) \,\big|\, \pi_{\mathrm{s}} \,\big] \\
& + (1 - \xi)\, \mathbb{E}_{\tilde{p}}\big[\, \alpha_{\mathrm{s}}(\tilde{p}) \mathbb{E}^{\mathrm{t}}[V_\theta(S-1, B-1, \tilde{\omega})] + (1 - \alpha_{\mathrm{s}}(\tilde{p}))\, V_\theta(S, B, \omega) \,\big|\, \pi_{\mathrm{b}} \,\big] \qquad \text{(A.3)}
\end{aligned}
$$

instead of by equation (3.4), and where V_θ^{e} satisfies equation (3.5).[17] It is convenient to set $V_{\mathrm{s}}(0, B, \omega) = V_{\mathrm{b}}(S, 0, \omega) = 0$ for all $S \in \mathcal{S}$ and $B \in \mathcal{B}$, so V_{s} and V_{b} have $\mathcal{S} \times \mathcal{B} \times \Omega$ as a domain.

The system of equations determining the continuation values of sellers and buyers has a unique solution by the standard fixed-point argument. Indeed, we can replace V_{b} by $W_{\mathrm{s}} \equiv 1 - V_{\mathrm{b}}$ and interpret the previous equations as an operator which maps any pair of functions $(V_{\mathrm{s}}, W_{\mathrm{s}}) : \mathcal{S} \times \mathcal{B} \times \Omega \to \mathbb{R}^2$ to another pair of similar functions. It is then easy to verify that such an operator satisfies the sufficient Blackwell conditions for a contraction.

We use the principle of optimality to define our equilibrium concept. More concretely, we say that $\{(\pi_\theta, \alpha_\theta)\}_{\theta \in \{\mathrm{s}, \mathrm{b}\}}$ is a *symmetric Markov perfect equilibrium* if the corresponding continuation values—solving the system of equations $\langle$(3.1), (A.1), (A.2), (A.3), (3.5)$\rangle$—are such that, for each state (S, B, ω) and $\theta \in \{\mathrm{s}, \mathrm{b}\}$, the pair $(\pi_\theta(S, B, \omega), \alpha_\theta(\cdot; S, B, \omega))$ maximizes the right-hand side of equation (A.1) if $\theta = \mathrm{s}$ and right-hand side of (A.2) if $\theta = \mathrm{b}$.

[17] As in the main text we keep the notation short by omitting the explicit dependence of λ and γ on the state (S, B, ω), and we use $\alpha_\theta(\tilde{p})$ to denote $\alpha_\theta(\tilde{p}; S, B, \omega)$.

A.2 Proofs of the results

Proof of Proposition 3.1

Proof. Fix an equilibrium, assuming it exists, and let α, V_s and V_b solve the system of equations $\langle(3.1),(A.1),(A.2),(A.3),(3.5)\rangle$ (where α is defined in equation (A.4)). Standard arguments imply that if there is a positive probability that offers made by a seller are accepted in state (S,B,ω), then the equilibrium probability that such offers are equal to $1 - V_b(S,B,\omega)$ is one. Similarly, an equilibrium offer by a buyer in state (S,B,ω) is accepted with positive probability in equilibrium if and only if it is equal to $V_s(S,B,\omega)$. Since these offers make the receiver of the offer indifferent between accepting them or not, it is without loss of generality (to prove existence of equilibria) to focus on equilibria where, in each state (S,B,ω) and for all $\theta \in \{s,b\}$, sellers offer $1 - V_b(S,B,\omega)$ and buyers offer $V_s(S,B,\omega)$ for sure, and a θ-trader accepts the equilibrium offer with some probability $\alpha_\theta(S,B,\omega)$. Thus, equations (A.1) and (A.2) can be replaced by equations (3.2) and (3.3). Note that the continuation values of a seller and a buyer only depend on α_b and α_s through

$$\alpha \equiv (1-\xi)\,\alpha_b + \xi\,\alpha_s \tag{A.4}$$

(see equation (A.3)), with the convention that $\alpha(S,B,\omega) = 0$ whenever $S = 0$ or $B = 0$, and so equation (A.3) can be replaced by equation (3.4). Hence, equations (3.1)–(3.5) determine the continuation payoffs in an equilibrium.

Fix some $\alpha \in [0,1]^{\mathcal{S}^* \times \mathcal{B}^* \times \Omega}$, interpreted as a putative equilibrium probability of trade. We can compute the equilibrium continuation values in each state by solving equations in (3.1)–(3.5), and let $V_s(\cdot;\alpha)$ and $V_b(\cdot;\alpha)$ denote the corresponding solutions. Note also that a seller and a buyer are indifferent on accepting the equilibrium offer at state (S,B,ω) if and only if $V_s(S,B,\omega;\alpha) + V_b(S,B,\omega;\alpha) = 1$. Hence, there is no $\theta \in \{s,b\}$ such that the θ-trader has a profitable deviation at a given state $(S,B,\omega) \in \mathcal{S}^* \times \mathcal{B}^* \times \Omega$ if and only if

$$\alpha(S,B,\omega) \in \begin{cases} \{0\} & \text{if } V_b(S,B,\omega;\alpha) + V_s(S,B,\omega;\alpha) > 1, \\ [0,1] & \text{if } V_b(S,B,\omega;\alpha) + V_s(S,B,\omega;\alpha) = 1, \\ \{1\} & \text{if } V_b(S,B,\omega;\alpha) + V_s(S,B,\omega;\alpha) < 1. \end{cases}$$

This can easily be seen by noting that, for example, $V_s(S,B,\omega;\alpha) + V_b(S,B,\omega;\alpha) > 1$ and that $\alpha_s(S,B,\omega) > 0$ (if, instead, $\alpha_s(S,B,\omega) < 1$ the argument is analogous). If a buyer makes the equilibrium offer (equal to $V_s(S,B,\omega;\alpha)$) at state (S,B,ω) he obtains

$$\begin{aligned} &\alpha_s\,(1 - V_s(S,B,\omega;\alpha)) + (1-\alpha_s)\,V_b(S,B,\omega;\alpha) \\ &\quad = V_b(S,B,\omega;\alpha) - \alpha_s\left(V_s(S,B,\omega;\alpha) + V_b(S,B,\omega;\alpha) - 1\right) < V_b(S,B,\omega;\alpha)\,. \end{aligned}$$

If, instead, he offers $V_s(S,B,\omega;\alpha) - \varepsilon$, for some $\varepsilon > 0$, the seller rejects the offer for sure, and so the buyer obtains $V_b(S,B,\omega;\alpha)$, which makes him strictly better off.

To conclude the proof of existence of equilibria, we define $A : [0,1]^{\mathcal{S}^* \times \mathcal{B}^* \times \Omega} \rightrightarrows [0,1]^{\mathcal{S}^* \times \mathcal{B}^* \times \Omega}$ as follows:

$$A(\alpha)(S,B,\omega) = \begin{cases} \{0\} & \text{if } V_{\mathrm{s}}(S,B,\omega;\alpha) + V_{\mathrm{b}}(S,B,\omega;\alpha) > 1, \\ [0,1] & \text{if } V_{\mathrm{s}}(S,B,\omega;\alpha) + V_{\mathrm{b}}(S,B,\omega;\alpha) = 1, \\ \{1\} & \text{if } V_{\mathrm{s}}(S,B,\omega;\alpha) + V_{\mathrm{b}}(S,B,\omega;\alpha) < 1. \end{cases}$$

Standard arguments apply to show that $A(\cdot)$ has a closed graph, and that $A(\alpha)$ is, for all $\alpha \in [0,1]$, non-empty and convex. Hence, the existence of equilibria follows from Kakutani's fixed point theorem.

Existence when $\bar{S} = \bar{B} = +\infty$. Fix some functions $\lambda, \gamma_{\mathrm{s}}, \gamma_{\mathrm{b}} : \mathbb{Z}_+^2 \times \Omega \to \mathbb{R}_{++} \times \mathbb{R}_+ \times \mathbb{R}_+$, where γ_{s} and γ_{b} are bounded. Consider a sequence $(\bar{S}_n, \bar{B}_n)_n$ strictly increasing in both arguments. For each n, we can construct a model with a finite state space as follows ("the n-th model"). In the n-th model, λ^n coincides with λ, now with domain $\mathcal{S}^n \times \mathcal{B}^n \times \Omega \equiv \{0, ..., \bar{S}_n\} \times \{0, ..., \bar{B}_n\} \times \Omega$. The arrival rates in the n-th model is, for each $(S,B,\omega) \in \mathcal{S}^n \times \mathcal{B}^n \times \Omega$, are

$$\gamma_{\mathrm{s}}^n(S,B,\omega) = \begin{cases} \gamma_{\mathrm{s}}(S,B,\omega) & \text{if } S < \bar{S}^n, \\ 0 & \text{if } S = \bar{S}^n, \end{cases} \quad \text{and} \quad \gamma_{\mathrm{b}}^n(S,B,\omega) = \begin{cases} \gamma_{\mathrm{s}}(S,B,\omega) & \text{if } B < \bar{B}^n, \\ 0 & \text{if } B = \bar{B}^n. \end{cases}$$

For each n, let α^n characterize an equilibrium of the n-th model. Fix some $\omega \in \Omega$ (the argument can be made for each value of the market conditions). Let $\mu : \mathbb{N} \to \mathbb{N}^2$ be a bijective ordering of $\mathbb{N}^2$. Initialize $(\alpha_0^n)_n = (\alpha^n)_n$. Then, for each $m \in \mathbb{N}$, we use $(\alpha_{m-1}^n)_n$ to recursively construct $(\alpha_m^n)_n$ as follows. Let $\underline{\alpha}_m$ be the minimum cluster point of $(\alpha_{m-1}^n(\mu(m),\omega))_n$ (recall that the set of cluster points of a sequence is closed). If there is an increasing subsequence of $(\alpha_{m-1}^n(\mu(m),\omega))_n$ converging to $\underline{\alpha}_m$, then we let $(\alpha_m^n)_n$ be the biggest subsequence of $(\alpha_{m-1}^n)_n$ such that $(\alpha_{m-1}^n(\mu(m),\omega))_n$ is increasing and converging to $\underline{\alpha}_m$. If no increasing subsequence of $(\alpha_{m-1}^n(\mu(m)))_n$ converging to $\underline{\alpha}_m$ exists, then there must exist a decreasing subsequence of $(\alpha_{m-1}^n(\mu(m),\omega))_n$ converging to $\underline{\alpha}_m$. In this case, $(\alpha_m^n)_n$ be the largest decreasing subsequence of $(\alpha_m^n)_n$ such that $(\alpha_{m-1}^n(\mu(m),\omega))_n$ is decreasing and converges to $\underline{\alpha}_m$.

Note that, as explained above, for each m there exists some $\underline{\alpha}_m \in [0,1]$ and a subsequence $(\alpha_m^n)_n$ of $(\alpha^n)_n$ such that $(\alpha_m^n(\mu(m'),\omega))_n$ converges to $\underline{\alpha}_{m'}$ for all $m' \in \{1, ..., m\}$. It is then easy to prove that, in fact, $\alpha : \mathbb{N}^2 \times \Omega \to [0,1]$ given by $\alpha(S,B,\omega) = \underline{\alpha}_{\mu^{-1}(S,B,\omega)}(S,B,\omega)$ characterizes an equilibrium in the infinite model. □

Proof of Proposition 3.2

Proof. The proof follows from the arguments in the main text. □

Proof of Lemma 4.1

Proof. Throughout the proof we fix a sequence $(k_n)_n$ tending to $+\infty$ and, for each n, an equilibrium for the model in which the matching rate is $\lambda = k_n\,\ell$. For each n and fixed state (S,B,ω), we let $V_{\theta,n} \equiv V_{\theta,n}(S,B,\omega)$ denote the continuation value of a θ-trader in the n-th equilibrium in state (S,B,ω), for $\theta \in \{\mathrm{s},\mathrm{b}\}$, and $\alpha_n \equiv \alpha_n(S,B,\omega)$ denote the probability of trade in a match in this equilibrium.

Let $\mathcal{M}^* \equiv \mathcal{S}^* \times \mathcal{B}^* \times \Omega$ denote the set of all states of the market with at least one seller and one buyer. Let $\mathcal{M}_1^n$, $\mathcal{M}_2^n$, and $\mathcal{M}_3^n$, denote, respectively, the first, the second, and the third, kinds of states described in the main text after the statement of Lemma 4.1 in the n-th equilibrium (so $\{\mathcal{M}_i^n\}_{i=1,2,3}$ is a partition of $\mathcal{M}^*$ for all n). Taking a subsequence if necessary, assume that for all n and for each state (S,B,ω) there is some $i = i(S,B,\omega) \in \{1,2,3\}$ such that $(S,B,\omega) \in \mathcal{M}^n_{i(S,B,\omega)}$ for all n in the subsequence. Assume, to ease notation and without loss of generality for the argument, that the aforementioned subsequence is equal to the original sequence. There are then three cases:

1. It is clear that for all $(S,B,\omega) \in \mathcal{M}_1^n$ for all n we have $V_n = 1$ for all n, so $V_n \to 1$.

2. Assume $(S,B,\omega) \in \mathcal{M}_2^n$ for all n and, without loss of generality, that $B = 1$ (the case $S = 1$ is analogous). Then, as indicated in the main text, a strategy available to the buyer is to wait until he makes the offer, and offer $V_{\mathrm{s},n}$ (if the state of the market has not changed before). This implies

$$V_{\mathrm{b},n} \geq \underbrace{\frac{(1-\xi)\,k_n\,\ell}{(1-\xi)\,k_n\,\ell + \gamma + r}}_{\to 1} (1 - V_{\mathrm{s},n}) + \underbrace{\frac{\gamma}{(1-\xi)\,k_n\,\ell + \gamma + r}}_{\to 0}\; V^{\mathrm{e}}_{\mathrm{b},n}\;.$$

Hence, since $V_{\mathrm{s},n} + V_{\mathrm{b},n} \leq 1$ for all n (by Proposition 3.2), we have $V_{\mathrm{s}} + V_{\mathrm{b}} \simeq 1$.

3. Assume finally that $(S,B,\omega) \in \mathcal{M}_3^n$ for all n and, without loss of generality, that $S \geq B > 1$ (the case $1 < S < B$ is analogous). Here we proceed by induction. Assume that $(S,B,\omega) \in \mathcal{M}_3^n$ is such that S is minimal among all states $(S',B',\omega') \in \mathcal{M}_3^n$ with $S' \geq B' > 1$. A strategy available to a seller is not trading until the next state, which implies

$$V_{\mathrm{s},n} \geq \underbrace{\frac{\frac{S-1}{S}\,k_n\,\ell}{\frac{S-1}{S}\,k_n\,\ell + \gamma + r}}_{\to 1}\; \mathbb{E}^{\mathrm{t}}[V_{\mathrm{s},n}(S{-}1, B{-}1, \tilde{\omega})] + \underbrace{\frac{\gamma}{\frac{S-1}{S}\,k_n\,\ell + \gamma + r}}_{\to 0}\; V^{\mathrm{e}}_{\mathrm{s},n}\;.$$

A buyer who follows the same strategy obtains

$$V_{\mathrm{b},n} \geq \underbrace{\frac{\frac{B-1}{B}\,k_n\,\ell}{\frac{B-1}{B}\,k_n\,\ell + \gamma + r}}_{\to 1}\; \mathbb{E}^{\mathrm{t}}[V_{\mathrm{b},n}(S{-}1, B{-}1, \tilde{\omega})] + \underbrace{\frac{\gamma}{\frac{B-1}{B}\,k_n\,\ell + \gamma + r}}_{\to 0}\; V^{\mathrm{e}}_{\mathrm{s},n}\;.$$

Hence, we have that $V \simeq \mathbb{E}^{t}[V(S-1,B-1,\tilde{\omega})]$. By the assumption that S is minimal we have $(S-1,B-1,\tilde{\omega}) \in \mathcal{M}_1^n \cup \mathcal{M}_2^n$ for all n and all $\tilde{\omega}$, and therefore $V(S-1,B-1,\tilde{\omega}) = 1$. This implies that $V = 1$. One can proceed recursively to show that, for all S, the result holds.

□

Proof Proposition 4.1

Proof. As in the proof of Lemma 4.1, throughout the proof we fix a sequence $(k_n)_n$ tending to $+\infty$ and, for each n, an equilibrium for the model in which the matching rate is $\lambda = k_n\,\ell$. For each n and fixed state (S,B,ω), we let $V_{\theta,n} \equiv V_{\theta,n}(S,B,\omega)$ denote the continuation value of a θ-trader in the n-th equilibrium in state (S,B,ω), for $\theta \in \{\mathrm{s},\mathrm{b}\}$, and $\alpha_n \equiv \alpha_n(S,B,\omega)$ denote the probability of trade in a match in this equilibrium.

Assume, without loss of generality for our arguments and taking a subsequence if necessary, that for each state $(S',B',\omega') \in \mathcal{S}^* \times \mathcal{B}^* \times \Omega$ the agreement rate at such a state, equal to $\alpha_n(S',B',\omega')\,k_n\,\ell(S',B',\omega')$, tends to some value $\delta(S',B',\omega') \in [0,+\infty]$ as n increases (with the convention that $\alpha_n(S',B',\omega') = 0$ when $S' = 0$ or $B' = 0$).

Preliminary result: We first note that using the standard analysis in Rubinstein (1982), we have that equation (4.6) holds for $V_{\mathrm{s},n}$. Indeed, when (S,B,ω) is such that $S=B=1$ we have

$$V_{\theta,n} = \tfrac{k_n\,\ell}{k_n\,\ell+\gamma+r}\left(\xi_\theta\,(1-V_{\bar{\theta},n}) + (1-\xi_\theta)\,V_{\theta,n}\right) + \tfrac{\gamma}{k_n\,\ell+\gamma+r}\,V^{\mathrm{e}}_{\theta,n}$$

for all $\theta \in \{\mathrm{s},\mathrm{b}\}$, where $\xi_b = 1-\xi$ and $\xi_s = \xi$. The previous equation coincides with the equation for the continuation payoff in a two-player Rubinstein bargaining where the "threat point" (i.e., value from not trading) for the θ-trader is $\tfrac{r}{\gamma+r}\,V^{\mathrm{e}}_{\theta,n}$. Solving for $V_{\mathrm{b},n}$ and $V_{\mathrm{s},n}$, we have

$$\lim_{n\to\infty} \left|\tfrac{r}{\gamma+r}\,\xi_\theta + \tfrac{\gamma}{\gamma+r}\,V^{\mathrm{e}}_{\theta,n} - V_{\theta,n}\right| = 0\ . \tag{A.5}$$

Definitions: For each n, define a function $\tilde{V}_{\theta,n} : \mathcal{S}\times\mathcal{B}\times\Omega \to [0,1]$, interpreted as the payoff of a θ-trader when she/he decides to trade only when the market is balanced, as follows. It is obtained solving equations (3.1), (3.4) and (3.5) (adding tildes to all V's), and instead of equations (3.2) and (3.3), we now require that $\tilde{V}^{\mathrm{m}}_{\theta,n} = \tilde{V}_{\theta,n}$ when $S \neq B$ (no trade when the market is imbalanced) and $\tilde{V}^{\mathrm{m}}_{\theta,n} = V^{\mathrm{m}}_{\theta,n}$ when $S = B$ (trade for sure when the market is balanced). Note that for $\theta = \mathrm{s}$ and $S \neq B$, equation (3.1) can be rewritten as equation (4.4) replacing "$\simeq$" by "$=$", adding tildes to all V's, and replacing α by α_n. We can obtain an analogous equation for $\tilde{V}_{\mathrm{b},n}$.

Our first goal is to show that, independently of the choice of the the sequence $(k_n)_n$ and a corresponding sequence equilibria,

$$\lim_{n\to\infty} |\tilde{V}_{s,n} - V_{s,n}| = 0 \text{ for all } S \geq B \quad\text{and}\quad \lim_{n\to\infty} |\tilde{V}_{\mathrm{b},n} - V_{\mathrm{b},n}| = 0 \text{ for all } S < B\ ;$$

or, in our notation, $\tilde{V}_{\mathrm{s}} \simeq V_{\mathrm{s}}$ for all $S \geq B$ and $\tilde{V}_{\mathrm{b}} \simeq V_b$ for all $S < B$. (As in the main text, "$\simeq$" means equal except for terms that go to 0 as n increases.) We then define $W_n \equiv W_n(S, B, \omega)$ to be equal to $\tilde{V}_{\mathrm{s},n}$ when $S \geq B$, and to be equal to $1 - \tilde{V}_{\mathrm{b},n}$ when $S < B$. Proving that W_n is approximated by the right-hand side of equation (4.3) as $n \to \infty$ proves then Proposition 4.1.

Begin fixing a state (S, B, ω) satisfying that $S = B \geq 1$. We aim at proving that for all $\varepsilon > 0$ there is some n such that $\left|\tilde{V}_{\mathrm{s},n} - V_{\mathrm{s},n}\right| < \varepsilon$. To see this, recall that by Proposition 3.2 there is immediate trade when the market is balanced and, by Lemma 4.1, $V_{\mathrm{b}} \simeq 1 - V_{\mathrm{s}}$. If $S = B = 1$ it is clear, by the definition of $\tilde{V}_s$, that $V_s \simeq V_s^{\mathrm{m}} \simeq \tilde{V}_s$. Proceeding similarly, we have that it is also the case that $W = \tilde{V}_{\mathrm{s}} \simeq V_{\mathrm{s}}$ because, as n increases, it is increasingly unlikely that an arrival happens before the market clears. If instead $S = B > 1$ we have

$$V_{\mathrm{s}} \simeq \tfrac{1}{S}\, V_{\mathrm{s}} + \tfrac{S-1}{S}\, \mathbb{E}^{\mathrm{t}}[V_{\mathrm{s}}(S-1, B-1, \tilde{\omega})] \quad \Rightarrow \quad V_{\mathrm{s}} \simeq \mathbb{E}^{\mathrm{t}}[V_{\mathrm{s}}(S-1, B-1, \tilde{\omega})]\ . \tag{A.6}$$

and similarly

$$\tilde{V}_{\mathrm{s}} \simeq \tfrac{1}{S}\, V_{\mathrm{s}} + \tfrac{S-1}{S}\, \mathbb{E}^{\mathrm{t}}[\tilde{V}_{\mathrm{s}}(S-1, B-1, \tilde{\omega})]\ .$$

Hence, if $S = B = 2$ it is clear that we also have $V_s \simeq V_s^{\mathrm{m}} \simeq \tilde{V}_s$. Proceeding recursively, we have that $V_s \simeq V_s^{\mathrm{m}} \simeq \tilde{V}_s$ holds for any $S = B > 0$. Hence, $V_s \simeq \tilde{V}_s = W_s$ whenever $S = B$.

For each n and state (S, B, ω), let $D_{\mathrm{s},n} \equiv D_{\mathrm{s},n}(S, B, \omega)$ denote $|V_{\mathrm{s},n} - \tilde{V}_{\mathrm{s},n}|$, and let $D_{\mathrm{b},n} \equiv D_{\mathrm{b},n}(S, B, \omega)$ denote $|V_{\mathrm{b},n} - \tilde{V}_{\mathrm{b},n}|$. Let also D_n denote $D_{\mathrm{b},n}$ when $S \geq B$ and $D_{\mathrm{s},n}$ when $S < B$. Assume, without loss of generality and for simplicity (considering a subsequence if necessary), assume that $D \equiv \lim_{n \to \infty} D_n$ is well defined for all states.

Case 1: Assume first $D = 0$ for all states. In this case, we have $W \simeq 1 - V_b$ for all $S \geq B$ and $W \simeq 1 - V_b$ for all $S < B$. Furthermore, when $S = B = 1$, we can rewrite equation (A.5) as

$$W \simeq \tfrac{r}{\gamma + r}\, \xi + \tfrac{\gamma}{\gamma + r}\, W^{\mathrm{e}}\ .$$

Hence, W satisfies equations (4.4)–(4.6) replacing all V_{s}'s by the corresponding W's. As it is argued in the main text, this implies that W satisfies equation (4.3).

Case 2: Now assume, for the sake of contradiction, that $D > 0$ for some states. We let $\bar{D}$ denote the maximal value of D among all states, and we assume without loss of generality that there is at least one state (S', B', ω') with $S' \geq B' > 0$ where $D(S', B', \omega') = \bar{D}$. We let (S, B, ω) be the state with such that $D(S, B, \omega) = \bar{D}$ such that S is minimal among all states (S', B', ω') with $S' \geq B' > 0$ where $D(S', B', \omega') = \bar{D}$.

We first rule out that $S = B$. We do this by assuming, for the sake of contradiction, that $S = B$. Hence, as we argued, $V_s \simeq \tilde{V}_s$. Note that this implies $D = 0$, a contradiction. Hence, we assume then, from now on, that $S > B$.

There are three cases:

1. Assume first that $\delta < \infty$. This is the case where the rate at which transactions happen remains finite as the bargaining frictions disappear. This implies that, for n large enough, we have $V_n = 1$. Also, since a seller is indifferent between trading or not we have

$$D_n = \frac{\frac{S-1}{S}\,\alpha_n\,k_n\,\ell}{\frac{S-1}{S}\,\alpha_n\,k_n\,\ell+\gamma+r}\,\mathbb{E}^{\mathrm{t}}[D_n(S-1,B-1,\tilde{\omega})] + \frac{\gamma}{\frac{S-1}{S}\,\alpha_n\,k_n\,\ell+\gamma+r}\,D_n^{\mathrm{e}}\ .$$

While the left hand side tends to $\bar{D}$, the right hand side tends to

$$\frac{\frac{S-1}{S}\,\delta}{\frac{S-1}{S}\,\delta+\gamma+r}\,\mathbb{E}^{\mathrm{t}}[D(S-1,B-1,\tilde{\omega})] + \frac{\gamma}{\frac{S-1}{S}\,\delta\gamma+r}\,D^{\mathrm{e}} \le \frac{\frac{S-1}{S}\,\delta+\gamma}{\frac{S-1}{S}\,\delta+\gamma+r}\,\bar{D} < \bar{D}\ .$$

This is a contradiction.

2. Assume now that $B = 0$ (and so $\ell(S,B,\omega) = 0$). In this case we have

$$D_n \le \frac{\gamma}{\gamma+r}\,D_n \quad \Rightarrow \quad D_n = 0\ .$$

This is, again, a contradiction.

3. We then have that, without loss of generality, we can assume $S > B > 0$ and that $\delta = \infty$. Note that this implies that equation (A.6) holds and, additionally,

$$\tilde{V}_{\mathrm{s}} \simeq \frac{1}{S}\,\tilde{V}_{\mathrm{s}} + \frac{S-1}{S}\,\mathbb{E}^{\mathrm{t}}[\tilde{V}_{\mathrm{s}}(S-1,B-1,\tilde{\omega})] \quad \Rightarrow \quad \tilde{V}_{\mathrm{s}} \simeq \mathbb{E}^{\mathrm{t}}[\tilde{V}_{\mathrm{s}}(S-1,B-1,\tilde{\omega})]\ .$$

This implies that

$$\bar{D} \le \mathbb{E}^{\mathrm{t}}[D(S-1,B-1,\tilde{\omega})]\ .$$

(The inequality is owed to the fact that the absolute value is a convex function.) This contradicts the assumption that S is minimal among all states (S',B',ω') with $S' \ge B' > 0$ where $D(S',B',\omega') = \bar{D}$.

$\square$

Proof of Corollary 4.1

Proof. As in the proof of Proposition 4.1, we fix a sequence $(k_n)_n$ tending to $+\infty$ and, for each n, an equilibrium for the model with $\lambda = k_n\,\ell$. Also, as in the proof of Proposition 4.1, we assume without loss of generality that for each state (S,B,ω), the values $V_{\mathrm{s},n}$, $V_{\mathrm{b},n}$ and $\alpha_n\,k_n\,\ell$ tend to some (state-dependent) values V_{s}, V_{b} and $\delta \in [0,+\infty]$ as n increases, respectively.

Fix some time t and state (S,B,ω). Assume, without loss of generality for our arguments, that $S > B$. For each $\Delta > 0$, we use $V_{\mathrm{s},\Delta} \equiv V_{\mathrm{s},\Delta}(S,B,\omega)$ denote the continuation value of the seller at time Δ if $(S_0,B_0,\omega_0) = (S,B,\omega)$ (which is a stochastic process). Then, $\mathbb{E}[V_{\mathrm{s},\Delta}]$ indicates the expected continuation value of the seller at time Δ if $(S_0,B_0,\omega_0) = (S,B,\omega)$. (Note that, since γ_{s}, γ_{b} and γ_{c} are independent of the equilibrium, and since the agreement rate converges to δ, the limit distribution of $(S_\Delta,B_\Delta,\omega_\Delta)$ as $n \to \infty$ is well defined for any $\Delta > 0$.)

We consider three cases:

1. Assume first that the state (S, B, ω) is such that $B = 0$. It is clear that, in this case,

$$\mathbb{E}[V_{\mathrm{s},\Delta}] = (1-\gamma\,\Delta)\,V_{\mathrm{s}}+\gamma\,\Delta\,V_{\mathrm{s}}^{\mathrm{e}}+O(\Delta^2) = \tilde{\mathbb{E}}[V_{\mathrm{s},\Delta}]+O(\Delta^2)\;.$$

Since, from equation (3.1), we have $r\,V_{\mathrm{s}} = \gamma\,(V_{\mathrm{s}}^{\mathrm{e}} - V_{\mathrm{s}})$, we have that

$$r\,V_{\mathrm{s}} = \frac{\mathbb{E}[V_{\mathrm{s},\Delta}] - V_{\mathrm{s}}}{\Delta} + O(\Delta) = \frac{\tilde{\mathbb{E}}[V_{\mathrm{s},\Delta}] - V_{\mathrm{s}}}{\Delta} + O(\Delta)\;.$$

2. Assume now that the state (S, B, ω) is such that $\delta < +\infty$. In this case, we have that

$$\mathbb{E}[V_{\mathrm{s},\Delta}] =(1-(\delta+\gamma)\,\Delta)\,V_{\mathrm{s}}+\delta\,\Delta\,\mathbb{E}^{\mathrm{t}}[V_{\mathrm{s}}(S-1,B-1,\tilde{\omega})]+\gamma_t\,\Delta\,V_{\mathrm{s}}^{\mathrm{e}}+O(\Delta^2)$$

as $\Delta \to 0$. The same equation holds if $\mathbb{E}$ is replaced by $\tilde{\mathbb{E}}$ on the left hand side and δ is replaced by $\frac{S-1}{S}\,\delta$ on the right-hand side. Also, we can use equation (3.1) to obtain

$$r\,V_{\mathrm{s}} = \tfrac{S-1}{S}\,\delta\,\mathbb{E}^{\mathrm{t}}[V_{\mathrm{s}}(S-1,B-1,\tilde{\omega}) - V_{\mathrm{s}}]+\gamma\,(V_{\mathrm{s}}^{\mathrm{e}} - V_{\mathrm{s}})\;.$$

It is then clear that $\frac{\tilde{\mathbb{E}}[V_{\mathrm{s},\Delta}]-V_{\mathrm{s}}}{\Delta} \simeq r\,V_{\mathrm{s}} + O(\Delta)$ as $\Delta \to 0$. Furthermore, it is easy to show (see the proof of Proposition 4.2) that $\mathbb{E}^{\mathrm{t}}[V_{\mathrm{s}}(S-1,B-1,\tilde{\omega}) - V_{\mathrm{s}}] = \frac{B\,S}{S-B}\,\frac{r}{\delta} > 0$, and so we have $\frac{\mathbb{E}[V_{\mathrm{s}+\Delta}]-V_{\mathrm{s}}}{\Delta} \succeq r\,V_{\mathrm{s}} + O(\Delta)$ as $\Delta \to 0$.

3. Assume finally that the state (S, B, ω) is such that $\delta = +\infty$. In this case, we have

$$V_{\mathrm{s}} = \mathbb{E}^{\mathrm{t}}[V_{\mathrm{s}}(S-1,B-1,\tilde{\omega})].$$

We consider two subcases:

   (a) Assume first that $B = 1$. This implies that, after there is a transaction, the new state has zero buyers, and then the analysis of part 1 of this proposition applies. Then, for each $\tilde{\omega}$,

$$r\,V_{\mathrm{s}}(S-1,B-1,\tilde{\omega}) = \frac{\mathbb{E}[V_{\mathrm{s},\Delta}(S-1,B-1,\tilde{\omega})] - V_{\mathrm{s}}(S-1,B-1,\tilde{\omega})}{\Delta} + O(\Delta)\;.$$

   Applying the expectation operator on both sides of the previous equation (with respect to $\tilde{\omega}$, using the distribution resulting from a transaction in state (S, B, ω)), we have

$$r\,\overbrace{\mathbb{E}^{\mathrm{t}}[V_{\mathrm{s}}(S-1,B-1,\tilde{\omega})]}^{=V_{\mathrm{s}}} = \frac{\overbrace{\mathbb{E}^{\mathrm{t}}[\mathbb{E}[V_{\mathrm{s},\Delta}(S-1,B-1,\tilde{\omega})]]}^{=\mathbb{E}[V_{\mathrm{s},\Delta}]} - \overbrace{\mathbb{E}^{\mathrm{t}}[V_{\mathrm{s}}(S-1,B-1,\tilde{\omega})]}^{=V_{\mathrm{s}}}}{\Delta}\Bigg] + O(\Delta)\;.$$

   The same argument applies replacing $\mathbb{E}$ by $\tilde{\mathbb{E}}$.

   (b) Now we proceed by induction. For each $B > 1$, we assume that the theorem applies for all states with lower number of buyers. Hence, for each realization of $\tilde{\omega}$, either $\delta(S-1,B-1,\tilde{\omega}) = +\infty$ (and the result applies then to state $(S-1,B-1,\tilde{\omega})$ by part

2 of this proof), or $\delta(S-1, B-1, \tilde{\omega}) < +\infty$, so the induction hypothesis dictates that the result holds. Then, we have

$$r\, V_{\mathrm{s}}(S-1, B-1, \tilde{\omega})] \preceq \frac{\mathbb{E}[V_{\mathrm{s},\Delta}(S-1, B-1, \tilde{\omega})] - V_{\mathrm{s}}(S-1, B-1, \tilde{\omega})}{\Delta}\Bigg] + O(\Delta)\ .$$

The previous inequality holds with equality "$\simeq$" if the expectation operator $\mathbb{E}$ is replaced by $\tilde{\mathbb{E}}$. Applying the expectation operator $\mathbb{E}^{\mathrm{t}}$ to each side of the previous equation, the result holds.

$\square$

Proof of Proposition 4.2

Proof. We assume that Condition 1 and, for the sake of contradiction, that there is a sequence $(k_n)_n$, a corresponding sequence of equilibria, and a sequence of states (S_n, B_n, ω_n) such that, in the n-th equilibrium, equilibrium offers are rejected with positive probability. For each state (S, B, ω), we use $V_{\mathrm{s},n}$ and $V_{\mathrm{b},n}$ to denote the continuation values of sellers and buyers in the n-th equilibrium, respectively, and α_n to denote the probability of acceptance of an equilibrium offer (so $\alpha_n(S_n, B_n, \omega_n) < 1$ for all n).

As in the proof Proposition 4.1 we assume, taking a subsequence if necessary assume, that for each state (S, B, ω), the continuation values $V_{\mathrm{b},n}$ and $V_{\mathrm{s},n}$ converge to some values $V_{\mathrm{b}} = V_{\mathrm{b}}(S, B, \omega)$ and $V_{\mathrm{s}} = V_{\mathrm{s}}(S, B, \omega)$, and the matching rates $\delta_n \equiv \alpha_n\, k_n\, \ell$ converge to some value $\delta \equiv \delta(S, B, \omega) \in [0, +\infty]$.

We first focus on characterizing the limit continuation value of seller, V_{s}, for states (S, B, ω) is such that $S > B \geq 1$. The equations are given by:

1. Consider first the case where $S > B \geq 0$ and $\delta < +\infty$. This implies that $B > 0$. Using equation (3.1) for both $\theta = \mathrm{s}, \mathrm{b}$, and using the fact that $V_{\mathrm{s},n} + V_{\mathrm{b},n} = 1$ if n is big enough, we can write

$$\begin{aligned} V_{\mathrm{s},n} &= \tfrac{\frac{S-1}{S}\,\delta_n}{\frac{S-1}{S}\,\delta_n+\gamma+r}\,\big(V_{\mathrm{s},n}(S-1, B-1, \omega)\big) + \tfrac{\gamma}{\frac{S-1}{S}\,\delta_n+\gamma+r}\,V^{\mathrm{e}}_{\mathrm{s},n} \\ &= \tfrac{r}{\frac{B-1}{B}\,\delta_n+\gamma+r} + \tfrac{\frac{B-1}{B}\,\delta_n}{\frac{B-1}{B}\,\delta_n+\gamma+r}\,\big(1-V_{\mathrm{b},n}(S-1, B-1, \omega)\big) + \tfrac{\gamma}{\frac{B-1}{B}\,\delta_n+\gamma+r}\,\big(1-V^{\mathrm{e}}_{\mathrm{s},n}\big)\ . \end{aligned}$$

(Note that there is no expectation when a transaction occurs because, by Condition 1(a), the market conditions do not change with transactions). Using Lemma 4.1 we have that $V_{\mathrm{b}} = 1 - V_{\mathrm{s}}$ for all states. Therefore, the previous two equalities can be rewritten, in the limit, as

$$V_{\mathrm{s}} = V_{\mathrm{s}}(S-1, B-1, \omega) - \tfrac{B\,S}{S-B}\,\tfrac{r}{\delta} = \tfrac{\gamma}{\gamma+r}\,V^{\mathrm{e}}_{\mathrm{s}} + \tfrac{r}{\gamma+r}\,\tfrac{(S-1)\,B}{S-B}\ . \tag{A.7}$$

It is then clear that there is no state where $\delta = 0$; that is, there is no state where trade occurs at a rate that becomes arbitrarily small as k increases. (The logic for this result is analogous to part 1 of Proposition 3.2.) Note that the second equality of equation (A.7) implies that, as indicated in part 3 of Proposition 3.2, traders on the long side of the market gain from other's transactions in states where is trade delay; i.e., when $V_{\mathrm{s}}(S-1, B-1, \omega) > V_{\mathrm{s}}$.

2. Consider now the case where $S > B \geq 1$ and $\delta = +\infty$. In this case, equation (4.1) implies that

$$V_{\mathrm{s}} = V_{\mathrm{s}}(S-1, B-1, \omega) \; .$$

3. Finally, for states where $B = 0$ we have

$$V_{\mathrm{s}} = \tfrac{\gamma}{\gamma+r} \, V_{\mathrm{s}}^{\mathrm{e}} \; .$$

For each state (S, B, ω) with $S > B \geq 1$, we define $\Delta \equiv V_{\mathrm{s}}(S-1, B-1, \omega) - V_{\mathrm{s}}$, and $\Delta = 0$ for each state (S, B, ω) with $S = B > 0$. Since, as we just showed, we have $V_{\mathrm{s}}(S-1, B-1, \omega) \geq V_{\mathrm{s}}$, it is the case that $\Delta \geq 0$ for all states with $S \geq B$. Let (S, B, ω) be a state which maximizes $\Delta(S', B', \omega')$ among all states with $S' \geq B' \geq 1$ and assume, for the sake of contradiction, that $\Delta > 0$ (so necessarily $S > B > 0$). If there are multiple states with this property, assume that (S, B, ω) is such that S is minimal among all of them.

Assume first that (S, B, ω) is such that $\delta = +\infty$. In this case, $V_{\mathrm{s}}(S-1, B-1, \omega) = V_{\mathrm{s}}$ (by part 2 in the previous argument). Hence, we have $\Delta = 0$, a contradiction. Then, it is necessarily the case that (S, B, ω) is such that $\delta < +\infty$. In this case, using equation (A.7), we have that

$$\begin{aligned}
\Delta = {} & \tfrac{\hat{\gamma}_{\mathrm{s}}}{\hat{\gamma}+r} \, V_{\mathrm{s}}(S, B-1, \omega) + \tfrac{\hat{\gamma}_{\mathrm{b}}}{\hat{\gamma}+r} \, V_{\mathrm{s}}(S-1, B, \omega) + \tfrac{\hat{\gamma}_{\mathrm{c}}}{\hat{\gamma}+r} \mathbb{E}^{\mathrm{c}}[V_{\mathrm{s}}(S-1, B-1, \tilde{\omega})] \\
& - \big(\tfrac{\gamma_{\mathrm{s}}}{\gamma+r} \, V_{\mathrm{s}}(S+1, B, \omega) + \tfrac{\gamma_{\mathrm{b}}}{\gamma+r} \, V_{\mathrm{s}}(S, B+1, \omega) + \tfrac{\gamma_{\mathrm{c}}}{\gamma+r} \, \mathbb{E}^{\mathrm{c}}[V_{\mathrm{s}}(S, B, \tilde{\omega})] + \tfrac{r}{\gamma+r} \, \tfrac{B\,(S-1)}{S-B} \big) \\
\leq {} & \tfrac{\gamma}{\gamma+r} \, \Delta \underbrace{- \tfrac{r}{\gamma+r} \, \tfrac{B\,(S-1)}{S-B}}_{\equiv(*)} \\
& + \underbrace{\big(\tfrac{\hat{\gamma}_{\mathrm{s}}}{\hat{\gamma}+r} - \tfrac{\gamma_{\mathrm{s}}}{\gamma+r} \big) V_{\mathrm{s}}(S+1, B, \omega) + \big(\tfrac{\hat{\gamma}_{\mathrm{b}}}{\hat{\gamma}+r} - \tfrac{\gamma_{\mathrm{b}}}{\gamma+r} \big) V_{\mathrm{s}}(S, B+1, \omega) + \big(\tfrac{\hat{\gamma}_{\mathrm{c}}}{\hat{\gamma}+r} - \tfrac{\gamma_{\mathrm{c}}}{\gamma+r} \big) \mathbb{E}^{\mathrm{c}}[V_{\mathrm{s}}(S, B, \tilde{\omega})]}_{\equiv(**)} \; ,
\end{aligned}$$

where the variables with a hat are evaluated at the state $(S-1, B-1, \omega)$ (note that, by Condition 1(a), $\hat{\gamma}_{\mathrm{c}} = \gamma_{\mathrm{c}}$). We will reach a contradiction (and then rule out that $\Delta > 0$) if the summation of the terms $(*)$ and $(**)$ in the previous equation is non-positive. Note first that

$$(**) \leq \max \big\{ \tfrac{\hat{\gamma}_{\mathrm{s}}}{\hat{\gamma}+r} - \tfrac{\gamma_{\mathrm{s}}}{\gamma+r}, 0 \big\} + \max \big\{ \tfrac{\hat{\gamma}_{\mathrm{b}}}{\hat{\gamma}+r} - \tfrac{\gamma_{\mathrm{b}}}{\gamma+r}, 0 \big\} + \max \big\{ \tfrac{\hat{\gamma}_{\mathrm{c}}}{\hat{\gamma}+r} - \tfrac{\gamma_{\mathrm{c}}}{\gamma+r}, 0 \big\} \leq \tfrac{r}{\gamma+r} \; ,$$

where the last inequality holds because of Condition 1(b). Therefore,

$$(*) + (**) \leq - \tfrac{r}{\gamma+r} \, \tfrac{B\,(S-1)}{S-B} + \tfrac{r}{\gamma+r} \leq 0 \; ,$$

and so the result holds.

We now prove that p exists satisfying the conditions in the statement. We use $p(\cdot,\cdot)$ to denote the solution of equations

$$p(N,\omega) = \tfrac{\gamma_{\mathrm{s}}}{\gamma+r}\, p(N{+}1,\omega){+}\tfrac{\gamma_{\mathrm{b}}}{\gamma+r}\, p(N{-}1,\omega){+}\tfrac{\gamma_{\mathrm{c}}}{\gamma+r}\, \mathbb{E}^{\mathrm{c}}[p(N,\tilde{\omega})] \qquad \text{if } N>0\,, \tag{A.8}$$

$$p(N,\omega) = \tfrac{r}{\gamma+r}{+}\tfrac{\gamma_{\mathrm{s}}}{\gamma+r}\, p(N{+}1,\omega){+}\tfrac{\gamma_{\mathrm{b}}}{\gamma+r}\, p(N{-}1,\omega){+}\tfrac{\gamma_{\mathrm{c}}}{\gamma+r}\, \mathbb{E}^{\mathrm{c}}[p(N,\tilde{\omega})] \qquad \text{if } N<0\,, \tag{A.9}$$

$$p(0,\omega) \;=\; \tfrac{r}{\gamma(1,1,\omega)+r}\,\xi{+}\tfrac{\gamma_{\mathrm{s}}(1,1,\omega)}{\gamma(1,1,\omega)+r}\, p(1,\omega){+}\tfrac{\gamma_{\mathrm{b}}(1,1,\omega)}{\gamma(1,1,\omega)+r}\, p(-1,\omega){+}\tfrac{\gamma_{\mathrm{c}}(1,1,\omega)}{\gamma(1,1,\omega)+r}\, \mathbb{E}^{\mathrm{c}}[p(0,\tilde{\omega})]\,. \tag{A.10}$$

where the arrival rates are evaluated at state $(N,0,\omega)$ when $N>0$ and at state $(0,-N,\omega)$ when $N<0$. The function $p(\cdot,\cdot)$ can be proven to be unique using standard fixed-point arguments similar to those in Section A.1. These equations approximate equations (4.4)–(4.6) replacing $V_{\mathrm{s}}(S,B,\omega)$ by $p(S-B,\omega)$, so it is clear that $V_{\mathrm{s}}(S,B,\omega) \simeq p(S-B,\omega)$ for all states (S,B,ω). $\square$

Proof of Corollary 4.2

Proof. Assume Conditions 1 and 2 hold. Since there is only one value ω for the market conditions, it is convenient to use $p(\cdot)$ to denote $p(\cdot,\omega)$ defined in Proposition 4.2.

For each $\bar{N} \geq 0$, one can rewrite equation (4.7) (recall that, under Condition 1, equation (4.7) holds also for the equilibrium measure) for all $N \geq \bar{N}$ as

$$p(N) = \mathbb{E}[e^{-r\,\bar{\tau}}\, p(\bar{N})|N_0{=}N]$$

where $\bar{\tau}$ is the stochastic time it takes the net supply to reach $\bar{N}$ for the first time. It is then clear, using for example $\bar{N}=0$, that $p(\cdot)$ is decreasing on $\{-\bar{B},...,\bar{S}\}$.

Consider now an increase on the discount rate from r_1 to r_2, with $r_1 < r_2$, and let $p^{r_i}(\cdot)$ denote the market price function for each r_i, $i=1,2$. Assume that $p^{r_1}(0) \geq p^{r_2}(0)$ (the reverse case is analogous). In this case, for all $N>0$ the price $p^{r_1}(N) > p^{r_2}(N)$. Indeed, using τ_0 to denote the (stochastic) time it takes for the market to become balanced (which is independent of r) and using equation (4.7) (recall that, under Condition 1, equation (4.7) holds also for the equilibrium measure), we can write

$$p^{r_1}(N) = \mathbb{E}\big[\, e^{-r_1\,\tau_0} \,\big|\, N_0{=}N\big]\, p^{r_1}(0) > \mathbb{E}\big[\, e^{-r_2\,\tau_0} \,\big|\, N_0{=}N\big]\, p^{r_2}(0) = p^{r_2}(N)\,. \tag{A.11}$$

Let $\bar{N}$ be the maximum value satisfying $p^{r_1}(\bar{N}) < p^{r_2}(\bar{N})$. Notice that equation (4.7) can be rewritten, for any $N \leq \bar{N} < 0$ and $i \in \{1,2\}$, as

$$p^{r_i}(N) = 1 - \mathbb{E}\big[\, e^{-r_i\,\bar{\tau}} \,\big|\, N_0{=}N\big]\, (1{-}p^{r_i}(\bar{N}))$$

where $\bar{\tau}$ is the first time where $N_t = \bar{N}$. It is then clear, using equation (A.11) and $p^{r_1}(\bar{N}) < p^{r_2}(\bar{N})$, that for all $N \leq \bar{N}$ we have $p^{r_1}(N) < p^{r_2}(N)$. Thus, in fact, $\bar{N}$ is such that

$$p^{r_1}(N) \geq p^{r_2}(N) \text{ for all } N > \bar{N} \text{ and } p^{r_1}(N) < p^{r_2}(N) \text{ for all } N \leq \bar{N}\,.$$

This property (and the fact that the ergodic distribution of N is independent of the discount rate) ensures that the distribution of $p^{r_2}(N)$ is a spread of $p^{r_1}(N)$.

Assume finally that $\gamma_{\mathrm{b}}(0,0,\omega) = \gamma_{\mathrm{b}}(1,1,\omega)$ and $\gamma_{\mathrm{s}}(0,0,\omega) = \gamma_{\mathrm{s}}(1,1,\omega)$. In this case, the limit ergodic distributions of N under both the equilibrium measure and the risk-neutral measure coincide. Let F be such a distribution. Then, the expected price under such a distribution is

$$\mathbb{E}[p(\tilde{N})|F] = \sum_{\tilde{N}\in\mathbb{Z}} F(\{\tilde{N}\})\, p(N)\ .$$

It is also the case that

$$\begin{aligned}\lim_{t\to\infty} \mathbb{E}[p_t] = \mathbb{E}[p(\tilde{N})|F] &= \lim_{t\to\infty} \mathbb{E}\left[\int_t^\infty e^{-r\,(s-t)}\,(\mathbb{I}_{N_s<0}+\xi\,\mathbb{I}_{N_s=0})\,r\,\mathrm{d}s\right] \\ &= \mathbb{E}\left[I_{\tilde{N}<0}+\xi\,\mathbb{I}_{\tilde{N}=0}\middle|F\right] = F(-\mathbb{N}) + \xi\,F(\{0\})\ ,\end{aligned}$$

where $-\mathbb{N}$ is the set of strictly negative integers. This proves that the ergodic mean of the market price is independent of r. $\square$

Proof of Propositon 4.3

Proof. **Proof of no delay.** Similar to the proof of Proposition 4.1, now fix a sequence $(k_n, M_n)_n$ tending to $(+\infty,+\infty)$ and, for each n, an equilibrium of the model where $\lambda = k_n\,\ell$ and $(\gamma_{\mathrm{b},n},\gamma_{\mathrm{s},n}) = M_n\,(\tilde{\gamma}_{\mathrm{b}},\tilde{\gamma}_{\mathrm{s}})$. For each state (S,B,ω) with $S,B>0$, we use $\tilde{\delta}_n(S,B,\omega)$ the trade rate at such state in the n-th equilibrium under the risk-neutral measure as in equation (4.2). Taking a subsequence if necessary, assume that $(\tilde{\delta}_n(S,B,\omega))_n$ tends to some limit $\tilde{\delta}(S,B,\omega) \in \mathbb{R}_+ \cup \{+\infty\}$ for all states (S,B,ω).

As in the main text, we use τ_0 to denote the stopping time until the market is balanced. Furthermore, for each state (S,B,ω), we denote

$$\phi_n(S,B,\omega) \equiv 1 - \tilde{\mathbb{E}}_n\left[e^{-r\,\tau_0}\middle|(S_0,B_0,\omega_0)=(S,B,\omega)\right] \in [0,1]$$

for each n, where the expectation is computed using the risk-neutral measure of the n-th equilibrium. Note that if $S=B$ then $\phi_n(S,B,\omega)=0$. Also, it satisfies the equation

$$\begin{aligned}\phi_n(S,B,\omega) = \tfrac{r}{\tilde{\delta}_n+\gamma+r} + \tfrac{\tilde{\delta}_n}{\tilde{\delta}_n+\gamma_n+r}\,\mathbb{E}^{\mathrm{t}}[\phi_n(S-1,B-1,\tilde{\omega})] \\ + \tfrac{\gamma_{\mathrm{s},n}}{\tilde{\delta}_n+\gamma_n+r}\,\mathbb{E}^{\mathrm{s}}[\phi_n(S+1,B,\tilde{\omega})] + \tfrac{\gamma_{\mathrm{b},n}}{\tilde{\delta}_n+\gamma_n+r}\,\mathbb{E}^{\mathrm{b}}[\phi_n(S,B+1,\tilde{\omega})] + \tfrac{\gamma_\omega}{\tilde{\delta}_n+\gamma_n+r}\,\mathbb{E}^{\mathrm{c}}[\phi_n(S,B,\tilde{\omega})]\end{aligned} \quad \text{(A.12)}$$

where, if $S=0$ or $B=0$, $\tilde{\delta}_n$ should be replaced with 0. Take a subsequence of our original sequence such that $(\phi_n(S,B,\omega))_n$ is converging to some $\phi(S,B,\omega)$ for all states of the world. Assume, for the sake of contradiction, that $\bar{\phi} \equiv \max_{(S',B',\omega')} \phi(S',B',\omega') > 0$. Assume $\phi(S',B',\omega') = \bar{\phi}$ for some state (S',B',ω') with $S' > B'$ (the other case is analogous).

Let (S, B, ω) such that $S > B$ satisfying, for all other states (S', B', ω') with $S' > B'$ and $\phi(S', B', \omega') = \bar{\phi}$, that (i) $S' - B' \geq S - B$, and that (ii) if $S' - B' = S - B$ then $S' > S$. Thus, (S, B, ω) has a minimal excess supply among all states with maximal ϕ and, among those with lowest excess supply, has minimal number of sellers. If $B = 0$ then we can write (A.12) as

$$\begin{aligned}\overbrace{\phi_n(S,B,\omega)}^{\to\bar{\phi}} &= \overbrace{\tfrac{r}{\gamma_n}\,(1-\phi_n(S,B,\omega))}^{\to 0}\\ &\quad+\tfrac{\gamma_{\mathrm{s},n}}{\gamma_n}\,\underbrace{\mathbb{E}^{\mathrm{s}}[\phi_n(S{+}1,B,\tilde{\omega})]}_{\to\leq\bar{\phi}}+\underbrace{\tfrac{\gamma_{\mathrm{b},n}}{\gamma}}_{>0}\,\underbrace{\mathbb{E}^{\mathrm{b}}[\phi_n(S,B{+}1,\tilde{\omega})]}_{\to<\bar{\phi}}+\tfrac{\gamma_\omega}{\gamma_n}\,\underbrace{\mathbb{E}^{\mathrm{c}}[\phi_n(S{+}1,B,\tilde{\omega})]}_{\to\leq\bar{\phi}}\,,\end{aligned}$$

where we used that $\tilde{\gamma}_{\mathrm{b}} > 0$ because $B < S$ (by Condition 2(b)) and that $S - (B{+}1) < S - B$. This is a clear contradiction. Assume, on the contrary, that $B > 0$. In this case, since $(S{-}1) - (B{-}1) = S{-}B$ we have $\mathbb{E}^{\mathrm{t}}[\phi(S{-}1, B{-}1, \tilde{\omega})] < \bar{\phi}$. We can write (A.12) as

$$\begin{aligned}\overbrace{\phi(S,B,\omega)}^{\to\bar{\phi}} &= \overbrace{\tfrac{r}{\tilde{\delta}_n+\gamma_n}\,(1-\phi(S,B,\omega))}^{\to 0}+\tfrac{\tilde{\delta}_n}{\tilde{\delta}_n+\gamma_n}\,\overbrace{\mathbb{E}^{\mathrm{t}}\phi_n(S{-}1,B{-}1,\tilde{\omega})]}^{\to<\bar{\phi}}\\ &\quad+\tfrac{\gamma_{\mathrm{s},n}}{\tilde{\delta}_n+\gamma_n}\,\underbrace{\mathbb{E}^{\mathrm{s}}[\phi_n(S{+}1,B,\tilde{\omega})]}_{\to\leq\bar{\phi}}+\tfrac{\gamma_{\mathrm{b},n}}{\tilde{\delta}_n+\gamma_n}\,\underbrace{\mathbb{E}^{\mathrm{b}}[\phi_n(S,B{+}1,\tilde{\omega})]}_{\to<\bar{\phi}}+\tfrac{\gamma_\omega}{\tilde{\delta}_n+\gamma_n}\,\underbrace{\mathbb{E}^{\mathrm{e}}[\phi_n(S,B,\tilde{\omega})]}_{\to<\bar{\phi}}\,.\end{aligned}$$

It is then clear that we reach, again, a contradiction.

The previous argument implies that $\phi(S, B, \omega) = 0$ for all states (S, B, ω); that is, the discounting until the market balances (under the risk-neutral measure) is 1. Using equation (4.7), we have then that $\lim_{n\to\infty} V_{\mathrm{s},n}(S, B, \omega) = \lim_{n\to\infty} V_{\mathrm{s},n}(1, 1, \omega)$ for all states (S, B, ω). Hence, we have $\lim_{n\to\infty} \frac{r}{\tilde{\delta}_n} = 0$ for all states. This implies that delay vanishes as $n \to 0$ also under the equilibrium measure.

Proof of single price for each ω. Assume now that Conditions 1 and 2(b) hold (i.e., we allow Ω to have more than one element). By the first part of this proof, the trade rate under the risk-neutral measure $\tilde{\delta}_n$ is such that $\lim_{n\to\infty} \frac{r}{\tilde{\delta}_n} = 0$ for all states. Hence, since the market conditions process does not change as n increases, the discounted time it takes for the distribution of the market composition to approximate the ergodic distribution of the market composition for a given initial ω_0 shrinks to 0. It is then clear that equation (4.9) follows from equation (4.3). Note that when Condition 2(a) holds we have that, by Proposition 4.1, p^* is equal to the probability that $S < B$ under the ergodic distribution plus ξ multiplied by the probability that $S = B$ under the ergodic distribution. □

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