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Bidding in Common-Value Auctions with an Uncertain Number of Competitors

Stephan Lauermann*
Andre Speit**

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*University of Bonn, s.lauermann@uni-bonn.de
**University of Bonn, aspeit@uni-bonn.de

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Bidding in Common-Value Auctions With an Uncertain Number of Competitors*

—includes extended appendix—

Stephan Lauermann[†] Andre Speit[‡]

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Abstract

This paper studies a first-price common-value auction in which bidders are uncertain about the number of their competitors. It shows that this uncertainty invalidates classic findings for common-value auctions with a known number of rival bidders (Milgrom and Weber 1982). In particular, the inference from winning is no longer monotonic, and a "winner's blessing" emerges at low bids. As a result, bidding strategies may not be strictly increasing but instead contain atoms. The location of the atoms is indeterminate, implying equilibrium multiplicity. Moreover, an equilibrium fails to exist when the expected number of competitors is large and the bid space is continuous. Therefore, we consider auctions on a grid. On a fine grid, high-signal bidders follow an essentially strictly increasing strategy whereas low-signal bidders pool on two adjacent bids on the grid. For the equilibrium characterization, we utilize a "communication extension" based on Jackson et al. (2002).

Keywords: common-value auctions, random player games, numbers uncertainty, Poisson games, endogenous tie-breaking, non-existence

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[†]University of Bonn, s.lauermann@uni-bonn.de

[‡]University of Bonn, aspeit@uni-bonn.de

1 Introduction

In most auctions, bidders are uncertain about the number of competitors they face:

- At auction houses such as Christie's and Sotheby's, personal attendance is in decline as bidders prefer to phone in or place their bids online. Therefore, bidders "[...] know even less about who they're bidding against, which in some cases can leave them wondering how high they should go."[1]

- eBay reveals the number of bidders who place a bid but does not disclose how many prospective bidders follow the auction. In particular, the platform does not display how many bidders are online to "snipe," that is, to place their bid in the last seconds of the auction (Roth and Ockenfels 2002).

- Considering auction-like trading mechanisms, the continuous order book at the New York Stock Exchange informs market participants about the stream of (un)filled buy and sell orders, but reveals neither the number nor the identity of (potential) buyers and sellers.

Although uncertainty about the number of competitors, or "numbers uncertainty", is ubiquitous, the subject has received little attention in the literature of auction theory. One reason may be its irrelevance in standard auction formats with pure private values: by a revenue-equivalence argument, equilibrium bids are just a weighted average of the bids that are optimal when the number of rival bidders is known (Krishna 2010, Chapter 3.2.2).

By contrast, in a common-value setting, numbers uncertainty significantly alters bidding behavior. Recall that when the number of rival bidders is known, the classic results going back to Milgrom and Weber (1982) establish that there exists a unique symmetric equilibrium in the first-price and second-price auctions, in which the bids are strictly increasing in the bidders' own value estimates. Uniqueness and strict monotonicity facilitate the revenue comparison of auction formats, welfare considerations (in general interdependent value settings), and empirical identification strategies. We show that these classic results no longer hold when the number of competitors is uncertain. Equilibria are generally not strictly increasing but contain atoms. The location of the atoms is often indeterminate, implying equilibrium multiplicity. Moreover, equilibrium payoffs are discontinuous at the atoms, invalidating standard methods for analyzing bidding behavior in these auctions. In particular, with a continuous bid space, equilibrium generally fails to exist.

To model an auction with numbers uncertainty, we start with a canonical common-value first-price auction. The value of the good is binary (high or low) and bidders receive conditionally independent and identically distributed signals, with higher signals indicating a higher value (affiliation). Each bidder simultaneously submits a bid, the highest bidder wins, and pays her bid. Ties are broken uniformly. The only difference from the textbook setting is that the number of (rival) bidders is not known, but instead

[1]The Wall Street Journal, "Why Auction Rooms Seem Empty These Days" `https://www.wsj.com/articles/with-absentee-bidding-on-the-rise-auction-rooms-seem-empty-these-days-1402683887` cf. Akbarpour and Li (2018)

a random variable which is assumed to be Poisson distributed. However, our results extend beyond this distributional assumption.

Numbers uncertainty affects bidding behavior with common values because it changes the value inference from winning. In a conventional common-value auction with a known number of bidders, the expected value conditional on winning is increasing in the relative position of the bid because a higher bid eases the "winner's curse." In fact, there is no winner's curse at the very top bid. This reduction reinforces price competition and implies the absence of pooling (atoms in the bid distribution). Note that at any bid below the top one, the winner's curse is more severe if there are more competitors.

With numbers uncertainty, winning is also informative about the number of rival bidders. In particular, winning with a low bid is more likely when there are fewer competitors which eases the winner's curse. Therefore, winning with a low bid is not necessarily bad news about the value of the good. In our model, the inference is U-shaped: intermediate bids are subject to the strongest winner's curse, while there is no winner's curse at the bottom or the top (Lemma 2 and 4).[2]

We show that every equilibrium is nondecreasing in the bidder's signal (Lemma 1), but the non-monotone inference implies that equilibria cannot be strictly increasing unless the expected number of competitors is sufficiently small (Propositions 1 and 2). Hence, the equilibrium bid distribution contains one or more atoms, as bidders with different signals pool on common bids. Numbers uncertainty incentivize bidders to pool because pooling shields them against the winner's curse: under a uniform tie-breaking rule, winning the auction with a bid that ties with positive probability is relatively more likely when there are fewer competitors, which reduces the negative inference from winning. An example in Appendix C.1 demonstrates that atoms already occur in very small auctions, namely when the expected number of rival bidders is larger than one.

The presence of atoms in the bid distribution substantially alters the analysis of the auction. First, the location of atoms is often indeterminate, as illustrated by two examples in Appendices C.2 and C.3. Second, atoms create discontinuities in the bidders' payoffs. As a result of these discontinuities, no equilibrium exists when the expected number of bidders is sufficiently large (Proposition 3).

If the bid space is discrete rather than continuous, equilibria do exist by standard arguments (Lemma 9). To study the resulting bidding behavior on a fine grid, we utilize a "communication extension" of the auction, based on Jackson et al. (2002). In the communication extension, bidders not only submit a monetary bid from the continuous bid space but also a message that indicates their "eagerness" to win, which is used to break ties. The communication extension is useful because, in contrast to the standard auction, the limit of any converging sequence of equilibria on the ever-finer grid corresponds to an equilibrium of the communication extension. Since such an equilibrium inherits the properties of the equilibria on the fine grid, we can use the equilibrium characterization of the communication extension in Proposition 4 to derive the equilibria on a fine grid (Proposition 5).

Qualitatively, any equilibrium on a fine grid with increments $d > 0$ consists of three

[2] The random number of competitors adds a second dimension of uncertainty. Thus, the value of the good is no longer affiliated with the first-order statistic of the signals.

regions: Bidders with high signals essentially follow a strictly increasing strategy (as the grid permits), while bidders with intermediate signals pool on some bid b_p, and bidders with low signals bid one increment below it, $b_p - d$.[3]

The equilibria are shaped by a severe winner's curse at b_p, and a "winner's blessing" that arises at bids below b_p, so that, at these bids, the expected value conditional on winning is significantly higher than b_p. This induces bidders with low signals to compete for the largest bid strictly below b_p. On the grid, this competition leads them to pool on $b_p - d$; on the continuous bid space, the non-existence of a largest bid below b_p implies the non-existence of an equilibrium.

We discuss the robustness of our results in Section 6. We argue that our findings do not depend on the Poisson distribution of the number of bidders, and that similar results hold in the second-price auction. Finally, we discuss the related literature on auctions with a non-constant number of bidders, especially recent contributions by Murto and Välimäki (2019) and Lauermann and Wolinsky (2018).

2 Model

A single, indivisible good is sold in a first-price, sealed-bid auction. The good's value is either high, v_h, or low, v_ℓ, with $v_h > v_\ell \geq 0$, depending on the unknown state of the world $\omega \in \{h, \ell\}$. The state is $\omega = h$ with probability ρ and $\omega = \ell$ with probability $1 - \rho$, where $\rho \in (0, 1)$. The number of bidders is a Poisson-distributed random variable with mean η, such that there are i bidders in the auction with probability $e^{-\eta} \frac{\eta^i}{i!}$. The realization of the variable is unknown to the bidders.

Every bidder receives a signal s from the compact set $[\underline{s}, \bar{s}]$. Conditional on the state, the signals are independent and identically distributed according to the cumulative distribution functions F_h and F_ℓ, respectively. Both distributions have continuous densities f_ω, and the likelihood ratio of these densities, $\frac{f_h(s)}{f_\ell(s)}$, satisfies the (weak) monotone likelihood ratio property: for all $s < s'$ it holds that $\frac{f_h(s)}{f_\ell(s)} \leq \frac{f_h(s')}{f_\ell(s')}$. Furthermore, $0 < \frac{f_h(\underline{s})}{f_\ell(\underline{s})} < \frac{f_h(\bar{s})}{f_\ell(\bar{s})} < \infty$, such that signals do contain information but never reveal the state perfectly. For convenience, let there be a unique neutral signal $\breve{s}$ at which $\frac{f_h(\breve{s})}{f_\ell(\breve{s})} = 1$.

Having received her signal, every bidder submits a bid b. Suppose that there is a reserve price at v_ℓ, and note that it is without loss to exclude bids above v_h, such that $b \in [v_\ell, v_h]$. The bidder with the highest bid wins the auction, receives the object, and pays her bid. Ties are broken uniformly. If there is no bidder, the good is not allocated. Bidders are risk neutral.

A discussion of the significance of the assumptions on the distribution of bidders, signals, the reserve price, and auction format can be found in Section 6. In the appendix, we allow the number of bidders to be state-dependent by considering state-dependent means η_ω. In fact, all results are shown in this more general setup.

[3]As the grid becomes fine, the two bids b_p and $b_p - d$ "merge." In the limit, both bids can no longer be separated, such that the outcome discretely changes, and low-signal bidders win with the same probability as intermediate-signal bidders. Hence, the limit strategy with $d = 0$ is generally not an equilibrium of the continuous bid space with the standard uniform tie-breaking rule. In contrast, the communication extension allows bidders with low and intermediate bids to send different messages, such that they can be differentiated.

It is useful to recall two special properties of the Poisson distribution prior to beginning the analysis. A detailed derivation and discussion can be found in Myerson (1998). First, when participating in the auction, a bidder does not change her belief regarding the number of other bidders in the auction. Therefore, her belief about the number of her competitors is again a Poisson distribution with mean η. This property is analogous to a stationary Poisson process, in which an event does not allow for inferences about the number of other events.

Second, the Poisson distribution implies that attention can be restricted to symmetric equilibria.[4] Since the Poisson distribution has an unbounded support, it draws bidders from a hypothetical infinite urn. Any individual bidder and, thus, any individual strategy are thereby drawn with zero probability. One could imagine that certain fractions of the population in the urn follow different strategies, such that those are encountered with positive probability. However, this would be equivalent to drawing the bidders first and having them mix between strategies afterward.

Accordingly, we consider symmetric strategies, which are functions mapping from the signals into the set of probability distributions over bids $\beta : [\underline{s}, \bar{s}] \to \Delta[v_\ell, v_h]$. Let $\pi_\omega(b; \beta)$ denote the probability of winning the auction with a bid b in state ω, if the rival bidders follow strategy β. Using Bayes' rule, the interim expected utility for a bidder with signal s choosing bid b is

$$\begin{aligned} U(b|s;\beta) = & \frac{\rho f_h(s)}{\rho f_h(s) + (1-\rho) f_\ell(s)} \pi_h(b;\beta)(v_h - b) \\ & + \frac{(1-\rho) f_\ell(s)}{\rho f_h(s) + (1-\rho) f_\ell(s)} \pi_\ell(b;\beta)(v_\ell - b). \end{aligned} \tag{1}$$

A strategy β^* is a *best response* to a strategy β, if, for (almost) all s, a bid $b \in \operatorname{supp} \beta^*(s)$ implies that $b \in \arg\max_{\hat{b} \in [v_\ell, v_h]} U(\hat{b}|s;\beta)$. Henceforth, we distinguish between claims that hold everywhere and almost everywhere only when it is central to the argument. Unless specified otherwise, results hold for almost all s. Two strategies are *equivalent* if they correspond to the same distributional strategy after merging all signals that share the same likelihood ratio $\frac{f_h}{f_\ell}$. Thus, equivalent strategies imply the same distribution over bids and utilities.

Lemma 1. *Let β be some strategy and β^* a best response to it. If the likelihood ratio $\frac{f_h}{f_\ell}$ is strictly increasing, then β^* is essentially*[5] *pure and nondecreasing. If the likelihood ratio is only weakly increasing, then there exists an equivalent best response $\hat{\beta}^*$ that is pure and nondecreasing.*

The proof is in the appendix. Higher bids improve the prospects of winning, which is desirable in the high state in which the winner turns a profit ($b \leq v_h$), but disadvantageous in the low state in which the winner incurs a loss ($b \geq v_\ell$). Thus, more optimistic bidders are willing to bid more aggressively. If the likelihood ratio $\frac{f_h}{f_\ell}$ is constant along

[4] This fits our aim of analyzing how uncertainty about the number of competitors rather than their identity affects the equilibrium bidding behavior.

[5] Up to a set of signals with measure zero.

some interval, the bids can be reordered along this interval.

We look for Bayes-Nash equilibria β^*, and, by Lemma 1, can restrict attention to pure and nondecreasing strategies. In the following, strategies are nondecreasing functions mapping signals into bids, $\beta : [\underline{s}, \bar{s}] \to [v_\ell, v_h]$.

3 Analysis of the standard auction

The analysis is structured into three parts. The first subsection focuses on the winning probability and inference from bids that never tie. We then use our findings to examine strictly increasing strategies, and show that there can be no strictly increasing equilibrium unless the expected number of bidders is sufficiently small. Hence, there have to be pooling bids—that is, atoms in the bid distribution. We investigate these atoms in the second subsection. Last, we argue that the atoms in the bid distribution necessarily prevent equilibrium existence.

3.1 Non-pooling bids

Fix some nondecreasing strategy β. A bid b is a *non-pooling bid* if it is selected with zero probability by any bidder. Given strategy β, this is the case if b is either not in the image of β, or if there is only a single signal s such that $\beta(s) = b$. In either situation, a bidder who chooses b wins whenever all of her competitors bid below b. Since β is nondecreasing, this implies that they all received lower signals than $\hat{s} = \sup\{s \colon \beta(s) \leq b\}$. Thus, the bidder wins in the event that $s_{(1)} \leq \hat{s}$, where $s_{(1)} = \sup\{s_{-i}\}$ is the highest of the competitors' signals. We employ the convention that $\sup\{\emptyset\} = -\infty$, which means that $s_{(1)} = -\infty$ in case there is no competitor. As a result, the generalized first-order statistic $s_{(1)}$ has a cumulative distribution function $F_{s_{(1)}}(s|\omega) = e^{-\eta(1-F_\omega(s))}$ for $s \in [\underline{s}, \bar{s}]$.[6] Since bid b wins whenever $s_{(1)} \leq \hat{s}$, it wins in state $\omega \in \{h, \ell\}$ with probability $\pi_\omega(b; \beta) = e^{-\eta(1-F_\omega(\hat{s}))}$.

A characteristic feature of common-value auctions is that winning is informative about the value of the good. When choosing a non-pooling bid, all that matters for this inference is the relative position of the bid, $\hat{s}$. Next, we analyze how this position $\hat{s}$ affects the conditional expected value, $\mathbb{E}[v|\text{win with } b; \beta] = \mathbb{E}[v|s_{(1)} \leq \hat{s}]$, with

$$\mathbb{E}[v|s_{(1)} \leq \hat{s}] = \frac{\rho e^{-\eta(1-F_h(\hat{s}))} v_h + (1-\rho) e^{-\eta(1-F_\ell(\hat{s}))} v_\ell}{\rho e^{-\eta(1-F_h(\hat{s}))} + (1-\rho) e^{-\eta(1-F_\ell(\hat{s}))}}. \tag{2}$$

Recall that $\breve{s}$ is the unique neutral signal, $\frac{f_h(\breve{s})}{f_\ell(\breve{s})} = 1$.

Lemma 2. *The conditional expected value $\mathbb{E}[v|s_{(1)} \leq \hat{s}]$ is strictly decreasing in $\hat{s}$ when $\hat{s} < \breve{s}$, has its unique global minimum at $\hat{s} = \breve{s}$, and is strictly increasing when $\hat{s} > \breve{s}$.*

[6] Conditional on state ω, any competitor (independently) receives a signal larger than $\hat{s}$ with probability $1 - F_\omega(\hat{s})$. By the decomposition and environmental equivalence properties of the Poisson distribution (Myerson (1998)), bidders believe that the number of rival bidders with signals larger than $\hat{s}$ is Poisson distributed with mean $\eta(1 - F_\omega(\hat{s}))$. The probability that $s_{(1)} \leq \hat{s}$ is the probability that there is no competitor with a signal above $\hat{s}$.

Proof. Note that $\frac{av_h+v_\ell}{a+1} > \frac{bv_h+v_\ell}{b+1}$ if and only if $a > b$. By (2), this means that $\mathbb{E}[v|s_{(1)} \leq \hat{s}]$ is strictly increasing if and only if $e^{\eta(F_h(\hat{s})-F_\ell(\hat{s}))}$ is strictly increasing. Its derivative is $e^{\eta(F_h(\hat{s})-F_\ell(\hat{s}))}\eta[f_h(\hat{s})-f_\ell(\hat{s})]$ and so $e^{\eta(F_h(\hat{s})-F_\ell(\hat{s}))}$ is increasing if and only if $f_h(\hat{s}) > f_\ell(\hat{s})$. The uniqueness of the neutral signal $\breve{s}$ where $f_h(\breve{s}) = f_\ell(\breve{s})$ and the monotone likelihood ratio property imply that $f_h(\hat{s}) < f_\ell(\hat{s})$ for $\hat{s} < \breve{s}$, and $f_h(\hat{s}) > f_\ell(\hat{s})$ for $\hat{s} > \breve{s}$. □

Lemma 2 implies that $\mathbb{E}[v|s_{(1)} \leq \hat{s}]$ is U-shaped in $\hat{s}$ with its minimum at $\breve{s}$. The intuition behind the shape may be explained best with the help of Figure 1:

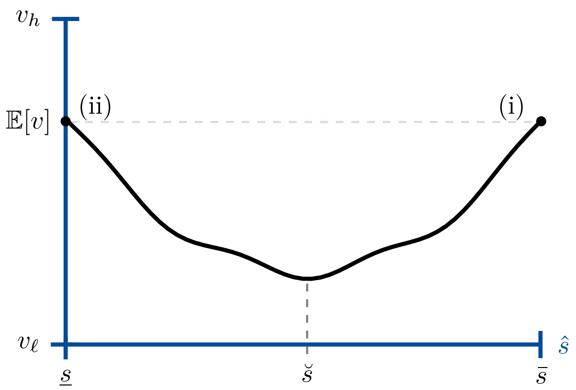


Figure 1: The conditional expected value $\mathbb{E}[v|s_{(1)} \leq \hat{s}]$.

First, consider point (i) on the top right, which marks $\mathbb{E}[v|s_{(1)} \leq \bar{s}]$. By definition, the highest signal, $s_{(1)}$, is always smaller than $\bar{s}$, independent of the state. Hence, the event that $s_{(1)} \leq \bar{s}$ is uninformative about the state and $\mathbb{E}[v|s_{(1)} \leq \bar{s}] = \mathbb{E}[v]$.

Second, consider point (ii) on the top left, denoting $\mathbb{E}[v|s_{(1)} \leq \underline{s}]$. The highest signal $s_{(1)}$ equals $\underline{s}$ with zero probability (the signal distribution has no atoms), while there are no competitor and $s_{(1)} = -\infty$ with positive probability. Consequently, $\mathbb{E}[v|s_{(1)} \leq \underline{s}] = \mathbb{E}[v|s_{(1)} = -\infty]$. Since the distribution of bidders is independent of the state, this event occurs with the same probability in both states. As a result, the event that $s_{(1)} \leq \underline{s}$ is also uninformative about the state and $\mathbb{E}[v|s_{(1)} \leq \underline{s}] = \mathbb{E}[v]$. Thus, there is no winner's curse at the bottom (ii) or at the top (i).

In the middle where $\hat{s} \in (\underline{s}, \bar{s})$, the winner's curse comes into play. With positive probability, there are competitor, all of which received signals below $\hat{s}$. These low signals are bad news about the value of the good. Consequently, for $\hat{s} \in (\underline{s}, \bar{s})$, the conditional expected value is smaller than the unconditional one, $\mathbb{E}[v|s_{(1)} \leq \hat{s}] < \mathbb{E}[v]$, with the global minimum at $\breve{s}$, where $f_h(\breve{s}) = f_\ell(\breve{s})$.

Observe that as η increases, the winner expects to face more rival bidders, such that the winner's curse grows more severe. For $\hat{s} \in (\underline{s}, \bar{s})$, it follows that $\mathbb{E}[v|s_{(1)} \leq \hat{s}] \overset{\eta\to\infty}{\to} v_\ell$.[7] At the boundaries $\underline{s}$ and $\bar{s}$, on the other hand, the inference is independent of η; therefore, $\mathbb{E}[v|s_{(1)} \leq s]$ converges in η to a $\sqcup$-shape.

[7]The monotone likelihood ratio property implies that $F_h(s) < F_\ell(s)$ for all $s \in (\underline{s}, \bar{s})$. Thus, $\eta(F_h(s) - F_\ell(s)) \to -\infty$ for all $s \in (\underline{s}, \bar{s})$ when $\eta \to \infty$. The convergence then follows by equation (2).

While the precise form of $\mathbb{E}[v|s_{(1)} \leq \hat{s}]$ follows from the Poisson distribution, the same effects are present under any distribution of bidders. Importantly, the non-monotonicity does not depend on the possibility that there is no rival bidder,[8] but is a consequence of the variation in the number of (rival) bidders. At any bid below the top, the winning bidder simultaneously updates her belief over two random variables: the number of competitors and their signal realization. Since these two can push the conditional expected value in opposite directions, the winning bidder's inference will generally not be monotone in $\hat{s}$. In other words, numbers uncertainty breaks the affiliation between the value of the good and the first-order statistic of (rivals') signals.

3.1.1 No strictly increasing equilibrium when η is large

The non-monotone inference from winning can substantially affect the equilibrium behavior of bidders. As a benchmark, consider the standard common-value auction with a fixed and known number of $n \geq 2$ bidders. In this setup, the inference is monotone, which implies that the unique symmetric equilibrium is strictly increasing.[9] When the numbers uncertainty causes a non-monotone inference, an equilibrium of this form, generally, does not exist.

Proposition 1. *When η is sufficiently large, no strictly increasing equilibrium exists.*

In Appendix C.1 we provide an example which shows that strictly increasing equilibria can fail to exist for η as low as 1. Here, we first give an intuitive, verbal argument before sketching out the critical steps of the proof, which is also relegated to the appendix.

Suppose to the contrary that there is a strictly increasing equilibrium β^* for an arbitrary large η arbitrary. In this case, a bidder with signal s, following the strategy β^* wins whenever $s_{(1)} \leq s$. Conditional on winning, and her own signal, she, therefore, expects the good to be of value $\mathbb{E}[v|\text{win with } \beta^*(s), s; \beta^*] = \mathbb{E}[v|s_{(1)} \leq s, s]$, with

$$\mathbb{E}[v|s_{(1)} \leq s, s] = \frac{\rho f_h(s) e^{-\eta(1-F_h(s))} v_h + (1-\rho) f_\ell(s) e^{-\eta(1-F_\ell(s))} v_\ell}{\rho f_h(s) e^{-\eta(1-F_h(s))} + (1-\rho) f_\ell(s) e^{-\eta(1-F_\ell(s))}}. \tag{3}$$

When η is large, the expected competition is fierce, which implies that equilibrium bids must be close to the expected value conditional on winning, $\beta^*(s) \approx \mathbb{E}[v|s_{(1)} \leq s, s]$. In addition to that, a large η makes the inference from winning more relevant for the expected value than the bidder's own signal. Consequently, when η is large, $\mathbb{E}[v|s_{(1)} \leq s, s]$ inherits the U-shape from $\mathbb{E}[v|s_{(1)} \leq s]$. Taken together, this means that $\beta^*(s)$ is decreasing below the neutral signal $\check{s}$, which is a contradiction.[10]

[8] For instance, if we consider a truncated Poisson distribution in which there are always at least $\underline{n} \geq 2$ bidders, $\mathbb{E}[v|s_{(1)} \leq \hat{s}]$ is still U-shaped when η is large. At the top, the inference from winning is unaffected by the truncation, and at the bottom, the winning bidder still updates her belief toward the lowest number of rival bidders possible, $\underline{n} - 1$. Thus, there is now a limited winner's curse at $\underline{s}$ which, however, does not depend on η. Since the winner's curse grows arbitrary large at any $\hat{s} \in (\underline{s}, \bar{s})$ when η increases, this results in the U-shape.

[9] There is an exception: If $\frac{f_h}{f_\ell}$ is constant along some interval at the bottom of the signal distribution, $[\underline{s}, s]$, then these signals choose the same bid (cf. Proposition 2 in Lauermann and Wolinsky (2017)).

[10] The crucial step of the proof is to show that $\beta^*(s)$ converges to $\mathbb{E}[v|s_{(1)} \leq s, s]$ quick enough, such that the U-shape of $\mathbb{E}[v|s_{(1)} \leq s, s]$ can be exploited. Otherwise, the argument might fail because $\mathbb{E}[v|s_{(1)} \leq s, s]$ converges to v_ℓ for all $s \in (\underline{s}, \bar{s})$.

To formalize this contradiction, fix three signals $s_- < s_\circ < s_+$ with $s_+ < \breve{s}$. The argument is structured into three steps: First, we derive an upper bound on the bid $\beta^*(s_+)$ from individual rationality (Step 1), and then a lower bound on $\beta^*(s_\circ)$ from the incentive constraints of s_- (Step 2). Step 3 shows that when η is large, the lower bound exceeds the upper bound.

Step 1. *An upper bound on $\beta^*(s_+)$ is given by*

$$\frac{\beta^*(s_+) - v_\ell}{v_h - \beta^*(s_+)} \leq \frac{\rho}{1-\rho} \frac{f_h(s_+)}{f_\ell(s_+)} \frac{e^{-\eta(1-F_h(s_+))}}{e^{-\eta(1-F_\ell(s_+))}}. \tag{4}$$

In equilibrium, it has to hold for any signal s that $\beta^*(s) \leq \mathbb{E}[v|\text{win with } \beta^*(s), s; \beta^*]$. Otherwise, the utility

$$U(\beta^*(s)|s;\beta^*) = \mathbb{P}[\text{win with } \beta^*(s)|s;\beta^*]\Big(\mathbb{E}[v|\text{win with } b, s; \beta^*] - \beta^*(s)\Big)$$

is negative, which cannot be the case in equilibrium, because a bid of v_ℓ guarantees a non-negative payoff. Using (3), the condition $\beta^*(s) \leq \mathbb{E}[v|\text{win with } \beta^*(s), s; \beta^*]$ can be rearranged to

$$\frac{\beta^*(s) - v_\ell}{v_h - \beta^*(s)} \leq \frac{\rho}{1-\rho} \frac{f_h(s)}{f_\ell(s)} \frac{\pi_h(\beta^*(s);\beta^*)}{\pi_\ell(\beta^*(s);\beta^*)}. \tag{5}$$

Now, inequality (4) follows from (5) with s_+ and $\pi_\omega(\beta^*(s_+);\beta^*) = e^{-\eta(1-F_\omega(s_+))}$ because β^* is a strictly increasing strategy.

Step 2. *A lower bound on $\beta^*(s_\circ)$ is given by*

$$\frac{\beta^*(s_\circ) - v_\ell}{v_h - \beta^*(s_\circ)} \geq \frac{\rho}{1-\rho} \frac{f_h(s_-)}{f_\ell(s_-)} \frac{e^{-\eta(1-F_h(s_\circ))}}{e^{-\eta(1-F_\ell(s_\circ))}} A(\eta), \tag{6}$$

where $A(\eta)$ is a decreasing function with $\lim_{\eta\to\infty} A(\eta) = 1$.

In equilibrium, a bidder with a signal s_- cannot have an incentive to deviate from $\beta^*(s_-)$ to $\beta^*(s_\circ)$, which implies that $U(\beta^*(s_-)|s_-;\beta^*) \geq U(\beta^*(s_\circ)|s_-;\beta^*)$. In the appendix, we show that this condition can be used to derive (6). Observe that when $A(\eta) = 1$, the inequality rearranges to $\beta^*(s_\circ) \geq \mathbb{E}[v|s_{(1)} \leq s_\circ, s_-]$. Since the argument holds for any $s_- < s_\circ$, $A(\eta) \to 1$ captures the observation that when η is large, bids have to be close to the expected value conditional on winning.

Step 3. *When η is sufficiently large, the upper bound on $\beta^*(s_+)$ expressed by (4) is smaller than the lower bound on $\beta^*(s_\circ)$ given by inequality (6).*

Since $\beta^*(s_+) > \beta^*(s_\circ)$ and $\frac{b-v_\ell}{v_h-b}$ is increasing in b, a necessary condition for both inequalities to hold simultaneously is that

$$\frac{\rho}{1-\rho} \frac{f_h(s_+)}{f_\ell(s_+)} \frac{e^{-\eta(1-F_h(s_+))}}{e^{-\eta(1-F_\ell(s_+))}} > \frac{\rho}{1-\rho} \frac{f_h(s_-)}{f_\ell(s_-)} \frac{e^{-\eta(1-F_h(s_\circ))}}{e^{-\eta(1-F_\ell(s_\circ))}} A(\eta).$$

This can be rearranged to

$$\frac{f_h(s_+)}{f_\ell(s_+)}\Big(\frac{f_h(s_-)}{f_\ell(s_-)}\Big)^{-1} > \frac{e^{-\eta(1-F_h(s_\circ))}}{e^{-\eta(1-F_\ell(s_\circ))}}\Big(\frac{e^{-\eta(1-F_h(s_+))}}{e^{-\eta(1-F_\ell(s_+))}}\Big)^{-1} A(\eta). \tag{7}$$

The fractions $\frac{e^{-\eta(1-F_h(s_\circ))}}{e^{-\eta(1-F_\ell(s_\circ))}}\Big(\frac{e^{-\eta(1-F_h(s_+))}}{e^{-\eta(1-F_\ell(s_+))}}\Big)^{-1}$ capture the difference in the inference from winning when $s_{(1)} \leq s_\circ$ instead of $s_{(1)} \leq s_+$. Since $s_\circ < s_+ < \breve{s}$, the signals are from the decreasing leg of $\mathbb{E}[v|s_{(1)} \leq s]$ such that the fraction is larger than one. In fact, the difference in inference grows without bound,[11]

$$\frac{e^{-\eta(1-F_h(s_\circ))}}{e^{-\eta(1-F_\ell(s_\circ))}}\Big(\frac{e^{-\eta(1-F_h(s_+))}}{e^{-\eta(1-F_\ell(s_+))}}\Big)^{-1} = e^{\eta([F_\ell(s_+)-F_\ell(s_\circ)]-[F_h(s_+)-F_h(s_\circ)])} \to \infty. \tag{8}$$

Since $A(\eta) \to 1$, this means that the right side of equation (7) becomes arbitrary large, while the left side stays constant. Hence, when η is large, the inference from winning (right side) dominates the inference from the signals (left side). This echoes the fact that $\mathbb{E}[v|s_{(1)} \leq s, s]$ becomes U-shaped as η grows. As a result, inequality (7) cannot hold, and β^* cannot be a strictly increasing equilibrium.

3.1.2 Unique strictly increasing equilibrium when η is small

When η is small, we can give necessary and sufficient conditions for the existence of a strictly increasing equilibrium. For $s, s' \in [\underline{s}, \bar{s}]$, let $F_{s_{(1)}}(s'|s)$ denote the expected cumulative distribution function of $s_{(1)}$ conditional on observing s,

$$F_{s_{(1)}}(s'|s) = \frac{\rho f_h(s)}{\rho f_h(s) + (1-\rho) f_\ell(s)} e^{-\eta(1-F_h(s'))} + \frac{(1-\rho) f_\ell(s)}{\rho f_h(s) + (1-\rho) f_\ell(s)} e^{-\eta(1-F_\ell(s'))},$$

and let $f_{s_{(1)}}(s'|s)$ be the associated density.

Proposition 2. *The ordinary differential equation*

$$\frac{\partial}{\partial s}\beta(s) = \Big(\mathbb{E}[v|s_{(1)} = s, s] - \beta(s)\Big)\frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)} \qquad \textit{with } \beta(\underline{s}) = v_\ell \tag{9}$$

has a unique solution $\hat{\beta}$.

(i) If $\hat{\beta}$ is strictly increasing, then it is the unique equilibrium in the class of strictly increasing equilibria.

(ii) If $\hat{\beta}$ is not strictly increasing, no strictly increasing equilibrium exists.

The proof is provided in the appendix.[12] The argument that no strictly increasing equilibrium exists made use of two effects of a large η: that the winner's curse determines the shape of $\mathbb{E}[v|s_{(1)} \leq s, s]$, and that competition is fierce. Lemma 3 shows that both of these conditions are necessary: when the expected value conditional on winning is monotone, or competition is lax, a strictly increasing equilibrium exists.

[11] $[F_\ell(s_+) - F_\ell(s_\circ)] - [F_h(s_+) - F_h(s_\circ)] = \int_{s_\circ}^{s_+}[f_\ell(z) - f_h(z)]dz \geq \int_{s_\circ}^{s_+} f_\ell(z)(1 - \frac{f_h(s_+)}{f_\ell(s_+)})dz > 0$ since $s_+ < \breve{s}$.

[12] Apart from the slightly different definition of $s_{(1)}$, this is the standard ODE in the literature cf. (Krishna 2010, Chapter 6.4).

Lemma 3. *A strictly increasing equilibrium exists if either*

(i) $\mathbb{E}[v|s_{(1)} = s, s]$ *is strictly increasing in s;*

(ii) or η *is sufficiently small.*

First, if $\mathbb{E}[v|s_{(1)} \leq s, s]$ is monotone, the existence problem described above does not arise. Even if bids are close to the conditional expected value, the bidding function can be strictly increasing. Indeed, there is a slightly tighter[13] condition, and a strictly increasing equilibrium exists if $\mathbb{E}[v|s_{(1)} = s, s]$ is strictly increasing in s. This is the case if and only if

$$2\Big(\frac{\partial}{\partial s}\frac{f_h(s)}{f_\ell(s)}\Big)\frac{f_\ell(s)}{f_h(s)} + \eta f_h(s) - \eta f_\ell(s) > 0 \text{ for a.e. } s \in [\underline{s}, \bar{s}]. \tag{10}$$

Observe that when $\frac{f_h}{f_\ell}$ is constant over some interval below the neutral signal $\breve{s}$, condition (10) is never fulfilled. However, even in this case, a strictly increasing equilibrium exists when η is small. If competition is very weak, bids stay far below the expected value conditional on winning. Therefore, the problem described in Section 3.1.1 does not arise, and a strictly increasing equilibrium exists.

3.1.3 Strictly increasing equilibria and the second-price auction

In a second-price auction, standard arguments imply that the equilibrium bid in a symmetric and strictly increasing equilibrium is the expected value conditional on being tied at the top, $\mathbb{E}[v|s_{(1)} = s, s]$. Thus, condition (10) is necessary and sufficient for the existence of a strictly increasing equilibrium, and no such equilibrium exists when η is large. Similar problems also arise for other distributions of the number of bidders: for instance, Harstad et al. (2008) provide an example in which the distribution is binary, and no strictly increasing equilibrium exists.

Wrapping up, Section 3.1 demonstrates that uncertainty over the number of competitors prevents the existence of a strictly increasing equilibrium unless η is sufficiently small. Combined with Lemma 1, this implies that if an equilibrium exists, it has to be piecewise flat. Next, we take a closer look at these flat parts to understand why bidders with different signals may have an incentive to pool on the same bid.

3.2 Pooling bids

Fix some nondecreasing strategy β, and suppose that $\beta(s) = b_p$ for all s from an interval, but $\beta(s) \neq b_p$ otherwise. We generally refer to the interval as a *pool*, to b_p as a *pooling bid* and, without loss, always consider the closure of interval which is denoted by $[s_-, s_+]$. We show by a simple computation (proof of Lemma 4) that the probability to win with b_p in state $\omega \in \{h, \ell\}$ is

$$\pi_\omega(b_p;\beta) = \frac{\mathbb{P}[s_{(1)} \in [s_-, s_+]\,|\omega]}{\mathbb{E}[\#s \in [s_-, s_+]\,|\omega]} = \frac{e^{-\eta(1-F_\omega(s_+))} - e^{-\eta(1-F_\omega(s_-))}}{\eta(F_\omega(s_+) - F_\omega(s_-))}, \tag{11}$$

[13] $\mathbb{E}[v|s_{(1)} \leq s, s]$ is strictly increasing if and only if $\mathbf{1}\Big(\frac{\partial}{\partial s}\frac{f_h(s)}{f_\ell(s)}\Big)\frac{f_\ell(s)}{f_h(s)} + \eta f_h(s) - \eta f_\ell(s) > 0$.

where "$\mathbb{E}[\#s \in [s_-, s_+]$" denotes the expected number of signal realizations from the interval $[s_-, s_+]$.

In Appendices C.2 and C.3 we provide two examples of equilibria with atoms. Bidders have an incentive to pool because it insures them against the winner's curse, meaning that the expected value conditional on winning with the pooling bid b_p is larger than the expected value conditional on winning with a bid marginally above b_p,

$$\mathbb{E}[v|\text{win with } b_p; \beta] > \lim_{\epsilon \to 0} \mathbb{E}[v|\text{win with } b_p + \epsilon; \beta] = \mathbb{E}[v|s_{(1)} \leq s_+].$$

If this wasn't he case, any bidder with a signal $s \in [s_-, s_+]$ would have an incentive to marginally overbid b_p, raising the expected profits conditional on winning.[14]

To gain intuition into how winning with b_p can ease the winner's curse compared to winning with a marginally higher bid, consider the following reasoning: With positive probability, multiple bidders tie on the pooling bid b_p such that the winner is decided by the uniform tie-breaking rule. Consequently, the bid b_p is more likely to win when there are fewer competitors who also choose b_p, that is, when there are fewer competitors with signals from $[s_-, s_+]$. If those signals are low, such that they are more likely to realize in the low state, this implies that b_p wins less often in the low state than in the high state. Since the bid marginally above b_p never ties, it loses this blessing.

For this insurance to work, the number of competitor has to be uncertain. Otherwise, winning more often when there are fewer competitors with signals from $[s_-, s_+]$ means winning more often when there are more competitors with signals below s_-. This exacerbates the winner's curse. When the number of bidders is Poisson distributed, the number of bidders with signals below s_- is independent of the number of bidders with signals from $[s_-, s_+]$. Therefore, winning with b_p is more advantageous than winning with a marginally higher bid whenever the expected number of (rival) bidders with signals from $[s_-, s_+]$, that is, $\eta[F_\omega(s_+) - F_\omega(s_-)]$, is larger in the low state than in the high state.

Formalizing these observations (proof is in the appendix) gives us the following Lemma.

Lemma 4. *Assume that β is some strategy for which there exists an interval $[s_-, s_+]$ and a bid b_p, such that $\beta(s) = b_p$ for all $s \in [s_-, s_+]$ and $\beta(s) < b_p < \beta(s')$ for all $s < s_- < s_+ < s'$.*

(i) If $\eta[F_h(s_+) - F_h(s_-)] < \eta[F_\ell(s_+) - F_\ell(s_-)]$, then

$$\mathbb{E}[v|s_{(1)} \leq s_-] > \mathbb{E}[v|\textit{win with } b_p; \beta] > \mathbb{E}[v|s_{(1)} \leq s_+]. \tag{12}$$

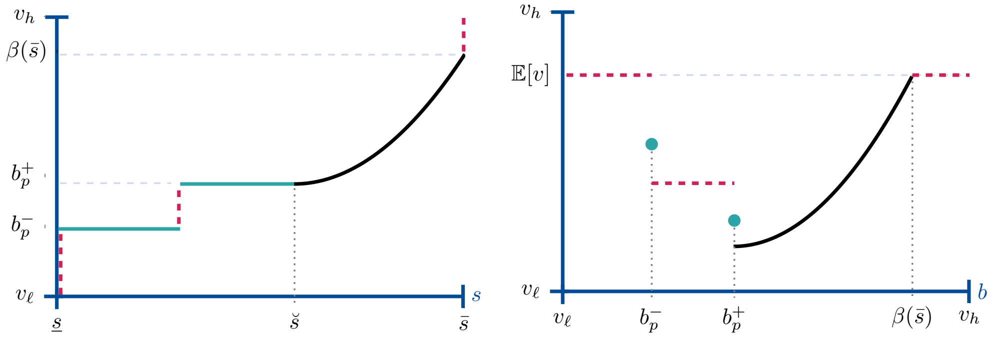
(ii) If $\eta[F_h(s_+) - F_h(s_-)] > \eta[F_\ell(s_+) - F_\ell(s_-)]$, then the inequalities in (12) reverse.

(iii) If β is an equilibrium strategy, then $\eta[F_h(s_+) - F_h(s_-)] < \eta[F_\ell(s_+) - F_\ell(s_-)]$.

Combined, Lemmas 2 and 4 imply that the inference from winning is always U-shaped in the bid. Suppose, for instance, that all competitors follow strategy β depicted

[14] Reviewing the argument for inequality (5) highlights that the inequality is, in fact, strict such that for all $s \in [s_-, s_+]$ it holds that $\beta(s) = b_p < \mathbb{E}[v|\text{win with } b_p, s; \beta]$, and winning more often raises the profit.

in the left panel of Figure 2 and consider the associated conditional expected value $\mathbb{E}[v|\text{win with } b; \beta]$ which is plotted in the right panel. Bids that are not in the image of β are colored pink (dashed), pooling bids are teal, and non-pooling bids are black.



A strategy $\beta(s)$ with two atoms

The expected value $\mathbb{E}[v|\text{win with b}; \beta]$

Figure 2: The inference from winning.

Going through the bids from low to high, first, consider the inference from winning with a bid below the image of β, that is, the first pink, dashed interval. These bids can only win when there is no rival bidder, which is why there is no winner's curse and the conditional expected value is just $\mathbb{E}[v]$.

The first bid in the image of β is the pooling bid b_p^-, which can win when there are rival bidders. As a result, winning with b_p^- is bad news about the value of the good, such that $\mathbb{E}[v|\text{win with } b_p^-; \beta] < \mathbb{E}[v]$. Further, because b_p^- is exclusively chosen by bidders with signals below the neutral signal $\breve{s}$, inequality (12) applies, and there is an even stronger winner's curse at bids between b_p^- and b_p^+ (second pink, dashed interval) than at b_p^-.

The next bid in the image of β, that is b_p^+, is again a pooling bid exclusively chosen by signals below $\breve{s}$. Thus, the winner's curse at b_p^+ is again stronger than at any lower bid, but a less severe one than winning with a marginally higher bid.

All bids above b_p^+ that are in the image of β are non-pooling bids and chosen by signals above $\breve{s}$. Thus, Lemma 2 applies, and the conditional expected value is strictly increasing above $\beta(\breve{s})$. At the top, there is no winner's curse since bids at or above $\beta(\bar{s})$ always win the auction.

When strategies contain atoms, the bidders' utilities are discontinuous in the bid. Winning with a pooling bid is discretely less likely than winning with a marginally higher bid, and since the probabilities change differently across states, the expected value conditional on winning with the pooling bid is discretely different, too. As a result, equilibria do not need to be unique. Instead of following a unique differential equation, they can consist of various mixtures of strictly increasing sections, pooling bids, and jumps. In Appendices C.2 and C.3 we provide two numerical examples of equilibrium multiplicity.

3.3 Non-existence of equilibria

In addition to the non-uniqueness, the discontinuities at atoms create an existence problem. In equilibrium, the U-shaped inference implies that there is an open set of bids below any pooling bid with a discretely higher expected value conditional on winning. When η is large, this induces bidders compete for the highest bid below the pooling bid, which prevents the existence of an equilibrium.

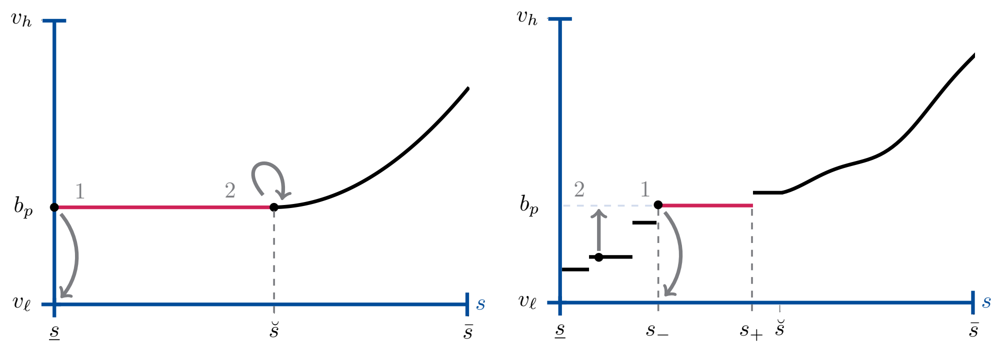
Proposition 3. *When η is sufficiently large, no equilibrium exists.*

Formally, this result is a corollary to Proposition 4 in the next section. However, the proof for Proposition 4 is fairly indirect. Thus, we sketch out the main idea here.

First, observe that Proposition 1 can be strengthened: when η is large, any strategy that is not locally constant below the neutral signal $\breve{s}$ can be excluded as an equilibrium. To be precise, for almost all $s < \breve{s}$, the winning probability must not converge to the probability of having the highest signal as η grows,

$$\lim_{\eta\to\infty} \frac{e^{-\eta(1-F_\omega(s))}}{\pi_\omega(\beta^*(s);\beta^*)} \neq 1 \quad \text{for } \omega \in \{h,\ell\},$$

such that bidders with almost all signals $s < \breve{s}$ tie with positive probability.[15] As a result, any candidate equilibrium must essentially be a step function below $s < \breve{s}$. In the following, we exclude two salient types of candidates: equilibria in which bidders with signals below $\breve{s}$ all pool on the same bid, and equilibria in which only bidders with a fixed, interior subset of signals $[s_-, s_+] \subset (\underline{s}, \breve{s}]$ pool. Figure 3 sketches out both types. The two arrows in each panel depict two possible deviations. We show that one of them has to be profitable when η is large, such that equilibria cannot take either form. While this still leaves a large set of equilibrium candidates—in particular, equilibria in which the boundaries of the pools change as η increases—it turns out that similar arguments can be used to exclude those, too.



(a) Candidate β^* with a single, large pool (b) Candidate β^* with an interior pool

Figure 3: Candidate equilibria.

(a) Single, large pool: To begin, we show that when η is large, there can be no equilibrium shaped like the one in the left panel of Figure 3.

[15] If the fraction does converge to one for a set of signals $\breve{s}$ with positive mass, the proof of Proposition 1 yields exactly the same contradiction for any three signals $s_-, s_\circ, s_+$ from this set.

Lemma 5. *For a sufficiently large η, there is no equilibrium β^* in which $\beta^*(\underline{s}) = \beta^*(\check{s})$ and $\beta^*(s) > \beta^*(\check{s})$ for all $s > \check{s}$.*

Suppose to the contrary that such a β^* exists for an arbitrary large η. Denote the pooling bid by $b_p = \beta^*(\underline{s}) = \beta^*(\check{s})$. The contradiction is derived in three steps. First, deviation 1 is used to derive an upper bound on b_p (Step 1), before deviation 2 is used to find a lower bound (Step 2). Last, Step 3 shows that when η is sufficiently large, the lower bound exceeds the upper bound, such that one deviation has to be profitable. As an abbreviation we use $\pi_\omega^\circ = \pi_\omega(b_p; \beta^*)$ and $\pi_\omega^+ = \lim_{\epsilon \searrow 0} \pi_\omega(b_p + \epsilon; \beta^*)$ for $\omega \in \{h, \ell\}$.[16]

Step 1. *By (5), individual rationality (deviation 1) for signal $\underline{s}$ implies that*

$$\frac{b_p - v_\ell}{v_h - b_p} \leq \frac{\rho}{1-\rho} \frac{f_h(\underline{s})}{f_\ell(\underline{s})} \frac{\pi_h^\circ}{\pi_\ell^\circ}. \tag{13}$$

Step 2. *There exists a function $B(\eta) < 1$ with $B(\eta) \to 1$ such that*

$$\frac{b_p - v_\ell}{v_h - b_p} \geq \frac{\rho}{1-\rho} \frac{f_h(\check{s})}{f_\ell(\check{s})} \frac{\pi_h^+}{\pi_\ell^+} B(\eta). \tag{14}$$

Signal $\check{s}$ has an incentive to deviate from b_p to a marginally higher bid (deviation 2), unless $U(b_p|\check{s}; \beta^*) \geq \lim_{\epsilon \searrow 0} U(b_p + \epsilon|\check{s}; \beta^*)$. Rearranging this inequality gives

$$\frac{b_p - v_\ell}{v_h - b_p} \geq \frac{\rho}{1-\rho} \frac{f_h(\check{s})}{f_\ell(\check{s})} \frac{\pi_h^+ - \pi_h^\circ}{\pi_\ell^+ - \pi_\ell^\circ}.$$

The bid marginally above b_p always wins when $s_{(1)} \leq \check{s}$, while b_p is also subject to a tie-break whenever there are competitors who also bid b_p. Since the expected number of competitors who choose b_p is $\eta F_\omega(\check{s})$, this means that $\frac{\pi_\omega^+}{\pi_\omega^\circ} \approx \eta F_\omega(\check{s})$. Because $\eta F_\omega(\check{s})$ grows in η without bound, this implies that $B(\eta) = \frac{\pi_h^+ - \pi_h^\circ}{\pi_\ell^+ - \pi_\ell^\circ} \left(\frac{\pi_h^+}{\pi_\ell^+} \right)^{-1} \to 1$ which gives the result. Observe that when $B(\eta) = 1$, equation (14) rearranges to $b_p \geq \mathbb{E}[v | s_{(1)} \leq \check{s}, \check{s}]$, meaning that b_p has to be at least the expected value conditional on winning with a marginally higher bid.

Step 3. *When η is sufficiently large, the lower bound (14) exceeds the upper bound (13). Thus, either deviation 1 or 2 is profitable.*

Combining inequalities (13) and (14) yields

$$\frac{\rho}{1-\rho} \frac{f_h(\underline{s})}{f_\ell(\underline{s})} \frac{\pi_h^\circ}{\pi_\ell^\circ} \geq \frac{\rho}{1-\rho} \frac{f_h(\check{s})}{f_\ell(\check{s})} \frac{\pi_h^+}{\pi_\ell^+} B(\eta).$$

By definition of the neutral signal $\frac{f_h(\check{s})}{f_\ell(\check{s})} = 1$, such that the inequality rearranges to

$$\frac{f_h(\underline{s})}{f_\ell(\underline{s})} \geq \frac{\pi_h^+}{\pi_\ell^+} \left(\frac{\pi_h^\circ}{\pi_\ell^\circ} \right)^{-1} B(\eta). \tag{15}$$

From $\frac{\pi_\omega^+}{\pi_\omega^\circ} \approx \eta F_\omega(\check{s})$, it follows that $\frac{\pi_h^+}{\pi_\ell^+} \left(\frac{\pi_h^\circ}{\pi_\ell^\circ} \right)^{-1} \to \frac{F_h(\check{s})}{F_\ell(\check{s})}$: the blessing from winning

[16] Explicitly, $\pi_\omega^+ = e^{-\eta(1 - F_\omega(s_+))}$ and $\pi_\omega^\circ = \frac{e^{-\eta(1-F_\omega(s_+))} - e^{-\eta(1-F_\omega(s_-))}}{\eta(F_\omega(s_+) - F_\omega(s_-))}$ for $\omega \in \{h, \ell\}$.

with b_p as opposed to a marginally higher bid is bounded and of order $\frac{F_h(\check{s})}{F_\ell(\check{s})}$. This blessing does not suffice to reconcile the lower bound (14) and the upper bound (13). Since $\frac{f_h(\underline{s})}{f_\ell(\underline{s})} < \frac{f_h(\bar{s})}{f_\ell(\bar{s})}$, the monotone likelihood ratio property implies that $\frac{f_h(\underline{s})}{f_\ell(\underline{s})} < \frac{F_h(\check{s})}{F_\ell(\check{s})}$.[17] Combined with the observation that $B(\eta) \to 1$, this means that condition (15) is violated when η is large. Thus, either deviation 1 or 2 is profitable, and there can be no equilibrium β^* in which all signals below $\check{s}$ pool on the same bid.

(b) Interior pool: Suppose now that there is an equilibrium with an "interior pool", even when η is arbitrary large. This type of equilibrium is depicted qualitatively in the right panel of Figure 3.

Lemma 6. *Fix any s_-, s_+ with $\underline{s} < s_- < s_+ \leq \check{s}$. When η is sufficiently large, there is no equilibrium β^* in which $\beta^*(s_-) = \beta^*(s_+)$, $\beta^*(s) < \beta^*(s_-)$ for all $s < s_-$ and $\beta^*(s) > \beta^*(s_+)$ for all $s > s_+$.*

Suppose to the contrary that even when η is arbitrary large such a β^* exists. Denote the pooling bid by $b_p = \beta^*(s_-) = \beta^*(s_+)$. We proceed in the same way as before and use the deviation 1 to derive an upper bound on b_p (Step 1) as well as deviation 2 to derive a lower bound on b_p (Step 2). Step 3 shows that the lower bound exceeds the upper bound when η is large, such that one of the deviations has to be profitable. As abbreviations, we use that $\pi_\omega^\circ = \pi_\omega(b_p; \beta^*)$ and $\pi_\omega^- = \lim_{\epsilon \searrow 0} \pi_\omega(b_p - \epsilon; \beta^*)$ for $\omega \in \{h, \ell\}$.[18]

Step 1. *By (5), individual rationality (deviation 1) for signal s_- implies that*

$$\frac{b_p - v_\ell}{v_h - b_p} \leq \frac{\rho}{1-\rho} \frac{f_h(s_-)}{f_\ell(s_-)} \frac{\pi_h^\circ}{\pi_\ell^\circ}. \tag{16}$$

Step 2. *There exists a function $E(\eta) > 1$ with $E(\eta) \to 1$ such that*

$$\frac{b_p - v_\ell}{v_h - b_p} \geq \frac{\rho}{1-\rho} \frac{f_h(\underline{s})}{f_\ell(\underline{s})} \frac{\pi_h^-}{\pi_\ell^-} E(\eta). \tag{17}$$

In equilibrium, no signal $s < s_-$ can have an incentive to deviate from $\beta^*(s)$ to any $b \in (\beta^*(s), b_p)$. In particular, there must not be an incentive to deviate a bid marginally below b_p (deviation 2), meaning that $U(\beta^*(s)|s; \beta^*) \geq \lim_{\epsilon \searrow 0} U(b_p - \epsilon | s; \beta^*)$. In the appendix, we use this condition for signal $\underline{s}$ to derive (17).

Step 3. *When η is sufficiently large, the lower bound (17) exceeds the upper bound (16). Thus, either deviation 1 or 2 is profitable.*

Combining inequalities (16) and (17) yields

$$\frac{\rho}{1-\rho} \frac{f_h(s_-)}{f_\ell(s_-)} \frac{\pi_h^\circ}{\pi_\ell^\circ} \geq \frac{\rho}{1-\rho} \frac{f_h(\underline{s})}{f_\ell(\underline{s})} \frac{\pi_h^-}{\pi_\ell^-} E(\eta).$$

[17] Since $\frac{f_h(\underline{s})}{f_\ell(\underline{s})} < \frac{f_h(\bar{s})}{f_\ell(\bar{s})}$, the monotone likelihood ratio property implies that $\frac{f_h(\underline{s})}{f_\ell(\underline{s})} < \frac{f_h(\check{s})}{f_\ell(\check{s})} = 1$. Because $\frac{f_h(\underline{s})}{f_\ell(\underline{s})} < 1$, and the densities are continuous $F_\ell(\check{s}) = \int_{\underline{s}}^{\check{s}} f_\ell(z)dz = \int_{\underline{s}}^{\check{s}} f_h(z)\frac{f_\ell(z)}{f_h(z)}dz < \int_{\underline{s}}^{\check{s}} f_h(z)\frac{f_\ell(\underline{s})}{f_h(\underline{s})}dz = \frac{f_\ell(\underline{s})}{f_h(\underline{s})}F_h(\check{s})$.

[18] Explicitly, $\pi_\omega^- = e^{-\eta(1-F_\omega(s_-))}$ and $\pi_\omega^\circ = \frac{e^{-\eta(1-F_\omega(s_+))} - e^{-\eta(1-F_\omega(s_-))}}{\eta(F_\omega(s_+) - F_\omega(s_-))}$ for $\omega \in \{h, \ell\}$.

This can be rearranged to

$$\frac{f_h(s_-)}{f_\ell(s_-)}\Big(\frac{f_h(\underline{s})}{f_\ell(\underline{s})}\Big)^{-1} \geq \frac{\pi_h^-}{\pi_\ell^-}\Big(\frac{\pi_h^\circ}{\pi_\ell^\circ}\Big)^{-1} E(\eta). \tag{18}$$

The product $\frac{\pi_h^-}{\pi_\ell^-}\Big(\frac{\pi_h^\circ}{\pi_\ell^\circ}\Big)^{-1}$ captures the difference in inference from winning with a bid marginally below b_p instead of b_p. From $s_+ < \breve{s}$ it follows that $\eta[F_h(s_+) - F_h(s_-)] < \eta[F_\ell(s_+) - F_\ell(s_-)]$ such that equation (12) of Lemma 4 implies that the product is larger than one: winning with a marginally lower bid reduces the winner's curse. In fact, this effect becomes arbitrarily strong,

$$\frac{\pi_h^-}{\pi_\ell^-}\Big(\frac{\pi_h^\circ}{\pi_\ell^\circ}\Big)^{-1} \approx \underbrace{e^{\eta([F_\ell(s_+)-F_\ell(s_-)]-[F_h(s_+)-F_h(s_-)])}}_{\to\infty \text{ by } (8)} \frac{F_h(s_+) - F_h(s_-)}{F_\ell(s_+) - F_\ell(s_-)} \to \infty.$$

By $E(\eta) \to 1$, this means that when η is large, the inference from winning on the right side of inequality (18) dominates the inference from the signals on the left side of inequality (18). As a result, the inequality cannot hold, and either deviation 1 or 2 has to be profitable.

Since the argument why candidate (b) cannot be an equilibrium contains some of the key incentives which shape the bidding behavior, it is useful to repeat it verbally: first, equilibrium bids can at most be the expected value conditional on winning, such that $\mathbb{E}[v|\text{win with } b_p, s_-; \beta^*]$ puts an upper bound on b_p (16). When η is large, there is a "winner's blessing" on bids below b_p, such that this upper bound is dwarfed by the expected value conditional on winning with any lower bid $b < b_p$. In particular, b_p has to be a lot smaller than the expected value conditional on winning with a marginally lower bid, which wins whenever $s_{(1)} \leq s_-$. Hence, the expected profits at this marginally lower bid are strictly positive. When η is large, a Bertrand competition emerges among bidders with signals below s_-: the rivals compete for the highest bid below b_p which maximizes their chances to win the auction, but is subject to a strictly smaller winner's curse than b_p. On the continuous bid space, a largest bid below b_p does not exist, such that no equilibrium exists.

As noted above, the arguments presented do not constitute a comprehensive proof. We restricted attention to pools which do not change in size as η increases and only considered equilibria in which the pools end at $\breve{s}$ and $s_+ < \breve{s}$, respectively. As it turns out, however, none of these simplifications are significant, and existence always fails due to an interior atom and the "open set problem" it creates. Naturally, this open set is a feature of the continuous bid space; when considering auctions on a grid, there is a maximal bid below any pooling bid, and an equilibrium exists. At the same time, however, a discrete bid space makes the equilibrium characterization more challenging. Therefore, we take an indirect approach and first analyze an extended auction on the continuous bid space, which will help us to characterize equilibria on a fine grid afterwards.

4 Communication extension

In this section, we augment the auction mechanism by a communication dimension similar to Jackson et al. (2002).[19] We denote this *communication extension* by $\Gamma^{\mathfrak{c}}$, whereas we label the standard auction mechanism by Γ. As we show in the next section, sequences of equilibria on an ever-finer grid converge to an equilibrium of the communication extension. Therefore, the communication extension always has an equilibrium, which we can use in Section 5 to characterize the equilibria on a fine grid.

In the communication extension, every bidder simultaneously selects three actions: a message space $M \subseteq [0,1]$, a message $m \in [0,1]$, and a bid $b \in [v_\ell, v_h]$. We consider strategies of the form $\sigma : [\underline{s}, \bar{s}] \to \mathcal{P}[0,1] \times \Delta\Big([0,1] \times [v_\ell, v_h]\Big)$ that map the signals into a message space and a distribution over messages and bids.[20]

The auction mechanism selects the winner according to the following rule: First, it checks whether all bidders report the same message space M; if not, the good is not allocated. Afterwards, it discards all bidders who report messages $m \notin M$. Among the remaining bidders, the good is allocated to the one with the highest bid. If multiple bidders tie on the highest bid, the tie is broken uniformly among those who report the highest message $m \in M$. The winner receives the object and pays her bid.

Denote the probability to win in state $\omega \in \{h, \ell\}$ with action-tuple (M, m, b) when all rival bidders follow strategy σ by $\pi^{\mathfrak{c}}_\omega(M, m, b; \sigma)$. Then, the interim expected utility for a bidder with signal s who selects (M, m, b) is

$$\begin{aligned} U^{\mathfrak{c}}(M, m, b|s; \sigma) = &\frac{\rho f_h(s)}{\rho f_h(s) + (1-\rho) f_\ell(s)} \pi^{\mathfrak{c}}_h(M, m, b; \sigma)(v_h - b) \\ &+ \frac{(1-\rho) f_\ell(s)}{\rho f_h(s) + (1-\rho) f_\ell(s)} \pi^{\mathfrak{c}}_\ell(M, m, b; \sigma)(v_\ell - b). \end{aligned} \tag{19}$$

A strategy σ^* is a *best response* to a strategy σ if for (almost) every s an action-tuple $(M, m, b) \in \operatorname{supp} \sigma^*(s)$ implies that $(M, m, b) \in \arg\max_{\hat{M}, \hat{m}, \hat{b}} U(\hat{M}, \hat{m}, \hat{b}|s; \sigma)$. As in the case of the standard auction, unless specified otherwise, all following claims hold for almost every s. Again, we look for symmetric Bayes-Nash equilibria, but restrict attention to *concordant* equilibria in which all bidders report the same message space M.[21]

Note that, conditional on a bid b, different messages m, m' may induce the same winning probability, such that they are equivalent. Two strategies are *m-equivalent*, if after merging all signals s that share the same likelihood ratio $\frac{f_h}{f_\ell}$, they correspond to the same distributional strategy, up to equivalent messages.

[19] We discuss the relation to Jackson et al. (2002) in footnote 21.

[20] We immediately restrict attention to equilibria in which all bidders choose the same M so that the restriction to pure strategies with respect to the message space is without further loss.

[21] The outcomes of concordant equilibria are a subset of the outcomes of solutions to the communication extension in Jackson et al. (2002). In their communication extension, the tie-breaking is part of the solution which can be interpreted as introducing the auctioneer as a player who selects the tie-breaking rule. By contrast, we fully specify the mechanism without introducing such an additional player. Roughly speaking, in our mechanism, the bidders report the tie-breaking rule. This is possible in our setting because a misreport can be punished by a uniformly worst outcome (no allocation) while such outcome may not exist in the more general payoff setting in Jackson et al. (2002).

Lemma 7. *Let σ^* be a concordant equilibrium of the communication extension. Then, there exists an m-equivalent, concordant equilibrium $\hat{\sigma}^*$ that is pure and where*

(i) bids b are nondecreasing in s;

(ii) for any given bid b, the report $m \in M$ is nondecreasing in s.

This implies that in both states $\omega \in \{h, \ell\}$

(a) $\pi^{\mathfrak{c}}_\omega(\hat{\sigma}^*(s); \hat{\sigma}^*)$ is nondecreasing in s;

(b) $\hat{\sigma}^*(s) = \hat{\sigma}^*(s')$ if and only if $\pi^{\mathfrak{c}}_\omega(\hat{\sigma}^*(s); \hat{\sigma}^*) = \pi^{\mathfrak{c}}_\omega(\hat{\sigma}^*(s'); \hat{\sigma}^*)$ for any $s, s' \in [\underline{s}, \bar{s}]$.

In essence, Lemma 7 is analogous to Lemma 1. Bidders with higher signals are more optimistic, select weakly higher bids/messages, and win weakly more often. If multiple signals induce the same belief, the actions can be reordered such that they are monotone, without altering the implied distribution of bids and (payoff-relevant) messages. Implication *(b)* establishes that the problem of equivalent messages can be ignored. If two distinct action-tuples are in the image of the strategy, then they win with different probabilities. This simplifies later statements.

Henceforth, we restrict attention to concordant strategies that are pure, in which bidders with higher signals win weakly more often and where *(b)* holds. We denote these by $\sigma : [\underline{s}, \bar{s}] \to \mathcal{P}[0,1] \times [0,1] \times [v_\ell, v_h]$.

We can now explicitly state the winning probabilities. To do so, fix some concordant strategy σ and functions μ and β such that $\sigma(s) = (M, \mu(s), \beta(s))$ for all s. Suppose a bidder chooses the action-tuple (M, m, b). If $m \in M$ and (M, m, b) is selected with zero probability by a competitor, then she wins whenever $s_{(1)} \leq \hat{s}$, where $\hat{s} = \sup(\{s \colon \beta(s) < b\} \cup \{s \colon \beta(s) = b \text{ and } \mu(s) < m\})$ is the highest signal that chooses a lower bid, or the same bid but lower message. This happens in state $\omega \in \{h, \ell\}$ with probability

$$\pi^{\mathfrak{c}}_\omega(M, m, b; \sigma) = e^{-\eta(1 - F_\omega(\hat{s}))}.$$

If $\sigma(s) = (M, m, b)$ for all $s \in [s_-, s_+]$, and $\sigma(s) \neq (M, m, b)$ for all other signals, then the action-tuple wins in state $\omega \in \{h, \ell\}$ with probability

$$\pi^{\mathfrak{c}}_\omega(M, m, b; \sigma) = \frac{e^{-\eta(1 - F_\omega(s_+))} - e^{-\eta(1 - F_\omega(s_-))}}{\eta(F_\omega(s_+) - F_\omega(s_-))}.$$

These probabilities are analogous to those in the standard auction, and are derived in the same manner.

If a bidder chooses an action-tuple (M', m', b) with $M' \neq M$, but $m' \in M'$ then she only wins when the deviation to M' is not detected. This is only the case when she is alone, which happens in state $\omega \in \{h, \ell\}$ with probability

$$\pi^{\mathfrak{c}}_\omega(M', m', b; \sigma) = e^{-\eta}.$$

If $M' = M$ but $m' \notin M$, the probability to win is zero.

To fix ideas, note that every equilibrium of the standard auction, Γ, is also an equilibrium of the communication extension, $\Gamma^{\mathfrak{c}}$. If all bidders report $M = \{0\}$ and

$m = 0$, the messages do not affect the outcome of the auction, and deviations from (M, m) are (weakly) dominated by bidding (M, m, v_ℓ); the lowest bid also only wins when the bidder is alone, but at the lowest possible cost. Thus, the winner is solely determined by the bids, and we only need to consider deviations in the bid. Obviously, this makes following the equilibrium strategy of Γ an equilibrium of $\Gamma^{\mathfrak{c}}$.

Since every equilibrium of the standard auction is an equilibrium of the communication extension, the set of equilibria of $\Gamma^{\mathfrak{c}}$ is a superset of the equilibria of Γ. Indeed, it can be a proper superset, because the communication extension always has an equilibrium.

Lemma 8. *The communication extension $\Gamma^{\mathfrak{c}}$ always has a concordant equilibrium.*

The result follows as a corollary to Lemmas 9 and 10 found in the next section. For now, we just take existence as given. Even though equilibria do not need to be unique, it is possible to characterize their form up to some ϵ environment around $\underline{s}$ and $\breve{s}$.

Proposition 4. *Fix any $\epsilon \in (0, \frac{\breve{s}-\underline{s}}{2})$. When η is sufficiently large (given ϵ), any concordant equilibrium σ^* of $\Gamma^{\mathfrak{c}}$ takes the following form:*
There are two disjoint, adjacent intervals of signals I, J such that

(i) $[\underline{s} + \epsilon, \breve{s} - \epsilon] \subset I \cup J$;

(ii) $\sigma^*(s_I) = (M, m_I, b_p)$ *for all* $s_I \in I$ *and* $\sigma^*(s_J) = (M, m_J, b_p)$ *for all* $s_J \in J$*, with* $m_I < m_J$;

(iii) there is no $m \in M$ *s.t.* $\pi^{\mathfrak{c}}_\omega(\sigma^*(s_I); \sigma^*) < \pi^{\mathfrak{c}}_\omega(M, m, b_p; \sigma^*) < \pi^{\mathfrak{c}}_\omega(\sigma^*(s_J); \sigma^*)$ *for* $\omega \in \{h, \ell\}$;

(iv) $\int_I \eta f_\omega(z) dz > \frac{1}{\epsilon}$, *and* $\int_J \eta f_\omega(z) dz > \frac{1}{\epsilon}$ *for* $\omega \in \{h, \ell\}$;

(v) on $s \in (\breve{s} + \epsilon, \bar{s}]$*, the bids are strictly increasing such that the message* m *is irrelevant.*

The proof is in the appendix. The following figure summarizes the results:

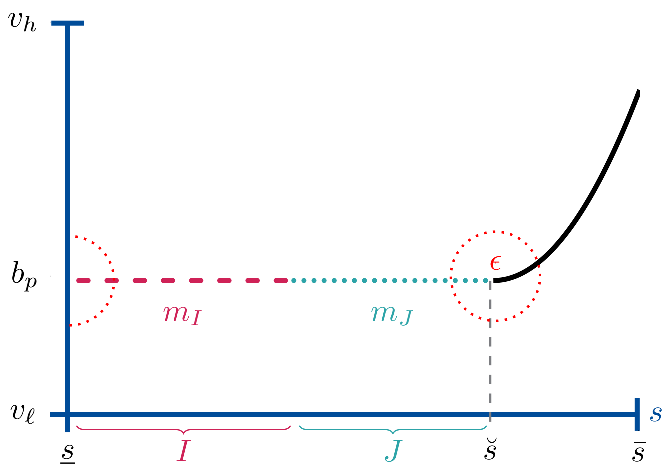


Figure 4: Equilibria σ^* of the communication extension.

By part *(i)*, there are two adjacent intervals I and J (pink/dashed and teal/dotted) that span the signals between $\underline{s} + \epsilon$ and $\breve{s} - \epsilon$. Bidders with signals from either interval bid b_p but separate by sending messages m_I and m_J, *(ii)*. Importantly, m_I and m_J are adjacent, meaning that there is no $m \in M$ with $m_I < m < m_J$. Thus, *(iii)* holds, and there is no action-tuple which wins more often than (M, m_I, b_p), but less often than (M, m_J, b_p). The intervals I and J can vary in length as η increases, but the expected number of bidders in both intervals grows without bound, as asserted by *(iv)*. Above $\breve{s} + \epsilon$, bids are strictly increasing and follow the ordinary differential equation (9) with the proper initial value, *(v)*. The message m is irrelevant in this region. Observe that Figure 4 is only a qualitative sketch: J may be contained in the ϵ-environment around $\breve{s}$, and the equilibrium may assume a different form within the ϵ-environments.

The form of the equilibrium is a direct consequence of the results in Section 3.3. There, we reasoned that in any equilibrium of the standard auction, bids cannot be strictly increasing below the neutral signal $\breve{s}$, and that $\underline{s}$ and $\breve{s}$ cannot pool. The logic behind these two results remains valid in the communication extension. Hence, there has to be an interior atom b_p on which bidders with intermediate signals, J, pool to insure against the winner's curse, as depicted in candidate equilibrium (b). Since the inference from winning is U-shaped (cf. Lemma 4 and Figure 2), compared to b_p, winning with any bid below b_p is a blessing for the conditional expected value. When η is large, this incentivizes bidders with low signals, I, to compete for the highest bid below b_p. In the standard auction, Γ, no such bid exists, such that no equilibrium exists. With an endogenous tie-breaking rule, the problem can be solved. By sending messages m_I and m_J, bidders with signals from I and J can differentiate themselves, while leaving no room for bidders with signals from I to marginally deviate upwards, as stated in part *(iii)*.

One immediate implication of Proposition 4 is that there can be no equilibrium in the standard auction (Proposition 3). By our earlier observation, all equilibria of Γ are also equilibria of the communication extension $\Gamma^{\mathfrak{c}}$, in which the message space is a singleton. Since Proposition 4 describes every equilibrium of $\Gamma^{\mathfrak{c}}$, and the intervals I and J cannot be separated without two distinct messages, Γ cannot have an equilibrium.

5 Standard auction on the grid

Consider a variation of the standard auction in which the bids are constrained to a set of $k \geq 2$ equidistant[22] bids

$$B_k = (v_\ell, v_\ell + d, ..., v_\ell + (k-2)d, v_h),$$

where $d = \frac{v_h - v_\ell}{k-1}$. We denote such an auction by $\Gamma(k)$.

Lemma 9. *Any auction on the grid $\Gamma(k)$ has an equilibrium in pure and nondecreasing strategies.*

The proof builds on Myerson (2000), and is in the appendix.[23] The monotonicity

[22] The assumption of equidistance is for expositional purposes only. The following results hold for any discretization, as long as the grid becomes dense on $[v_\ell, v_h]$ as $k \to \infty$.

[23] Since best responses are monotonic, the proof can be simplified, and applies even if the likelihood

directly follows from Lemma 1, which does not rely on the form of the bid space.

While the discretization solves the existence problem, the discontinuous bid space makes the equilibrium characterization more challenging. Here, the communication extension, $\Gamma^{\mathfrak{c}}$, proves helpful. We are going to show that the equilibria on a fine grid must have the same structure as the equilibria of $\Gamma^{\mathfrak{c}}$. A first step shows that the limit of a converging sequence of equilibria on the ever-finer grid can be represented as a concordant equilibrium of the communication extension. For a deterministic population, this corresponds to a special case of Theorem 2 in Jackson et al. (2002).

Lemma 10. *Consider any sequence of auctions on the ever-finer grid* $(\Gamma(k))_{k\in\mathbb{N}}$ *and any corresponding sequence of equilibria* $(\beta^*_k)_{k\in\mathbb{N}}$. *There exists a subsequence of auctions* $(\Gamma(n))_{n\in\mathbb{N}}$ *with equilibria* $(\beta^*_n)_{n\in\mathbb{N}}$ *and a concordant equilibrium* σ^* *of* $\Gamma^{\mathfrak{c}}$, *such that, for all* $s \in [\underline{s}, \bar{s}]$,

(i) $\sigma^*(s) = (M, \mu(s), \lim_{n\to\infty} \beta^*_n(s))$ *for some* M *and function* $\mu : [\underline{s}, \bar{s}] \to M$;

(ii) $\lim_{n\to\infty} \pi_\omega(\beta^*_n(s); \beta^*_n) = \pi^{\mathfrak{c}}_\omega(\sigma^*(s); \sigma^*)$ *for* $\omega \in \{h, \ell\}$,

and, therefore,

(iii) $\lim_{n\to\infty} U(\beta^*_n(s)|s; \beta^*_n) = U^{\mathfrak{c}}(\sigma^*(s)|s; \sigma^*)$.

The proof is in the appendix. Combined with Lemma 9, Lemma 10 establishes the existence of equilibria of $\Gamma^{\mathfrak{c}}$ (Lemma 8). Next, we compare the structure of equilibria on the ever-finer grid with the corresponding limit equilibrium of the communication extension.

Lemma 11. *Consider a sequence of auctions on the ever-finer grid* $(\Gamma(n))_{n\in\mathbb{N}}$ *with corresponding equilibria* $(\beta^*_n)_{n\in\mathbb{N}}$ *that converge to an equilibrium of* $\Gamma^{\mathfrak{c}}$, *denoted* σ^*, *in the sense of Lemma 10. Then it holds for (almost) any two signals* $s_- < s_+$ *that*

(i) $\sigma^*(s_-) = \sigma^*(s_+)$, *if and only if* $\beta^*_n(s_-) = \beta^*_n(s_+)$ *for any* n *sufficiently large;*

(ii) $\sigma^*(s_-) \neq \sigma^*(s_-)$, *if and only if* $\beta^*_n(s_-) < \beta^*_n(s_+)$ *for any* n *sufficiently large.*

Due to this close relationship, equilibria on a fine grid have to be similar to those of the communication extension. Thus, the characterization from Proposition 4 can be used to derive properties of equilibria on a fine grid.

Proposition 5. *Fix any* $\epsilon \in (0, \frac{\breve{s}-\underline{s}}{2})$. *When* η *is sufficiently large (given* ϵ*) and* k *is sufficiently large (given* ϵ *and* η*), any equilibrium* β^* *of* $\Gamma(k)$ *takes the following form: There are two disjoint, adjacent intervals of signals* I, J *such that*

(i) $[\underline{s} + \epsilon, \breve{s} - \epsilon] \subset I \cup J$;

(ii) $\beta^*(s_I) = b$ *for all* $s_I \in I$ *and* $\beta^*(s_J) = b + d$ *for all* $s_J \in J$;

(iii) $\int_I \eta f_\omega(z)dz > \frac{1}{\epsilon}$, *and* $\int_J \eta f_\omega(z)dz > \frac{1}{\epsilon}$ *for* $\omega \in \{h, \ell\}$;

(iv) on $s \in (\breve{s} + \epsilon, \bar{s}]$, *the bids tie with a probability smaller than* ϵ.[24]

ratio $\frac{f_h(s)}{f_\ell(s)}$ contains jumps.

[24] Take any $\hat{s} > \breve{s} + \epsilon$ and let $\hat{b} = \beta^*(\hat{s})$. If there exists an interval $[s_-, s_+]$ such that $\beta^*(s) = \hat{b}$ for all $s \in [s_-, s_+]$, then $\eta \int_{s_-}^{s_+} f_\omega(z)dz < \epsilon$ for $\omega \in \{h, \ell\}$.

Proposition 5 describes the discrete analog of the equilibria of the communication extension. Again, the result is summarized best with the help of a figure:

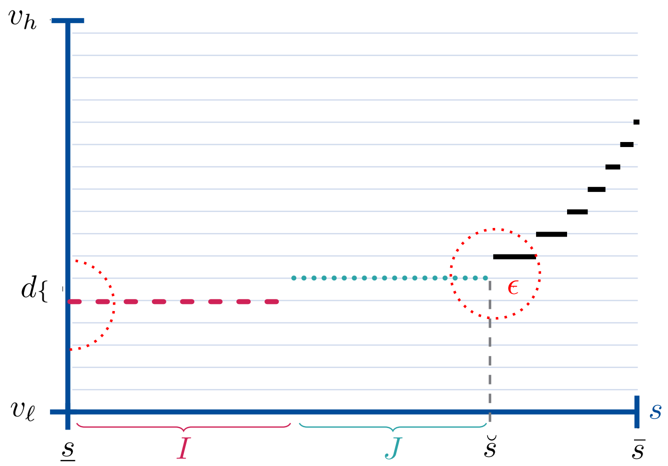


Figure 5: Equilibria β^* of the auction on the grid.

There are two adjacent intervals I and J (pink/dashed and teal/dotted). By *(i)*, any signal between $\underline{s} + \epsilon$ and $\breve{s} - \epsilon$ is part of one of the two intervals. By *(ii)*, bidders with signals from interval I pool on a bid b_p, while bidders on the interval J select the next bid on the grid, $b_p + d$. The intervals can vary in length as η increases, but the expected number of bidders in both intervals grows without bound, *(iii)*. Assertion *(iv)* states that there are no significant atoms above $\breve{s} + \epsilon$; in fact, the bidding function becomes smooth and strictly increasing as grid $d \to 0$.

The characterization highlights why the standard auction, Γ, is not the limit of the auctions on an arbitrarily fine grid. As $d \to 0$, the difference between the two pooling bids b_p and $b_p + d$ vanishes. In the limit, I and J can no longer be separated such that they win with the same probability, and the utility changes discontinuously. Therefore, the limit of a sequence of equilibria on the ever-finer grid is generally not an equilibrium of the limit auction Γ.[25] However, the limit outcome can be represented as an equilibrium of $\Gamma^{\mathfrak{c}}$, because the tie-breaking rule can be chosen to preserve the different winning probabilities in I and J. Thereby, equilibria of $\Gamma^{\mathfrak{c}}$ inherit the characteristics of equilibria on a fine grid, which is why the communication extension can be used to characterize the equilibria on a fine grid.

We now turn to the proof of Proposition 5:

Proof. Suppose that for every k at least one of the properties *(i)*-*(iv)* is violated. Then, there exists a sequence of auctions on the ever-finer grid $(\Gamma(k))_{k\in\mathbb{N}}$ with equilibria $(\beta_k^*)_{k\in\mathbb{N}}$, along which one property never holds. By Lemma 10, this sequence has a subsequence $(\beta_n^*)_{n\in\mathbb{N}}$ converging to an equilibrium of the communication extension, σ^*. When η is large, strategy σ^* takes the form detailed in Proposition 4. We use this

[25] In the limit, the strategy becomes roughly the one we ruled out in candidate equilibrium (a) of Section 3.3 in which all signals below the neutral signal $\breve{s}$ pool on a single bid.

form of σ^* and the convergence of $(\beta_n^*)_{n\in\mathbb{N}}$ to find contradictions for the violations of properties *(i)*-*(iv)* for infinitely many n.

First, consider property *(iv)*. If the bids in σ^* are strictly increasing over some interval, so is $\beta^* = \lim_{n\to\infty} \beta_n^*$. Thus, when n is sufficiently large (d sufficiently small), the bids tie with a probability smaller than ϵ on $s \in (\breve{s}+\epsilon, \bar{s}]$, and property *(iv)* cannot be violated.

Next, consider the intervals I and J of σ^*, and fix some $s_I \in int(I)$ and $s_J \in int(J)$. Further, define $I^n = \{s\colon \beta_n^*(s) = \beta_n^*(s_I)\}$ as well as $J^n = \{s\colon \beta_n^*(s) = \beta_n^*(s_J)\}$. By Lemma 11, $I^n \to I$ and $J^n \to J$. Thus, property *(iii)* cannot be violated when n is large.

What remains to be shown is that $\beta_n^*(s_I)+d = \beta_n^*(s_J)$ when n is sufficiently large *(ii)*. If this is the case, then $(\underline{s}+\epsilon, \breve{s}-\epsilon) \subset I^n \cup J^n$, such that property *(i)* follows, completing the proof. Suppose to the contrary that $\beta_n^*(s_I) + d < \beta_n^*(s_J)$ for every n. Then, it follows from $I^n \to I$ and $J^n \to J$, that $\lim_{n\to\infty} \pi_\omega(\beta_n^*(s_I) + d; \beta_n^*) = e^{-\eta(1-F_\omega(\hat{s}))}$. Since strategy β_n^* is an equilibrium, $U(\beta_n^*(s_n)|s_n;\beta_n^*) \geq U(\beta_n^*(s_I) + d|s_n;\beta_n^*)$ for all $s_n \in I^n \cup J^n$. Hence, Lemma 10 implies that

$$\lim_{n\to\infty} U(\beta_n^*(s)|s;\beta_n^*) = U^{\mathfrak{c}}(\sigma^*(s)|s;\sigma^*) \geq \lim_{n\to\infty} U(\beta_n^*(s_I) + d|s;\beta_n^*) \quad \forall s \in I \cup J.$$

This means that in σ^*, bidders prefer $\sigma^*(s_I)$ or $\sigma^*(s_J)$ over some hypothetical action-tuple that wins whenever $s_{(1)} \leq \hat{s}$. Thus, there could be an $m \in M$ with $m_I < m < m_J$ because bidders would not deviate to such a message. This is a contradiction to property *(iii)* of Proposition 4, which completes the proof. □

The proof illustrates how the communication extension can be employed to characterize equilibria on a fine grid. This contrasts with standard auctions on the continuous bidding space that cannot handle non-vanishing atoms in the equilibrium bid distribution, thereby acting as an equilibrium refinement. The communication extension is, hence, the "correct" mechanism to analyze auctions on the fine grid.

6 Discussion

6.1 State-dependent competition

One natural modification of the model is the introduction of state-dependent participation, expressed by a state-dependent mean η_ω. This extension combines numbers uncertainty with the deterministic but state-dependent participation in Lauermann and Wolinsky (2017). When the number of bidders depends on the state, being solicited to participate in the auction contains information about the state. Conditional on participation, a bidder updates her belief to

$$\mathbb{P}[\omega = h|\text{participation}] = \frac{\rho\eta_h}{\rho\eta_h + (1-\rho)\eta_\ell}.$$

Further, knowledge of the number of competitor now has an additional effect. Apart from determining the intensity of the winner's curse, it is also directly informative about the state. This changes the inference from winning, and, thus, the form of $\mathbb{E}[v|s_{(1)} \leq \hat{s}]$. Specifically, consider the effect state-dependent participation has on the inference at the bottom, $\mathbb{E}[v|s_{(1)} \leq \underline{s}]$. As we argued in Section 3.1, if $s_{(1)} \leq \underline{s}$, then there is no competitor. When participation is state dependent, this is either good news about the value of the good ($\eta_h < \eta_\ell$) or bad news ($\eta_h > \eta_\ell$). Thus, there is either a winner's blessing, or winner's curse at the bottom. As long as $\frac{\eta_h}{\eta_\ell} \in (\frac{f_\ell(\bar{s})}{f_h(\bar{s})}, \frac{f_\ell(\underline{s})}{f_h(\underline{s})})$, however, this effect does not change the general shape of the conditional expected value: $\mathbb{E}[v|s_{(1)} \leq \hat{s}]$ is decreasing in $\hat{s}$ when $\eta_h f_h(\hat{s}) < \eta_\ell f_\ell(\hat{s})$, has its minimum where $\frac{\eta_h}{\eta_\ell}\frac{f_h(\hat{s})}{f_\ell(\hat{s})} = 1$, and is increasing when $\eta_h f_h(\hat{s}) > \eta_\ell f_\ell(\hat{s})$. As a result, state-dependent participation leaves our results mostly unaltered. One merely needs to replace $f_\omega(s)$ with $\eta_\omega f_\omega(s)$ in every expression and redefine the neutral signal $\breve{s}$ such that $\frac{\eta_h}{\eta_\ell}\frac{f_h(\breve{s})}{f_\ell(\breve{s})} = 1$. In the appendix, we prove every result for this more general case. Only when $\frac{\eta_h}{\eta_\ell} \notin (\frac{f_\ell(\bar{s})}{f_h(\bar{s})}, \frac{f_\ell(\underline{s})}{f_h(\underline{s})})$ such that no neutral signal $\breve{s}$ exists are Propositions 3, 4, and 5 vacuous. If $\frac{\eta_h f_h(\underline{s})}{\eta_\ell f_\ell(\underline{s})} \geq 1$, claim *(iii)* of Proposition 2 ensures the existence of a strictly increasing strategy; by Lemma 4, this is the only symmetric equilibrium.[26] If, on the other hand, $\frac{\eta_h f_h(\bar{s})}{\eta_\ell f_\ell(\bar{s})} < \frac{f_h(\underline{s})}{f_h(\underline{s})}$ and η_h, η_ℓ are sufficiently large, then there exists an equilibrium in which every bidder selects the same bid.

6.2 Distribution of the number of bidders

Generally, numbers uncertainty breaks the affiliation between the first-order statistic of bidders' signals and the value of the good. Without affiliation, however, one cannot expect the equilibrium strategy to be strictly increasing. At the same time, the lack of affiliation creates room for atoms in the bid distribution. Thus, neither result hinges on the distributional assumption. The Poisson distribution only serves as a transparent example to illustrate the effects because it allows for closed-form solutions and is characterized by a single parameter. It is not clear, however, whether there is a class other than Poisson for which the equilibrium existence necessarily fails. At the very least, the Poisson distribution is not a "knife-edge" case, in the sense that we can truncate the distribution to always have at least $\underline{n} \geq 2$ bidders (cf. footnote 8), or marginally change the probabilities without changing the results.

6.3 Signal structure

The assumption of a unique neutral signal $\breve{s}$ is for convenience only. If there is an interval of signals along which $f_h(s) = f_\ell(s)$, the propositions just become lengthier.[27] Also, unboundedly informative signals leave our results unchanged but complicate some proofs.

While all results are given for continuous densities, we can allow for a finite number of jumps in f_h and f_ℓ. This nests problems with a finite number of discrete signals, because these can be modeled as intervals of signals sharing the same likelihood ratio. When the densities are discontinuous, all results except of Propositions 3, 4, and 5 still apply. The equilibrium characterizations and non-existence result, however, rely an interval of

[26] Compare Lauermann et al. (2018).

[27] For example, in Proposition 4 the bids are constant between $\underline{s} + \epsilon$ and $\inf\{s\colon f_h(s) = f_\ell(s)\} - \epsilon$, and strictly increasing at or above $\sup\{s\colon f_h(s) = f_\ell(s)\} + \epsilon$.

signals S such that $\frac{f_h(s)}{f_\ell(s)} \leq 1$, but $\frac{f_h(\underline{s})}{f_\ell(\underline{s})} \frac{F_\ell(s)}{F_h(s)} < \frac{f_h(s)}{f_\ell(s)}$ for all $s \in S$. If there is no such interval, and η is sufficiently large, an equilibrium of the form of candidate equilibrium (a) exists: all signals below $\check{s}$ pool on the same bid, and all higher signals follow a strictly increasing strategy. Note that this is always true when signals are binary, which makes this signal structure a special case.[28]

6.4 Reserve price

The assumption of a reserve price at v_ℓ is used in the proof of Lemma 1, which shows that, without loss, any equilibrium strategy is nondecreasing. If η is sufficiently large, the assumption can be dropped. As η increases, the probability of being alone in the auction vanishes, such that, by Bertrand logic, bidders with signals above some $\underline{s} + \epsilon$ choose a bid at or above v_ℓ and follow a nondecreasing strategy.[29] We prove the result formally in Lemma 12 in the appendix. Alternatively, if one assumes that the good is only allocated when there are at least two bidders, or if one truncates the Poisson distribution at $\underline{n} \geq 2$ (cf. 6.2), the Bertrand logic applies, and all equilibrium bids have to be above v_ℓ. The condition of a minimal amount of competition leaves our results qualitatively unaltered.

6.5 Second-price auction

As noted in Section 3.1.3, whenever η is sufficiently large, there is no strictly increasing equilibrium in the second-price auction because condition (10) does not hold. Thus, any equilibrium bid distribution necessarily contains atoms, which are problematic for the standard auctions. In fact, one can check that when η is sufficiently large, no nondecreasing equilibrium exists in the second-price auction, either. However, it is possible to construct an analogous communication extension for the second-price auction that captures the bidding behavior on a fine grid.

6.6 Related literature

There is a small literature on numbers uncertainty in private-value auctions, notably Matthews (1987), McAfee and McMillan (1987), and Harstad et al. (1990), studying, e.g., the interaction of numbers uncertainty and risk aversion.

Moreover, there is a recent strand of literature on common-value auctions and non-constant numbers of bidders. Murto and Välimäki (2019) consider a common-value auction with costly entry.[30] After observing a binary signal, potential bidders have to decide whether to pay a fee to bid in the auction. When the pool of potential bidders is arbitrary large, the number of participating bidders is Poisson distributed with a signal-dependent mean. The signal dependent entry decision precludes atoms in the bid distribution, which enables revenue comparisons.

[28] Lauermann and Wolinsky (2017) make use of this fact.

[29] If, to the contrary, the reserve price is $0 < v_\ell$, participation is state dependent with $\eta_h << \eta_\ell$ and if η_h, η_ℓ are small, then equilibrium strategies can be strictly decreasing. In this case, bidders with high signals expect less competition and are, therefore, bid less. Bidders with signal $\bar{s}$ bid 0, betting to be alone in the auction.

[30] Auctions with endogenous entry are also examined by, among others, Levin and Smith (1994) and Harstad (1990).

In Lauermann and Wolinsky (2017, 2018) the participation is deterministic, but state dependent (due to a solicitation decision by an informed auctioneer). The interest is in how the outcome of a large first-price auction is affected by the ratio of bidders in the high to the low state. If this the ratio is sufficiently high, the outcome resembles the usual outcome in large auctions, whereas, when the ratio is small, there are necessarily atoms at the top. Atoms are the result of a "participation curse" that arises when there are fewer bidders in the high than in the low state. The atom at the top prevents information aggregation.

In a setting with many goods, Harstad et al. (2008) and Atakan and Ekmekci (2019) consider the effect of numbers uncertainty on the information aggregation properties of a k-th price auction (Pesendorfer and Swinkels 1997). In Harstad et al. (2008), the distribution of bidders is exogenously given. They find that even if the equilibrium strategy is strictly increasing (which aids aggregation), information aggregation fails unless the numbers uncertainty is negligible. They also provide an example in which equilibrium is not strictly increasing, but they do not study this question further. Atakan and Ekmekci (2019) assume that bidders have a type-dependent outside option such that the numbers uncertainty arises endogenously and is correlated with the state, showing that this also upsets information aggregation.

7 Conclusion

We have studied a canonical common-value auction in which the bidders are uncertain about the number of their competitors. This numbers uncertainty invalidates classic findings for common-value auctions (Milgrom and Weber 1982). In particular, it breaks the affiliation between the first-order statistic of the signals and the value of the good. As a consequence, bidding strategies are generally not strictly increasing but contain atoms. The location of the atoms is indeterminate, implying equilibrium multiplicity. Moreover, no equilibrium exists in the standard auction on the continuous bid space when the expected number of bidders is sufficiently large.

Many of the known failures of equilibrium existence in auctions require careful crafting of the setup, and rely on a discrete type space to generate atoms in the bid distribution (cf. Maskin and Riley (2000), Jackson (2009)). By contrast, we identify a failure of equilibrium existence in an otherwise standard auction setting in which the type space is continuous, and atoms in the bid distribution arise endogenously.

We solve the existence problem by analyzing auctions on the grid, which we then characterize with the help of a communication extension based on Jackson et al. (2002). While previous applications of the communication extension used it largely to provide abstract existence proofs, we show how it can be utilized as a solution method.

The communication extension captures all limit outcomes of equilibria on the ever-finer grid. Hence, equilibria on the fine grid have to share the characteristic properties of the equilibria of the communication extension. In particular, we show the emergence of an interior atom and a "winner's blessing" at bids below it. This incentivizes bidders with low signals to compete for the highest bid below the atom. Since such a bid does not exist on the continuous bidding space, none of the equilibria of the communication extension are compatible with the uniform tie-breaking of the standard auction.

Pooling and the equilibrium multiplicity that arise from numbers uncertainty have interesting implications. For example, even though the model is purely competitive, bidders with low signals engage in cooperative behavior to reduce the winner's curse. Contrary to a common-value auction with affiliation, they have an incentive to coordinate on certain bids. Consequently, equilibria resemble collusive behavior, even though they are the outcome of independent, utility-maximizing behavior of the bidders.[31] Moreover, the presence of atoms in the bid distribution invalidates empirical identification strategies that rely on the bidder's first-order condition (cf. Athey and Haile (2007)) and, hence, on a strictly increasing bidding strategy.

Future research may examine the consequences of pooling and equilibrium multiplicity for classic questions such as revenue comparisons across auction formats. Since atoms arise at the bottom of the bid distribution, they are particularly relevant for the determination of the optimal reserve price. Finally, with atoms, the auction sometimes fails to sell to the bidder with the highest signal, suggesting negative welfare consequences in general interdependent value settings with a small private-value component.

[31] Compare also Lauermann and Wolinsky (2017).

Appendix A Overview

The appendix is divided into five parts. After this overview and some general comments (A) follow the proofs skipped in the body of the text (B). The last section (C) contains the numerical examples.

Maintained assumptions: All proofs are given with state-dependent means, η_ω. To that end, we redefine $\breve{s}\colon \frac{\eta_h f_h(\breve{s})}{\eta_\ell f_\ell(\breve{s})} = 1$ and sometimes restate the claims for this more general case, which is when we asterisk them. For convenience, we distinguish between claims that hold everywhere and almost everywhere only when it is central to the argument—unless specified otherwise, results hold for almost all s. Further, we always only consider pure and monotone equilibria (cf. Lemma 1 and 7).

As a reminder for the reader, we restate the most important symbols:

$\omega \in \{h, \ell\}$	states of the world	ρ	prior probability $\omega = h$
η_ω	mean of the number of bidders	v_ω	value of the good
β	standard strategy	$b \in [v_\ell, v_h]$	bid
$s \in [\underline{s}, \bar{s}]$	signals	$s_{(1)}$	highest (other) signal
f_ω	signal density	F_ω	signal cdf
$M \subseteq [0,1]$	message space	$m \in M$	message
σ	comm. extension strategy	$\breve{s}$	$\breve{s} : \frac{\eta_h f_h(\breve{s})}{\eta_\ell f_\ell(\breve{s})} = 1$

The interim expected utility of the standard on the continuous bid space Γ and on the grid $\Gamma(k)$ is:

$$U(b|s;\beta) = \frac{\rho\eta_h f_h(s)}{\rho\eta_h f_h(s) + (1-\rho)\eta_\ell f_\ell(s)}\pi_h(b;\beta)(v_h - b) + \frac{(1-\rho)\eta_\ell f_\ell(s)}{\rho\eta_h f_h(s) + (1-\rho)\eta_\ell f_\ell(s)}\pi_\ell(b;\beta)(v_\ell - b). \tag{20}$$

The interim expected utility in the communication extension $\Gamma^{\mathfrak{c}}$ is:

$$U^{\mathfrak{c}}(M,m,b|s;\sigma) = \frac{\rho\eta_h f_h(s)}{\rho\eta_h f_h(s) + (1-\rho)\eta_\ell f_\ell(s)}\pi_h^{\mathfrak{c}}(M,m,b;\sigma)(v_h - b) + \frac{(1-\rho)\eta_\ell f_\ell(s)}{\rho\eta_h f_h(s) + (1-\rho)\eta_\ell f_\ell(s)}\pi_\ell^{\mathfrak{c}}(M,m,b;\sigma)(v_\ell - b). \tag{21}$$

Appendix B Proofs

Lemma 1. *Let β be some strategy and β^* a best response to it. If the likelihood ratio $\frac{f_h}{f_\ell}$ is strictly increasing, then β^* is essentially*[32] *pure and nondecreasing. If the likelihood ratio is only weakly increasing, then there exists an equivalent best response $\hat{\beta}^*$ that is pure and nondecreasing.*

Proof.

Step 1. *If $b' > b \geq v_\ell$ and $U(b'|s;\beta) \geq U(b|s;\beta)$, then $U(b'|s';\beta) \geq U(b|s';\beta)$ for $s' > s$. The second inequality is strict if $\frac{f_h(s')}{f_\ell(s')} > \frac{f_h(s)}{f_\ell(s)}$.*

Since $b' > b \geq v_\ell$ it follows that $(v_\ell - b') < (v_\ell - b) \leq 0$. Because the winning probability π_ω is weakly increasing in the bid and never zero (the bidder is alone with positive probability), $\pi_\omega(b';\beta) \geq \pi_\omega(b;\beta) \geq \pi_\omega(v_\ell;\beta) > 0$. Together, this yields $\pi_\ell(b';\beta)(v_\ell - b') < \pi_\ell(b;\beta)(v_\ell - b) \leq 0$. Hence, $U(b'|s;\beta) \geq U(b|s;\beta)$ requires that $\pi_h(b';\beta)(v_h - b') > \pi_h(b;\beta)(v_h - b)$. Rearranging $U(b'|s;\beta) \geq U(b|s;\beta)$ gives

$$\frac{\rho\eta_h f_h(s)}{(1-\rho)\eta_\ell f_\ell(s)}[\pi_h(b';\beta)(v_h - b') - \pi_h(b;\beta)(v_h - b)] \geq \pi_\ell(b;\beta)(v_\ell - b) - \pi_\ell(b';\beta)(v_\ell - b').$$

If $s' > s$ is such that $\frac{f_h(s')}{f_\ell(s')} > \frac{f_h(s)}{f_\ell(s)}$, the left side is strictly larger for s', and thus $U(b'|s',\beta) > U(b|s',\beta)$.

Step 2. *The set of interim beliefs that imply indifference between two bids, $L = \{\frac{f_h(s)}{f_\ell(s)} : \exists b, b' \text{ with } b \neq b' \text{ and } U(b|s;\beta) = U(b'|s;\beta)\}$, is countable.*

By construction, $\forall l \in L$ there exist two bids $b^l_- < b^l_+$ such that a bidder $s^l : \frac{f_h(s)}{f_\ell(s)} = l$ is indifferent between these two bids, $U(b^l_-|s^l;\beta) = U(b^l_+|s^l;\beta)$. Furthermore, there exists a $q^l \in \mathbb{Q}$ s.t. $b^l_- < q^l < b^l_+$. By Step 1, $b^l_+ \leq b^{l'}_-$ for all $l < l'$, which implies that $q^l < q^{l'}$. Because $\mathbb{Q}$ is countable, so is L.

Step 3. *Fix any strategy β. If the likelihood ratio $\frac{f_h}{f_\ell}$ is constant on some interval I, then there is an equivalent strategy $\hat{\beta}$ which is pure and nondecreasing over I and equal to β at every other signal.*

Compare Pesendorfer and Swinkels (1997) footnote 8.

Now combine the steps to prove the Lemma. First, suppose that the MLRP holds strictly. Then, for every element $l \in L$, there is only a single signal s_l such that $\frac{f_h(s_l)}{f_\ell(s_l)} = l$ which is indifferent between two bids and may mix. Since L is countable, the set of signals which potentially mix has zero measure and we can assign them the lowest bid in the support of their strategies. The resulting strategy is pure and, by Step 1, nondecreasing. Since the strategy is only changed on a set of measure zero, the resulting distribution of bids is unchanged.

Next, suppose that signal structure is such that it contains intervals I along which the likelihood ratio is constant. In this case, apply Step 3 sequentially to any such I and receive a strategy which is pure and nondecreasing. Furthermore, the reordering leaves the distribution of bids and thereby outcomes and utilities unaltered. □

[32] Up to a set of signals with measure zero.

Lemma 2*. *The expected value* $\mathbb{E}[v|s_{(1)} \leq \hat{s}]$ *is strictly decreasing in* $\hat{s}$ *when* $\hat{s} < \breve{s}$*, has its unique global minimum at* $\hat{s} = \breve{s}$ *and is strictly increasing when* $\hat{s} > \breve{s}$.

Proof. Applying the proof from Lemma 2 verbatim implies the claim, after adjusting (2), by replacing η with η_ω. $\square$

Proposition 1*. *Holding* $\frac{\eta_h}{\eta_\ell} = l < \frac{f_\ell(\underline{s})}{f_h(\underline{s})}$ *fixed, when* η_h *is sufficiently large , no strictly increasing equilibrium exists.*

Proof. Suppose to the contrary that a strictly increasing equilibrium β^* exists when η_h is arbitrarily large. Fix three signals $s_- < s_\circ < s_+$ with $s_+ < \breve{s}$. The argument is structured into three steps: First, Step 1 derives an upper bound on the bid $\beta^*(s_+)$, and Step 2 a lower bound on $\beta^*(s_\circ)$. Step 3 shows that when η_h is sufficiently large, the lower bound exceeds the upper bound which completes the proof.

Step 1. *An upper bound on* $\beta^*(s_+)$ *is given by*

$$\frac{\beta^*(s_+) - v_\ell}{v_h - \beta^*(s_+)} \leq \frac{\rho}{1-\rho} \frac{\eta_h f_h(s_+)}{\eta_\ell f_\ell(s_+)} \frac{e^{-\eta_h(1-F_h(s_+))}}{e^{-\eta_\ell(1-F_\ell(s_+))}}. \tag{22}$$

In any equilibrium and for any signal s, it has to hold that $\beta^*(s) \leq \mathbb{E}[v|\text{win with } \beta^*(s), s; \beta^*]$. Otherwise, the utility

$$U(\beta^*(s)|s;\beta^*) = \mathbb{P}[\text{win with } \beta^*(s)|s;\beta^*]\Big(\mathbb{E}[v|\text{win with } b, s; \beta^*] - \beta^*(s)\Big)$$

is negative. Since a bid of v_ℓ guarantees a non-negative payoff, there would be profitable deviation. Using (3), the condition $\beta^*(s) \leq \mathbb{E}[v|\text{win with } \beta^*(s), s; \beta^*]$ can be rearranged to

$$\frac{\beta^*(s) - v_\ell}{v_h - \beta^*(s)} \leq \frac{\rho}{1-\rho} \frac{\eta_h f_h(s)}{\eta_\ell f_\ell(s)} \frac{\pi_h(\beta^*(s);\beta^*)}{\pi_\ell(\beta^*(s);\beta^*)}. \tag{23}$$

Replacing s by s_+ and using that $\pi_\omega(\beta^*(s_+);\beta^*) = e^{-\eta_\omega(1-F_\omega(s_+))}$ provides inequality.

Step 2. *A lower bound on* $\beta^*(s_\circ)$ *is given by*

$$\frac{\beta^*(s_\circ) - v_\ell}{v_h - \beta^*(s_\circ)} \geq \frac{\rho}{1-\rho} \frac{\eta_h f_h(s_-)}{\eta_\ell f_\ell(s_-)} \frac{e^{-\eta_h(1-F_h(s_\circ))}}{e^{-\eta_\ell(1-F_\ell(s_\circ))}} A(\eta_h), \tag{24}$$

and where $A(\eta_h)$ *is a decreasing function with* $\lim_{\eta_h \to \infty} A(\eta_h) = 1$.

In equilibrium, there is no profitable deviation, such that $U(\beta^*(s_-)|s_-;\beta^*) \geq U(\beta^*(s_\circ)|s_-;\beta^*)$ that is

$$\frac{\rho\eta_h f_h(s_-)\pi_h^-(\beta^*(s_-);\beta^*)(v_h - \beta^*(s_-)) + (1-\rho)\eta_\ell f_\ell(s_-)\pi_\ell^-(\beta^*(s_-);\beta^*)(v_\ell - \beta^*(s_-))}{\rho\eta_h f_h(s_-) + (1-\rho)\eta_\ell f_\ell(s_-)}$$
$$\geq \frac{\rho\eta_h f_h(s_-)\pi_h(\beta^*(s_\circ);\beta^*)(v_h - \beta^*(s_\circ)) + (1-\rho)\eta_\ell f_\ell(s_-)\pi_\ell(\beta^*(s_\circ);\beta^*)(v_\ell - \beta^*(s_\circ))}{\rho\eta_h f_h(s_-) + (1-\rho)\eta_\ell f_\ell(s_-)}.$$

Since $\beta^*(s_-) \geq v_\ell$ a necessary condition for the inequality is that

$$\begin{aligned}&\rho\eta_h f_h(s_-)\pi_\ell(\beta^*(s_\circ);\beta^*)(v_h - v_\ell)\\ &\geq \rho\eta_h f_h(s_-)\pi_h(\beta^*(s_\circ);\beta^*)(v_h - \beta^*(s_\circ)) + (1-\rho)\eta_\ell f_\ell(s_-)\pi_\ell(\beta^*(s_\circ);\beta^*)(v_\ell - \beta^*(s_\circ)).\end{aligned}$$

Rearranging the inequality gives a lower bound on $\beta^*(s_\circ)$

$$\frac{\beta^*(s_\circ) - v_\ell}{v_h - \beta^*(s_\circ)} \geq \frac{\rho}{1-\rho}\frac{\eta_h f_h(s_-)}{\eta_\ell f_\ell(s_-)}\frac{\pi_h(\beta^*(s_\circ);\beta^*)}{\pi_\ell(\beta^*(s_\circ);\beta^*)}\Big(1 - \frac{\pi_h(\beta^*(s_-);\beta^*)}{\pi_h(\beta^*(s_\circ);\beta^*)}\frac{v_h - v_\ell}{v_h - \beta^*(s_\circ)}\Big). \quad (25)$$

Because $s_\circ > s_-$ and $\eta_h \to \infty$, it follows that $\frac{\pi_h(\beta^*(s_-);\beta^*)}{\pi_h(\beta^*(s_\circ);\beta^*)} = e^{-\eta_h(F_h(s_\circ)-F_h(s_-))} \to 0$. Thus, $A(\eta_h) = 1 - \frac{\pi_h(\beta^*(s_-);\beta^*)}{\pi_h(\beta^*(s_\circ);\beta^*)}\frac{v_h - v_\ell}{v_h - \beta^*(s_\circ)} \to 1$ unless $\beta^*(s_\circ) \to v_h$. If $\beta^*(s_\circ) \to v_h$, however, $\beta^*(s_+) \to v_h$. In this case, the left side of inequality (22) grows without bound, while $e^{-\eta_h(1-F_h(s_+))}/e^{-\eta_\ell(1-F_\ell(s_+))} < 1$ such that the right side stays bounded. As a result, inequality (22) is always be violated when η_h is large, such that it is without loss to restrict attention to the case in which $A(\eta_h) \to 1$.

Step 3. *When η_h is sufficiently large, the upper bound on $\beta^*(s_+)$ expressed by (24) is smaller than the lower bound on $\beta^*(s_\circ)$ given by inequality (22).*

Since $\frac{b - v_\ell}{v_h - b}$ is increasing in b, a necessary condition for both inequalities to hold simultaneously is that

$$\frac{\rho}{1-\rho}\frac{\eta_h f_h(s_+)}{\eta_\ell f_\ell(s_+)}\frac{e^{-\eta_h(1-F_h(s_+))}}{e^{-\eta_\ell(1-F_\ell(s_+))}} > \frac{\rho}{1-\rho}\frac{\eta_h f_h(s_-)}{\eta_\ell f_\ell(s_-)}\frac{e^{-\eta_h(1-F_h(s_\circ))}}{e^{-\eta_\ell(1-F_\ell(s_\circ))}}A(\eta_h).$$

This can be rearranged to

$$\frac{f_h(s_+)}{f_\ell(s_+)}\Big(\frac{f_h(s_-)}{f_\ell(s_-)}\Big)^{-1} > \frac{e^{-\eta_h(1-F_h(s_\circ))}}{e^{-\eta_\ell(1-F_\ell(s_\circ))}}\Big(\frac{e^{-\eta_h(1-F_h(s_+))}}{e^{-\eta_\ell(1-F_\ell(s_+))}}\Big)^{-1}A(\eta_h). \quad (26)$$

The fractions $\frac{e^{-\eta_h(1-F_h(s_\circ))}}{e^{-\eta_\ell(1-F_\ell(s_\circ))}}\Big(\frac{e^{-\eta_h(1-F_h(s_+))}}{e^{-\eta_\ell(1-F_\ell(s_+))}}\Big)^{-1} = e^{\eta_h[F_h(s_+)-F_h(s)]-\eta_\ell[F_\ell(s_+)-F_\ell(s)]} \to 0$, because

$$\begin{aligned}\eta_h[F_h(s_+) - F_h(s_\circ)] - \eta_\ell[F_\ell(s_+) - F_\ell(s_\circ)] &= \int_{s_\circ}^{s_+}[1 - \frac{\eta_\ell f_\ell(s)}{\eta_h f_h(s)}]\eta_h f_h(s)ds\\ &< \underbrace{\eta_h}_{\to\infty}\int_{s_\circ}^{s_+}[1 - \frac{\eta_\ell f_\ell(s_+)}{\eta_h f_h(s_+)}]f_h(s)ds \to -\infty,\end{aligned}$$

where we use that $\frac{\eta_h f_h(s_+)}{\eta_\ell f_\ell(s_+)} < \frac{\eta_h f_h(\check{s})}{\eta_\ell f_\ell(\check{s})} = 1$ is a constant. Because $A(\eta_h) \to 1$, and $\frac{f_h(s_+)}{f_\ell(s_+)}\frac{f_\ell(s_\circ)}{f_h(s_\circ)}$ is constant, this implies that equation (26) cannot hold when η_h is large. This is a contradiction. □

Proposition 2*. *The ordinary differential equation*

$$\frac{\partial}{\partial s}\beta(s) = \Big(\mathbb{E}[v|s_{(1)} = s, s] - \beta(s)\Big)\frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)} \qquad \textit{with } \beta(\underline{s}) = v_\ell \quad (27)$$

has a unique solution, denoted by $\hat{\beta}$.

(i) If $\hat{\beta}$ is strictly increasing, then it is the unique equilibrium in the class of strictly increasing equilibria.

(ii) If $\hat{\beta}$ is not strictly increasing, no strictly increasing equilibrium exists.

Proof. For $s, \hat{s} \in [\underline{s}, \bar{s}]$, let $F_{s_{(1)}}(\hat{s}|s)$ denote the cdf of $s_{(1)}$ conditional on observing s, and let $f_{s_{(1)}}$ be the associated density

$$F_{s_{(1)}}(\hat{s}|s) = \frac{\rho\eta_h f_h(s)e^{-\eta_h(1-F_h(\hat{s}))} + (1-\rho)\eta_\ell f_\ell(s)e^{-\eta_\ell(1-F_\ell(\hat{s}))}}{\rho\eta_h f_h(s) + (1-\rho)\eta_\ell f_\ell(s)}, \tag{28}$$

$$f_{s_{(1)}}(\hat{s}|s) = \frac{\rho\eta_h^2 f_h(s)f_h(\hat{s})e^{-\eta_h(1-F_h(\hat{s}))} + (1-\rho)\eta_\ell^2 f_\ell(s)f_\ell(\hat{s})e^{-\eta_\ell(1-F_\ell(\hat{s}))}}{\rho\eta_h f_h(s) + (1-\rho)\eta_\ell f_\ell(s)}. \tag{29}$$

Note that because signal distribution is atomless, the probability that there is no (other) bidder, $s_{(1)} = -\infty$, conditional on observing signal s is $\mathbb{P}[s_{(1)} = -\infty|s] = F_{s_{(1)}}(\underline{s}|s)$. As a result, for $\hat{s} \in [\underline{s}, \bar{s}]$ it holds that $F_{s_{(1)}}(\hat{s}|s) = \int_{\underline{s}}^{\hat{s}} f_{s_{(1)}}(z|s)dz + F_{s_{(1)}}(\underline{s}|s)$.

As a further abbreviation define $v(\hat{s}|s) = \mathbb{E}[v|s_{(1)} = \hat{s}, s]$, that is

$$v(\hat{s}|s) = \begin{cases} \frac{\rho\eta_h^2 f_h(s)f_h(\hat{s})e^{-\eta_h(1-F_h(\hat{s}))}v_h + (1-\rho)\eta_\ell^2 f_\ell(s)f_\ell(\hat{s})e^{-\eta_\ell(1-F_\ell(\hat{s}))}v_\ell}{\rho\eta_h^2 f_h(s)f_h(\hat{s})e^{-\eta_h(1-F_h(\hat{s}))} + (1-\rho)\eta_\ell^2 f_\ell(s)f_\ell(\hat{s})e^{-\eta_\ell(1-F_\ell(\hat{s}))}} & \text{if } \hat{s} \in [\underline{s}, \bar{s}] \\ \frac{\rho\eta_h f_h(s)e^{-\eta_h}v_h + (1-\rho)\eta_\ell f_\ell(s)e^{-\eta_\ell}v_\ell}{\rho\eta_h f_h(s)e^{-\eta_h} + (1-\rho)\eta_\ell f_\ell(s)e^{-\eta_\ell}} & \text{if } \hat{s} = -\infty. \end{cases} \tag{30}$$

If β is strictly increasing and continuous, $\pi_\omega(b;\beta) = \mathbb{P}[s_{(1)} \leq \beta^{-1}(b)|\omega;\beta]$ for all b in β's support. As a result, for all b in the support, the expected utility (20) can be rewritten as

$$U(b|s;\beta) = \int_{\underline{s}}^{\beta^{-1}(b)=s} \Big[v(z|s) - b\Big] f_{s_{(1)}}(z|s)dz + \Big[v(-\infty|s) - b\Big] F_{s_{(1)}}(\underline{s}|s). \tag{31}$$

Step 1. *If β is a strictly increasing equilibrium, then β is differentiable and solves the ODE $\frac{\partial\beta(s)}{\partial s} = \Big(\mathbb{E}[v|s_{(1)} = s, s] - \beta(s)\Big)\frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)}$ with $\beta(\underline{s}) = v_\ell$.*

Suppose β is a strictly increasing equilibrium (economizing on the $*$), then it is continuous. If β would jump upwards, any bid just above a jump would be dominated by a bid just below the jump, which wins with the same probability but at a lower price. By the same reason, $\beta(\underline{s}) = v_\ell$.

Take any point $s \in (\underline{s}, \bar{s})$ and show that β is differentiable at this point. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence converging to s from below. Then, the sequence with elements $b_n = \beta(s_n)$ converges to $b = \beta(s)$ from below, too. Because $b_n < b$ is a best response for $s_n < s$, it follows that $U(b_n|s_n;\beta) \geq U(b|s_n;\beta)$. Using (31), gives

$$\begin{aligned}\int_{\underline{s}}^{\beta^{-1}(b_n)=s_n} &\Big[v(z|s_n) - b_n\Big] f_{s_{(1)}}(z|s_n)dz + \Big[v(-\infty|s_n) - b_n\Big] F_{s_{(1)}}(\underline{s}|s_n) \\ &\geq \int_{\underline{s}}^{\beta^{-1}(b)=s} \Big[v(z|s_n) - b\Big] f_{s_{(1)}}(z|s_n)dz + \Big[v(-\infty|s_n) - b\Big] F_{s_{(1)}}(\underline{s}|s_n),\end{aligned}$$

which can be rearranged to

$$\int_{\underline{s}}^{s_n} \Big[b - b_n\Big] f_{s_{(1)}}(z|s_n)dz + \Big[b - b_n\Big] F_{s_{(1)}}(\underline{s}|s_n) \geq \int_{s_n}^{s} \Big[v(z|s_n) - b\Big] f_{s_{(1)}}(z|s_n)dz.$$

Dividing by $s - s_n > 0$, as well as $F_{s_{(1)}}(s|s_n) = \int_{\underline{s}}^{s} f_{s_{(1)}}(z|s_n)dz + F_{s_{(1)}}(\underline{s}|s_n) > 0$ and taking the lim inf yields

$$\liminf_{n\to\infty} \frac{b - b_n}{s - s_n} \geq \liminf_{n\to\infty} \frac{1}{s - s_n} \int_{s_n}^{s} \Big[v(z|s_n) - b\Big] \frac{f_{s_{(1)}}(z|s_n)}{F_{s_{(1)}}(s|s_n)} dz.$$

By inspection of equations (29) and (30), the continuity of f_h and f_ℓ ensures that $v(z|s_n)$, $f_{s_{(1)}}(z|s_n)$ and, thereby, $F_{s_{(1)}}(s|s_n)$ are continuous in both arguments such that

$$\liminf_{n\to\infty} \frac{b - b_n}{s - s_n} \geq [v(s|s) - b] \frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)}. \tag{32}$$

At the same time, bid b is a best response for signal s, implying that $U(b_n|s;\beta) \leq U(b|s;\beta)$, which rearranges to

$$\begin{aligned} &\int_{\underline{s}}^{\beta^{-1}(b_n)=s_n} \Big[v(z|s) - b_n\Big] f_{s_{(1)}}(z|s)dz + \Big[v(-\infty|s) - b_n\Big] F_{s_{(1)}}(\underline{s}|s) \\ &\qquad \leq \int_{\underline{s}}^{\beta^{-1}(b)=s} \Big[v(z|s) - b\Big] f_{s_{(1)}}(z|s)dz + \Big[v(-\infty|s) - b\Big] F_{s_{(1)}}(\underline{s}|s). \end{aligned}$$

Repeating the steps as before, but taking the lim sup instead, yields

$$\limsup_{n\to\infty} \frac{b - b_n}{s - s_n} \leq [v(s|s) - b] \frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)}, \tag{33}$$

and because lim inf $\leq$ lim sup, it follows from equations (32) and (33) that

$$\lim_{n\to\infty} \frac{b - b_n}{s - s_n} = \lim_{n\to\infty} \frac{\beta(s) - \beta(s_n)}{s - s_n} = [v(s|s) - \beta(s)] \frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)}.$$

We can repeat the construction for any sequence of signals and bids which converges from above instead of below and obtain the same result. Therefore, β is differentiable and can be written as (replacing v)

$$\frac{\partial\beta(s)}{\partial s} = \Big(\mathbb{E}[v|s_{(1)} = s, s] - \beta(s)\Big) \frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)},$$

or, fully spelled out for future reference,

$$\frac{\partial\beta(s)}{\partial s} = \frac{\rho\eta_h^2 f_h(s)^2 e^{-\eta_h(1-F_h(s))}(v_h - \beta(s)) + (1-\rho)\eta_\ell^2 f_\ell(s)^2 e^{-\eta_\ell(1-F_\ell(s))}(v_\ell - \beta(s))}{\rho\eta_h f_h(s) e^{-\eta_h(1-F_h(s))} + (1-\rho)\eta_\ell f_\ell(s) e^{-\eta_\ell(1-F_\ell(s))}}. \tag{34}$$

Step 2. *If β is strictly increasing and solves the ODE $\frac{\partial\beta(s)}{\partial s} = \Big(\mathbb{E}[v|s_{(1)} = s, s] - \beta(s)\Big) \frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)}$ with initial value $\beta(\underline{s}) = v_\ell$, then β is an equilibrium.*

Suppose that β is strictly increasing and solves the ODE. We want to show that $U(\beta(s)|s;\beta) \geq U(\beta(s')|s;\beta)$ for all $s' \in [\underline{s}, \bar{s}]$. This suffices because $\beta(\underline{s}) = v_\ell$ denotes the lower bound of bids and any bid $b > \beta(\bar{s})$ is dominated by bidding $\beta(\bar{s})$, which also always wins but at a lower cost. We show that $U(\beta(s)|s;\beta) \geq U(\beta(s')|s;\beta)$ by proving that $\frac{\partial U(\beta(s')|s;\beta)}{\partial s'} \geq 0$ for all $s' < s$ and $\frac{\partial U(\beta(s')|s;\beta)}{\partial s'} \leq 0$ for all $s' > s$ such that the utility is hump-shaped with a global maximum for signal s at $\beta(s)$.

Substituting b by $\beta(s')$ in the utility function (31) and taking the derivative wrt. s' yields (note that β is differentiable by the assumption of the step)

$$\frac{\partial}{\partial s'} U(\beta(s')|s;\beta) = \Big([v(s'|s) - \beta(s')] \frac{f_{s_{(1)}}(s'|s)}{F_{s_{(1)}}(s'|s)} - \beta'(s')\Big) F_{s_{(1)}}(s'|s),$$

which is positive if and only if

$$[v(s'|s) - \beta(s')] \frac{f_{s_{(1)}}(s'|s)}{F_{s_{(1)}}(s'|s)} > \beta'(s').$$

Because β solves the ODE $\beta'(s') = [v(s'|s') - \beta(s')] \frac{f_{s_{(1)}}(s'|s')}{F_{s_{(1)}}(s'|s')}$, this means that $\frac{\partial}{\partial s'} U(\beta(s')|s, \beta)$ is positive if and only if

$$[v(s'|s) - \beta(s')] \frac{f_{s_{(1)}}(s'|s)}{F_{s_{(1)}}(s'|s)} > [v(s'|s') - \beta(s')] \frac{f_{s_{(1)}}(s'|s')}{F_{s_{(1)}}(s'|s')}.$$

Fully expanded, the left side of the equation becomes (c.f. equations (28)-(30))

$$\frac{\rho \eta_h f_h(s) e^{-\eta_h(1-F_h(s'))}}{\rho \eta_h f_h(s) e^{-\eta_h(1-F_h(s'))} + (1-\rho)\eta_\ell f_\ell(s) e^{-\eta_\ell(1-F_\ell(s'))}} \underbrace{\eta_h f_h(s')(v_h - \beta(s'))}_{>0}$$
$$+ \frac{(1-\rho)\eta_\ell f_\ell(s) e^{-\eta_\ell(1-F_\ell(s'))}}{\rho \eta_h f_h(s) e^{-\eta_h(1-F_h(s'))} + (1-\rho)\eta_\ell f_\ell(s) e^{-\eta_\ell(1-F_\ell(s'))}} \underbrace{\eta_\ell f_\ell(s')(v_\ell - \beta(s'))}_{<0}.$$

As a result, the expression is nondecreasing in s, and strictly increasing in s if $\frac{f_h(s)}{f_\ell(s)}$ is increasing. This means that

$$[v(s'|s) - \beta(s')] \frac{f_{s_{(1)}}(s'|s)}{F_{s_{(1)}}(s'|s)} > [v(s'|s') - \beta(s')] \frac{f_{s_{(1)}}(s'|s')}{F_{s_{(1)}}(s'|s')}$$

if and only if $\frac{f_h(s')}{f_\ell(s')} < \frac{f_h(s)}{f_\ell(s)}$. It follows that

- $\frac{\partial}{\partial s'} U(\beta(s')|s, \beta) > 0$ for all $s' < s : \frac{f_h(s')}{f_\ell(s')} < \frac{f_h(s)}{f_\ell(s)}$,
- $\frac{\partial}{\partial s'} U(\beta(s')|s, \beta) < 0$ for all $s' > s : \frac{f_h(s')}{f_\ell(s')} > \frac{f_h(s)}{f_\ell(s)}$,
- $\frac{\partial}{\partial s'} U(\beta(s')|s, \beta) = 0$ for all $s' : \frac{f_h(s')}{f_\ell(s')} = \frac{f_h(s)}{f_\ell(s)}$,

such that $\beta(s)$ is a global maximizer for s.

Step 3. *$\hat{\beta}$ is a strictly increasing equilibrium if an only if it is strictly increasing, solves the ODE $\frac{\partial \hat{\beta}(s)}{\partial s} = \Big(\mathbb{E}[v|s_{(1)} = s, s] - \beta(s)\Big) \frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)}$ with initial value $\hat{\beta}(\underline{s}) = v_\ell$. If $\hat{\beta}$ is* *an equilibrium, it is unique in the class of strictly increasing equilibria. Thus, if $\hat{\beta}$ is not strictly increasing, no strictly increasing equilibrium exists.*

Because the signal densities are continuous, the likelihood ratio $\frac{f_h}{f_\ell}$, bids, and values v_ω are bounded and $F_{s_{(1)}}(s|s) > 0$, the ODE $\frac{\partial \hat{\beta}(s)}{\partial s} = [\mathbb{E}[v|s_{(1)} = s, s] - \hat{\beta}(s)] \frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)}$ is Lipschitz continuous (c.f. (29) and (30)). Thus, there exists a unique solution to the initial value problem $\beta(\underline{s}) = v_\ell$. Combining this with Step 1 (necessary condition) and 2 (sufficient condition), the result follows. $\square$

Lemma 3*. *A strictly increasing equilibrium exists if either*

(i) $\mathbb{E}[v|s_{(1)} = s, s]$ *is strictly increasing in* s,

(ii) or η_h *is sufficiently small.*

Proof. Proposition 2* shows that a strictly increasing equilibrium exists if and only if the unique solution $\hat{\beta}$ to the ODE (27) is strictly increasing. Thus, we have to show that $\hat{\beta}$ is strictly increasing.

Step 1. *If* $\mathbb{E}[v|s_{(1)} = s, s]$ *is strictly increasing in* s, *then* $\hat{\beta}$ *is strictly increasing. This is the case if and only if* $2\Big(\frac{\partial}{\partial s}\frac{f_h(s)}{f_\ell(s)}\Big)\frac{f_\ell(s)}{f_h(s)} + \eta_h f_h(s) - \eta_\ell f_\ell(s) > 0$ *for almost all* s.

Since $\frac{\eta_h f_h(\underline{s})}{\eta_\ell f_\ell(\underline{s})} > 0$, it follows that $\mathbb{E}[v|s_{(1)} = \underline{s}, \underline{s}] = v(\underline{s}, |\underline{s}) > v_\ell$. In combination with the initial value $\hat{\beta}(\underline{s}) = v_\ell$, this means that $\hat{\beta}'(\underline{s}) > 0$ (c.f. (27)). Because the densities f_h and f_ℓ are continuous, so is $\hat{\beta}$ and $\hat{\beta}'$. Thus, $\hat{\beta}'$ can only become negative if it intersects the 0 from above. In that case, there exists a $\hat{s}$ such that $\hat{\beta}'(\hat{s}) = 0$, meaning that $v(\hat{s}|\hat{s}) - \hat{\beta}(\hat{s}) = 0$. Since $\hat{\beta}'(\hat{s}) = 0$, marginally increasing $\hat{s}$ will not change $\hat{\beta}$. Hence, the marginal change of $v(\hat{s}|\hat{s})$ decides whether $\hat{\beta}'$ is just tangent, or intersects the 0 at $\hat{s}$. Thus, it suffices that $\mathbb{E}[v|s_{(1)} = \hat{s}, \hat{s}] = v(\hat{s}|\hat{s})$ is strictly increasing in $\hat{s} \in (\underline{s}, \bar{s})$.

The expected value $v(s|s)$ is increasing at (almost) every s if and only if $\frac{f_h(s)^2 e^{-\eta_h(1-F_h(s))}}{f_\ell(s)^2 e^{-\eta_\ell(1-F_\ell(s))}}$, is increasing in s (cf. (30)). Differentiating with respect to s yields

$$2\Big(\frac{\partial}{\partial s}\frac{f_h(s)}{f_\ell(s)}\Big)\frac{f_h(s)}{f_\ell(s)}\frac{e^{-\eta(1-F_h(s))}}{e^{-\eta(1-F_\ell(s))}} + \frac{f_h(s)^2}{f_\ell(s)^2}\frac{e^{-\eta(1-F_h(s))}e^{-\eta(1-F_\ell(s))}}{(e^{-\eta(1-F_\ell(s))})^2}(\eta_h f_h(s) - \eta_\ell f_\ell(s)) > 0$$

Dividing by $\frac{e^{-\eta_h(1-F_h(s))}}{e^{-\eta_\ell(1-F_\ell(s))}} > 0$ and $\frac{f_h(s)^2}{f_\ell(s)^2} > 0$ yields the result. Since $\frac{f_h}{f_\ell}$ is monotone, it is differentiable almost everywhere.

Step 2. *When* η_h *is sufficiently small,* $\hat{\beta}$ *is strictly increasing.*

If $\frac{\eta_h}{\eta_\ell} > \frac{f_\ell(\underline{s})}{f_h(\underline{s})}$ then $\eta_h f_h(s) \geq \eta_\ell f_\ell(s)$ for all s. By Step 1, a strictly increasing equilibrium exists. Thus, we focus on the case when $\frac{\eta_h}{\eta_\ell} = l \leq \frac{f_\ell(\underline{s})}{f_h(\underline{s})}$. Then

$$\begin{aligned} v(s|s) &= \frac{\rho \eta_h^2 f_h(s)^2 e^{-\eta_h(1-F_h(s))} v_h + (1-\rho)\eta_\ell^2 f_\ell(s)^2 e^{-\eta_\ell(1-F_\ell(s))} v_\ell}{\rho \eta_h^2 f_h(s)^2 e^{-\eta_h(1-F_h(s))} + (1-\rho)\eta_\ell^2 f_\ell(s)^2 e^{-\eta_\ell(1-F_\ell(s))}} \\ &\overset{\eta_h \to 0}{\to} \frac{\rho\, l^2 f_h(s)^2 v_h + (1-\rho) f_\ell(s)^2 v_\ell}{\rho\, l^2 f_h(s)^2 + (1-\rho) f_\ell(s)^2} =: \phi(s) \geq \phi(\underline{s}) > v_\ell. \end{aligned}$$

Using that $\hat{\beta}(s) \geq v_\ell$ and equation (34), $\hat{\beta}'(s)$ can be bounded above by $\eta_h f_h(s)(v_h - v_\ell)$. Therefore, $\hat{\beta}(s) = \int_{\underline{s}}^{s} \hat{\beta}'(z)dz + v_\ell < \phi(\underline{s})$ when η_h is sufficiently small. Thus, if η_h, η_ℓ are both small, $\hat{\beta}'(s) = [v(s|s) - \hat{\beta}(s)]\frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)} \geq [\phi(\underline{s}) - \hat{\beta}(s)]\frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)} > 0$ for all s. □

Lemma 4*. *Assume that β is some strategy for which there exists an interval $[s_-, s_+]$ and a bid b_p, such that $\beta(s) = b_p$ for all $s \in [s_-, s_+]$ and $\beta(s) < b_p < \beta(s')$ for all $s < s_- < s_+ < s'$. Then b_p wins with probability*

$$\pi_\omega(b_p; \beta) = \frac{\mathbb{P}(s_{(1)} \in [s_-, s_+]\,|\omega)}{\mathbb{E}[\#s \in [s_-, s_+]\,|\omega]} = \frac{e^{-\eta_\omega(1-F_\omega(s_+))} - e^{-\eta_\omega(1-F_\omega(s_-))}}{\eta_\omega(F_\omega(s_+) - F_\omega(s_-))} \qquad \text{for } \omega \in \{h, \ell\}.$$

Furthermore,

(i) If $\eta_h[F_h(s_+) - F_h(s_-)] < \eta_\ell[F_\ell(s_+) - F_\ell(s_-)]$, then

$$\mathbb{E}[v|s_{(1)} \leq s_-] > \mathbb{E}[v|\text{win with } b_p; \beta] > \mathbb{E}[v|s_{(1)} \leq s_+]. \tag{35}$$

(ii) If $\eta_h[F_h(s_+) - F_h(s_-)] > \eta_\ell[F_\ell(s_+) - F_\ell(s_-)]$, then the inequalities in (35) reverse.

(iii) If β is an equilibrium strategy, then $\eta_h[F_h(s_+) - F_h(s_-)] < \eta_\ell[F_\ell(s_+) - F_\ell(s_-)]$.

Proof.

Step 1. *$\pi_\omega(b_p; \beta) = \frac{\mathbb{P}[s_{(1)} \in [s_-, s_+]\,|\omega]}{\mathbb{E}[\#s \in [s_-, s_+]\,|\omega]} = \frac{e^{-\eta_\omega(1-F_\omega(s_+))} - e^{-\eta_\omega(1-F_\omega(s_-))}}{\eta_\omega(F_\omega(s_+) - F_\omega(s_-))}$ for $\omega \in \{h, \ell\}$.*

$$\begin{aligned}
\pi_\omega(b_p; \beta) &= \mathbb{P}(\text{no bid} > b_p|\omega)\sum_{n=0}^{\infty}\frac{1}{n+1}\mathbb{P}(\text{n rivals bid } b_p|\omega) \\
&= e^{-\eta_\omega(1-F_\omega(s_+))}\Big(\sum_{n=0}^{\infty}\frac{1}{n+1}e^{-\eta_\omega(F_\omega(s_+)-F_\omega(s_-))}\frac{[\eta_\omega(F_\omega(s_+) - F_\omega(s_-))]^n}{n!}\Big) \\
&= e^{-\eta_\omega(1-F_\omega(s_+))}\Big(\sum_{n=0}^{\infty}e^{-\eta_\omega(F_\omega(s_+)-F_\omega(s_-))}\frac{[\eta_\omega(F_\omega(s_+) - F_\omega(s_-))]^n}{(n+1)!}\Big) \\
&= \frac{e^{-\eta_\omega(1-F_\omega(s_+))}}{\eta_\omega(F_\omega(s_+) - F_\omega(s_-))}\Big(\sum_{n=1}^{\infty}e^{-\eta_\omega(F_\omega(s_+)-F_\omega(s_-))}\frac{[\eta_\omega(F_\omega(s_+) - F_\omega(s_-))]^n}{n!}\Big) \\
&= \frac{e^{-\eta_\omega(1-F_\omega(s_+))}}{\eta_\omega(F_\omega(s_+) - F_\omega(s_-))}\Big(\sum_{n=1}^{\infty}\mathbb{P}(\text{n rivals bid } b_p|\omega)\Big) \\
&= \frac{e^{-\eta_\omega(1-F_\omega(s_+))}}{\eta_\omega(F_\omega(s_+) - F_\omega(s_-))}\Big(1 - e^{-\eta_\omega(F_\omega(s_+)-F_\omega(s_-))}\Big) \\
&= \frac{e^{-\eta_\omega(1-F_\omega(s_+))} - e^{-\eta_\omega(1-F_\omega(s_-))}}{\eta_\omega(F_\omega(s_+) - F_\omega(s_-))}.
\end{aligned}$$

The numerator is $\mathbb{P}[s_{(1)} \in [s_-, s_+]|\omega]$ and the denominator is the expected number of signals from $[s_-, s_+]$ in state ω i.e. $\mathbb{E}[\#s \in [s_-, s_+]\,|\omega]$.

Step 2. *If $\eta_h[F_h(s_+) - F_h(s_-)] < \eta_\ell[F_\ell(s_+) - F_\ell(s_-)]$, then $\mathbb{E}[v|s_{(1)} \leq s] > \mathbb{E}[v|\text{win with } b_p; \beta] > \mathbb{E}[v|s_{(1)} \leq s_+]$. If $\eta_h[F_h(s_+) - F_h(s_-)] > \eta_\ell[F_\ell(s_+) - F_\ell(s_-)]$, the inequalities reverse.*

For any two events ϕ and ϕ', $\mathbb{E}[v|\phi] > \mathbb{E}[v|\phi']$ if and only if $\frac{\mathbb{P}[\phi|h]}{\mathbb{P}[\phi|\ell]} > \frac{\mathbb{P}[\phi'|h]}{\mathbb{P}[\phi'|\ell]}$. Therefore, we have to show that when $\eta_h[F_h(s_+) - F_h(s_-)] < \eta_\ell[F_\ell(s_+) - F_\ell(s_-)]$ it holds that

$$\frac{e^{-\eta_h(1-F_h(s_-))}}{e^{-\eta_\ell(1-F_\ell(s_-))}} > \frac{\frac{e^{-\eta_h(1-F_h(s_+))}-e^{-\eta_h(1-F_h(s_-))}}{\eta_h[F_h(s_+)-F_h(s_-)]}}{\frac{e^{-\eta_\ell(1-F_\ell(s_+))}-e^{-\eta_\ell(1-F_\ell(s_-))}}{\eta_\ell[F_\ell(s_+)-F_\ell(s_-)]}} > \frac{e^{-\eta_h(1-F_h(s_+))}}{e^{-\eta_\ell(1-F_\ell(s_+))}}. \tag{36}$$

As an abbreviation, define $x_\omega = \eta_\omega[F_\omega(s_+) - F_\omega(s_-)]$ for $\omega \in \{h, \ell\}$. Dividing the left inequality of (36) by $\frac{e^{-\eta_h(1-F_h(s_-))}}{e^{-\eta_\ell(1-F_\ell(s_-))}}$, it becomes

$$1 > \frac{\frac{e^{x_h}-1}{x_h}}{\frac{e^{x_\ell}-1}{x_\ell}},$$

which holds because $\frac{e^z-1}{z}$ is strictly increasing in z.

If, on the other hand, the right inequality of (36) is divided by $\frac{e^{-\eta_h(1-F_h(s_+))}}{e^{-\eta_\ell(1-F_\ell(s_+))}}$, it becomes

$$\frac{\frac{1-e^{x_h}}{x_h}}{\frac{1-e^{x_\ell}}{x_\ell}} > 1,$$

which is true because $\frac{1-e^z}{z}$ is strictly decreasing in z.

Step 3. *β can only be an equilibrium if $\eta_h[F_h(s_+) - F_h(s_-)] < \eta_\ell[F_\ell(s_+) - F_\ell(s_-)]$.*

Suppose to the contrary that β is an equilibrium, but $\eta_h[F_h(s_+) - F_h(s_-)] \geq \eta_\ell[F_\ell(s_+) - F_\ell(s_-)]$.[33] Consider a deviation to $b + \epsilon$ by any $s \in [s_-, s_+]$. There are two possibilities:

First, $b_p + \epsilon$ can be a pooling bid meaning that there exists an interval of signals $[s'_-, s'_+]$ such that $\forall s \in [s'_-, s'_+]$ it holds that $\beta(s) = b_p + \epsilon$ and $\neq b_p + \epsilon$ otherwise. Since $\check{s} < s_+ \leq s'_-$, this implies that $\eta_h[F_h(s'_+) - F_h(s'_-)] \geq \eta_\ell[F_\ell(s'_+) - F_\ell(s'_-)]$, and thus

$$\mathbb{E}[v|\text{win with } b_p{+}\epsilon; \beta] \overset{\text{Step 2}}{\geq} \mathbb{E}[v|s_{(1)} \leq s'_-] \overset{\text{Lemma 2*}}{\geq} \mathbb{E}[v|s_{(1)} \leq s_+] \overset{\text{Step 2}}{\geq} \mathbb{E}[v|\text{win with } b_p; \beta].$$

If $b_p + \epsilon$ is not played with positive probability, then it wins whenever $s_{(1)} \leq y$ for some $y \geq s^+$, which means that $\mathbb{E}[v|\text{win with } b_p{+}\epsilon, s; \beta] = \mathbb{E}[v|s_{(1)} \leq y, s]$. This implies that

$$\mathbb{E}[v|\text{win with } b_p + \epsilon; \beta] = \mathbb{E}[v|s_{(1)} \leq y] \overset{\text{Lemma 2*}}{\geq} \mathbb{E}[v|s_{(1)} \leq s_+] \overset{\text{Step 2}}{\geq} \mathbb{E}[v|\text{win with } b_p; \beta].$$

In either case, it follows that $\mathbb{E}[v|\text{win with } b_p + \epsilon, s; \beta] \geq \mathbb{E}[v|\text{win with } b_p, s; \beta] \geq b_p$, where the latter inequality follows by individual rationality (cf. (23)). Since a deviation to $b_p + \epsilon$ discretely increases the winning probability by avoiding the tiebreak, is always profitable for ϵ sufficiently small. Thus, β cannot be an equilibrium when $\eta_h[F_h(s_+) - F_h(s_-)] \geq \eta_\ell[F_\ell(s_+) - F_\ell(s_-)]$ which proves the last assertion. $\square$

Lemma 5*. *Holding $\frac{\eta_h}{\eta_\ell} = l \in (\frac{f_\ell(\bar{s})}{f_h(\bar{s})}, \frac{f_\ell(\underline{s})}{f_h(\underline{s})})$ fixed, when η_h is sufficiently large there is no equilibrium β^* in which $\beta^*(\underline{s}) = \beta^*(\check{s})$ and $\beta^*(s) > \beta^*(\check{s})$ for all $s > \check{s}$.*

[33]Note that because $\check{s} : \frac{\eta_h f_h(\check{s})}{\eta_\ell f_\ell(\check{s})} = 1$, it follows from the MLRP that $s_+ > \check{s}$.

Proof. Suppose to the contrary that the claim is violated when η_h is arbitrarily large and denote $b_p = \beta^*(\underline{s}) = \beta^*(\breve{s})$. The contradiction is derived in three steps. First, deviation 1 is used to derive an upper bound on b_p (Step 1), before deviation 2 is used to find a lower bound (Step 2). Last, Step 3 shows that when η_h is sufficiently large, the lower bound exceeds the upper bound, such that one deviation has to be profitable. As an abbreviation we use for either $\omega \in \{h, \ell\}$ that

$$\pi_\omega^\circ = \pi_\omega(b_p; \beta^*) = e^{-\eta_\omega(1-F_\omega(s_+))}$$
$$\pi_\omega^+ = \lim_{\epsilon \searrow 0} \pi_\omega(b_p + \epsilon; \beta^*) = \frac{e^{-\eta_\omega(1-F_\omega(s_+))} - e^{-\eta_\omega(1-F_\omega(s_-))}}{\eta_\omega(F_\omega(s_+) - F_\omega(s_-))}.$$

Step 1. *By (23), individual rationality (deviation 1) for signal $\underline{s}$ implies that*

$$\frac{b_p - v_\ell}{v_h - b_p} \leq \frac{\rho}{1-\rho} \frac{\eta_h f_h(\underline{s})}{\eta_\ell f_\ell(\underline{s})} \frac{\pi_h^\circ}{\pi_\ell^\circ}. \tag{37}$$

Step 2. *There exists a function $B(\eta_h) < 1$ with $B(\eta_h) \to 1$ such that*

$$\frac{b_p - v_\ell}{v_h - b_p} \geq \frac{\rho}{1-\rho} \frac{\eta_h f_h(\breve{s})}{\eta_\ell f_\ell(\breve{s})} \frac{\pi_h^+}{\pi_\ell^+} B(\eta_h). \tag{38}$$

Signal $\breve{s}$ has an incentive to deviate from b_p to a marginally higher bid (deviation 2), unless $U(b_p|\breve{s}; \beta^*) \geq \lim_{\epsilon \searrow 0} U(b_p + \epsilon|\breve{s}; \beta^*)$, that is

$$\frac{\rho\eta_h f_h(\breve{s})\pi_h^\circ(v_h - b_p) + (1-\rho)\eta_\ell f_\ell(\breve{s})\pi_\ell^\circ(v_\ell - b_p)}{\rho\eta_h f_h(\breve{s}) + (1-\rho)\eta_\ell f_\ell(\breve{s})} \geq \frac{\rho\eta_h f_h(\breve{s})\pi_h^+(v_h - b_p) + (1-\rho)\eta_\ell f_\ell(\breve{s})\pi_\ell^+(v_\ell - b_p)}{\rho\eta_h f_h(\breve{s}) + (1-\rho)\eta_\ell f_\ell(\breve{s})}$$

Rearranging this inequality gives

$$\frac{b_p - v_\ell}{v_h - b_p} \geq \frac{\rho}{1-\rho} \frac{\eta_h f_h(\breve{s})}{\eta_\ell f_\ell(\breve{s})} \frac{\pi_h^+ - \pi_h^\circ}{\pi_\ell^+ - \pi_\ell^\circ}.$$

The last fraction can be replaced by

$$\frac{\pi_h^+ - \pi_h^\circ}{\pi_\ell^+ - \pi_\ell^\circ} = \frac{\pi_h^+ - \pi_h^\circ}{\pi_\ell^+ - \pi_\ell^\circ}\left(\frac{\pi_h^+}{\pi_\ell^+}\right)^{-1}\left(\frac{\pi_h^+}{\pi_\ell^+}\right) = \frac{1 - \frac{1-e^{-\eta_h F_h(\breve{s})}}{\eta_h F_h(\breve{s})}}{1 - \frac{1-e^{-\eta_\ell F_\ell(\breve{s})}}{\eta_\ell F_\ell(\breve{s})}}\left(\frac{\pi_h^+}{\pi_\ell^+}\right) = B(\eta_h)\frac{\pi_h^+}{\pi_\ell^+},$$

where $B(\eta_h) = \frac{1 - \frac{1-e^{-\eta_h F_h(\breve{s})}}{\eta_h F_h(\breve{s})}}{1 - \frac{1-e^{-\eta_\ell F_\ell(\breve{s})}}{\eta_\ell F_\ell(\breve{s})}} \to 1$, because $\eta_\omega F_\omega(\breve{s}) \to \infty$ for $\omega \in \{h, \ell\}$.

Step 3. *When η_h is sufficiently large, the lower bound (38) exceeds the upper bound (37). Thus, either deviation 1 or 2 is profitable.*

Combing inequalities (37) and (38) yields

$$\frac{\rho}{1-\rho} \frac{\eta_h f_h(\underline{s})}{\eta_\ell f_\ell(\underline{s})} \frac{\pi_h^\circ}{\pi_\ell^\circ} \geq \frac{\rho}{1-\rho} \frac{\eta_h f_h(\breve{s})}{\eta_\ell f_\ell(\breve{s})} \frac{\pi_h^+}{\pi_\ell^+} B(\eta_h).$$

Because $\frac{\eta_h f_h(\check{s})}{\eta_\ell f_\ell(\check{s})} = 1$, this rearranges to

$$\frac{\eta_h f_h(\underline{s})}{\eta_\ell f_\ell(\underline{s})} \geq \frac{\pi_h^+}{\pi_\ell^+}\Big(\frac{\pi_h^\circ}{\pi_\ell^\circ}\Big)^{-1} B(\eta_h). \tag{39}$$

Observe that it follows from $\eta_\omega F_\omega(\check{s}) \to \infty$ for $\omega \in \{h, \ell\}$ that

$$\frac{\pi_h^+}{\pi_\ell^+}\Big(\frac{\pi_h^\circ}{\pi_\ell^\circ}\Big)^{-1} = \frac{\pi_\ell^\circ}{\pi_\ell^+}\frac{\pi_h^+}{\pi_h^\circ} = \Big(\frac{1-e^{-\eta_\ell F_\ell(\check{s})}}{\eta_\ell F_\ell(\check{s})}\Big)\Big(\frac{1-e^{-\eta_h F_h(\check{s})}}{\eta_h F_h(\check{s})}\Big)^{-1} \to \frac{\eta_h F_h(\check{s})}{\eta_\ell F_\ell(\check{s})}.$$

Since $\frac{f_h(\underline{s})}{f_\ell(\underline{s})} < \frac{f_h(\bar{s})}{f_\ell(\bar{s})}$, the MLRP implies that $\frac{\eta_h f_h(\underline{s})}{\eta_\ell f_\ell(\underline{s})} < \frac{\eta_h f_h(\check{s})}{\eta_\ell f_\ell(\check{s})} = 1$. Because $\frac{\eta_h f_h(\underline{s})}{\eta_\ell f_\ell(\underline{s})} < 1$, and the densities are continuous $\eta_\ell F_\ell(\check{s}) = \int_{\underline{s}}^{\check{s}} \eta_\ell f_\ell(z)dz = \int_{\underline{s}}^{\check{s}} \eta_h f_h(z)\frac{\eta_\ell f_\ell(z)}{\eta_h f_h(z)}dz < \int_{\underline{s}}^{\check{s}} \eta_h f_h(z)\frac{\eta_\ell f_\ell(\underline{s})}{\eta_h f_h(\underline{s})}dz = \frac{\eta_\ell f_\ell(\underline{s})}{\eta_h f_h(\underline{s})}F_h(\check{s})$. Combined with $B(\eta_h) \to 1$, this means that the right side of condition (39) is larger than the left side when η_h is sufficiently large. Thus, when η_h is large, either deviation 1 or 2 is profitable and there can be no equilibrium β^* in which all signals below $\check{s}$ pool. $\square$

Lemma 6*. *Fix the ratio $\frac{\eta_h}{\eta_\ell} = l \in (\frac{f_\ell(\bar{s})}{f_h(\bar{s})}, \frac{f_\ell(\underline{s})}{f_h(\underline{s})})$ and any s_-, s_+ with $\underline{s} < s_- < s_+ \leq \check{s}$. When η_h is sufficiently large, there is no equilibrium β^* in which $\beta^*(s_-) = \beta^*(s_+)$, $\beta^*(s) < \beta^*(s_-)$ for all $s < s_-$ and $\beta^*(s) > \beta^*(s_+)$ for all $s > s_+$.*

Proof. Suppose to the contrary that when η_h is arbitrary large such an equilibrium β^* exists and denote $b_p = \beta^*(s_-) = \beta^*(s_+)$. We proceed in the same way as before and use the potential deviation 1 to derive an upper bound on b_p (Step 1) as well as deviation 2 to derive a lower bound on b_p (Step 2). Step 3 shows that the lower bound exceeds the upper bound when η_h is large, such that one of the deviations has to be profitable. As an abbreviation we use for either $\omega \in \{h, \ell\}$ that

$$\begin{aligned}
\pi_\omega^- &= \lim_{\epsilon \searrow 0} \pi_\omega(b_p - \epsilon; \beta^*) = e^{-\eta(1-F_\omega(s_-))} \\
\pi_\omega^\circ &= \pi_\omega(b_p; \beta^*) = \frac{e^{-\eta(1-F_\omega(s_+))} - e^{-\eta(1-F_\omega(s_-))}}{\eta(F_\omega(s_+) - F_\omega(s_-))}
\end{aligned}$$

Step 1. *By (23), individual rationality (deviation 1) for signal s_- implies that*

$$\frac{b_p - v_\ell}{v_h - b_p} \leq \frac{\rho}{1-\rho}\frac{\eta_h f_h(s_-)}{\eta_\ell f_\ell(s_-)}\frac{\pi_h^\circ}{\pi_\ell^\circ}. \tag{40}$$

Step 2. *There exists a function $E(\eta_h) > 1$ with $E(\eta_h) \to 1$ such that*

$$\frac{b_p - v_\ell}{v_h - b_p} \geq \frac{\rho}{1-\rho}\frac{\eta_h f_h(\underline{s})}{\eta_\ell f_\ell(\underline{s})}\frac{\pi_h^-}{\pi_\ell^-}E(\eta_h). \tag{41}$$

In equilibrium no signal $s < s_-$ has an incentive to deviate from $\beta^*(s)$ to any $b \in (\beta^*(s), b_p)$. In particular, there is no incentive to deviate to a bid marginally below b_p (deviation 2), that is, $U(\beta^*(s)|s; \beta^*) \geq \lim_{\epsilon \searrow 0} U(b_p - \epsilon|s; \beta^*)$. Using equation (25), this rearranges to

$$\frac{b_p - v_\ell}{v_h - b_p} \geq \frac{\rho}{1-\rho} \frac{\eta_h f_h(s)}{\eta_\ell f_\ell(s)} \frac{\pi_h^-}{\pi_\ell^-} \Big(1 - \frac{\pi_h(\beta^*(s); \beta^*)}{\pi_h^-} \frac{v_h - v_\ell}{v_h - b_p}\Big). \tag{42}$$

Observe that the right side of equation (42) is decreasing in $\pi_h(\beta^*(s); \beta^*)$. Thus, we can find the most conservative lower bound on b_p by bounding $\pi_h(\beta^*(s); \beta^*)$ from above.

Consider now $\underline{s}$ and note that monotonicity implies that

$$\pi_h(\beta^*(\underline{s}); \beta^*) \leq \frac{e^{-\eta_h(1-F_h(s_-))} - e^{-\eta_h(1-F_h(\underline{s}))}}{\eta_h[F_h(s_-) - F_h(\underline{s})]} =: \frac{e^{-\eta_h(1-F_h(s_-))} - e^{-\eta_h}}{\eta_h F_h(s_-)} \bar{\pi}_h.$$

Thus, if we plug $\underline{s}$ into equation (42), the lower bound becomes

$$\frac{b_p - v_\ell}{v_h - b_p} \geq \frac{\rho}{1-\rho} \frac{\eta_h f_h(\underline{s})}{\eta_\ell f_\ell(\underline{s})} \frac{\pi_h^-}{\pi_\ell^-} \Big(1 - \frac{\bar{\pi}_h}{\pi_h^-} \frac{v_h - v_\ell}{v_h - b_p}\Big).$$

By inspection, $\frac{\bar{\pi}_h}{\pi_h^-} \to 0$ such that $E(\eta_h) = 1 - \frac{\pi_h}{\pi_h^-} \frac{v_h - v_\ell}{v_h - b_p} \to 1$, unless $b_p \to v_h$. If $b_p \to v_h$, however, then (40) is always violated when η_h is sufficiently large, because $\frac{\pi_h^\circ}{\pi_\ell^\circ} \leq 1$. Thus, it is without loss to suppose that $E(\eta_h) \to 1$.

Step 3. *When η_h is sufficiently large, the lower bound (41) exceeds the upper bound (40). Thus, either deviation 1 or 2 is profitable.*

Combing inequalities (40) and (41) yields

$$\frac{\rho}{1-\rho} \frac{\eta_h f_h(s_-)}{\eta_\ell f_\ell(s_-)} \frac{\pi_h^\circ}{\pi_\ell^\circ} \geq \frac{\rho}{1-\rho} \frac{\eta_h f_h(\underline{s})}{\eta_\ell f_\ell(\underline{s})} \frac{\pi_h^-}{\pi_\ell^-} E(\eta_h).$$

This can be rearranged to

$$\frac{f_h(s_-)}{f_\ell(s_-)} \Big(\frac{f_h(\underline{s})}{f_\ell(\underline{s})}\Big)^{-1} \geq \frac{\pi_h^-}{\pi_\ell^-} \Big(\frac{\pi_h^\circ}{\pi_\ell^\circ}\Big)^{-1} E(\eta_h). \tag{43}$$

Note that the fractions $\frac{\pi_h^-}{\pi_\ell^-} \Big(\frac{\pi_h^\circ}{\pi_\ell^\circ}\Big)^{-1}$ can be expanded to

$$\begin{aligned}\frac{\pi_h^-}{\pi_\ell^-} \Big(\frac{\pi_h^\circ}{\pi_\ell^\circ}\Big)^{-1} &= \frac{\pi_\ell^\circ}{\pi_\ell^-} \frac{\pi_h^-}{\pi_h^\circ} = \frac{e^{\eta_\ell[F_\ell(s_+) - F_\ell(s_-)]} - 1}{\eta_\ell[F_\ell(s_+) - F_\ell(s_-)]} \Big(\frac{e^{\eta_h[F_h(s_+) - F_h(s_-)]} - 1}{\eta_h[F_h(s_+) - F_h(s_-)]}\Big)^{-1} \\ &= \frac{1 - e^{-\eta_\ell[F_\ell(s_+) - F_\ell(s_-)]}}{1 - e^{-\eta_h[F_h(s_+) - F_h(s_-)]}} e^{-\eta_h[F_h(s_+) - F_h(s_-)] + \eta_\ell[F_\ell(s_+) - F_\ell(s_-)]} \frac{\eta_\ell[F_\ell(s_+) - F_\ell(s_-)]}{\eta_h[F_h(s_+) - F_h(s_-)]}.\end{aligned}$$

Since $\eta_\omega[F_\omega(s_+) - F_\omega(s_-)] \to \infty$ in either state $\omega \in \{h, \ell\}$, the first fraction, $\frac{1 - e^{-\eta_\ell[F_\ell(s_+) - F_\ell(s_-)]}}{1 - e^{-\eta_h[F_h(s_+) - F_h(s_-)]}} \to 1$. Further, the last fraction $\frac{\eta_\ell[F_\ell(s_+) - F_\ell(s_-)]}{\eta_h[F_h(s_+) - F_h(s_-)]}$ is a constant. We now show that $-\eta_h[F_h(s_+) - F_h(s_-)] + \eta_\ell[F_\ell(s_+) - F_\ell(s_-)] \to \infty$, such that the expression whole expression grows without bound:

$$\eta_\ell[F_\ell(s_+) - F_\ell(s_-)] - \eta_h[F_h(s_+) - F_h(s_-)] = \int_{s_-}^{s_+} [1 - \frac{\eta_h f_h(z)}{\eta_\ell f_\ell(z)}]\eta_\ell f_\ell(z)dz$$

$$> \underbrace{\eta_\ell}_{\to\infty} \int_{s_-}^{s_+} \underbrace{[1 - \frac{\eta_h f_h(s_+)}{\eta_\ell f_\ell(s_+)}]}_{(1-l\frac{f_h(s_+)}{f_\ell(s_+)})>0,\text{ constant}} f_\ell(z)dz \to \infty.$$

Because $E(\eta_h) \to 1$, this means that the right side of equation (43) grows without bound, while the left side stays constant and bounded. Hence, when η_h is sufficiently large, (43) is violated which is a contradiction. □

Proposition 3*. *Holding $\frac{\eta_h}{\eta_\ell} = l < \frac{f_\ell(\underline{s})}{f_h(\underline{s})}$ fixed, when η_h is sufficiently large, no equilibrium exists.*

Proof. As in the case with $\eta_h = \eta_\ell$, the Proposition follows as a corollary to Proposition 4* □

Lemma 7. *Let σ^* be a concordant equilibrium of the communication extension. Then, there exists an m-equivalent, concordant equilibrium $\hat{\sigma}^*$ that is pure and where*

(i) bids b are nondecreasing in s;

(ii) for any given bid b, the report $m \in M$ is nondecreasing in s.

This implies that in both states $\omega \in \{h, \ell\}$

(a) $\pi^{\mathfrak{c}}_\omega(\hat{\sigma}^*(s); \hat{\sigma}^*)$ is nondecreasing in s;

(b) $\hat{\sigma}^*(s) = \hat{\sigma}^*(s')$ if and only if $\pi^{\mathfrak{c}}_\omega(\hat{\sigma}^*(s); \hat{\sigma}^*) = \pi^{\mathfrak{c}}_\omega(\hat{\sigma}^*(s'); \hat{\sigma}^*)$ for any $s, s' \in [\underline{s}, \bar{s}]$.

Proof. Since we only deal with concordant strategies, all signals report the same message space M. Further, reporting a different space is weakly dominated by reporting M, any $m \in M$ and bidding v_ℓ. To keep notation cleaner, we, hence, drop the explicit reference to M from all expressions.

Step 1. *Consider any concordant strategy σ and two actions (m, b) and (m', b') s.t. $\pi^{\mathfrak{c}}_h(m', b'; \sigma) > \pi^{\mathfrak{c}}_h(m, b; \sigma)$. If $U^{\mathfrak{c}}(m', b'|s; \sigma) \geq U^{\mathfrak{c}}(m, b|s; \sigma)$, then $U^{\mathfrak{c}}(m', b'|s'; \sigma) \geq U^{\mathfrak{c}}(m, b|s'; \sigma)$ for $s' > s$. The second inequality is strict if and only if $\frac{f_h(s')}{f_\ell(s')} > \frac{f_h(s)}{f_\ell(s)}$.*

Note that $\pi^{\mathfrak{c}}_h(m', b'; \sigma) > \pi^{\mathfrak{c}}_h(m, b; \sigma)$ implies that $\pi^{\mathfrak{c}}_\ell(m', b'; \sigma) > \pi^{\mathfrak{c}}_\ell(m, b; \sigma)$ since the winning probabilities are isomorphic across states.

From $\pi^{\mathfrak{c}}_h(m', b'; \sigma) > \pi^{\mathfrak{c}}_h(m, b; \sigma)$, it follows that $b' \geq b \geq v_\ell$ which implies that $(v_\ell - b') \leq (v_\ell - b) \leq 0$. If $b' = b$, then $\pi^{\mathfrak{c}}_h(m', b'; \sigma)(v_h - b') > \pi^{\mathfrak{c}}_h(m', b; \sigma)(v_h - b)$. If $b' > b$, on the other hand, $\pi^{\mathfrak{c}}_\ell(m', b'; \sigma) > \pi^{\mathfrak{c}}_\ell(m, b; \sigma)$ implies that $\pi^{\mathfrak{c}}_\ell(m', b'; \sigma)(v_\ell - b') < \pi^{\mathfrak{c}}_\ell(m, b; \sigma)(v_\ell - b)$. In this case, $U^{\mathfrak{c}}(m', b'|s; \sigma) \geq U^{\mathfrak{c}}(m, b|s; \sigma)$ also requires that $\pi^{\mathfrak{c}}_h(m', b'; \sigma)(v_h - b') > \pi^{\mathfrak{c}}_h(m, b; \sigma)(v_h - b)$.

Rearranging $U^{\mathfrak{c}}(m',b'|s;\sigma) \geq U^{\mathfrak{c}}(m,b|s;\sigma)$ yields

$$\frac{\rho\eta_h f_h(s)}{(1-\rho)\eta_\ell f_\ell(s)}[\pi_h^{\mathfrak{c}}(m',b';\sigma)(v_h-b') - \pi_h^{\mathfrak{c}}(m,b;\sigma)(v_h-b)]$$
$$\geq \pi_\ell^{\mathfrak{c}}(m,b;\sigma)(v_\ell-b) - \pi_\ell^{\mathfrak{c}}(m',b';\sigma)(v_\ell-b').$$

If $s' > s$ is such that $\frac{f_h(s')}{f_\ell(s')} > \frac{f_h(s)}{f_\ell(s)}$, the left side is strictly larger for s' and thus $U^{\mathfrak{c}}(m',b'|s;\sigma) > U^{\mathfrak{c}}(m,b|s;\sigma)$.

Step 2. *The set of interim beliefs that imply indifference between two actions which win with different probabilities,* $L = \{\frac{f_h(s)}{f_\ell(s)} : \exists (m,b),(m',b')$ *with* $\pi_h^{\mathfrak{c}}(m,b;\hat{\sigma}^*) \neq \pi_h(m',b';\hat{\sigma}^*)$ *and* $U^{\mathfrak{c}}(m,b|s;\sigma^*) = U^{\mathfrak{c}}(m',b'|s;\sigma^*)\}$, *is countable.*

By construction, $\forall l \in L$ there exist two tuples (m_-^l, b_-^l), (m_+^l, b_+^l) with $\pi_h^{\mathfrak{c}}(m_-^l, b_-^l;\hat{\sigma}^*) < \pi_h^{\mathfrak{c}}(m_+^l, b_+^l;\hat{\sigma}^*)$ such that signals $s^l : \frac{f_h(s)}{f_\ell(s)} = l$ are indifferent between these two bids, $U^{\mathfrak{c}}(m_-^l, b_-^l|s^l;\sigma^*) = U^{\mathfrak{c}}(m_+^l, b_+^l|s^l;\sigma^*)$. Furthermore, there exists a $q^l \in \mathbb{Q}$ s.t. $\pi_h^{\mathfrak{c}}(m_-^l, b_-^l;\hat{\sigma}^*) < q^l < \pi_h^{\mathfrak{c}}(m_+^l, b_+^l;\hat{\sigma}^*)$. By Step 1, $\pi_h^{\mathfrak{c}}(m_+^l, b_+^l;\hat{\sigma}^*) \leq \pi_h^{\mathfrak{c}}(m_-^{l'}, b_-^{l'};\hat{\sigma}^*)$ for all $l < l'$, which implies that $q^l < q^{l'}$. Because $\mathbb{Q}$ is countable, so is L.

Step 3. *Let σ^* be a concordant equilibrium. Then, there exists a m-equivalent, concordant equilibrium $\hat{\sigma}^*$ with the following property: If (m,b) and (m',b') are from in the support of $\hat{\sigma}^*$, and $\pi_h^{\mathfrak{c}}(m,b;\hat{\sigma}^*) = \pi_h^{\mathfrak{c}}(m',b';\hat{\sigma}^*)$, then $(m,b) = (m',b')$.*

If (m,b) and (m',b') are in the support of σ^*, and $\pi_h^{\mathfrak{c}}(m,b;\sigma^*) = \pi_h^{\mathfrak{c}}(m',b';\sigma^*)$ (and thereby $\pi_\ell^{\mathfrak{c}}(m,b;\sigma) = \pi_\ell^{\mathfrak{c}}(m',b';\sigma)$), then $b = b'$. Otherwise, the action-tuple with the higher bid is dominated and, hence, cannot be part of a best response.

If (m,b) and (m',b) are in the support of σ^*, and $\pi_h^{\mathfrak{c}}(m,b;\sigma^*) = \pi_h^{\mathfrak{c}}(m',b';\sigma^*)$, then the report m is, conditional on b, irrelevant. Thus, any (m',b) can be replaced by (m,b) without altering winning probabilities or payoffs, receiving a m-equivalent equilibrium $\hat{\sigma}^*$ which has the asserted properties.

Using Step 3, in equilibrium, any winning probability can be identified with a unique message/bid combination (m,b). By Step 1, bidders with higher beliefs choose actions tuples which win with higher probabilities and by Step 2 there are at most countably many beliefs which are indifferent between multiple action-tuples. We, thus, can proceed as in Lemma 1 and reorder the actions in such a way, that the strategies are pure and the probability to win is nondecreasing in s. In the resulting strategy $\hat{\sigma}^*$ bids are nondecreasing in the signal, and, given a bid, the reports are nondecreasing in the signal. $\square$

Lemma 8. *The communication extension $\Gamma^{\mathfrak{c}}$ always has a concordant equilibrium.*

Proof. Take any sequence of games on an ever-finer grid $(\Gamma(k))_{k\in\mathbb{N}}$. By Lemma 9, for any grid size k, a pure, nondecreasing equilibrium exists. By Lemma 10, there is a concordant equilibrium of the communication extension. $\square$

Proposition 4*. *Fix any* $\frac{\eta_h}{\eta_\ell} = l \in (\frac{f_\ell(\bar{s})}{f_h(\bar{s})}, \frac{f_\ell(\underline{s})}{f_h(\underline{s})})$ *and* $\epsilon \in (0, \frac{\check{s}-\underline{s}}{2})$*. When* η_h *is sufficiently large (given* ϵ*), any concordant equilibrium* σ^* *of* $\Gamma^{\mathfrak{c}}$ *takes the following form: There are two disjoint, adjacent intervals of signals* I, J *such that*

(i) $[\underline{s} + \epsilon, \check{s} - \epsilon] \subset I \cup J$*;*

(ii) $\sigma^*(s_I) = (M, m_I, b_p)$ *for all* $s_I \in I$ *and* $\sigma^*(s_J) = (M, m_J, b_p)$ *for all* $s_J \in J$*, with* $m_I < m_J$*;*

(iii) there is no $m \in M$ *s.t.* $\pi^{\mathfrak{c}}_\omega(\sigma^*(s_I); \sigma^*) < \pi^{\mathfrak{c}}_\omega(M, m, b_p; \sigma^*) < \pi^{\mathfrak{c}}_\omega(\sigma^*(s_J); \sigma^*)$ *for* $\omega \in \{h, \ell\}$*;*

(iv) $\int_I \eta f_\omega(z)dz > \frac{1}{\epsilon}$*, and* $\int_J \eta f_\omega(z)dz > \frac{1}{\epsilon}$ *for* $\omega \in \{h, \ell\}$*;*

(v) on $s \in (\check{s} + \epsilon, \bar{s}]$*, the bids are strictly increasing such that the message* m *is irrelevant.*

Proof. Consider a sequence of communication extensions $(\Gamma^{\mathfrak{c}}_n)_{n\in\mathbb{N}}$, where $\frac{\eta^n_h}{\eta^n_\ell} = l \in (\frac{f_\ell(\bar{s})}{f_h(\bar{s})}, \frac{f_\ell(\underline{s})}{f_h(\underline{s})})$ and $\eta^n_h, \eta^n_\ell \to \infty$. By Lemma 8, there exists an equilibrium for each n denoted (economizing on the $*$) σ_n. For any σ_n, we adopt the notation that $\sigma_n(s) = (M_n, \mu_n(s), \beta_n(s))$ for some M_n and functions $\mu_n : [\underline{s}, \bar{s}] \to M_n$ and $\beta_n : [\underline{s}, \bar{s}] \to [v_\ell, v_h]$. Since we look at concordant equilibria, we drop the explicit reference M_n, unless its central to the argument.

Step 1. *Fix two signals* $s_- < s_+$ *with* $s_+ > \check{s}$*. If* $\beta_n(s_-) = \beta_n(s_+)$ *on exactly* $[s_-, s_+]$*, then,* $s_- < \check{s}$*. Further, there is a signal* $s_\circ$ *with* $s_- \leq s_\circ < \check{s}$ *such that* $\sigma_n(s_\circ) = \sigma_n(s_+)$*.*

First, suppose that $\mu_n(s)$ is strictly increasing on some sub-interval $[s'_-, s'_+] \subseteq [s_-, s_+]$. Then, any $\hat{s} \in [\hat{s}'_-, \hat{s}'_+]$ wins whenever $s_{(1)} \leq \hat{s}$, that is with probability $e^{-\eta^n_\omega(1-F_\omega(\hat{s}))}$. Since the conditional expected value $\mathbb{E}[v|s_{(1)} \leq s, s]$ is strictly increasing above $\check{s}$, it has to be that $s'_+ \leq \check{s}$. Otherwise, any signal $s' \in [\check{s}, s'_+)$ would have a strict incentive to mimic s'_+, winning more often and receiving a higher expected value.[34]

Since $\mu_n(s)$ cannot be strictly increasing above $\check{s}$, there has to be a signal $s_\circ < s_+$, such that $\mu_n(s) = \mu_n(s_+)$ for all $s \in [s_\circ, s_+]$, meaning that $\sigma_n(s) = \sigma_n(s_+)$ for all $s \in [s_\circ, s_+]$.

Now suppose that $s_\circ \geq \check{s}$ such that $\eta^n_h[F_h(s_+) - F_h(s_\circ)] < \eta^n_\ell[F_\ell(s_+) - F_\ell(s_\circ)]$. By inspection, the argument in the proof of Step 2 of Lemma 4* is also valid in the communication extension. Thus, there is a profitable deviation for all $s \in [s_\circ, s_+]$. By choosing bid marginally above $\beta_n(s) = \beta_n(s_+)$ and an arbitrary report $m \in M_n$, signal s wins more often and receives a good of a higher expected value.

Step 2. *Fix any* $\epsilon > 0$*. If* n *is sufficiently large,* $\beta_n(s)$ *is strictly increasing on* $(\check{s} + \epsilon, \bar{s}]$ *and* $\mu_n(s)$ *is irrelevant on that interval.*

Suppose to the contrary that there is a sequence of equilibria $(\sigma_n)_{n\in\mathbb{N}}$ for which the claim is violated: For any n, there is an interval of signals $[s^n_-, s^n_+]$ with $s^n_+ > \check{s} + \epsilon$ along which the bid is constant and equal to b_n. By Step 0, there is a signal $s^n_\circ < \check{s}$ such that $\sigma_n(s^n_\circ) = \sigma_n(s^n_+) =: (m_n, b_n)$. Without loss, let $s^n_\circ = s^n_-$. Note that by construction, $\beta_n(s) > b_n$ for all $s > s^+_n$.

[34] Individual rationality would imply that $\beta_n(s_-) = \beta_n(s') \leq \mathbb{E}[s_{(1)} \leq s', s'] < \mathbb{E}[v|s_{(1)} \leq s'_+, s']$.

The proof now follows by contradiction, which is structured into three parts: First, Substep 1 derives an upper bound on b_n, and Substep 2 a lower bound on b_n. Then, Substep 3 shows that n sufficiently large the lower bound exceeds the upper bound which completes the proof.

Note that because $\beta_n(s) > b_n$ for all $s > s_n^+$, a bid marginally above b_n wins the auction whenever $s_{(1)} \leq s_+^n$, independent of the signal $m \in M_n$. To simplify notation, we abbreviate the implied winning probabilities from bidding (m_n, b_n) and bidding marginally more in state $\omega \in \{h, \ell\}$ by

$$\begin{aligned}\pi_\omega^n &= \pi_\omega^{\mathfrak{c}}(m_n, b_n; \sigma_n) = \frac{e^{-\eta_\omega^n(1-F_\omega(s_+^n))} - e^{-\eta_\omega^n(1-F_\omega(s_-^n))}}{\eta_\omega^n(F_\omega(s_+^n) - F_\omega(s_-^n))},\\ \pi_\omega^{+,n} &= \lim_{\epsilon\to 0} \pi_\omega^{\mathfrak{c}}(m_n, b_n + \epsilon; \sigma_n) = e^{-\eta_\omega^n(1-F_\omega(s_+^n))}.\end{aligned}$$

Substep 1. *An upper bound on b_n is given by*

$$\frac{b_n - v_\ell}{v_h - b_n} \leq \frac{\rho\eta_h^n f_h(s_-^n)}{(1-\rho)\eta_\ell^n f_\ell(s_-^n)} \frac{\pi_h^n}{\pi_\ell^n}. \tag{44}$$

The individual rationality argument (23) remains unaltered in the communication extension. Since $\beta_n(s_-^n) = b_n$, this means that $b_n \leq \mathbb{E}[v|\text{win with } (m_n, b_n), s_-^n; \sigma_n]$ which rearranges to equation (44).

Substep 2. *A lower bound on b_n is given by*

$$\frac{b_n - v_\ell}{v_h - b_n} \geq \frac{\rho}{1-\rho} \frac{\eta_h^n f_h(s_+^n)}{\eta_\ell^n f_\ell(s_+^n)} \frac{\pi_h^{+,n} - \pi_h^n}{\pi_\ell^{+,n} - \pi_\ell^n}. \tag{45}$$

Since σ_n is an equilibrium, there can not be a profitable deviation. In particular, $U^{\mathfrak{c}}(m_n, b_n|s_+^n; \sigma_n) \geq \lim_{\epsilon\to 0} U^{\mathfrak{c}}(m_n, b_n + \epsilon|s_+^n; \sigma_n)$, that is,

$$\begin{aligned}&\frac{\rho\eta_h^n f_h(s_+^n)\pi_h^n(v_h - b_n) + (1-\rho)\eta_\ell^n f_\ell(s_+^n)\pi_\ell^n(v_\ell - b_n)}{\rho\eta_h^n f_h(s_+^n) + (1-\rho)\eta_\ell^n f_\ell(s_+^n)}\\ &\qquad \geq \frac{\rho\eta_h^n f_h(s_+^n)\pi_h^{+,n}(v_h - b_n) + (1-\rho)\eta_\ell^n f_\ell(s_+^n)\pi_\ell^{+,n}(v_\ell - b_n)}{\rho\eta_h^n f_h(s_+^n) + (1-\rho)\eta_\ell^n f_\ell(s_+^n)},\end{aligned}$$

which rearranges to (45).

Substep 3. *When n is sufficiently large, the upper bound on b_n expressed by (44) is smaller than the lower bound on b_n given by inequality (45).*

Combining inequalities (44) and (45) yields

$$\frac{\rho\eta_h^n f_h(s_-^n)}{(1-\rho)\eta_\ell^n f_\ell(s_-^n)} \frac{\pi_h^n}{\pi_\ell^n} \geq \frac{\rho}{1-\rho} \frac{\eta_h^n f_h(s_+^n)}{\eta_\ell^n f_\ell(s_+^n)} \frac{\pi_h^{+,n} - \pi_h^n}{\pi_\ell^{+,n} - \pi_\ell^n},$$

which rearranges to

$$\frac{\eta_h^n f_h(s_-^n)}{\eta_\ell^n f_\ell(s_-^n)} \geq \frac{\eta_h^n f_h(s_+^n)}{\eta_\ell^n f_\ell(s_+^n)} \frac{\frac{\pi_h^{+,n}}{\pi_h^n} - 1}{\frac{\pi_\ell^{+,n}}{\pi_\ell^n} - 1}.$$

Note that because $s^n_+ > \check{s}$ it follows that $\frac{\eta^n_h f_h(s^n_+)}{\eta^n_\ell f_\ell(s^n_+)} > 1$, such that is has to hold that

$$\frac{\eta^n_h f_h(s^n_-)}{\eta^n_\ell f_\ell(s^n_-)} > \frac{\frac{\pi^{+,n}_h}{\pi^n_h} - 1}{\frac{\pi^{+,n}_\ell}{\pi^n_\ell} - 1}. \tag{46}$$

Since $s^n_- < \check{s} < \check{s} + \epsilon \leq s^n_+$, it has to be true that $\eta_\omega[F_\omega(s^n_+) - F_\omega(s^n_-)] \to \infty$ for $\omega \in \{h, \ell\}$. Thus, it follows from

$$\frac{\pi^{+,n}_\omega}{\pi^n_\omega} = \frac{\eta_\omega[F_\omega(s^n_+) - F_\omega(s^n_-)]}{1 - e^{-\eta^n_\omega}[F_\omega(s^n_+) - F_\omega(s^n_-)]} \quad \text{that} \quad \lim_{n\to\infty} \frac{\frac{\pi^{+,n}_h}{\pi^n_h} - 1}{\frac{\pi^{+,n}_\ell}{\pi^n_\ell} - 1} = \lim_{n\to\infty} \frac{\eta^n_h[F_h(s^n_+) - F_h(s^n_-)]}{\eta^n_\ell[F_\ell(s^n_+) - F_\ell(s^n_-)]}.$$

Further, the MLRP implies that

$$\begin{aligned} \eta^n_h[F_h(s^n_+) - F_h(s^n_-)] &= \eta^n_h \int_{s^n_-}^{s^n_+} f_h(z)dz = \int_{s^n_-}^{s^n_+} \eta^n_\ell f_\ell(z) \frac{\eta^n_h f_h(z)}{\eta^n_\ell f_\ell(z)} dz \\ &\geq \int_{s^n_-}^{s^n_+} \eta^n_\ell f_\ell(z) \frac{\eta^n_h f_h(s^n_-)}{\eta^n_\ell f_\ell(s^n_-)} dz = \frac{\eta^n_h f_h(s^n_-)}{\eta^n_\ell f_\ell(s^n_-)} \eta^n_\ell[F_\ell(s^n_+) - F_\ell(s^n_-)], \end{aligned}$$

which rearranges to $\frac{\eta^n_h[F_h(s^n_+) - F_h(s^n_-)]}{\eta^n_\ell[F_\ell(s^n_+) - F_\ell(s^n_-)]} \geq \frac{\eta^n_h f_h(s^n_-)}{\eta^n_h f_h(s^n_-)}$. Thus equation (46) is necessarily violated when n is sufficiently large.

Step 3. *Fix any $\epsilon \in (0, \frac{\check{s} - \underline{s}}{2})$. For every n sufficiently large, $\nexists(M, m, b)$ s.t. $\pi^{\mathfrak{c}}_\omega(\sigma_n(\underline{s} + \epsilon); \sigma_n) < \pi^{\mathfrak{c}}_\omega(M, m, b; \sigma_n) < \pi^{\mathfrak{c}}_\omega(\sigma_n(\check{s} - \epsilon); \sigma_n)$ for $\omega \in \{h, \ell\}$. As a result, $\beta_n(s)$ is constant on $[\underline{s} + \epsilon, \check{s} - \epsilon]$.*

Since the winning probabilities are isomorphic across states, if the claim is violated in one state, it is also violated in the other. Suppose to the contrary, that there exists an $\epsilon \in (0, \frac{\check{s} - \underline{s}}{2})$ and a subsequence of equilibria $(\sigma_n)_{n\in\mathbb{N}}$ for which such a deviation denoted by (M'_n, m'_n, b'_n) exists. It follows immediately that $M'_n = m_n$. Otherwise, the deviation only wins when the bidder is alone. Henceforth, let $M_n = M'_n$ and ignore it.

The rest of the for this step is structured into three steps which yield a contradiction. First, Substep 1 derives an upper bound on b'_n, and Substep 2 a lower bound. Then, Substep 3 shows that for large n, the lower exceeds the upper bound, which yields the contradiction. To simplify notation, define signals $s^n_- = \sup\{s \colon \sigma_n(s) = \sigma_n(\underline{s} + \epsilon)\}$ and $s^n_+ = \inf\{s \colon \sigma_n(s) = \sigma_n(\check{s} - \epsilon)\}$ and fix some signal $s_{++} > \check{s}$, such that $\frac{\eta^n_h[F_h(s_{++}) - F_h(\check{s} - \epsilon)]}{\eta^n_\ell[F_\ell(s_{++}) - F_\ell(\check{s} - \epsilon)]} < 1$.

Substep 1. *An upper bound on b'_n is given by*

$$\frac{b'_n - v_\ell}{v_h - b'_n} \leq \frac{\rho\eta^n_h f_h(s_{++})}{(1 - \rho)\eta^n_\ell f_\ell(s_{++})} \frac{e^{-\eta^n_h(1 - F_h(s_{++}))}}{e^{-\eta^n_\ell(1 - F_\ell(s_{++})}}. \tag{47}$$

By Step 2, $s_{++} > \check{s}$ does not pool when n is sufficiently large, and, hence, wins whenever $s_{(1)} \leq s_{++}$, such that $\pi^{\mathfrak{c}}_\omega(\sigma_n(s_{++}); \sigma_n) = e^{-\eta^n_\omega(1 - F_\omega(s_{++}))}$ for $\omega \in \{h, \ell\}$. Since the individual rationality argument for equation (23) remains unaltered in the communication extension, an upper bound on $\beta_n(s_{++})$ is given by

$$\frac{\beta_n(s_{++}) - v_\ell}{v_h - \beta_n(s_{++})} \leq \frac{\rho \eta_h^n f_h(s_{++})}{(1-\rho)\eta_\ell^n f_\ell(s_{++})} \frac{e^{-\eta_h^n(1-F_h(s_{++}))}}{e^{-\eta_\ell^n(1-F_\ell(s_{++})}}.$$

Because the left side of the inequality is increasing in b_n and $\beta_n(s_{++}) \geq b'_n$, the upper bound (47) follows.

Substep 2. *A lower bound on b'_n is given by*

$$\frac{b'_n - v_\ell}{v_h - b'_n} \geq \frac{\rho}{1-\rho} \frac{\eta_h^n f_h(\underline{s})}{\eta_\ell^n f_\ell(\underline{s})} \frac{\pi_h^{\mathfrak{c}}(m'_n, b'_n; \sigma_n)}{\pi_\ell^{\mathfrak{c}}(m'_n, b'_n; \sigma_n)} Q(n), \tag{48}$$

where $Q(n)$ is an increasing function with $\lim_{n\to\infty} Q(n) = 1$.

First, we want to find a lower bound on $\pi_\omega^{\mathfrak{c}}(\sigma_n(\underline{s}); \sigma_n)$. By monotonicity, this probability is maximal if $\sigma_n(\underline{s}) = \sigma_n(\underline{s} + \epsilon)$. Further, if $\pi_\omega^{\mathfrak{c}}(\sigma_n(\underline{s}+\epsilon); \sigma_n) = \pi_\omega^{\mathfrak{c}}(\sigma_n(\underline{s}); \sigma_n)$, then it attains the highest value in either state $\omega \in \{h, \ell\}$ if all signals up to s_-^n pool on the same action-tuple, that is if $\sigma_n(\underline{s}+\epsilon) = \sigma_n(s)$ for $s < s_-^n$. As a result,

$$\pi_\omega^{\mathfrak{c}}(\sigma_n(\underline{s}); \sigma_n) \leq \frac{e^{-\eta_\omega^n(1-F_\omega(s_-^n))} - e^{-\eta_\omega}}{\eta_\omega^n F_\omega(s_-^n)} =: \pi_\omega^{-,n}.$$

Given this lower bound and because $\beta(\underline{s}) \geq v_\ell$, we can bound

$$U^{\mathfrak{c}}(\sigma_n(\underline{s})|\underline{s}; \sigma_n) \leq \frac{\rho\eta_h^n f_h(\underline{s})\pi_h^{-,n}(v_h - v_\ell) + (1-\rho)\eta_\ell^n f_\ell(\underline{s})\pi_\ell^{-,n}(v_\ell - v_\ell)}{\rho\eta_h^n f_h(\underline{s}) + (1-\rho)\eta_\ell^n f_\ell(\underline{s})}. \tag{49}$$

In equilibrium, it has to hold that $U^{\mathfrak{c}}(\sigma_n(\underline{s})|\underline{s}; \sigma_n) \geq U^{\mathfrak{c}}(m_n, b'_n|\underline{s}; \sigma_n)$. Using the lower bound on (49), this rearranges to

$$\frac{b'_n - v_\ell}{v_h - b'_n} \geq \frac{\rho}{1-\rho} \frac{\eta_h^n f_h(\underline{s})}{\eta_\ell^n f_\ell(\underline{s})} \frac{\pi_h^{\mathfrak{c}}(m'_n, b'_n; \sigma_n)}{\pi_\ell^{\mathfrak{c}}(m'_n, b'_n; \sigma_n)} \Big(1 - \frac{\pi_h^{-,n}}{\pi_h^{\mathfrak{c}}(m'_n, b'_n; \sigma_n)} \frac{v_h - v_\ell}{v_h - b'_n}\Big).$$

Note that (m'_n, b'_n) wins at least whenever $s_{(1)} \leq s_-^n$, such that $\pi_h^{\mathfrak{c}}(m'_n, b'_n; \sigma_n) \geq e^{-\eta_h(1-F_h(s_-^n))}$. Thus $\pi_h^{-,n}/\pi_h^{o,n} \to 0$ and $1 - \frac{\pi_h^{-,n}}{\pi_h^{\mathfrak{c}}(m'_n, b'_n; \sigma_n)} \frac{v_h - v_\ell}{v_h - b'_n} \to 1$, unless $b'_n \to v_h$. In this case, however, inequality (47) can never hold for infinitely many n, because the left side grows without bound while $e^{-\eta_h^n(1-F_h(s_{++}))}/e^{-\eta_\ell^n(1-F_\ell(s_{++}))} \leq 1$ such that the right side stays bounded. Thus, it is without loss to assume that $Q(n) = 1 - \frac{\pi_h^{-,n}}{\pi_h^{\mathfrak{c}}(m'_n, b'_n; \sigma_n)} \frac{v_h - v_\ell}{v_h - b'_n} \to 1$ which proves the step.

Substep 3. *When n is sufficiently large, the upper bound on b'_n expressed by inequality (47) is smaller than the lower bound on b'_n given by inequality (48).*

Combining inequalities (47) and (48) yields

$$\frac{\rho\eta_h^n f_h(s_{++})}{(1-\rho)\eta_\ell^n f_\ell(s_{++})} \frac{e^{-\eta_h^n(1-F_h(s_{++}))}}{e^{-\eta_\ell^n(1-F_\ell(s_{++})}} \geq \frac{\rho}{1-\rho} \frac{\eta_h^n f_h(\underline{s})}{\eta_\ell^n f_\ell(\underline{s})} \frac{\pi_h^{\mathfrak{c}}(m'_n, b'_n; \sigma_n)}{\pi_\ell^{\mathfrak{c}}(m'_n, b'_n; \sigma_n)} Q(n),$$

which rearranges to

$$\frac{f_h(s_{++})}{f_\ell(s_{++})}\frac{f_\ell(\underline{s})}{f_h(\underline{s})} \geq \Big(\frac{e^{-\eta_h^n(1-F_h(s_{++}))}}{e^{-\eta_\ell^n(1-F_\ell(s_{++})}}\Big)^{-1}\frac{\pi_h^{\mathfrak{c}}(m_n,b_n;\sigma_n)}{\pi_\ell^{\mathfrak{c}}(m_n,b_n;\sigma_n)}Q(n). \tag{50}$$

Observe that because $\pi_\omega^{\mathfrak{c}}(m_n,b_n;\sigma_n) < e^{-\eta_\omega^n(1-F_\omega(s_+^n))}$ for $\omega \in \{h,\ell\}$, inequality (36) and $s_+^n < \check{s}$ imply that

$$\frac{\pi_h^{\mathfrak{c}}(m_n,b_n;\sigma_n)}{\pi_\ell^{\mathfrak{c}}(m_n,b_n;\sigma_n)} \geq \frac{e^{-\eta_h^n(1-F_h(s_+^n))}}{e^{-\eta_\ell^n(1-F_\ell(s_+^n))}} \geq \frac{e^{-\eta_h^n(1-F_h(\check{s}-\epsilon))}}{e^{-\eta_\ell^n(1-F_\ell(\check{s}-\epsilon))}}.$$

Thus, a necessary condition for inequality (50) is that

$$\frac{f_h(s_{++})}{f_\ell(s_{++})}\frac{f_\ell(\underline{s})}{f_h(\underline{s})} \geq \Big(\frac{e^{-\eta_h^n(1-F_h(s_{++}))}}{e^{-\eta_\ell^n(1-F_\ell(s_{++})}}\Big)^{-1}\frac{e^{-\eta_h^n(1-F_h(\check{s}-\epsilon))}}{e^{-\eta_\ell^n(1-F_\ell(\check{s}-\epsilon))}}Q(n).$$

However, the definition s_{++} ensures that

$$\begin{aligned}&\eta_\ell^n[F_\ell(s_{++})-F_\ell(\check{s}-\epsilon)]-\eta_h^n[F_h(s_{++})-F_h(\check{s}-\epsilon)]\\ =&\eta_\ell^n[F_\ell(s_{++})-F_\ell(\check{s}-\epsilon)][1-\frac{\eta_h^n[F_h(s_{++})-F_h(\check{s}-\epsilon)]}{\eta_\ell^n[F_\ell(s_{++})-F_\ell(\check{s}-\epsilon)]}]\to\infty,\end{aligned}$$

such that the right side grows without bound, while the left stays constant. Thus, inequality (50) is violated when n is sufficiently large, which proves the claim.

Step 4. *Fix any $\epsilon \in (0,\frac{\check{s}-\underline{s}}{2})$. For every n sufficiently large, there are two disjoint, adjacent intervals I_n and J_n with $[\underline{s}+\epsilon,\check{s}-\epsilon] \subset I_n \cup J_n$. Signals $s_I \in I_n$ choose $\sigma_n(s_I) = (M_n, m_I^n, b_n)$ and signals $s_J \in J_n$ choose $\sigma_n(s_J) = (M_n, m_J^n, b_n)$. There is no $m_n \in M_n$ s.t. $m_I^n < m_n < m_J^n$, which implies that $\nexists(M,m,b)$ s.t. $\pi_\omega^{\mathfrak{c}}(\sigma_n(s_I);\sigma_n) < \pi_\omega^{\mathfrak{c}}(M,m,b;\sigma_n) < \pi_\omega^{\mathfrak{c}}(\sigma_n(s_J);\sigma_n)$ for $\omega \in \{h,\ell\}$. Last, the expected number of bidders in both intervals is larger than $\frac{1}{\epsilon}$.*

Fix any $\epsilon > 0$ sufficiently small, such that $\frac{f_h(\underline{s}+\epsilon)}{f_\ell(\underline{s}+\epsilon)}\frac{F_\ell(\check{s}-\epsilon)}{F_h(\check{s}-\epsilon)} < \frac{\eta_h f_h(\check{s}-\epsilon)}{\eta_\ell f_\ell(\check{s}-\epsilon)}$. Notice that such an ϵ exists, because $\frac{f_h(\underline{s})}{f_\ell(\underline{s})}\frac{F_\ell(\check{s})}{F_h(\check{s})} < 1 = \frac{\eta_h^n f_h(\check{s})}{\eta_\ell^n f_\ell(\check{s})}$ and the expressions are continuous in its arguments.

Define $(m_n^I, b_n) = \sigma_n(\underline{s}+\epsilon)$ and let $I_n = \{s\colon (\mu_n(s),\beta_n(s)) = (m_n^I,b_n)\}$ be the interval of signals which choose the same action-tuple as $\underline{s}+\epsilon$. Further, let $m_n^J = \inf\{m \in M_n : m > m_n^I\}$ be the next higher report from M_n and $J_n = \{s\colon (\mu_n(s),\beta_n(s)) = (m_n^J,b_n)\}$ be the interval of signals which choose the same bid as $\underline{s}+\epsilon$, but this higher report.[35]

By Step 3 $[\underline{s}+\epsilon,\check{s}-\epsilon] \subset I_n \cup J_n$ and, apart from the last, all the other properties follow by construction. Thus, we only need to check that $\int_{I_n}\eta_\omega^n f_\omega(s)ds, \int_{J_n}\eta_\omega^n f_\omega(s)ds \to \infty$ for $\omega \in \{h,\ell\}$. Observe that it suffices to show the convergence in state h.[36]

[35] It might be the case that there is no $m \in M_n : m > m_n^I$. In this case, all $s > \sup I_n$ choose a bid $\beta_n(s) > b_n$, such that choosing a $m > m_n^I$ is equivalent to choosing a marginally higher bid. Thus, it is without loss to assume that the report exists and, if necessary, approximate the action-tuple by choosing a marginally higher bid.

[36] By MLRP $\frac{\int_I \eta_h f_h(s)ds}{\int_I \eta_\ell f_\ell(s)ds} \in [\frac{\eta_h f_h(\underline{s})}{\eta_\ell f_\ell(\underline{s})}, \frac{\eta_h f_h(\bar{s})}{\eta_\ell f_\ell(\bar{s})}]$ for any interval I. Since the likelihood ratios are

First, consider interval I_n with bounds denoted $s_-^{I,n} = \inf I_n$ and $s_+^{I,n} = \sup I_n$ and suppose to the contrary that $\eta_h^n(F_h(s_+^{I,n}) - F_h(s_+^{I,n})) \not\to \infty$. Then, there exists a subsequence along which $s_+^{I,n} - s_-^{I,n} \to 0$, which, by construction, means that $s_+^{I,n}, s_-^{I,n} \to \underline{s} + \epsilon$. This implies, however, that when n is sufficiently large $\pi_h^{\mathfrak{c}}(\sigma_n(\underline{s} + \frac{\epsilon}{2}); \sigma_n) < \pi_h^{\mathfrak{c}}(\sigma_n(\underline{s} + \epsilon); \sigma_n) < \pi_h^{\mathfrak{c}}(\sigma_n(\breve{s} - \frac{\epsilon}{2}); \sigma_n)$ which is a contradiction to Step 3. Thus, $\int_{I_n} \eta_h^n f_h(s) ds \to \infty$.

Second, consider interval J_n with bounds denoted $s_-^{J,n} = \inf J_n = \sup I_n$ and $s_+^{J,n} = \sup J_n$.[37] and suppose to the contrary that $\eta_h^n(F_h(s_+^{J,n}) - F_h(s_+^{J,n})) \not\to \infty$. In this case, there is a subsequence along which, $s_-^{J,n}, s_+^{J,n}$ converge to some common limit s^J. Without loss, let the original sequence be this subsequence. Notice that it cannot be that $s^J < \breve{s} - \epsilon$. Otherwise, $\pi_h^{\mathfrak{c}}(\sigma_n(\underline{s} + \epsilon); \sigma_n) < \pi_h^{\mathfrak{c}}(m_n^J, b_n; \sigma_n) < \pi_h^{\mathfrak{c}}(\sigma_n(\breve{s} - \epsilon); \sigma_n)$, which is a contradiction to Step 3. Since the same is true for any $\epsilon' < \epsilon$ and $s^J < \breve{s} - \epsilon'$, it follows that $s^J \geq \breve{s}$. In the following, we only concentrate on this remaining case.

We, hence, suppose that $s_-^{J,n}, s_+^{J,n}$ converge to some $s^J \geq \breve{s}$, such that $\eta_h^n(F_h(s_+^{J,n}) - F_h(s_+^{J,n})) \not\to \infty$. The contradiction is created in four steps. First, Substep 0 shows that the inference from winning with (m_n^J, b_n) is approximately the same as from winning whenever $s_{(1)} \leq s_+^{I,n}$. Then Substep 1 derives an upper bound on b_n, and Substep 2 a lower bound. Substep 3 shows that when n sufficiently large, which yields a contradiction. To simplify notation, abbreviate the probabilities to win with action-tuple $(m_n^{I,n}, b_n)$ by $\pi_\omega^{I,n} = \pi_\omega^{\mathfrak{c}}(m_n^{I,n}, b_n); \sigma_n)$ and with action-tuple $(m_n^{J,n}, b_n)$ by $\pi_\omega^{J,n} = \pi_\omega^{\mathfrak{c}}(m_n^{J,n}, b_n); \sigma_n)$ for $\omega \in \{h, \ell\}$.

Substep 0.

$$\frac{\pi_h^{J,n}}{\pi_\ell^{J,n}} = \frac{e^{-\eta_h^n(1-F_h(s_-^{J,n}))}}{e^{-\eta_h^n(1-F_h(s_-^{J,n}))}} D(n), \tag{51}$$

where $D(n)$ is a function with $\lim_{n\to\infty} D(n) = 1$.

First, if $\eta_h^n[F_h(s_+^{J,n}) - F_h(s_-^{J,n})] \to 0$, then $\eta_\ell^n[F_\ell(s_+^{J,n}) - F_\ell(s_-^{J,n})] \to 0$ (cf footnote 36) such that

$$\begin{aligned}
D(n) = \frac{\pi_h^{J,n}}{\pi_\ell^{J,n}} \Big(\frac{e^{-\eta_h^n(1-F_h(s_-^{J,n}))}}{e^{-\eta_\ell^n(1-F_\ell(s_-^{J,n}))}} \Big)^{-1} &= \frac{\frac{e^{-\eta_h^n(1-F_h(s_+^{J,n}))} - e^{-\eta_h^n(1-F_h(s_-^{J,n}))}}{\eta_h^n(F_h(s_+^{J,n}) - F_h(s_-^{J,n}))}}{\frac{e^{-\eta_\ell^n(1-F_\ell(s_+^{J,n}))} - e^{-\eta_\ell^n(1-F_\ell(s_-^{J,n}))}}{\eta_\ell^n(F_\ell(s_+^{J,n}) - F_\ell(s_-^{J,n}))}} \Big(\frac{e^{-\eta_h^n(1-F_h(s_-^{J,n}))}}{e^{-\eta_\ell^n(1-F_\ell(s_-^{J,n}))}} \Big)^{-1} \\
&= \frac{\frac{e^{\eta_h^n(F_h(s_+^{J,n}) - F_h(s_-^{J,n}))} - 1}{\eta_h^n(F_h(s_+^{J,n}) - F_h(s_-^{J,n}))}}{\frac{e^{\eta_\ell^n(F_\ell(s_+^{J,n}) - F_\ell(s_-^{J,n}))} - 1}{\eta_\ell^n(F_\ell(s_+^{J,n}) - F_\ell(s_-^{J,n}))}} \to 1. \quad \text{by l'Hospital.}
\end{aligned}$$

Second, if $\eta_h^n[F_h(s_+^{J,n}) - F_h(s_-^{J,n})] \to k$ with $k \in (0, \infty)$, then $s^J = \breve{s}$. Otherwise, there is a signal $s^J > \breve{s}$ which ties with positive probability, which is at odds with Step 3 when n is sufficiently large. If $s^J = \breve{s}$ and $\eta_h^n[F_h(s_+^{J,n}) - F_h(s_-^{J,n})] \not\to 0$ then

bounded, the claim follows.

[37] If J_n is empty, set $s_-^{J,n} = s_+^{J,n} = \sup I_n$

$\eta_h^n[F_h(s_+^{J,n}) - F_h(s_-^{J,n})] - \eta_\ell^n[F_\ell(s_+^{J,n}) - F_\ell(s_-^{J,n})] \to 0$,[38] and $\frac{\eta_h^n[F_h(s_+^{J,n})-F_h(s_-^{J,n})]}{\eta_\ell^n[F_\ell(s_+^{J,n})-F_\ell(s_-^{J,n})]} \to 1$, such that

$$D(n) = \frac{\pi_h^{J,n}}{\pi_\ell^{J,n}}\Big(\frac{e^{-\eta_h^n(1-F_h(s_-^{J,n}))}}{e^{-\eta_\ell^n(1-F_\ell(s_-^{J,n}))}}\Big)^{-1} = \frac{\frac{e^{-\eta_h^n(1-F_h(s_+^{J,n}))}-e^{-\eta_h^n(1-F_h(s_-^{J,n}))}}{\eta_h^n(F_h(s_+^{J,n})-F_h(s_-^{J,n}))}}{\frac{e^{-\eta_\ell^n(1-F_\ell(s_+^{J,n}))}-e^{-\eta_\ell^n(1-F_\ell(s_-^{J,n}))}}{\eta_\ell^n(F_\ell(s_+^{J,n})-F_\ell(s_-^{J,n}))}}\Big(\frac{e^{-\eta_h^n(1-F_h(s_-^{J,n}))}}{e^{-\eta_\ell^n(1-F_\ell(s_-^{J,n}))}}\Big)^{-1}$$

$$= \frac{\eta_\ell^n[F_\ell(s_+^{J,n}) - F_\ell(s_-^{J,n})]}{\eta_h^n[F_h(s_+^{J,n}) - F_h(s_-^{J,n})]}\frac{e^{\eta_h^n[F_h(s_+^{J,n})-F_h(s_-^{J,n})]-\eta_\ell^n[F_\ell(s_+^{J,n})-F_\ell(s_-^{J,n})]} - e^{-\eta_\ell^n[F_\ell(s_+^{J,n})-F_\ell(s_-^{J,n})]}}{1-e^{-\eta_\ell^n[F_\ell(s_+^{J,n})-F_\ell(s_-^{J,n})]}} \to 1.$$

Substep 1. *An upper bound on b_n is given by*

$$\frac{b_n - v_\ell}{v_h - b_n} \le \frac{\rho\eta_h^n f_h(\underline{s}+\epsilon)}{(1-\rho)\eta_\ell^n f_\ell(\underline{s}+\epsilon)}\frac{\pi_h^{I,n}}{\pi_l^{I,n}}. \tag{52}$$

The individual rationality argument for equation (23) remains unaltered in the communication extension. Applied to signal $\underline{s}+\epsilon$, which chooses (b_n, m_n^I), it provides the inequality.

Substep 2. *A lower bound on b_n is given by*

$$\frac{b_n - v_\ell}{v_h - b_n} \ge \frac{\rho\eta_h^n f_h(s_+^{I,n})}{(1-\rho)\eta_\ell^n f_\ell(s_+^{I,n})}\frac{\pi_h^{J,n} - \pi_h^{I,n}}{\pi_\ell^{J,n} - \pi_l^{I,n}}. \tag{53}$$

Consider signal $s_+^{I,n} = s_-^{J,n}$ which is indifferent (if J_n is non-empty) or prefers (if J_n is empty) (m_n^I, b_n) over (m_n^J, b_n). Then $U^{\mathfrak{c}}(m_n^I, b_n|s_+^{I,n};\sigma_n) \ge U^{\mathfrak{c}}(m_n^J, b_n|s_+^{I,n};\sigma_n)$ implies that

$$\frac{\rho\eta_h^n f_h(s_+^{I,n})\pi_h^{I,n}(v_h - b_n) + (1-\rho)\eta_\ell^n f_\ell(s_+^{I,n})\pi_\ell^{I,n}(v_\ell - b_n)}{\rho\eta_h^n f_h(s_+^{I,n}) + (1-\rho)\eta_\ell^n f_\ell(s_+^{I,n})}$$
$$\ge \frac{\rho\eta_h^n f_h(s_+^{I,n})\pi_h^{J,n}(v_h - b_n) + (1-\rho)\eta_\ell^n f_\ell(s_+^{I,n})\pi_\ell^{J,n}(v_\ell - b_n)}{\rho\eta_h^n f_h(s_+^{I,n}) + (1-\rho)\eta_\ell^n f_\ell(s_+^{I,n})},$$

which rearranges to inequality (53).

Substep 3. *When n is sufficiently large, the lower bound (53) exceeds the upper bound (52).*

Combining equations (52) and (53) yields

$$\frac{\rho\eta_h^n f_h(\underline{s}+\epsilon)}{(1-\rho)\eta_\ell^n f_\ell(\underline{s}+\epsilon)}\frac{\pi_h^{I,n}}{\pi_l^{I,n}} \ge \frac{\rho}{1-\rho}\frac{\eta_h^n f_h(s_+^{I,n})}{\eta_\ell^n f_\ell(s_+^{I,n})}\frac{\pi_h^{J,n} - \pi_h^{I,n}}{\pi_\ell^{J,n} - \pi_\ell^{I,n}},$$

[38] If $\eta_\omega^n(F_\omega(s_+^{J,n}) - F_\omega(s_-^{J,n})) \not\to \infty$, then $\eta_h^n[F_h(s_+^{J,n}) - F_h(s_-^{J,n})] - \eta_\ell^n[F_\ell(s_+^{J,n}) - F_\ell(s_-^{J,n})] = \int_{[s_-^{J,n}, s_+^{J,n}]}\eta_h^n f_h(s) - \eta_\ell^n f_\ell(s)ds \le \int_{[s_-^{J,n}, s_+^{J,n}]}(\frac{\eta_h^n f_h(s_+^{J,n})}{\eta_\ell^n f_\ell(s_+^{J,n})} - 1)\eta_\ell^n f_\ell(s)ds = (\frac{\eta_h^n f_h(s_+^{J,n})}{\eta_\ell^n f_\ell(s_+^{J,n})} - 1)\eta_\ell^n[F_\ell(s_+^{J,n}) - F_\ell(s_-^{J,n})] \to 0$, since $\frac{\eta_h^n f_h(s_+^{J,n})}{\eta_\ell^n f_\ell(s_+^{J,n})} \to \frac{\eta_h^n f_h(s^J)}{\eta_\ell^n f_\ell(s^J)} = \frac{\eta_h^n f_h(\check{s})}{\eta_\ell^n f_\ell(\check{s})} = 1$ which bounds the limit from above. The bound from below follows by using $\int_{[s_-^{J,n}, s_+^{J,n}]}\eta_h^n f_h(s) - \eta_\ell^n f_\ell(s)ds \ge \int_{[s_-^{J,n}, s_+^{J,n}]}(\frac{\eta_h^n f_h(s_-^{J,n})}{\eta_\ell^n f_\ell(s_-^{J,n})} - 1)\eta_\ell^n f_\ell(s)ds$.

which rearranges to

$$\frac{f_h(\underline{s}+\epsilon)}{f_\ell(\underline{s}+\epsilon)}\Big(\frac{f_h(s_+^{I,n})}{f_\ell(s_+^{I,n})}\Big)^{-1} \geq \frac{\frac{\pi_h^{J,n}}{\pi_h^{I,n}}-1}{\frac{\pi_\ell^{J,n}}{\pi_\ell^{I,n}}-1}. \tag{54}$$

Since $s_+^{I,n} = s_-^{J,n} \to \check{s}$ it has to hold in either state $\omega \in \{h,\ell\}$ that $\eta_\omega^n(F_\omega(s_+^{n,I}) - F_\omega(\underline{s}+\epsilon)) \to \infty$. Combined with the observation that

$$\pi_\omega^{I,n} \leq \frac{e^{-\eta_\omega^n(1-F_\omega(s_+^{I,n}))}}{\eta_\omega^n(F_\omega(s_+^{n,I}) - F_\omega(\underline{s}+\epsilon))} \text{ and } \pi_\omega^{J,n} \geq e^{-\eta_\omega^n(1-F_\omega(s_+^{I,n}))} \text{ this implies that } \frac{\pi_\omega^{I,n}}{\pi_\omega^{J,n}} \to 0.$$

Hence, the right side of inequality (54) converges to

$$\lim_{n\to\infty}\frac{\frac{\pi_h^{J,n}}{\pi_h^{I,n}}-1}{\frac{\pi_\ell^{J,n}}{\pi_\ell^{I,n}}-1} = \lim_{n\to\infty}\frac{\pi_h^{J,n}}{\pi_l^{J,n}}\Big(\frac{\pi_h^{I,n}}{\pi_l^{I,n}}\Big)^{-1} \overset{\text{Step 1}}{=} \lim_{n\to\infty} D(n)\frac{e^{-\eta_h^n(1-F_h(s_-^{J,n}))}}{e^{-\eta_h^n(1-F_h(s_-^{J,n}))}}\Big(\frac{\pi_h^{I,n}}{\pi_l^{I,n}}\Big)^{-1}$$

$$= \lim_{n\to\infty}\frac{e^{-\eta_h^n(1-F_h(s_-^{J,n}))}}{e^{-\eta_h^n(1-F_h(s_-^{J,n}))}}\frac{\eta_\ell^n[F_\ell(s_+^{I,n})-F_\ell(s_-^{I,n})]}{\eta_h^n[F_h(s_+^{I,n})-F_h(s_-^{I,n})]}\frac{e^{-\eta_\ell^n(1-F_\ell(s_+^{I,n}))}-e^{-\eta_\ell^n(1-F_\ell(s_-^{I,n}))}}{e^{-\eta_h^n(1-F_h(s_+^{I,n}))}-e^{-\eta_h^n(1-F_h(s_-^{I,n}))}}$$

$$= \lim_{n\to\infty}\frac{\eta_\ell^n[F_\ell(s_+^{I,n})-F_\ell(s_-^{I,n})]}{\eta_h^n[F_h(s_+^{I,n})-F_h(s_-^{I,n})]}\frac{1-e^{-\eta_\ell^n(F_\ell(s_+^{I,n})-F_\ell(s_-^{I,n}))}}{1-e^{-\eta_h^n(F_h(s_+^{I,n})-F_h(s_-^{I,n}))}} = \lim_{n\to\infty}\frac{\eta_\ell^n[F_\ell(s_+^{I,n})-F_\ell(s_-^{I,n})]}{\eta_h^n[F_h(s_+^{I,n})-F_h(s_-^{I,n})]}.$$

Since $s_+^{I,n} = s_-^{J,n} \to s^J \geq \check{s}$, when n is sufficiently large, the MLRP implies that $\frac{\eta_\ell^n[F_\ell(s_+^{I,n})-F_\ell(s_-^{I,n})]}{\eta_h^n[F_h(s_+^{I,n})-F_h(s_-^{I,n})]} \leq \frac{\eta_\ell^n F_\ell(s_-^{I,n})}{\eta_h^n F_h(s_-^{I,n})} < \frac{\eta_\ell^n F_\ell(\check{s}-\epsilon)}{\eta_h^n F_h(\check{s}-\epsilon)}$. Furthermore, chose $\epsilon > 0$ s.t. $\frac{\eta_h^n f_h(\underline{s}+\epsilon)}{\eta_\ell^n f_\ell(\underline{s}+\epsilon)}\frac{\eta_\ell^n F_\ell(\check{s}-\epsilon)}{\eta_h^n F_h(\check{s}-\epsilon)} < \frac{\eta_h^n f_h(\check{s}-\epsilon)}{\eta_\ell^n f_\ell(\check{s}-\epsilon)} < \frac{\eta_h^n f_h(s_+^{I,n})}{\eta_\ell^n f_\ell(s_+^{I,n})}$ such that $\frac{f_h(\underline{s}+\epsilon)f_\ell(s_+^{I,n})}{f_\ell(\underline{s}+\epsilon)f_h(s_+^{I,n})} < \frac{\eta_\ell^n[F_\ell(s_+^{I,n})-F_\ell(s_-^{I,n})]}{\eta_h^n[F_h(s_+^{I,n})-F_h(s_-^{I,n})]}$ when n is sufficiently large. Thus, inequality (54) is violated when n is large, which means that the lower bound on b_n (53) exceeds the upper bound (52), such that b_n cannot exist. Since $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence of equilibria, it, therefore, cannot be that $\eta_h^n(F_h(s_+^{J,n}) - F_h(s_+^{J,n})) \not\to \infty$. □

Lemma 9. *Any auction on the grid $\Gamma(k)$ has an equilibrium in pure and nondecreasing strategies.*

Proof. Denote the bid space with $k \geq 2$ equidistant bids by B_k. Existence is shown by a fixed point argument on the distribution of bids. Since those are Poisson distributed and thereby fully described by the mean, we look at the compact set of vectors

$$\Lambda = \Big\{ \Big(\lambda(b_1|h) \quad ... \quad \lambda(b_k|h) \quad \lambda(b_1|\ell) \quad ... \quad \lambda(b_k|\ell)\Big) : \sum_{b\in B_k}\lambda(b|\omega) = \eta_\omega\Big\} \subset R^{n\times 2},$$

where $\lambda(b|\omega)$ denotes the expected number of bids b in state ω.

Let $F : \Lambda \rightrightarrows \mathcal{P}(\Lambda)$ be the correspondence which maps any λ into the set of vectors $\{\tilde{\lambda}\}$ that are induced by a pure and nondecreasing best response $\beta : [\underline{s}, \bar{s}] \to B_k$ meaning that $\tilde{\lambda}(b|\omega) = \int_{\beta^{-1}(b)} \eta_\omega f_\omega(s)ds$ for all $b \in B_k$, and $\beta(s) \in \arg\max_b U(b|s, \lambda)$ for almost all s. Here, $U(b|s, \lambda)$ is the interim expected utility from bidding b, given the bidders signal s and distribution of (other) bids described by the Poisson parameter λ which fully determines the probability to win with the bid b.

Because Λ is compact, to apply Kakutani's Fixed-Point Theorem we need to show that $F(\lambda)$ is non-empty, convex valued and that F has a closed graph.

$F(\lambda)$ is non-empty because on the finite set there exists a best response for any signal s. By Lemma 1, these best responses can be reordered, such that the resulting β is pure and nondecreasing.

To show that $F(\lambda)$ is convex valued, consider $\tilde{\lambda}$ and $\tilde{\lambda}'$ from its image. We have to show that $\forall \alpha \in [0,1]$, $\alpha\tilde{\lambda} + (1-\alpha)\tilde{\lambda}' = \tilde{\lambda}^* \in F(\lambda)$. $\tilde{\lambda}$ and $\tilde{\lambda}'$ are induced by two best responses $\tilde{\beta}$ and $\tilde{\beta}'$. Consider a mixed strategy, which follows $\tilde{\beta}$ with probability α and $\tilde{\beta}'$ with probability $1-\alpha$. Such a strategy is optimal for the bidders and result in a distribution of bids $\tilde{\lambda}^*$. By Lemma 1, there is a pure, nondecreasing strategy inducing the same distribution and utilities. Thus $\tilde{\lambda}^* \in F(\lambda)$.

What remains to be shown is that F has a closed graph. Take any two sequences $\lambda_n \to \lambda$ and $\tilde{\lambda}_n \to \tilde{\lambda}$ where $\tilde{\lambda}_n \in F(\lambda_n)$. We have to show that $\tilde{\lambda} \in F(\lambda)$. For every λ_n there is a nondecreasing best response β_n inducing $\tilde{\lambda}_n$. By Helly's Selection Theorem, there is a point-wise converging subsequence of those β_n with a nondecreasing limit β. Obviously, β induces $\tilde{\lambda}$. Furthermore, because $U(b|s,\lambda_n)$ is continuous in both λ_n and b, β is a best response to λ. Thus, F has a closed graph.

Kakutani's Fixed-Point Theorem guarantees an equilibrium vector $\lambda \in \Lambda$ and by construction, there exists a pure, nondecreasing strategy β which is a best response and induces this λ. Thus, β is a pure, nondecreasing and symmetric equilibrium. □

Lemma 10. *Consider any sequence of auctions on the ever-finer grid* $(\Gamma(k))_{k\in\mathbb{N}}$ *and any corresponding sequence of equilibria* $(\beta_k^*)_{k\in\mathbb{N}}$. *There exists a subsequence of auctions* $(\Gamma(n))_{n\in\mathbb{N}}$ *with equilibria* $(\beta_n^*)_{n\in\mathbb{N}}$ *and a concordant equilibrium* σ^* *of* $\Gamma^\mathfrak{c}$, *such that, for all* $s \in [\underline{s}, \bar{s}]$,

(i) $\sigma^*(s) = (M, \mu(s), \lim_{n\to\infty} \beta_n^*(s))$ *for some* M *and function* $\mu : [\underline{s}, \bar{s}] \to M$*;*

(ii) $\lim_{n\to\infty} \pi_\omega(\beta_n^*(s); \beta_n^*) = \pi_\omega^\mathfrak{c}(\sigma^*(s); \sigma^*)$ *for* $\omega \in \{h, \ell\}$,

and, therefore,

(iii) $\lim_{n\to\infty} U(\beta_n^*(s)|s; \beta_n^*) = U^\mathfrak{c}(\sigma^*(s)|s; \sigma^*)$.

Proof. Take the sequence of auctions on the ever-finer grid $(\Gamma(k))_{k\in\mathbb{N}}$, denote the sequence of respective bid spaces by $(B_k)_{k\in\mathbb{N}}$ and equilibria by $(\beta_k)_{k\in\mathbb{N}}$ (economizing on the $*$). For every k, denote the on-path winning probability in the high state by $\pi_h^k(s) = \pi_h(\beta_k(s); \beta_k)$ and define an auxiliary function $\delta_k(b) = \max\{b' \in B_k \colon b' \leq b\}$.

Since $(\beta_k)_{k\in\mathbb{N}}$, $(\pi_h^k)_{k\in\mathbb{N}}$ and $(\delta_k)_{k\in\mathbb{N}}$ are sequences of nondecreasing functions, by Helly's Selection Theorem, there is a subsequence along which these functions converge pointwise to some nondecreasing limit β, π_h and δ, respectively. We denote this subsequence by n and, henceforth, consider it exclusively.

Construct M by including $m \in M$ if and only if there exists a signal $s \in [\underline{s}, \bar{s}]$ such that $\pi_h(s) = m$. Further, define function $\sigma^*(s) = (M, \pi_h(s), \beta(s))$ for every s.

By construction, properties *(i)* and *(ii)* of Lemma 10 are fulfilled. Steps 1 and 2 proceed by showing properties *(iii)* and *(iv)*, before Steps 3 and 4 show that σ^* is an equilibrium of communication extension.

Step 1. $\pi^{\mathfrak{c}}_\omega(\sigma^*(s); \sigma^*) = \lim_{n\to\infty} \pi_\omega(\beta_n(s); \beta_n)$ *for every* s *and* $\omega \in \{h, \ell\}$.

We focus on state h, the result follows for ℓ because the winning probabilities are isomorphic across states. Fix any $\hat{s} \in [\underline{s}, \bar{s}]$ and define the sets $W_n = \{s \colon \pi^n_h(s) < \pi^n_h(\hat{s})\}$, $T_n = \{s \colon \pi^n_h(s) = \pi^n_h(\hat{s})\}$ and $L_n = \{s \colon \pi^n_h(s) > \pi^n_h(\hat{s})\}$. Furthermore, define $W = \{s \colon \pi_h(s) < \pi_h(\hat{s})\}$, and T, L respectively. Because π^n_h is nondecreasing and converges pointwise, $W_n \to W, T_n \to T$ and $L_n \to L$.

Given strategy β_n, signal $\hat{s}$ wins against signals from the set W_n, loses against signals L_n and ties with signals from T_n. We want to show that, under strategy σ^*, signal $\hat{s}$ wins against signals from the set W, loses against signals L and ties with signals from T. If this is true, the convergence of the sets and atomless signal distribution ensures that the winning probabilities converge.

Fix any $s_L \in L$. When n is sufficiently large, $s_L \in L_n$. Further, it follows from $\pi^n_h(\hat{s}) < \pi^n_h(s_L)$ that $\beta_n(\hat{s}) < \beta_n(s_L)$. This, and the convergence of β_n implies that $\beta(\hat{s}) \le \beta(s)$. Further, by definition of L and M, $\mu(s_L) = \pi_h(s_L) > c(\hat{s}) = \pi_h(\hat{s})$. Thus, either s_L chooses a higher bid, and/or a higher report than $\hat{s}$. Thus, $\hat{s}$ never wins the auction when s_L is present.

The symmetric argument can be made for all signals $s_W \in W$ such that signal $\hat{s}$ following wins against signals from $s_W \in W$.

Last, fix any $s_T \in T$. Again, when n is sufficiently large, $s_T \in T_n$ meaning that $\pi^n_h(s_T) = \pi^n_h(\hat{s})$. This implies that $\beta_n(s_T) = \beta_n(\hat{s})$ for all n large, which means that in the limit $\beta(s_T) = \beta(\hat{s})$. Further, by definition of T and M, $\mu(s_T) = \pi_h(s_T) = c(\hat{s}) = \pi_h(\hat{s})$. Thus, s_T and $\hat{s}$ choose the same bid and same report, $\sigma^*(s_T) = \sigma^*(\hat{s})$ and tie.

Step 2. *For every* s*, it holds that*

$$U(\beta_n(s)|s; \beta_n) \to U^{\mathfrak{c}}(\sigma^*(s)|s; \sigma^*) = U^{\mathfrak{c}}(M, \pi_h(\sigma^*(s); \sigma^*), \beta(s)|s; \sigma^*).$$

Since $\pi_\omega(\beta_k(s); \beta_k) \to \pi^{\mathfrak{c}}_\omega(\sigma^*(s); \sigma^*)$ in both states $\omega \in \{h, \ell\}$ and for every s, and because $\beta_n(s)$ converges to $\beta(s)$ for every s, the convergence is immediate from equation (21).

Step 3. *For every* (b, m) *with* $m \in M$*, there is a sequence of bids* $(b_n)_{n\in\mathbb{N}}$ *with* $b_n \in B_n$*, such that* $b_n \to b$ *and* $\pi_\omega(b_n; \beta_n) \to \pi^{\mathfrak{c}}_\omega(m, b; \sigma^*)$ *in either state* $\omega \in \{h, \ell\}$.

Since M is kept fixed throughout the proof, it is dropped from the expressions for ease of notation. Further, we focus on state h. For state ℓ the result follows because the winning probabilities are isomorphic across states. By construction of M, there exists a signal s_m such that $\pi^{\mathfrak{c}}_h(\sigma^*(s_m); \sigma^*) = m$. The proof is structured into three cases:

Case 1: $\pi_h^{\mathfrak{c}}(m, b; \sigma^*) = m$

If $\sigma^*(s_m) = (m, b)$, then it follows from $\pi_h(\beta_n(s_m); \beta_n) = \pi_h^n(s_m) \to m$ and $\beta_n(s_m) \to b$, that $(\beta_n(s_m))_{n\in\mathbb{N}}$ is the desired sequence.

If $\sigma^*(s_m) \neq (m, b)$ because $b > \beta(s_m)$, it follows that $\pi_h^{\mathfrak{c}}(m, \beta(s_m); \sigma^*) = P(s_1 \leq s_m|h)$. Otherwise, there is a non-trivial interval of signals $I = \{s \colon \sigma^*(s) = (m, \beta(s_m))\}$, which is outbid by (m, b) such that (m, b) wins strictly more often. The same is true, if there is a non-trivial interval $I = \{s \colon b > \beta(s) > \beta(s_m)\}$. Thus, $\beta(s') > b$ for all $s' > s_m$. Take any $\lambda \in (0, 1)$ and consider $b_n = \delta_n(\lambda b + (1 - \lambda)\beta(s'))$. For any n sufficiently large, $\beta_n(s_m) < b_n < \beta_n(s')$, such that in the limit

$$\pi_h^{\mathfrak{c}}(m, b; \sigma^*) = \mathbb{P}[s_1 \leq s_m|h] \leq \lim_{n\to\infty} \pi_h^n(b_n; \beta_n) \leq \mathbb{P}[s_1 \leq s'|h].$$

Since this is true for any $s' > s_m$ and λ arbitrary close to 1, the desired sequence exists.

If $\sigma^*(\tilde{s}) \neq (m, b)$ because $b < \beta(\tilde{s})$ the construction can be repeated symmetrically.

Case 2: $\pi_h^{\mathfrak{c}}(m, b; \sigma^*) < m$

In this case (m, b) wins, whenever there is no bid above b. Let $s_+ = \inf\{s \colon \beta(s) > b\}$. Then, $\pi_h^{\mathfrak{c}}(m, b; \sigma^*) = \mathbb{P}[s_{(1)} \leq s_+|h]$ and for all $s' > s_+$ it holds that $\beta(s') > b$. Take any $\lambda \in (0, 1)$ and consider $b_n = \delta_n(\lambda b + (1 - \lambda)\beta(s'))$. For any n sufficiently large, $\beta_n(s_+) < b_n < \beta_n(s')$, such that in the limit

$$\mathbb{P}[s_{(1)} \leq s_+|h] \leq \lim_{n\to\infty} \pi_h^n(b_n; \beta_n) \leq \mathbb{P}[s_1 \leq s'|h].$$

Since this is true for any $s' > s_+$ and λ arbitrary close to 1, the desired sequence exists.[39]

Case 3: $\pi_h^{\mathfrak{c}}(m, b; \sigma^*) > m$

The proof is symmetric to Case 2, with an approximation from below.

Step 4. *σ^* is a concordant equilibrium of the communication extension $\Gamma^{\mathfrak{c}}$.*

First, deviations to a different M' and/or a $m' \notin M$ are dominated by reporting M, an arbitrary $m \in M$ and bidding v_ℓ. This action-tuple wins at least whenever the bidder is alone and generates a strictly positive profit in the high state. Thus, we restrict attention to deviations (m', b') where $m' \in M$ and, for ease of notation, do not explicitly reference M in the expressions.

Suppose now there is a signal $\hat{s}$ and a profitable deviation (m', b') such that $U^{\mathfrak{c}}(m', b'|\hat{s}; \sigma^*) > U^{\mathfrak{c}}(\sigma^*(\hat{s})|\hat{s}; \sigma^*)$. By Step 3, there exists a sequence of bids $(b_n)_{n\in\mathbb{N}}$ with $b_n \to b'$, such that $\pi_\omega(b_n; \beta_n) \to \pi_\omega^{\mathfrak{c}}(m', b'; \sigma^*)$ in either state $\omega \in \{h, \ell\}$. This means that $U(b_n|\hat{s}; \beta_n) \to U^{\mathfrak{c}}(m', b'|\hat{s}; \sigma^*)$. But then $U(\beta_n(\hat{s})|\hat{s}; \beta_n) \to U^{\mathfrak{c}}(\sigma^*(\hat{s})|\hat{s}; \sigma^*)$ (Step 2) implies that when n is sufficiently large, a deviation from $\beta_n(\hat{s})$ to b_n must have been profitable for $\hat{s}$.[40] This is a contradiction. $\square$

Lemma 11. *Consider a sequence of auctions on the ever-finer grid $(\Gamma(n))_{n\in\mathbb{N}}$ with corresponding equilibria $(\beta_n^*)_{n\in\mathbb{N}}$ that converge to an equilibrium of $\Gamma^{\mathfrak{c}}$, denoted σ^*, in the sense of Lemma 10. Then it holds for (almost) any two signals $s_- < s_+$ that*

[39] Note that if $s_+ = \bar{s}$, then $\lim_{n\to\infty} \pi_h^n(b_n; \beta_n) \to 1$ with any $b_n = \delta_n(\lambda b + (1 - \lambda)v_h)$.

[40] Observe that $\beta_n(\hat{s}) \neq b_n$ for infinitely many n. Otherwise, $\lim_{n\to\infty} \pi_h(b_n; \beta_n) = \lim_{n\to\infty} \pi_h(\beta_n(\hat{s}); \beta_n) = \pi_h(\hat{s})$ and, by construction, $(m', b') = \sigma^*(\hat{s})$.

(i) $\sigma^*(s_-) = \sigma^*(s_+)$*, if and only if* $\beta_n^*(s_-) = \beta_n^*(s_+)$ *for any* n *sufficiently large;*

(ii) $\sigma^*(s_-) \neq \sigma^*(s_-)$*, if and only if* $\beta_n^*(s_-) < \beta_n^*(s_+)$ *for any* n *sufficiently large.*

Proof. The "if" part of the statement follows directly by (ii) and (iii) of Lemma 10. Thus, we only show that "only if" part.

Step 1. *If* $\sigma^*(s_-) = \sigma^*(s_+)$*, then* $\beta_n^*(s_-) = \beta_n^*(s_+)$ *for every* n *sufficiently large.*

Suppose to the contrary that there is a sequence along which $\beta_n^*(s_-) \neq \beta_n^*(s_+)$. Since any β_n^* is nondecreasing, it has to hold that $\{s\colon \beta_n^*(s) \in [\beta_n^*(s_-), \beta_n^*(s_+)]\} \not\to \emptyset$. We show that combined, these two conditions imply that $|\pi_h(\beta_n^*(s_+); \beta_n^*) - \pi_h(\beta_n^*(s_-); \beta_n^*)| \not\to 0$.

If $\{s\colon \beta_n^*(s) \in (\beta_n^*(s_-), \beta_n^*(s_+))\} \not\to \emptyset$, this follows immediately. Otherwise, either $\{s\colon \beta_n^*(s) = \beta_n^*(s_-)\} \not\to \emptyset$, in which case $\pi_h(\beta_n^*(s_+); \beta_n^*)$ stays bounded above $\pi_h(\beta_n^*(s_-); \beta_n^*)$, because it wins the uniform tiebreak on $\beta_n^*(s_-)$ with certainty; and/or $\{s\colon \beta_n^*(s) = \beta_n^*(s_+)\} \not\to \emptyset$, in which case $\pi_h(\beta_n^*(s_-); \beta_n^*)$ stays bounded below $\pi_h(\beta_n^*(s_+); \beta_n^*)$ because $\beta_n^*(s_-)$ only wins when no bid at or above $\beta_n^*(s_+)$ is made.

If $|\pi_h(\beta_n^*(s_+); \beta_n^*) - \pi_h(\beta_n^*(s_-); \beta_n^*)| \not\to 0$, it follows that $|\pi_h(\beta_n^*(s_+); \beta_n^*) - \pi_h^{\mathfrak{c}}(\sigma^*(s_+); \sigma^*)| + |\pi_h(\beta_n^*(s_-); \beta_n^*) - \pi_h^{\mathfrak{c}}(\sigma^*(s_-); \sigma^*)| \geq |\pi_h(\beta_n^*(s_+); \beta_n^*) - \pi_h(\beta_n^*(s_-); \beta_n^*)| \not\to 0$. This is a contradiction to property *(iii)* of Lemma 10, however, which implies that if $\sigma^*(s_-) = \sigma^*(s_+)$, then $\pi_h(\beta_n^*(s_-); \beta_n^*)$ and $\pi_h(\beta_n^*(s_+); \beta_n^*)$ converge to some common limit $\pi_h^{\mathfrak{c}}(\sigma^*(s_-); \sigma^*)$.

Step 2. *If* $\sigma^*(s_-) \neq \sigma^*(s_+)$*, then* $\beta_n^*(s_-) < \beta_n^*(s_+)$ *for every* n *sufficiently large.*

Suppose to the contrary that the claim is not true. Then $\beta_n^*(s_-) = \beta_n^*(s_+)$ for infinitely many n, and hence $\beta^*(s_-) = \beta^*(s_+)$. Further, it implies that $\lim_{n\to\infty} \pi_h(\beta_n^*(s_-); \beta_n^*) = \lim_{n\to\infty} \pi_h(\beta_n^*(s_+); \beta_n^*)$. Since $\lim_{n\to\infty} \pi_h(\beta_n^*(s); \beta_n^*) = \pi_h^{\mathfrak{c}}(\sigma^*(s); \sigma^*)$ for all s, this means that $\pi_h^{\mathfrak{c}}(\sigma^*(s_-); \sigma^*) = \pi_h^{\mathfrak{c}}(\sigma^*(s_+); \sigma^*)$.

Denote the report which strategy σ^* assigns to signal s by $\mu(s)$. Because $\beta^*(s_-) = \beta^*(s_+)$, Lemma 7 implies that $\mu(s_-) \leq \mu(s_+)$. We now show that it cannot be that $\mu(s_-) < \mu(s_+)$, because then $\pi_h^{\mathfrak{c}}(\sigma^*(s_-); \sigma^*) < \pi_h^{\mathfrak{c}}(\sigma^*(s_+); \sigma^*)$. If $\{s \in [s_-, s_+]\colon \mu(s) \in (\mu(s_-), \mu(s_+))\}$ has positive mass, then $\pi_h^{\mathfrak{c}}(\sigma^*(s_-); \sigma^*) < \pi_h^{\mathfrak{c}}(\sigma^*(s_+); \sigma^*)$ follows immediately. Otherwise, either $\{s \in [s_-, s_+]\colon \mu(s) = \mu(s_-)\}$ has positive mass, in which case $\pi_h^{\mathfrak{c}}(\mu(s_+), \beta^*(s_-); \sigma^*) > \pi_h^{\mathfrak{c}}(\mu(s_-), \beta^*(s_-); \sigma^*)$, because $\mu(s_+)$ wins the uniform tiebreak on $(\mu(s_-), \beta^*(s_-))$ with certainty; and/or $\{s \in [s_-, s_+]\colon \mu(s) = \mu(s_-)\}$ has positive mass, in which case $\pi_h^{\mathfrak{c}}(\mu(s_-), \beta^*(s_-); \sigma^*) < \pi_h^{\mathfrak{c}}(\mu(s_+), \beta^*(s_-); \sigma^*)$, because $\mu(s_-)$ never wins when an action-tuple $(\mu(s_+), \beta^*(s_-))$ is played.

Thus, $\mu(s_-) = \mu(s_+)$, such that $\sigma^*(s_-) = \sigma^*(s_+)$, which is a contradiction. □

Proposition 5*. *Fix any* $\frac{\eta_h}{\eta_\ell} = l \in (\frac{f_\ell(\bar{s})}{f_h(\bar{s})}, \frac{f_\ell(\underline{s})}{f_h(\underline{s})})$ *and any* $\epsilon \in (0, \frac{\check{s}-\underline{s}}{2})$*. When* η_h *is sufficiently large (given* ϵ*) and* k *sufficiently large (given* ϵ *and* η_h*), any equilibrium* β^* *of* $\Gamma(k)$ *takes the following form: There are two disjoint, adjacent intervals of signals* I, J *such that*

(i) $[\underline{s} + \epsilon, \check{s} - \epsilon] \subset I \cup J$*;*

(ii) $\beta^*(s_I) = b$ *for all* $s_I \in I$ *and* $\beta^*(s_J) = b + d$ *for all* $s_J \in J$*;*

(iii) $\int_I \eta_\omega f_\omega(z)dz > \frac{1}{\epsilon}$*, and* $\int_J \eta_\omega f_\omega(z)dz > \frac{1}{\epsilon}$ *for* $\omega \in \{h, \ell\}$*;*

(iv) on $s \in (\breve{s} + \epsilon, \bar{s}]$*, the bids tie with a probability smaller than* ϵ*.*[41]

Proof. Applying the proof for Proposition 5 verbatim gives the result. □

Lemma 12. *Suppose that there is no reserve price at* v_ℓ*, such that bids* $b \in [0, v_h]$*. Fix any* $\epsilon > 0$ *and* $\frac{\eta_h}{\eta_\ell} = l \leq 1$*. Let* β^* *be an equilibrium. When* η_h *is sufficiently large, there exists an alternative equilibrium* $\hat{\beta}^*$ *which is pure, and nondecreasing above* $\underline{s} + \epsilon$*. Furthermore,* $\hat{\beta}^*(s) \geq v_\ell$ *for all* $\underline{s} + \epsilon$*.*

Proof.

Step 1. *If* b_p *is a pooling bid, then* $b_p > v_\ell$

If $b_p \leq v_\ell$ is a pooling bid, then winning with b_p results in a (weak) profit in both states. Thus, a marginally higher bid strictly raises profits by discretely raising the probability to win.

Step 2. *If* $b' > b$ *and* $b' \geq v_\ell$ *and* $U(b'|s; \beta^*) \geq U(b|s; beta^*)$*, then* $U(b'|s'; \beta^*) \geq U(b|s'; \beta^*)$ *for all* $s' > s$*. The second inequality is strict if* $\frac{f_h(s')}{f_\ell(s')} > \frac{f_h(s)}{f_\ell(s)}$*.*

If $b \geq v_\ell$, the result follows by Step 1 of Lemma 1.

If $b < v_\ell$, then $U(b'|s; \beta^*) \geq U(b|s; beta^*)$ rearranges to

$$\frac{\rho \eta_h f_h(s)}{(1-\rho)\eta_\ell f_\ell(s)}[\pi_h(b'; \beta^*)(v_h - b') - \pi_h(b; \beta^*)(v_h - b)] \geq \pi_\ell(b; \beta^*)(v_\ell - b) - \pi_\ell(b'; \beta^*)(v_\ell - b')$$

Since $b' \geq v_\ell > b$, it follows that $\pi_\ell(b; \beta^*)(v_\ell - b) > 0$, while $\pi_\ell(b'; \beta^*)(v_\ell - b') < 0$. Thus, both sides of the inequality are positive. If $s' > s$ and $\frac{f_h(s')}{f_\ell(s')} > \frac{f_h(s)}{f_\ell(s)}$, the left side is strictly larger for s', and thus $U(b'|s', \beta^*) > U(b|s', \beta^*)$.

Define $s_- := \sup\{s\colon \sup\{\text{supp } \beta^*(s)\} \leq v_\ell\}$ as the highest signal which only chooses bids below v_ℓ, and let $s_+ := \inf\{s\colon \inf\{\text{supp } \beta^*(s)\} > v_\ell\}$ be the lowest signal which only chooses bids above v_ℓ. Note that if all bids are above v_ℓ, Lemma 1 applies and we are done. If all bids are below v_ℓ, let $s_+ = \bar{s}$.

If $\frac{f_h(s)}{f_\ell(s)}$ is strictly increasing, Steps 1 and 2 imply that $s_- = s_+$. We denote this signal by $\hat{s}$.

If $\frac{f_h(s)}{f_\ell(s)}$ is not strictly increasing and $s_- \neq s_+$, then $\frac{f_h(s_-)}{f_\ell(s_-)} = \frac{f_h(s_+)}{f_\ell(s_+)}$. In this case, we can apply Step 3 of Lemma 1 to the interval $I = \{s\colon \frac{f_h(s)}{f_\ell(s)} = \frac{f_h(s_-)}{f_\ell(s_-)}\}$ and reorder β^* such that it is pure and nondecreasing along I. Given the reordered β^*, it has to hold that $s_- = s_+ = \hat{s}$.

In either case, we conclude that, without loss, there is a unique $\hat{s} \in [\underline{s}, \bar{s}]$ such that all signals above $\bar{s}$ exclusively choose bids above v_ℓ and all signals below $\hat{s}$ exclusively choose lower bids. As a result, $\pi_\omega(v_\ell; \beta^*) = e^{-\eta_\omega(1-F_\omega(\hat{s}))}$ in state $\omega \in \{h, \ell\}$. Further, we can apply Lemma 1 to all signals above $\hat{s}$, such that the resulting $\hat{\beta}^*$ is pure and nondecreasing with $\hat{\beta}^*(s) \geq v_\ell$ above $\hat{s}$.

[41] Take any $\hat{s} > \breve{s} + \epsilon$ and let $\hat{b} = \beta^*(\hat{s})$. If there exists an interval $[s_-, s_+]$ such that $\beta^*(s) = \hat{b}$ for all $s \in [s_-, s_+]$, then $\eta_\omega \int_{s_-}^{s_+} f_\omega(z)dz < \epsilon$ for $\omega \in \{h, \ell\}$.

What remains to be shown is that $\hat{s} < \underline{s} + \epsilon$ when η_h is sufficiently large. Suppose to the contrary that $\hat{s} \geq \underline{s} + \epsilon$, even when η_h arbitrary large. Since bids below v_ℓ cannot be atoms, and there bids below v_ℓ in the support of $\hat{\beta}^*$, the lowest bid in the support has to be 0. Thus, there is a signal $s_0 \leq \hat{s}$ which bids 0 with positive probability. This can only be an equilibrium if a deviation to v_ℓ is not profitable, that is

$$\begin{aligned}
& U(0|s_0; \hat{\beta}^*) \geq U(v_\ell|s_0; \hat{\beta}^*) \\
\iff & \rho \eta_h f_h(s_0) e^{-\eta_h} v_h + (1-\rho)\eta_\ell f_\ell(s_0) e^{-\eta_\ell} v_\ell \geq \rho \eta_h f_h(s_0) e^{-\eta_h(1-F_h(s_0))}(v_h - v_\ell) \\
\iff & \rho \frac{\eta_h}{\eta_\ell} f_h(s_0) v_h + (1-\rho) f_\ell(s_0) e^{\eta_h[1-\frac{\eta_\ell}{\eta_h}]} v_\ell \geq \rho \frac{\eta_h}{\eta_\ell} f_h(s_0) e^{\eta_h F_h(s_0)}(v_h - v_\ell).
\end{aligned}$$

$\frac{\eta_h}{\eta_\ell} = l \leq 1$, the left side stays bounded as $\eta_h \to \infty$, while the right side grows without bound when $\hat{s} \not\to \underline{s}$. This results in a contradiction which completes the proof.

□

Appendix C Online—Numerical examples

C.1 No strictly increasing when $\eta > 1$

Lemma 13. *Suppose that $v_\ell = 0$, $v_h = 1$ and both states are equally likely. For any $\eta > 1$, there are signal distributions such that no strictly increasing equilibrium exists.*

Proof. Without loss, let the signal space be $[0, 1]$. In a strictly increasing equilibrium, the lowest bid equals the reserve price $v_\ell = 0$. Otherwise, the lowest signal, $s = 0$, can lower her bid, win in the same situations (when she is alone) but pay less. Suppose that $\frac{f_h(s)}{f_\ell(s)}$ is constant on $s \in [0, \frac{1}{2}]$, such that bidders with these signals are essentially identical and have to be indifferent about each other's bids, i.e.

$$\begin{aligned}
& U(0|0; \beta) = U(\beta(s)|s; \beta) \quad \forall s \in [0, \frac{1}{2}] \\
\iff & \frac{\rho f_h(0)\pi_h(0; \beta)}{\rho f_h(0) + (1-\rho) f_\ell(0)} = \frac{\rho f_h(s)\pi_h(\beta(s); \beta)(1-\beta(s)) + (1-\rho) f_\ell(s)\pi_\ell(\beta(s); \beta)(-\beta(s))}{\rho f_h(s) + (1-\rho) f_\ell(s)}.
\end{aligned}$$

Note that $f_\omega(s) = f_\omega(0)$ for all $s \in [0, \frac{1}{2}]$, $\omega \in \{h, \ell\}$ and $\rho = \frac{1}{2}$, such that we can rearrange the fraction to

$$\begin{aligned}
\iff \beta(s) &= \frac{f_h(s)}{f_\ell(s)} \frac{\pi_h(\beta(s); \beta) - \pi_h(\beta(0); \beta)}{\pi_\ell(\beta(s); \beta) + \frac{f_h(s)}{f_\ell(s)} \pi_h(\beta(s); \beta)} \\
&= \frac{f_h(s)}{f_\ell(s)} \frac{1 - e^{-\eta F_h(s)}}{e^{\eta(F_\ell(s) - F_h(s))} + \frac{f_h(s)}{f_\ell(s)}}.
\end{aligned}$$

The derivative of β is positive on $s \in [0, \frac{1}{2}]$ if (note that $\frac{\partial}{\partial s} \frac{f_h(s)}{f_\ell(s)} = 0$ on $[0, \frac{1}{2}]$)

$$\frac{f_h(s)}{f_\ell(s)} \frac{\eta f_h(s) e^{-\eta F_h(s)}(e^{\eta(F_\ell(s) - F_h(s))} + \frac{f_h(s)}{f_\ell(s)}) - \eta(f_\ell(s) - f_h(s)) e^{\eta(F_\ell(s) - F_h(s))}(1 - e^{-\eta F_h(s)})}{(e^{\eta(F_\ell(s) - F_h(s))} + \frac{f_h(s)}{f_\ell(s)})^2} \geq 0$$

which rearranges to

$$\frac{f_h(s)}{f_\ell(s)} \geq e^{\eta F_\ell(s)}(\frac{f_\ell(s)}{f_h(s)} - 1 - \frac{f_\ell(s)}{f_h(s)} e^{-\eta F_h(s)}). \tag{55}$$

Suppose now that $f_h(s), f_\ell(s)$ are constant for $s \leq \frac{1}{2}$, and consider the point $s = \frac{1}{2}$. At this point, the inequality (55) becomes

$$\frac{f_h(0.5)}{f_\ell(0.5)} \geq e^{\eta 0.5 f_\ell(0.5)}(f_\ell(0.5)\frac{1 - e^{-\eta 0.5 f_h(0.5)}}{f_h(0.5)} - 1).$$

If, for $s \leq \frac{1}{2}$, density $f_\ell(s) \in (\frac{1}{\eta 0.5}, 2)$ and $f_h(s)$ becomes arbitrary small, the left side converges to zero, while

$$\lim_{f_h(0.5)\to 0} \frac{1 - e^{-\eta 0.5 f_h(0.5)}}{f_h(0.5)} = 0.5\eta,$$

such that right sides remains bounded above 0. Thus, the inequality is violated if f_h is sufficiently small. The densities can above $\frac{1}{2}$ be chosen freely as long $F_\omega(1) = 1$ and the MLRP holds. □

C.2 Equilibria with atoms—binary signals

In this subsection, we construct an example of equilibrium multiplicity in the context of a binary signal structure. While this violates our standing assumption that signal densities are continuous, the example is more transparent. We give an example with continuous densities in the next subsection.

Suppose that $v_\ell = 0$, $v_h = 1$ and both states are equally likely. Let the signal space be $[0, 1]$ and suppose that

$$f_h(s) = \begin{cases} \frac{1}{2} & s \in [0, \frac{1}{2}) \\ \frac{3}{2} & s \in [\frac{1}{2}, 1] \end{cases} \qquad\qquad f_\ell(s) = \begin{cases} \frac{3}{2} & s \in [0, \frac{1}{2}) \\ \frac{1}{2} & s \in [\frac{1}{2}, 1], \end{cases}$$

such that the likelihood ratio is constant and equal to $\frac{1}{3}$ on $s \leq \frac{1}{2}$ and $\frac{3}{1}$ on $s > \frac{1}{2}$.

By inspection of inequality (55), when $\eta = 3$, no strictly increasing equilibrium exists. However, there is an equilibrium in which all signals $s < \frac{1}{2}$ pool, while all higher signals follow a strictly increasing strategy. In particular, let $\beta^*(s) = 0.036$ for all $s < \frac{1}{2}$ suppose that it follows the ODE (34) with initial value $\beta^*(0.5) = 0.036$ for $s \geq \frac{1}{2}$. To ensure that this is an equilibrium, low signal bidders $s < \frac{1}{2}$ must have no incentive to deviate to 0 or a bid marginally above 0.0036. Further, $\beta^*(s)$ has to be strictly increasing above 0.5, and that high signals $s \geq \frac{1}{2}$ have to prefer the high bids over the pooling bid 0.036.

By simple computation, for $s < \frac{1}{2}$

$$U(0|s;\beta^*) = \frac{\frac{1}{2}e^{-3}v_h + \frac{3}{2}e^{-3}v_\ell}{\frac{1}{2} + \frac{3}{2}} = \frac{e^{-3}}{4} < 0.0125,$$

whereas

$$U(0.036|s;\beta^*) = \frac{\frac{1}{2}\frac{e^{-3(1-0.5\cdot 0.5)}-e^{-3}}{3\cdot 0.5\cdot 0.5}[0.964] + \frac{3}{2}\frac{e^{-3(1-0.5\cdot 1.5)}-e^{-3}}{3\cdot 0.5\cdot 1.5}[-0.036]}{\frac{1}{2}+\frac{3}{2}} > 0.0127,$$

such that a deviation to any bid $b \in [0, 0.036)$ is not profitable for low signal bidders. At the same time, the utility from bidding marginally above 0.036 is

$$\lim_{\epsilon \searrow 0} U(0.036+\epsilon|s;\beta^*) = \frac{\frac{1}{2}e^{-3(1-0.5\cdot 0.5)}[0.964] + \frac{3}{2}e^{-3(1-0.5\cdot 1.5)}[-0.036]}{\frac{1}{2}+\frac{3}{2}} < 0.0127,$$

such that this is no profitable deviation, either.

Since $f_h(s) > f_\ell(s)$ for all $s \geq \frac{1}{2}$, the ODE (34) is strictly increasing above $\frac{1}{2}$ if $\beta^*(0.5) = 0.036 \leq \mathbb{E}[v|s_{(1)} = 0.5, 0.5]$. This is fulfilled because

$$\mathbb{E}[v|s_{(1)} = 0.5, 0.5] = \frac{(\frac{3}{2})^2 e^{-3(1-0.5\cdot 0.5)}}{(\frac{3}{2})^2 e^{-3(1-0.5\cdot 0.5)} + (\frac{1}{2})^2 e^{-3(1-0.5\cdot 1.5)}} > 0.6.$$

Last, for $s \geq \frac{1}{2}$ it holds that

$$U(0.036|s;\beta^*) = \frac{\frac{3}{2}\frac{e^{-3(1-0.5\cdot 0.5)}-e^{-3}}{3\cdot 0.5\cdot 0.5}[0.964] + \frac{1}{2}\frac{e^{-3(1-0.5\cdot 1.5)}-e^{-3}}{3\cdot 0.5\cdot 1.5}[-0.036]}{\frac{3}{2}+\frac{1}{2}} < 0.06,$$

and

$$\lim_{\epsilon \searrow 0} U(0.036+\epsilon|s;\beta^*) = \frac{\frac{3}{2}\frac{e^{-3(1-0.5\cdot 0.5)}-e^{-3}}{3\cdot 0.5\cdot 0.5}[0.964] + \frac{1}{2}\frac{e^{-3(1-0.5\cdot 1.5)}-e^{-3}}{3\cdot 0.5\cdot 1.5}[-0.036]}{\frac{3}{2}+\frac{1}{2}} > 0.07,$$

such that all high signals prefer to follow the ODE (34).

Hence, β^* is an equilibrium. Further, all inequalities are strict, and utilities are continuous in the payment, such that there is a continuum of equilibria with different pooling bids around 0.036.

C.3 Equilibria with atoms—continuous signals

In this subsection we construct another example of equilibrium multiplicity. Suppose that $v_\ell = 0$, $v_h = 1$ and both states are equally likely. Let the signal space be $[0, 1]$ and suppose that for $s \in [0.36, 0.37]$

$$f_h(s) = \frac{2s}{100} \qquad f_\ell(s) = \frac{3-4s}{100}$$
$$F_h(s) = \frac{199+16s^2}{1600} \qquad F_\ell(s) = \frac{395+24s-16s^2}{800},$$

as well as $\frac{f_h(0)}{f_\ell(0)} = \frac{1}{4}$. The details of the rest of the distribution is arbitrary, as long as the MLRP is fulfilled. Let $\eta = 7$.

We want to show that there is a continuum of equilibria which has an atom at the bottom and is strictly increasing above. An equilibrium of this form, β^*, is characterized by a cutoff $\hat{s}$ and a bid b_p, such that $\beta^*(0) = \beta^*(\hat{s}) = b_p$ whereas $\beta^*(s)$ follows ODE (34) for $s > \hat{s}$ with initial value $\beta^*(\hat{s}) = b_p$. We restrict attention to equilibria where

$\hat{s} \in [0.361, 0.365]$. A combination $(\hat{s}, b_p)$ describes an equilibrium if bidders with signal $\hat{s}$ are indifferent between bidding b_p and marginally higher bid. Further, bidders with the lowest signal $s = 0$ can have no incentive to deviate to 0, and that $\beta^*(s)$ has to be strictly increasing above $\hat{s}$.

For $\hat{s} \in [0.361, 0.365]$ a bidder with signal $\hat{s}$ is indifferent between b_p and a marginally higher bid if $U(b_p|\hat{s}; \beta^*) = \lim_{\epsilon \searrow 0} U(b_p + \epsilon|\hat{s}; \beta^*)$, which rearranges to

$$b_p = \frac{f_h(\hat{s})(e^{-7(1-F_h(\hat{s}))} - \frac{e^{-7(1-F_h(\hat{s}))}-e^{-7}}{7F_h(\hat{s})})}{f_h(\hat{s})(e^{-7(1-F_h(\hat{s}))} - \frac{e^{-7(1-F_h(\hat{s}))}-e^{-7}}{7F_h(\hat{s})}) + f_\ell(\hat{s})(e^{-7(1-F_\ell(\hat{s}))} - \frac{e^{-7(1-F_\ell(\hat{s}))}-e^{-7}}{7F_\ell(\hat{s})})}.$$

Plugging in $F_h(\hat{s}), F_\ell(\hat{s})$, the indifference gives rise to an increasing function $b_p(\hat{s})$ with $b_p(0.361) = 0.0151769$, and $b_p(0.365) = 0.0154979$.

Bidders with signal $s = 0$ prefer $b_p(\hat{s})$ over 0 if $U(0|0; \beta^*) \leq U(b_p(\hat{s})|0; \beta^*)$. This rearranges to

$$b_p(\hat{s}) \leq \frac{\frac{1}{4}(\frac{e^{-7(1-F_h(\hat{s}))}-e^{-7}}{7F_h(\hat{s})} - e^{-7})}{\frac{1}{4}\frac{e^{-7(1-F_h(\hat{s}))}-e^{-7}}{7F_h(\hat{s})} + \frac{e^{-7(1-F_\ell(\hat{s}))}-e^{-7}}{7F_\ell(\hat{s})}},$$

where the right side is larger than 0.0155 for $\hat{s} \in (0.361, 0.365)$. Thus, the condition is never violated.

Next, one can check that $\mathbb{E}[v|s_{(1)} = s, s]$ is increasing on $s \in [\hat{s}, 1]$ because inequality (10) holds. Hence, the ODE (34) is strictly increasing on $[\hat{s}, 1]$ if $b_p(\hat{s}) \leq \mathbb{E}[v|s_{(1)} = \hat{s}, \hat{s}]$. This is the case if

$$b_p(\hat{s}) \leq \frac{f_h(\hat{s})^2 e^{-7(1-F_h(\hat{s}))}}{f_h(\hat{s})^2 e^{-7(1-F_h(\hat{s}))} + f_\ell(\hat{s})^2 e^{-7(1-F_\ell(\hat{s}))}}.$$

The right side is increasing in $\hat{s}$, and equal to 0.0152206 for $\hat{s} = 0.361$ and equal to 0.0158707 for $\hat{s} = 0.365$. Indeed, one can check that the upper bound is never violated for $\hat{s} \in (0.361, 0.365)$

Hence, we constructed a continuum of equilibria: For any $\hat{s} \in (0.361, 0.365)$, there is an equilibrium in which all signals $s \leq \hat{s}$ pool on $b_p(\hat{s})$, and all higher signals follow a strictly increasing strategy.

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