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Strategies under Strategic Uncertainty

Helene Mass*

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*University of Bonn, hmass@uni-bonn.de

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Strategies under Strategic Uncertainty

Helene Mass*

Abstract

I investigate the decision problem of a player in a game of incomplete information who faces uncertainty about the other players' strategies. I propose a new decision criterion — the rational maximin criterion — which works in two steps. First, I assume common knowledge of rationality and eliminate all strategies which are not rationalizable. Second, I apply the maximin criterion. Using this decision criterion, one can derive predictions about outcomes and recommendations for players facing strategic uncertainty. I analyze applications to first-price auctions, contests, and bilateral trade.

JEL classification: D81, D82, D83

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1 Introduction

Consider a game and a player who has to decide on her strategy. It is crucial for her decision to form a conjecture about the other players' strategies. Under the traditional approach the player conjectures that her opponents play according to a Nash equilibrium of the given fame. However, the player may consider more strategies than one particular Nash equilibrium as possible and does not form a unique conjecture. In other words, the player faces strategic uncertainty and cannot derive an optimal strategy. In order to address this issue, I propose a new decision criterion — *the rational*

*University of Bonn, hmass@uni-bonn.de
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maximin criterion — which allows to derive recommendations for an agent who faces strategic uncertainty. The decision criterion works in two steps: First, I assume common knowledge of rationality and eliminate all actions which are not best replies. That is, the set of the other players' possible strategies is restricted to the set of rationalizable strategies. Second, I apply the maximin criterion.

This criterion can be applied to any game of complete or incomplete information and allows to derive recommendations for a player facing strategic uncertainty. I will present applications to first-price auctions, contests and bilateral trade. The novelty of the rational maximin criterion is that it allows to derive meaningful recommendations while making minimal assumptions. The fact that rational players interact in a game with commonly known rules already allows for a meaningful restriction of the set of possible conjectures about other players' strategies. Restricting the set further would require additional common knowledge assumptions. Not making use of this fact would add non-rational conjectures to the set possibly causing overly-pessimistic beliefs and foregone profits.

In their seminal papers Pearce (1984) and Bernheim (1984) have challenged the concept of Nash equilibria. As stated by Pearce (1984), "some Nash equilibria are intuitively unreasonable and not all reasonable strategy profiles are Nash equilibria". They argue that in a one-shot interaction a player will best reply to Nash equilibrium strategies only if she is certain that the other players will employ these strategies. That is, players need to deduce unique conjectures about their opponents' strategies. Similarly, Renou and Schlag (2010) argue that "common knowledge of conjectures" is required in order to justify Nash equilibria as a decision criterion.[1]

However, a player may consider more than one strategy of the other players' as possible, even if everyone is rational and there is common knowledge of this. For example, this can occur under the existence of multiple Nash equilibria without one being focal or salient as claimed by

[1]Bernheim (1984) indicates that a Nash equilibrium is the result of some underlying dynamic process which allows the players to coordinate on a Nash equilibrium. This process, however, is not modeled in one-shot interaction games. In fact, a repeated game would be a new game with possibly different equilibria. Without this dynamic process players may not coordinate on a Nash equilibrium.

Bernheim (1984). But also if there exists a unique Nash equilibrium, a player may not be certain that it is played by her opponents because they have a multitude of rational strategies to choose from. If a player does not observe or does not deduce a unique conjecture about the other players' strategies, she faces strategic uncertainty and cannot derive an optimal strategy. As a solution for this issue I propose the rational maximin criterion which aims at providing a unique conjecture about other players' strategies for a player in a given game. I will now discuss the rational maximin criterion in more detail.

So far, I argued that a player may not know which strategies are adopted by her opponents. But a player may not consider all strategies of the other players as possible. The fact that rational players interact strategically given some commonly known rules of a game (e.g. the rules of a first-price auction), already implies a restriction on the set of possible strategies. Therefore, in the first step of the decision criterion I propose to consider strategies which a player can deduce only from common knowledge of rationality. By definition, a player is rational if her action is a best reply given her type, the commonly known type distribution and a conjecture about the other players' strategies.[2] A strategy which a player assumes to be played by another rational player has to be rational as well, i.e., the action prescribed by a strategy for a given type has to be a best reply given her type, the commonly known type distribution and a conjecture about the other players' strategies. This reasoning continues ad infinitum. Pearce (1984) and Bernheim (1984) (and Battigalli and Siniscalchi (2003b) for games of incomplete information) show that common knowledge of rationality is equivalent to bidders playing *rationalizable strategies*. These are strategies which survive the iterated elimination of actions which are not best replies to some strategy which consists of actions which have not been eliminated in previous elimination rounds.

Since rationalizable strategies are not necessarily unique, an additional step is needed in order to derive a unique conjecture. In the second step I apply the maximin criterion as in Gilboa and Schmeidler (1989). The

[2]Throughout this paper I assume common knowledge of type distributions. The rational maximin criterion could be also extended to uncertainty about type distributions. However, this is beyond the scope of this paper.

application of the maximin criterion can be modeled as a simultaneous two-player zero-sum game against an adverse nature. Therefore, also the rational maximin criterion can be modeled this way. Consider a given game and a player applying the rational maximin criterion. This player plays a zero-sum game against an adverse nature where the adverse nature's action space is restricted to all rationalizable strategy profiles of the other players. The action space of the player is identical to the player's action space in the given game. The utility of the player in the zero-sum game is the expected utility induced from the utility function in the given game, the player's action, her type, and the other players' strategy profile chosen by the adverse nature. The adverse nature's utility is the player's utility multiplied by -1.

The model as a zero-sum game against an adverse nature allows for a subjective belief interpretation of the rational maximin criterion. That is, a player chooses a subjective belief about the other players' strategies and acts optimally given this subjective belief. The first step of the decision criterion determines the set from which a player chooses her subjective belief. The second step determines how the subjective belief is chosen from this set. The subjective belief is given by the adverse nature's equilibrium strategy, in the following called *rational maximin belief*. I will call an action which is a best reply to the rational maximin belief a *rational maximin action*. In order to distinguish the Nash equilibrium in the simultaneous game between a player and the adverse nature and the Nash equilibrium which may exist in a given game , I will refer to the Nash equilibrium in the former case as a *rational maximin equilibrium*.

The following two examples illustrate two different reasons for why strategic uncertainty can occur and how the rational maximin criterion applies under strategic uncertainty. In the first example there exist multiple Nash equilibria without one being salient. In the second example a salient Nash equilibrium exists but is not the unique rationalizable action. In particular, the salient Nash equilibrium is not compatible with actions derived from the maximin utility or minimax regret criterion.

For the first example consider a sender who has to deposit a package either in places A, B or C. A receiver has to decide to which places she sends one or two drivers in order to pick up the package. If the package is picked up, sender and receiver earn each a utility of P and zero otherwise. In addition, the receiver faces a cost of c if a driver travels to place A or B and a cost of $\tilde{c}$ if a driver travels to place C. The game is summarized in the following payoff table:

	A	B	C	AB	AC	BC
A	$P; P-c$	$0; -c$	$0; -\tilde{c}$	$P; P-2c$	$P; P-c-\tilde{c}$	$0; -c-\tilde{c}$
B	$0; -c$	$P; P-c$	$0; -\tilde{c}$	$P; P-2c$	$0; -c-\tilde{c}$	$P; P-c-\tilde{c}$
C	$0; -c$	$0; -c$	$P; P-\tilde{c}$	$0; -2c$	$P; P-c-\tilde{c}$	$P; P-c-\tilde{c}$

It is common knowledge that it holds $P - \tilde{c} < -c$ and $P - 2c > 0$. Assume that sender and receiver interact only once. The pure strategy Nash equilibria in this game are $(A; A)$, $(B; B)$. Moreover, a Nash equilibrium is given if the sender mixes between A and B with probability $\frac{1}{2}$ and the receiver chooses AB. Due to the multiplicity of equilibria, the players may be uncertain which equilibrium strategy to follow, in particular, if they interact for the first time. If c is very small relatively to P, the latter equilibrium seems to be the intuitive one.

Given the multiplicity of the equilibria and the resulting uncertainty, the players may apply the maximin utility criterion. The application of this criterion leaves both players indifferent between actions A and B. The maximin criterion does not yield to action AB for the receiver since by choosing AB she would face the risk that the sender deposits the package in C, leaving the receiver with the costs of two drivers $-2c$. However, the result of the maximin criterion changes after assuming common knowledge of rationality, i.e., if the rational maximin criterion is applied. Excluding actions which are not best replies leads to the elimination of strategies C, AC and BC for the receiver, leading to the elimination of action C for the sender:

	A	B	AB
A	$P; P-c$	$0; -c$	$P; P-2c$
B	$0; -c$	$P; P-c$	$P; P-2c$

Now the maximin criterion leads to action AB for the receiver. In other words, if the receiver anticipates that the sender anticipates that she will never send a driver to C, the application of the maximin criterion leads to action AB. Thus, in this setting the rational maximin criterion serves as an equilibrium selection device which chooses the intuitive equilibrium.

As a second example consider the following payoff table. It illustrates the decision problem of the row player player who is uncertain about which of the possible rationalizable actions her opponent will choose:

	X	Y	Z
A	10;10	0;9	0,0
B	5;1	5;9	0,0
C	4;1	4;9	4;0
D	1;10	6;9	0;0

The unique Nash equilibrium in pure strategies, (A, X), is focal in the sense that it is the social optimum and leads to the highest possible utility for both players. However, a rational column player can also choose Y instead of X. Action Y is rationalizable and moreover, the application of the maximin or the minimax regret criterion would lead to action Y for the column player. In other words, the column player may prefer to get a utility of 9 with certainty instead of aiming for the utility of 10 and risking to get a utility of 1. Given this uncertainty about the column player's strategy, the row player may resort to the application of the maximin criterion. This leads to action C which ensures a utility of 4 for the row player. However, the row player can anticipate that action Z is strictly dominated for the column player. After the elimination of this action, C becomes strictly dominated for the row player. The iterated elimination of actions which are not best replies, i.e., the elimination of actions Z and C, leads to the following payoff table:

	X	Y
A	$10; 10$	$0; 9$
B	5;1	5;9
D	1;10	6;9

Now the application of the maximin criterion leads to action B for the row player. That is, after anticipating that the column player will never play Z, the row player can ensure a utility of 5 instead a utility of 4. Note that the rational maximin criterion prescribes the row player to play B with probability one. However, playing B with probability one is not part of an equilibrium. Therefore, in contrast to the previous example, the rational maximin action is not part of an equilibrium.

These examples show how the rational maximin criterion provides recommendations under strategic uncertainty. Moreover, they show why players may not be able to form unique conjectures about their opponents' strategies and why the application of the maximin utility criterion alone may cause forgone profits. In the rest of this section I will summarize the results of the applications to first-price auctions, Tullock contests and bilateral trade.

The rational maximin belief of a bidder in a first-price auction is that her opponent places the highest rationalizable bid given her valuation. As a result, the bidder never expects to bid against an equal or higher type and resorts to win against a lower type with certainty by placing the highest rationalizable bid of a lower type. If every bidder applies the criterion, the outcome is efficient.

The rational maximin belief of a player in a Tullock contest is that her opponent exerts the highest rationalizable effort given her cost type. In the case of complete information the highest rationalizable effort coincides with the equilibrium effort. Thus, the recommendation derived from the rational maximin criterion (rational maximin action) coincides with the equilibrium. In the case of incomplete information with two types the highest rationalizable efforts are strictly higher than equilibrium efforts. The rational maximin action for the low-cost type is higher than in equilibrium while the rational maximin action of the high-cost type is lower. Tis result indicates whether a contest designer should maintain or dissolve strategic uncertainty, depending on whether in the given economic setting lowest or highest efforts are relevant.

Finally, I apply the rational maximin criterion to a bilateral trade setting. Buyer and seller simultaneously submit a bid and a reserve price. If the bid exceeds the reserve price, trade takes place with a price which is a weighted average of bid and reserve price. I consider an incomplete informations setting where buyer and seller have each two different types. Strategic uncertainty causes buyer and seller to expect zero utility. As a consequence, under the rational maximin criterion they are indifferent between a continuum of actions. The extent of strategic uncertainty shouldn't come as a surprise since a continuum of equilibria exists causing a continuum of strategies, including extreme ones, to be rationalizable. Depending on the parameter constellation it may be beneficial for the buyer to commit to bid a minimum amount. This reduces the seller's strategic uncertainty making the seller's actions more predictable for the buyer. This implies that at least the buyer with a high valuation expects a strictly positive utility. Analogously, it may be beneficial for the seller to commit to set a maximal reserve price.

Relation to the literature

I will structure the literature review into three different strands: the literature on decision criteria for strategic uncertainty, on the maximin criterion, and the literature on rationalizability.

Several papers propose equilibrium concepts for players facing strategic uncertainty.[3] Bich (2016) proposes the concept of *prudent equilibrium* for players who face strategic uncertainty in a given game. A prudent player maximizes her minimum utility given that the other players choose actions in a neighborhood around a given strategy profile. Strategic uncertainty is parametrized by a function assigning different weights to different deviations. In contrast to this paper, some knowledge about the other players' strategy is required. This comment also applies to other equilibrium concepts under

[3] I will leave out papers which propose concepts not compatible with common knowledge of rationality.

strategic uncertainty, e.g. quantile response equilibrium.

Dow and da Costa Werlang (1994), Marinacci (2000), and Eichberger and Kelsey (2000) study equilibria where players can have different degrees of uncertainty (or uncertainty aversion), modeled by Choquet expected utility with a non-additive probability measure. Klibanoff (1993) studies equilibrium outcomes where players maximize their minimum payoff given a set of possible conjectures about the opponent's strategies. Renou and Schlag (2010) analyze strategic uncertainty using the minimax regret criterion where the set of possible strategies is restricted exogenously. Kasberger and Schlag (2017) use the minimax regret criterion in first-price auctions and allow for the possibility that a bidder can impose bounds on the other bidders' bids or valuation distributions. For example, they consider the case where a bidder can impose a lower bound on the highest bid.[4] The crucial difference of this strand of literature to this paper is that the set of possible conjectures about the other players' strategies is either determined by some additional knowledge about equilibrium play or is exogenously given. Inostroza and Pavan (2017) use a concept similar to the rational maximin criterion in global games where a prime minister expects the agents to act according to the "most aggressive rationalizable profile".

The axiomatization of the rational maximin criterion is provided in Gilboa and Schmeidler (1989). In Bergemann and Schlag (2008) both criteria are applied to a monopoly pricing problem where a seller faces uncertainty about the buyer's valuation distribution. Since the seller knows that the buyer will obtain the good if the price is equal or lower than her valuation, the seller does not face strategic uncertainty. In Lo (1998), Salo and Weber (1995), and Chen et al. (2007) the rational maximin criterion is applied to first-price auctions under distributional uncertainty. These three papers use

[4] In their literature on robust mechanism design Dirk Bergemann and Stephen Morris consider the problem of a social planner facing uncertainty about the players' actions. In Bergemann and Morris (2005) a social planner can circumvent uncertainty about the players' strategies by choosing ex-post implementable mechanisms. Bergemann and Morris (2013) provide predictions in games independent of the specification of the information structure. In order to do so, they characterize the set of Bayes correlated equilibria. An application of this concept to first-price auctions is carried out in Bergemann et al. (2017). In contrast to this strand of literature, I study the strategic uncertainty a player is facing in a given game.

Bayes-Nash equilibria as a solution concept, that is, the issue of strategic uncertainty is not addressed. Bose et al. (2006) derive the optimal auction in a setting where seller and bidders may face different degrees of ambiguity, that is, they may face different sets of possible valuation distributions. Carrasco et al. (2018) consider a seller facing a single buyer. The set of distributions the seller considers to be possible is determined by a given support and mean. In these two papers strategic uncertainty is not an issue since the seller chooses a strategy-proof direct mechanism. A crucial difference between the application of the maximin criterion in the papers mentioned and in this paper is that the application to *strategic uncertainty* allows for an endogenous restriction of the set of possible conjectures while under *distributional uncertainty* and exogenous restriction is required.

The concept of rationalizable strategies has been first introduced by Bernheim (1984) and Pearce (1984) for games with complete information. Battigalli and Siniscalchi (2003b) extend rationalizability to games of incomplete information. Bergemann and Morris (2017) propose a concept of rationalizability in settings with incomplete information where agents may not know their own payoff-type. An application to first-price auctions has been carried out by Dekel and Wolinsky (2001). They apply rationalizable strategies to a first-price auction with discrete private valuations and discrete bids. They present a condition on the distribution of types under which the only rationalizable action is to bid the highest bid below valuation. Battigalli and Siniscalchi (2003a) assume that valuation distributions in a first-price auction are common knowledge but not the strategies of the bidders. They characterize the set of rationalizable actions under the assumption of *strategic sophistication*, which implies common knowledge of rationality and of the fact that bidders with positive bids win with positive probability. They find that for a bidder with a given valuation θ all bids in an interval $(0, b^{max}(\theta))$ are rationalizable where $b^{max}(\theta)$ is higher than the Bayes-Nash equilibrium bid. Kashaev and Salcedo (2019) propose conditions under which one can determine whether observed data could have been generated by any solution concept stronger than rationalizability, including the rational maximin criterion.

2 Model

Underlying game of incomplete information The starting point of the model is a game of incomplete information which is denoted by $\left(\{1,\ldots,I\},\Theta,A,\{u_i\}_{i\in\{1,\ldots,I\}}\right)$ where $\{1,\ldots,I\}$ is the set of players and for every $i\in\{1,\ldots,I\}$, $A_i\subseteq\mathbb{R}$ is the set of possible actions and $\Theta_i\subseteq\mathbb{R}$ is the set of possible privately known types of player i. A and Θ are defined by $A=A_1\times\ldots\times A_I$ and $\Theta=\Theta_1\times\ldots\times\Theta_I$. A *pure strategy* of player i is a mapping

$$\beta_i:\Theta_i\to A_i$$

$$\theta_i\mapsto a_i.$$

The set S_i is the set of all pure strategies of player i. A *(mixed) strategy* of player i is a mapping

$$\beta_i:\Theta_i\to\Delta A_i$$

$$\theta_i\mapsto a_i$$

where ΔA_i is the set of probability distributions on A_i. In the following $g_{\theta_i}^{\beta_i}$ will denote the density of the bid distribution $\beta_i(\theta_i)$ and $supp(\beta_i(\theta_i))$ its support.[5] Let

$$u_i:A\times\Theta_i\to\mathbb{R}$$

$$(a_1,\ldots,a_I,\theta_i)\mapsto u_i(a_1,\ldots,a_I,\theta_i)$$

denote the utility function of player i. That is, I consider a setting with private valuations.

Although the rational maximin criterion could be extended to distributional uncertainty in a natural way, in this paper I focus only on strategic uncertainty. That is, the players' type distribution is common knowledge.[6]

[5] A pure strategy can be interpreted as distribution of bids which puts probability weight 1 on one bid. I abuse notation since in the case of a pure strategy, $\beta_i(\theta_i)$ denotes an element in A_i while in the case of a (mixed) strategy $\beta_i(\theta_i)$ denotes an element in ΔA_i. However, in the following it will be clear whether β_i is a pure or a mixed strategy.

[6] Under distributional uncertainty a player does not know the type distribution of her opponents but instead considers a set of possible type distributions. In an extension of the rational maximin criterion to distributional uncertainty the adverse nature would choose not only a rationalizable strategy profile but also a type distribution of the other players.

For a given type distribution

$$F : \Theta \to [0, 1]$$

and a profile of strategies $(\beta_1, \ldots, \beta_n)$ the expected utility of a player i is defined by [7]

$$\begin{aligned} &U_i\left(\theta_i, \beta_i\left(\theta_i\right), \beta_{-i}, F_{-i}\right) \\ &= \int_{\theta_{-i}} \int_{a_{-i}} u_i\left(a_1, \ldots, a_i, \ldots, a_I, \theta_i\right) \prod_{j \neq i} g_{\theta j}^{\beta_j}\left(a_j\right) d\theta_{-j} dF_{-i}\left(\theta_{-i} | \theta_i\right) d\theta_{-i}. \end{aligned} \quad (1)$$

Action space of adverse nature In order to formalize the rational maximin criterion, a new player, denoted by n, is introduced, representing the adverse nature a player i applying the criterion faces. Players i and n play a two-player simultaneous zero-sum game, called *game under strategic uncertainty.* The first step of a formal description of this game is the definition of the adverse nature's action space. Under common knowledge of rationality the adverse nature is restricted to choose from rationalizable strategy profiles.

Rationalizable strategies The assumption of common knowledge of rationality leads to the following reasoning. Every player i maximizes her expected utility given her type, the type distribution F and a conjecture about the other players' strategies. The strategy which player i assumes is played by some player $j \neq i$ has also to be compatible with common knowledge of rationality. Therefore, for every possible type of player j, the action prescribed by the strategy assumed by player i maximizes j's expected utility given her type, the type distribution F and a conjecture about the other players' strategies. Again, player j's conjecture has to be compatible with common knowledge of rationality. This reasoning continues ad infinitum.[8]

[7] For a vector $(v_1, \ldots, v_I)$ I denote by v_{-i} the vector $(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_I)$.

[8] As stated above, under strategic uncertainty a rational player acts optimally given a conjecture about the other players' strategies (and a conjecture about the other players' type distributions if also distributional uncertainty is present). Instead of "conjecture" other terms have been used in economic literature, e.g. belief, subjective

Given the type of a player, an action which is compatible with common knowledge of rationality is called *rationalizable.* Battigalli and Siniscalchi (2003b) have shown that it is equivalent to define a rationalizable action for a given type as follows.

Definition 1.

(i) *Let $i \in \{1, \ldots, I\}$ be a player and $\theta_i \in \Theta_i$ be a type of player i. The set of* rationalizable actions *for player i is defined as follows. Set $RS_i^1(\theta_i) := A_i$. Assume that for $k \in \mathbb{N}$ the set $RS_i^k(\theta_i)$ is already defined. Then the set $RS_i^{k+1}(\theta_i)$ is defined as the set of all elements a_i in A_i for which there exists a strategy profile β_{-i} of the other players such that it holds*

(I) $a_j \in supp\left(\beta_j\left(\theta_j\right)\right)$ *for* $\theta_j \in \Theta_j \;\Rightarrow\; a_j \in RS_j^k\left(\theta_j\right)$ *for all* $j \neq i$

(II) $a_i \in \underset{a_i' \in A_i}{\operatorname{argmax}}\; U_i\left(\theta_i, a_i', \beta_{-i}, F_{-i}\right)$

and $RS_i(\theta_i)$ is given by

$$RS_i\left(\theta_i\right) = \bigcap_{k \geq 1} RS_i^k\left(\theta_i\right).$$

(ii) *A strategy β_i of a player i is* rationalizable *if for every $\theta_i \in \Theta_i$ every action a_i with $a_i \in supp\left(\beta_i\left(\theta_i\right)\right)$ is rationalizable, i.e., an element of $RS_i\left(\theta_i\right)$.*

(iii) *For a player i let RS_{-i} be the set of rationalizable strategies of all players besides player i.*

The intuition behind this definition is that an action for a player which is consistent with common knowledge of rationality, i.e., a rationalizable action, is an action which survives the iterated elimination of actions which are not best replies. An action is a best reply if it maximizes the player's expected utility given her type, the commonly known type distribution F and a conjecture about the other players' strategies which prescribe actions

prior, assumption, assessment ect. I use the term conjecture as proposed in Bernheim (1984).

that have not been eliminated yet. A strategy is rationalizable if all actions in its support are rationalizable.

The definition of rationalizable strategies allows for a formal definition of the adverse nature's action space and therefore for a formal definition of the simultaneous game against the adverse nature – the game under strategic uncertainty.

Simultaneous game against adverse nature The following definition summarizes all components describing a game under strategic uncertainty.

Definition 2. *A* game under strategic uncertainty *consists of an underlying game of incomplete information, denoted by*

$$\left(\{1,\ldots,I\},\Theta,A,\{u_i\}_i\in\{1,\ldots,I\}\right),$$

a player $i\in\{1,\ldots,I\}$ applying the rational maximin criterion, and a player n. Player i chooses a strategy

$$\beta_i:\Theta_i\to\Delta A_i.$$

and player n chooses for every type of player i a strategy of the other players:[9]

$$\beta_n:\Theta_i\to RS_{-i}$$
$$\theta_i\mapsto\beta_{-i}^{\theta_i}.$$

Here the superscript θ_i indicates that the adverse nature chooses the other players' strategies depending on player i's type, i.e., the game under strategic uncertainty is a game under complete information. The utility of a player $i\in\{i_1,\ldots,i_k\}$ is given by

$$U_i\left(\theta_i,\beta_i\left(\theta_i\right),\beta_{-i}^{n_i,\theta_i},F_{-i}\right)$$

which is defined as in (1) and is induced by the utility function of player i

[9]I assume that the adverse nature is restricted to choose uncorrelated strategies. However, this restriction will turn out to be without loss for the applications.

in the underlying game of incomplete information, denoted by u_i:

$$u_i : A \times \Theta_i \to \mathbb{R}.$$

$$(a_1, \dots, a_I, \theta_i) \mapsto u_i (a_1, \dots, a_I, \theta_i).$$

The utility of player nature is given by

$$-\sum_{j=1}^{k} U_i \left(\theta_i, \beta_i (\theta_i), \beta_{-i}^{n_i, \theta_i}, F_{-i} \right).$$

Since the other players' strategies the adverse nature chooses for a player $i \in \{1, \dots, I\}$, are not observed by a player $j \neq i$, the adverse nature faces an independent minimization problem for every player applying the rational maximin utility criterion. Equivalently, one could introduce an additional adverse nature for every player applying the rational maximin criterion. Therefore, for a given player the application of the criterion does not depend on whether it is applied by other players as well.

Note that after specifying the player applying the rational maximin criterion, a given game of incomplete information uniquely defines a game under strategic uncertainty. Thus, throughout the remainder of the paper it will be assumed that after specifying the underlying game of incomplete information, a game under strategic uncertainty is given without explicitly stating all its ingredients.

Now it is possible to define a rational maximin strategy in a game under strategic uncertainty which can be seen as a recommendation for a player facing strategic uncertainty.

Definition 3. *In a game under strategic uncertainty for a player i a strategy*

$$RM_i : \Theta_i \to \Delta A_i$$

is a rational maximin strategy *if there exists a Nash equilibrium in the simultaneous game between nature and player i such that RM_i is player i's equilibrium strategy. The action (or mixed strategy) given by $RM_i (\theta_i)$ is called a* rational maximin action for type *θ_i.*

The Nash equilibrium in the simultaneous game between nature and *player i is called* rational maximin equilibrium*.*

As described above, such a maximin strategy has two properties. First, if a player would not choose an action according to a maximin strategy, then there would exist a rationalizable strategy of the other players under which the player's expected utility is lower than under the action prescribed by a maximin strategy. Second, the strategy chosen by nature can be interpreted as the player's subjective belief about the state of the world against which she maximizes her expected utility given her type. The second property is formalized in the following definition.

Definition 4. *In a game under strategic uncertainty let β_n be the adverse nature's equilibrium strategy. A* rational maximin belief *of player i with valuation θ_i is defined as*

$$\beta_n(\theta_i) = \beta^{\theta}_{-i},$$

that is, the adverse nature's rational maximin equilibrium strategy evaluated at θ_i.

Note that neither the rational maximin belief of player nor her rational maximin strategy are necessarily unique. However, every best reply of a player i to any rational maximin belief (or equivalently every rational maximin strategy) induces the same expected utility for player i.

3 Rationalizable strategies

This section collects sufficient conditions for actions to be rationalizable which will be useful in the rest of the paper. The first condition is presented in the following lemma which states that best replies to rationalizable strategies are again rationalizable.

Lemma 1. *In a game of incomplete information let $i \in \{1, \ldots, I\}$ be a player with valuation θ_i and for every $j \in \{1, \ldots, I\} \setminus \{i\}$ let β_j be a rationalizable strategy for player j. Let $a_i \in A_i$ be a best reply to β_{-i}, i.e., it holds that*

$$a_i \in \underset{a'_i \in A_i}{\operatorname{argmax}} \; U_i(\theta_i, a'_i, \beta_{-i}, F_{-i}),$$

then $a_i \in RS_i(\theta_i)$, that is, a_i is a rationalizable action for player i with valuation θ_i.

This lemma follows from the definition of rationalizable strategies. Formally, one can show per induction that for every $k \in \mathbb{N}$ and for every $a_i \in supp\left(\beta_i\left(\theta_i\right)\right)$ it holds that a_i is an element in $RS_i^k\left(\theta_i\right)$.

Recall that a rational maximin strategy for a player is the equilibrium strategy played by the player in a game under strategic uncertainty, i.e., in the simultaneous zero-sum game between a player and the adverse nature. A direct consequence of Lemma 1 is that rational maximin strategies are rationalizable.

Corollary 1. *In a game under strategic uncertainty let β_i be a rational maximin strategy. Then β_i is a rationalizable strategy for player i.*

Besides providing an additional sufficient condition for strategies to be rationalizable, this result is desirable since it ensures that the strategy prescribed by the rational maximin criterion is compatible with common knowledge of rationality which is assumed throughout the paper.

Another sufficient condition for an action to be rationalizable is that it is played in a Bayes-Nash equilibrium which is formalized in the following definition and proposition.

Definition 5. *In a game of incomplete information a strategy profile $(\beta_1, \ldots \beta_I)$ together with a joint type distribution $\hat{F}$ is a* Bayes-Nash equilibrium with a common prior *if for every $i \in \{1, \ldots, I\}$, every $\theta_i \in \Theta_i$ and every $a_i \in A_i$ such that $a_i \in supp\left(\beta_i\left(\theta_i\right)\right)$ it holds that*

$$a_i \in \underset{a_i' \in A_i}{\operatorname{argmax}}\ U_i\left(\theta_i, a_i', \beta_{-i}, \hat{F}_{-i}\right).$$

That is, every player maximizes her expected utility given the other players' strategies and the other players' commonly known type distributions.

Proposition 1. *Let the profile of strategies $(\beta_1, \ldots, \beta_I)$ together with a joint type distribution $\hat{F}$ constitute a Bayes-Nash equilibrium with a common prior of a game of incomplete information. Then for every $i \in \{1, \ldots, I\}$ the strategy β_i is rationalizable.*

It follows from Proposition 1 and Lemma 1 that a best reply to strategies played in a Bayes-Nash equilibrium is rationalizable. This constitutes another sufficient condition for an action to be rationalizable:

Corollary 2. *Let the profile of strategies $(\beta_1, \ldots, \beta_I)$ together with a joint type distribution $\hat{F}$ constitute a Bayes-Nash equilibrium with a common prior of a game of incomplete information. Let $i \in \{1, \ldots I\}$ be a player with valuation θ_i and let $a_i \in A_i$ be a best reply to β_{-i} and some distribution of the other players' types $F_{-i}{}' \in \Delta_{\Theta_{-i}}$, i.e., it holds that*

$$a_i \in \operatorname*{argmax}_{a_i' \in A_i} U_i\left(\theta_i, a_i', \beta_{-i}, F_{-i}'\right),$$

then $a_i \in RS_i(\theta_i)$, that is, a_i is a rationalizable action for player i with valuation θ_i.

After presenting the formal model and sufficient conditions for strategies to be rationalizable, we can now turn to the analysis of applications. I study three applications: first-price auctions, Tullock contests and bilateral trade. Instead of a rigorous analysis of all applications, I rather provide an analysis of simplified models with few types. This allows to explain in detail how the decision criterion applies and what the main intuition of the results is.

4 First-price auctions under strategic uncertainty

In this section I analyze how the proposed decision criterion applies to first-price auctions. As mentioned above, a game of incomplete information uniquely defines a game under strategic uncertainty, i.e., a game between a player applying the rational maximin criterion and an adverse nature. Thus, in order to specify the general model for first-price auctions, it is left to formally describe the underlying game of incomplete information.

Underlying game of incomplete information I consider a simplified model with two bidders A and B who draw their types from a finite type space Θ where the lowest type is normalized to zero. Every bidder chooses a bid on some (arbitrarily fine) bid grid $\mathscr{G}$ on an interval $[0, G]$ with $G \geq \max\{\theta \mid \theta \in \Theta\}$.[10]

For every $b \in (0, G]$ with $b > 0$ there exists a predecessor in $\mathscr{G}$ denoted by

$$b^{-} = \max_{b' \in \mathscr{G}} b' < b$$

and for every $b \in [0, G)$ with $b < G$ there exists a successor in $\mathscr{G}$ denoted by

$$b^{+} = \min_{b' \in \mathscr{G}} b' > b.$$

Thus, a strategy of a bidder is given by

$$\beta : \Theta \rightarrow \Delta \mathscr{G}.$$

In a first-price auction the bidders submit bids, the bidder with the highest bid wins the object and pays her bid. In addition, it holds an efficient tie-breaking rule.[11] Formally, the utility of bidder i with valuation θ_i and bid b_i given bidder j's bid b_j is denoted by

$$u_i\left(\theta_i, b_i, b_j\right) = \begin{cases} \theta_i - b_i & \text{if } b_i > b_j \\ \theta_i - b_i & \text{if } b_i = b_j \text{ and } \theta_i > \theta_j \\ \frac{1}{2}\left(\theta_i - b_i\right) & \text{if } b_i = b_j \text{ and } \theta_i = \theta_j \\ 0 & \text{if } b_i = b_j \text{ and } \theta_i < \theta_j \\ 0 & \text{if } b_i < b_j \end{cases}$$

where θ_j denotes the valuation of bidder j.

The bidders' valuations are identically and independently distributed

[10] A finite grid is used for the set of all possible bids instead of the interval $[0, G]$ because of the following reason: assume bidders A and B have the same valuation θ. If bidder A bids some amount $b < \theta$, one has to identify the smallest bid which is strictly higher than b since this would be the unique best reply of bidder B. This allows a more formal analysis than using expressions like "bidding an arbitrarily small amount more than b". The grid is assumed to be finite in order to ensure that any subset of the bid grid is compact.

[11] The core statements in the results do not depend on the choice of the tie-breaking rule, i.e., under a random tie-breaking rule for every bidder and every valuation the bid prescribed by the maximin strategy would change by at most one step on the bid grid.

according to a distribution function

$$F : \Theta \to [0, 1]$$

with probability density function

$$f : \Theta \to [0, 1].$$

I assume that the bid grid is sufficiently fine and on every type there is sufficiently much weight such that tying with another bidder is never optimal.

Application of the rational maximin criterion Assume that bidder A applies the proposed decision criterion. By definition, bidder A plays a simultaneous zero-sum game against an adverse nature where the adverse nature chooses a rationalizable strategy of bidder B. In any equilibrium of this game, i.e., in any rational maximin equilibrium, the adverse nature chooses a strategy such that bidder B bids as high as possible and hence, the adverse nature chooses a strategy such that bidder B bids the highest rationalizable bid given her type. In other words, according to the rational maximin belief of bidder A, bidder B bids the highest rationalizable bid given her type.[12]

In order to compute the highest rationalizable bid for every type, the following lemma will be useful. It can be shown by double induction with respect to types and bids.[13]

Lemma 2. *For every type $\theta_k \in \Theta$ it holds that every bid $b > \theta_k$ is not rationalizable for type θ_k.*

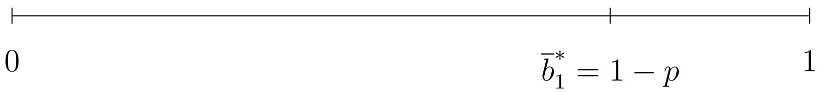
Application of the rational maximin criterion - two types In order to gain some intuition on how the rational maximin criterion applies to

[12]Note that for the adverse nature it is not a weakly dominant strategy to pick this strategy of bidder B. If bidder A would bid above her valuation, then a best reply of the adverse nature would be that bidder B bids zero causing bidder A to have negative utility. However, in a rational maximin equilibrium bidder A never bids above her own valuation.

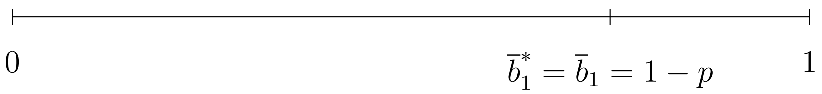
[13]The existence of a unique highest rationalizable bid for every type follows from the compactness of $\overline{B}$.

first-price auctions, I start the analysis with the simplest possible case: a setting with two bidders A and B where it is common knowledge that they can have either a valuation of zero with probability p or a valuation of 1 with probability $1-p$ where $p \in (0,1)$.

It will be convenient to look at the unique Bayes-Nash equilibrium first. In this equilibrium, both 0-types bid zero and both 1-types mix on the interval $[0, \overline{b}_1^*]$ with $\overline{b}_1^* = 1-p$.



By Lemma 2, for type 1 it is not rationalizable to bid above 1. However, it is also not rationalizable for a 1-type to bid close to 1. If a bidder with valuation 1 bids zero, she gets an expected utility of p, since by Lemma 2 she can be certain to win against type 0. Hence, bidding too close to the own valuation cannot be rational for a 1-type. Thus, even if a bidder with valuation 1 wins with probability 1 , it cannot be rationalizable to bid above $1-p$. Bidding zero yields to an expected utility of p for the 1-type while winning with probability 1 and bidding above $1-p$ yields to an expected utility less than p. Since $1-p$ is played in a Bayes-Nash equilibrium, it follows from Proposition 1 that $1-p$ is rationalizable. Hence, this has to be the highest rationalizable bid, denoted by $\overline{b}_1$.



In other words, the rational maximin belief of both types of bidder A is that type zero of bidder B bids zero and type 1 bids $\overline{b}_1$. This allows for a computation of the rational maximin strategy which is a best reply to this belief: type zero bids zero and type 1 bids also zero, which is denoted by $RM_A(0) = 0$ and $RM_A(1) = 0$. Type 1 of bidder A cannot bid above $\overline{b}_1$ since this is not rationalizable by definition. This bid is only rationalizable given the belief that a bidder wins with probability 1 if placing this bid and therefore tying with type 1 of bidder B is not rationalizable for bidder A either. As a result, type 1 of bidder A resorts to win against type 0 of bidder B.

For the case with two possible valuations the highest rationalizable bid of a bidder with the higher valuation coincides with the highest bid played in the unique Bayes-Nash equilibrium. We will see below that with more than two valuations the highest rationalizable bid of a type is strictly higher than the highest bid played in the unique Bayes-Nash equilibrium.

Application of the rational maximin criterion - three types Now we turn to the setting with two bidders who can have type 0 with probability p, type θ with probability q and type 1 with probability $1-p-q$ where $0<\theta<1$. The analysis of this case requires more complicated techniques than the setting with two types. These techniques are also applicable to a setting with arbitrarily many finite types. The rest of the section is structured as follows: first I will prove two lemmata which provide additional results about which bids are rationalizable. Then I will establish the highest rationalizable bids for every type. As a corollary, one can derive the rational maximin action for every type.

For the rest of the section let $\overline{b}_\theta$ denote the highest rationalizable bid of type θ and $\overline{b}_1$ denote the highest rationalizable bid of type 1.

Lemma 3. *Bidding zero is rationalizable for types θ and 1.*

Proof. Assume a bidder with type θ conjectures that her opponent employs the following strategy: type zero bids zero, type θ bids $\overline{b}_\theta$ and type 1 bids $\overline{b}_1$. The unique best reply to this strategy profile is to bid zero. Thus, bidding zero for type θ is a best reply to a strategy which is rationalizable by definition and it follows from Lemma 1 that bidding zero is rationalizable for type θ.

Assume a bidder with type 1 conjectures that her opponent employs the following strategy: types zero and θ bid zero and type 1 bids $\overline{b}_1$. Since this is a rationalizable strategy and bidding zero is a best reply, it follows from Lemma 1 that bidding zero is rationalizable for type 1. □

Lemma 4. *Every bid in $\left[0,\overline{b}_\theta\right]$ is rationalizable for type θ and every bid in $\left[0,\overline{b}_1\right]$ is rationalizable for type 1.*

Proof. First, I will show by induction with respect to the bids in the grid that every bid in $\left[0,\overline{b}_1\right]$ is rationalizable for type 1. By Lemma 3, zero is rationalizable for type 1. Assume that it has been already shown that every bid smaller or equal than b is rationalizable. Then it is to show that b^+ is also rational if $b^+ < \bar{b}_1$.

Let a bidder with valuation 1 conjecture that her opponent employs the following strategy: type zero bid zero, type θ bids $\bar{b}_\theta$ and type 1 bids b. By the induction assumption, this is a rationalizable strategy. If b^+ is not a best reply to this strategy, it holds that $1 - b^+ < \max\left\{(p+q)\left(1-\bar{b}_\theta\right), p\right\}$. This implies that no bid higher than b^+ can be rationalizable which is a contradiction to $b^+ < \bar{b}_1$. Thus, b^+ is a best reply to this strategy and since the strategy is rationalizable, it follows from Lemma 1 that b^+ is rationalizable. This completes the induction step.

Second, I will show by induction that every bid in $\left[0, \bar{b}_\theta\right]$ is rationalizable for type θ. By Lemma 3, zero is rationalizable for type θ. Assume that it has been already shown that every bid smaller or equal than b is rationalizable. Then it is to show that b^+ is also rational if $b^+ < \bar{b}_\theta$.

Let a bidder with valuation 1 conjecture that her opponent employs the following strategy: type zero bid zero and types θ and 1 bid b. Du to the first step it is rationalizable for type 1 to bid b and due to the induction assumption it is rationalizable for type θ to bid b. If b^+ is not a best reply to this strategy, it holds that $\theta - b^+ < p\theta$. Analogously as in the first step, a contradiction follows which completes the induction step and the proof. □

Finally, the following proposition determines the highest rationalizable bids for all types:

Proposition 2. *It holds that*[14]

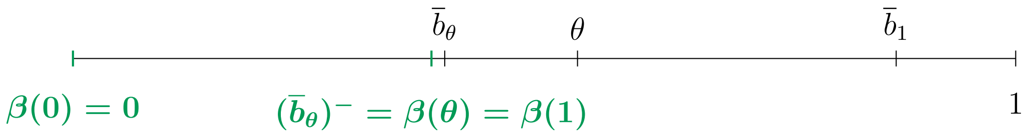
$$\bar{b}_\theta = \theta - p\theta$$

and

$$\bar{b}_1 = 1 - \max\left\{(p+q)(1-\theta+p\theta), p\right\}.$$

Proof. The bid $\bar{b}_\theta$ is a best reply to the following strategy which is illustrated in the following graph: type zero bids zero, types θ and 1 bid $\left(\bar{b}_\theta\right)^-$, i.e., the predecessor of $\bar{b}_\theta$.

[14]Since the bid grid is arbitrarily fine, for simplicity I assume that $\theta - p\theta$ is on the bid grid. Otherwise, $\bar{b}_\theta$ would be the highest grid element which is smaller than $\theta - p\theta$. Similarly, I assume that $1 - \max\left\{(p+q)(1-\theta+p\theta), p\right\}$ is on the bid grid.

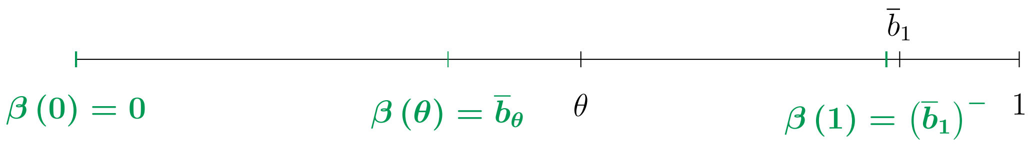


It follows from Lemma 3 that this strategy is rationalizable. Given that $\bar{b}_\theta$ is the highest rationalizable bid, any other strategy does not induce $\bar{b}_\theta$ as a best reply. Intuitively, the strategy inducing $\bar{b}_\theta$ has to maximize the incentives to place this bid. Therefore, types θ and 1 bid just below $\bar{b}_\theta$ while type zero bids the only rationalizable bid, which is zero. Thus, $\bar{b}_\theta$ is determined by the equation

$$\theta - \bar{b}_\theta = p\theta,$$

i.e., it makes the θ-type indifferent between winning with probability one while bidding $\bar{b}_\theta$ and deviating to bidding zero.

Similarly, the strategy inducing $\bar{b}_1$ as a best reply has to maximize the incentives to place this bid and at the same time minimize the incentives to deviate. Thus, the following strategy induces $\bar{b}_1$: type 1 bids $\left(\bar{b}_1\right)$, type θ bids $\bar{b}_\theta$ and type zero bids zero.



The bid of type 1 maximizes the incentives to bid $\bar{b}_1$. Since it is not rationalizable for type θ to place such a high bid, the bid of type θ has to be as high as possible in order to minimize the incentives to deviate to it. Hence type θ places the highest rationalizable bid. Again, type zero has to bid zero. Thus, $\overline{b}_1$ is determined by the equation

$$1 - \overline{b}_1 = \max\left\{(p+q)\left(1 - \overline{b}_\theta\right), p\right\} = \max\left\{(p+q)(1 - \theta + p\theta), p\right\},$$

i.e., it makes the 1-type indifferent between winning with probability 1 while bidding $1 - \overline{b}_1$ and the most profitable deviation. The most profitable deviation is made as unprofitable as possible since all types below 1 bid their highest rationalizable bid. □

Given the rational maximin belief of a bidder, one can determine her rational maximin strategy which is a best reply to this belief and is denoted by RM.

Corollary 3. *It holds that* $RM(0) = 0$, $RM(\theta) = 0$, *and*

$$RM(1) = \max\left\{(p+q)\left(1 - \overline{b}_\theta\right), p\right\}.$$

That is, type zero bids zero, a θ-type does not expect to win against a θ- or a 1-type and bids zero. A 1-type does not expect to win against another 1-type and depending on p and q bids the highest rationalizable bid of a lower type.

I conclude this section by stating some insights which hold for all values of θ and p, q and can be also extended to a setting with an arbitrarily finite number of bidders and types:

Observation 1. *(i) The rational maximin belief of a bidder is not necessarily unique. For example, any strategy where bidders with strictly positive valuation place strictly positive bids, constitutes a rational maximin belief for type zero. However, the conjecture that a bidder places the highest rationalizable bid given her type is the unique conjecture which constitutes a rational maximin belief for every type.*

(ii) The rational maximin strategy of a bidder is unique and a pure strategy (except for a set of parameters with measure zero).

(iii) If every bidder applies the rational maximin criterion, the resulting outcome is efficient, i.e., every bidder who wins the good with positive probability has the maximum valuation.

(iv) With more than two types the highest rationalizable bid of any type besides zero is strictly higher than the highest bid played in the unique Bayes-Nash equilibrium if the probability weight on type zero is sufficiently low. The highest rationalizable bid of type θ is induced by the case that the θ-type wins with probability one which is not possible in the Bayes-Nash equilibrium. The highest rationalizable bid of type 1 is induced by the case where type θ bids her highest rationalizable bid. Therefore, the incentives to deviate to win against type θ are smaller than in the Bayes-Nash equilibrium and the highest rationalizable bid of type 1 is higher than the highest bid played in the Bayes-Nash equilibrium. If the weight on type zero is too high, then the highest rationalizable bids of all types become zero.

(v) The highest rationalizable bid of a type makes this type indifferent between bidding this bid while winning with probability one and the most profitable deviation the highest rationalizable bid of a lower type. The rational maximin action for a type is the most profitable deviation to the highest rationalizable bid of a lower type. This implies that although a bidder with a given type never expects to win against an equal type and resorts to win again lower types, given her rational maximin belief, the bidder expects the same utility as her opponent with an equal type.

5 Contests under strategic uncertainty

Now I will discuss the application to contests. If one would model contests as an all-pay auction, the results would be similar as for first-price auctions: every bidder never expects to win against an equal or higher type and bids the highest rationalizable bid of a lower type. Thus, I analyze Tullock contests where every bidder wins the prize with a probability which is influenced by her effort. As before, we begin by specifying the underlying game of incomplete information.

Underlying game of incomplete information There are two players competing in a contest where the winner gets a prize which is normalized to 1. A player i chooses her effort $e_i \in \mathbb{R}^+$ and her type $c_i \in \Theta_i$ determines the costs she has to spend on exerting effort. If player i exerts effort e_i and player j exerts effort e_j, the probability that player i wins the prize is given by

$$\frac{e_i}{e_i + e_j}.$$

Thus, a strategy of player i is given by

$$\beta_i : \Theta_i \to \mathbb{R}^+$$

and her utility function by

$$u_i : \Theta_i \times \left(\mathbb{R}^+ \times \mathbb{R}^+\right) \to \mathbb{R}^+$$

$$u_i\left(c_i, e_i, e_j\right) = \frac{e_i}{e_i + e_j} - c_i e_i.$$

5.1 Contests with complete information

Application of the rational maximin criterion In order to gain some intuition on how the rational maximin criterion applies, first I carry out the application where $\Theta_i = \Theta_j = c$ with $c > 0$, i.e., I consider a contest with complete information. Since the utility of a player decreases in the other player's effort, the maximin belief of a player is that her opponent exerts the highest rationalizable effort. Thus, the following calculations aim at computing the highest rationalizable effort.

Given that player j exerts effort e_j, the optimal reaction of player i, denoted by e_1^* is determined by the equation

$$\frac{e_j}{\left(e_i + e_j\right)^2} = c$$

which gives

$$e_i^* = \frac{\sqrt{e_j}}{\sqrt{c}} - e_j.$$

This function is illustrated in the following graph for the parameter $c = 0.1$:

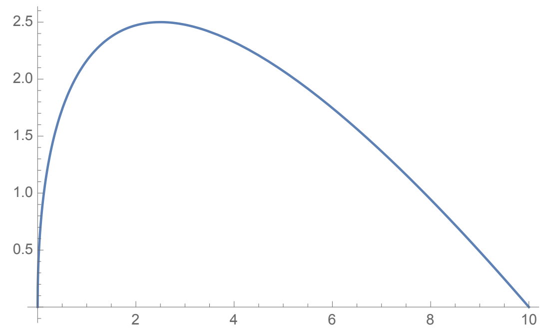


Figure 1: Best reply function

This graph is single-peaked and obtains its maximum at (2.5;2.5). More generally, it holds that for strictly positive c the corresponding graph is single peaked. The effort of player i is maximized at

$$e_j = \frac{1}{4c}$$

which induces an effort of $e_i = \frac{1}{4c}$. Thus, if an effort of $\frac{1}{4c}$ would be rationalizable, this would be the highest rationalizable effort. Solving for the equilibrium gives the unique equilibrium $\left(\frac{1}{4c}; \frac{1}{4c}\right)$. It follows from Proposition 1 that the effort $\frac{1}{4c}$ is rationalizable and is therefore the highest rationalizable effort. Thus, in the case of incomplete information a direct application of the definition of rationalizable actions is not necessary. However, as we will see below, a direct application is also tractable.

Therefore, the maximin belief of a player is that her opponent exerts an effort of $\frac{1}{4c}$ and her best reply to this belief is to also exert an effort of $\frac{1}{4c}$. As a consequence, the action prescribed by the rational maximin criterion coincides with the equilibrium action.

Rationalizable efforts For a complete analysis I will characterize the set of all rationalizable efforts. In order to do so, I will directly follow the definition of rationalizable actions, i.e., iteratively eliminate actions which are not best replies. As already established above, the first round eliminates all efforts above $\frac{1}{4c}$. All efforts below this amount are not eliminated because, as is illustrated by the graph, all efforts in the interval $(0, \frac{1}{4c}]$ are best replies to efforts on the interval $(0, \frac{1}{4c}]$. In other words, the function e_i^* is bijection from $(0, \frac{1}{4c}]$ to $(0, \frac{1}{4c}]$.

The existence of this bijection also implies that after the first round the process of eliminating non-best replies stops. Hence, the set of rationalizable efforts is given by $\left(0, \frac{1}{4c}\right]$. However, only the effort $\frac{1}{4c}$ can occur in equilibrium since the best reply to any lower effort e is higher than e.

We can conclude that in a Tullock contest with complete information the unique equilibrium is *strategically stable*: if a player would deviate to an effort below the equilibrium effort, she could possibly get a higher utility but would also face the risk of getting a lower expected utility while the expected utility in equilibrium is obtained for sure.

5.2 Contest with incomplete information

Now I turn to the case with two possible cost types, i.e., it holds that $\Theta_i = \Theta_j = \{c_L, c_H\}$ with $0 < c_L < c_H$. I assume that both types occur with probability $\frac{1}{2}$. Thus, if player j with type c_k exerts effort e_k^j for $c_k \in \{c_L, c_H\}$, the expected utility of player i with type $c_i \in \{c_L, c_H\}$ is given by

$$u_i\left(c_i, e_i, e_L^j, e_H^j\right) = \frac{1}{2}\left(\frac{e_i}{e_i + e_L^j} P - c_i e_i\right) - \frac{1}{2}\left(\frac{e_i}{e_i + e_H^j} - c_i e_i\right). \qquad (2)$$

Highest rationalizable efforts As before, the expected utility of a player decreases in the effort of her opponent. Therefore, the maximin belief of a player is that her opponent exerts the highest rationalizable effort given her type. Thus, the next step is the calculation of the highest rationalizable efforts. As will become clear, in case of incomplete information the highest rationalizable effort will not be part of an equilibrium. Therefore, in contrast to the case of complete information, I will directly apply the definition of rationalizable actions. To keep the analysis simple, this section is limited to the analysis of the case with two types. However, the rational maximin criterion can be applied to more types using similar techniques.

Proposition 3. *The highest rationalizable effort of type c_L, denoted by $\bar{e}_L$, is given by the unique strictly positive solution of the following equation:*

$$\frac{\frac{1}{4c_H}}{\left(\frac{1}{4c_H} + \bar{e}_L\right)^2} + \frac{1}{4\bar{e}_L} = 2c_L. \qquad (3)$$

The highest rationalizable effort of type c_H is given by $\frac{1}{4c_H}$.

Below I give a rather informal and intuitive proof. The formal proof is relegated to the appendix.

Proof. Consider a player i with a high cost c_H. Recall that in the case of one type the highest rationalizable effort of type c_H was

$$\frac{1}{4c_H}.$$

I claim that in case of two types the highest rationalizable effort of type c_H cannot be higher than $\frac{1}{4c_H}$. Assume that there exists an effort $\hat{e}_i > \frac{1}{4c_H}$ which is a best reply for player i to some strategy of player j denoted by $\left(e_L^j, e_H^j\right)$. Independent of player j's strategy the terms

$$\frac{\hat{e}_i}{\hat{e}_i + e_L^j} - c_H \hat{e}_i$$

and

$$\frac{\hat{e}_i}{\hat{e}_i + e_H^j} - c_H \hat{e}_i$$

would increase if instead of $\hat{e}_i$ one would plug in some effort e_i' with $\frac{1}{4c_H} \leq e_i' < \hat{e}_i$. Thus, the whole expression in (2) would increase. Hence, $\hat{e}_i$ with $\hat{e}_i > \frac{1}{4c_H}$ cannot be a best reply.

As a consequence, we can eliminate all efforts above $\frac{1}{4c_H}$ for the high-cost type and analogously all efforts above $\frac{1}{4c_L}$ for the low-cost type. Any effort $e \leq \frac{1}{4c_i}$ remains a best reply for type c_i. In order to show this, it is sufficient to find a strategy of the opponent which induces e as a best reply. Recall from the complete information case that the reaction function of the high-cost type is a bijection on the interval $\left(0, \frac{1}{4c_i}\right]$. Thus, if the opponent's both types exert an effort which is the inverse of the reaction function

$$\frac{\sqrt{e_j}}{\sqrt{c_i}} - e_j$$

on the interval $\left(0, \frac{1}{4c_H}\right]$, we have a strategy which makes e a best reply. To summarize, after the first round of elimination the best replies for a type c_i with $c_i \in c_L, c_H$ is given by $\left(0, \frac{1}{4c_i}\right]$. The highest rationalizable efforts after the first round of elimination, denoted by $\overline{r}_1(c_L)$ and $\overline{r}_1(c_H)$, and the strategies inducing these efforts are illustrated in the following figure:

$$\begin{tikzcd}[row sep=small] & e(c_L)=\frac{1}{4c_L} \\ \boxed{\overline{r}_1(c_L)=\frac{1}{4c_L}} \arrow[ur] \arrow[dr] & \\ & e(c_H)=\frac{1}{4c_L} \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] & e(c_L)=\frac{1}{4c_H} \\ \boxed{\overline{r}_1(c_H)=\frac{1}{4c_H}} \arrow[ur] \arrow[dr] & \\ & e(c_H)=\frac{1}{4c_H} \end{tikzcd}$$

After the first elimination round the highest rationalizable bid for type c_L is $\frac{1}{4c_L}$ and this bid is a best reply to the low type and the high type exerting effort $\frac{1}{4c_L}$ (the analogous explanation holds for the high-cost type).

In the second round of elimination nothing changes for the high-cost type. All actions which are needed to induce efforts in the interval $\left(0, \frac{1}{4c_H}\right]$ have not been eliminated in the first round. However, the effort $\frac{1}{4c_L}$ is not a best reply for the low-cost type anymore since it can be induced only if both of the opponent's types exert effort $\frac{1}{4c_L}$. All efforts higher than $\frac{1}{4c_H}$ have been eliminated for the high-cost type.

$$\begin{tikzcd}[row sep=small] & e(c_L)=\frac{1}{4c_L} \\ \boxed{\overline{r}_1(c_L)=\frac{1}{4c_L}} \arrow[ur] \arrow[dr] & \\ & e(c_H)=\frac{1}{4c_L}>\frac{1}{4c_H} \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] & e(c_L)=\frac{1}{4c_H} \\ \boxed{\overline{r}_1(c_H)=\frac{1}{4c_H}} \arrow[ur] \arrow[dr] & \\ & e(c_H)=\frac{1}{4c_H} \end{tikzcd}$$

Thus, the new highest possible effort for the low-cost type is given by the best reply to the strategy where the low-cost type exerts effort $\frac{1}{4c_L}$ and the high-cost type exerts effort $\frac{1}{4c_H}$. This leads to a new highest best reply for the low-cost type, denoted by e'_L. Thus, after the second round of elimination the rationalizable efforts for the high-cost type are given by $\left(0, \frac{1}{4c_H}\right]$ and the rationalizable effort for the low-cost types are given by $(0, e'_L]$.

$$\begin{tikzcd}[row sep=small] & e(c_L)=\frac{1}{4c_L} \\ \boxed{\overline{r}_2(c_L)=e'_L} \arrow[ur] \arrow[dr] & \\ & e(c_H)=\frac{1}{4c_H} \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] & e(c_L)=\frac{1}{4c_H} \\ \boxed{\overline{r}_2(c_H)=\frac{1}{4c_H}} \arrow[ur] \arrow[dr] & \\ & e(c_H)=\frac{1}{4c_H} \end{tikzcd}$$

This implies that in the next round of elimination the effort e'_L is not a best reply anymore since $\frac{1}{4c_L}$ has been eliminated in the second round.

$$\begin{tikzcd} & e\left(c_L\right) = \frac{1}{4c_L} > e'_L \\ \boxed{\overline{r}_2\left(c_L\right) = e'_L} \arrow[ur] \arrow[dr] & \\ & e\left(c_H\right) = \frac{1}{4c_L} \end{tikzcd} \qquad \begin{tikzcd} & e\left(c_L\right) = \frac{1}{4c_H} \\ \boxed{\overline{r}_2\left(c_H\right) = \frac{1}{4c_H}} \arrow[ur] \arrow[dr] & \\ & e\left(c_H\right) = \frac{1}{4c_H} \end{tikzcd}$$

This process will continue until it reaches a highest best reply for the low-cost type, denoted by $\bar{e}_L$ such that the best reply of the low-cost type to the strategy where the high-cost type exerts effort $\frac{1}{4c_H}$ and the low-cost type exerts effort $\bar{e}_L$, is equal to $\bar{e}_L$. An effort $\bar{e}_L$ which fulfills this condition is the highest rationalizable effort for the low-cost type.

$$\begin{tikzcd} & e\left(c_L\right) = \bar{e}_L \\ \boxed{\overline{r}_\infty\left(c_L\right) = \bar{e}_L} \arrow[ur] \arrow[dr] & \\ & e\left(c_H\right) = \frac{1}{4c_L} \end{tikzcd} \qquad \begin{tikzcd} & e\left(c_L\right) = \frac{1}{4c_H} \\ \boxed{\overline{r}_\infty\left(c_H\right) = \frac{1}{4c_H}} \arrow[ur] \arrow[dr] & \\ & e\left(c_H\right) = \frac{1}{4c_H} \end{tikzcd}$$

Formally, $\bar{e}_L$ is determined by the equation

$$\frac{\frac{1}{4c_H}}{\left(\frac{1}{4c_H} + \bar{e}_L\right)^2} + \frac{\bar{e}_L}{\left(\bar{e}_L + \bar{e}_L\right)^2} = 2c_L.$$

$\square$

Application of the rational maximin criterion After establishing the highest rationalizable efforts for every type, we are in the position to apply the rational maximin criterion.

Corollary 4. *It holds that* $RM\left(c_L\right) = \bar{e}_L$ *and* $RM\left(c_H\right)$ *is determined by the equation*

$$\frac{\frac{1}{4c_H}}{\left(\frac{1}{4c_H} + RM\left(c_H\right)\right)^2} + \frac{\bar{e}_L}{\left(\bar{e}_L + RM\left(c_H\right)\right)^2} = 2c_H.$$

The rational maximin actions of the low-cost and high-cost types are best replies to the strategy where the opponent exerts the highest rationalizable efforts, $\bar{e}_L$ and $\frac{1}{4c_H}$.

Comparison to equilibrium strategies A shown in Fey (2008), in equilibrium the low-cost type exerts effort

$$e_L^* = \frac{4\frac{c_L}{c_H} + \left(1 + \frac{c_L}{c_H}\right)^2}{8c_L\left(1 + \frac{c_L}{c_H}\right)^2}$$

and the high-cost type exerts effort

$$e_H^* = \frac{4\frac{c_L}{c_H} + \left(1 + \frac{c_L}{c_H}\right)^2}{8c_H\left(1 + \frac{c_L}{c_H}\right)^2}.$$

Proposition 4. *It holds that $e_L^* < RM\left(c_L\right)$ and $e_H^* > RM\left(c_H\right)$.*

Since equilibrium efforts are rationalizable, they cannot be higher than the highest rationalizable efforts. Since $\bar{e}_L$ is not only the highest rationalizable effort for the low-cost type but also the effort prescribed by the rationalizable maximin criterion, the equilibrium effort is lower or equal than the rational maximin effort.

For the high-cost type the comparison of the equilibrium effort and the effort prescribed by the rational maximin criterion is not straightforward. The equilibrium effort is a best reply to e_L^* and e_H^* while the rational maximin effort is a best reply to $\bar{e}_L$ and $\frac{1}{4c_H}$. Recall that the effort of a high-cost type is maximized if both types of her opponent exert effort $\frac{1}{4c_H}$. Thus, the fact that $e_H^* < \frac{1}{4c_H}$ has the effect to increase the rational maximin effort in comparison to the equilibrium effort while $e_L^* < \bar{e}_L$ causes to decrease it since e_L^* is closer to $\frac{1}{4c_H}$. Intuitively, the equilibrium effort is induced by the strategy where the low-cost type exerts more than $\frac{1}{4c_H}$ and the high-cost types exerts less while the rational maximin effort is induced by the strategy where the high-cost type exerts effort $\frac{1}{4c_H}$ and the low-cost type exerts a higher effort. Thus, the strategy inducing the equilibrium effort is more balanced leading to a higher effort than the rational maximin effort.

I conclude this section by stating some insights.

Observation 2. *(i) The rational maximin belief of a player is that her opponent exerts the highest rationalizable effort and therefore does not* *depend on her type.*

(ii) *The rational maximin belief and the rational maximin strategy of a player are unique. The rational maximin strategy is a pure strategy.*

(iii) *Due to the single-peakedness of the best-reply functions in the complete information cases for types c_L and c_H, it holds that $RM\left(c_L\right) = \bar{e}_L < \frac{1}{4c_L}$ and $RM\left(c_H\right) < \frac{1}{4c_H}$.*

(iv) *If every player applies the rational maximin criterion, the outcome is ex-ante efficient, i.e., the player with the lowest cost exerts the highest effort and hence wins the prize with the highest probability.*

6 Bilateral trade under strategic uncertainty

Finally, I investigate the application of the rational maximin criterion to bilateral trade. As before, the analysis begins with specifying the underlying game of incomplete information.

Underlying game of incomplete information There is a seller and a buyer who act simultaneously. The seller produces a good either at low costs c_L or high costs c_H. The buyer has either a low valuation v_L or a high valuation v_H for the good with $c_L < v_L < c_H < v_H$. The seller proposes a reserve price $r \in \mathbb{R}^+$ and the buyer proposes a bid $b \in R^+$. If the bid is below the price, trade does not take place and both players get a utility of zero. If the bid is above the price, trade does take place at price $p = \alpha\left(b + r\right)$ with $\alpha \in (0, 1)$. That is, the seller's strategy is given by

$$\beta_s : \{c_L, c_H\} \to \mathbb{R}^+$$

and her utility is given by

$$u_s = \begin{cases} 0 & \text{if } b < p \\ p - c & \text{if } b \geq p \end{cases}$$

while the buyer's strategy is given by

$$\beta_b : \{v_L, v_H\} \to \mathbb{R}^+$$

and her utility is given by

$$u_b = \begin{cases} 0 & \text{if } b < p \\ v - p & \text{if } b \geq p \end{cases}.$$

Application of the rational maximin criterion In contrast to first-price auctions and contests, in a bilateral trade setting the rational maximin belief of the buyer and seller is so pessimistic that they do not expect a positive utility. This leads to a continuum of rational maximin strategies.

Proposition 5. *In a rational maximin equilibrium buyer and seller expect a utility of zero. For a buyer with valuation $v \in \{v_L, v_H\}$ every bid in $[c_L, v]$ is a rational maximin strategy. For a seller with cost $c \in \{c_L, c_H\}$ every reserve price in $[c, v_H]$ is a rational maximin strategy.*

Proof. The following strategy profile constitutes a Bayes-Nash equilibrium and due to Proposition 1 is therefore rationalizable:

$$r(c_L) = r(c_H) = v_H, \quad \beta(v_L) = \beta(v_H) = c_L. \tag{4}$$

Thus, it is a feasible strategy of the adverse nature to choose $r(c_L) = r(c_H) = v_H$ as the strategy of the seller. Hence, the rational maximin belief of the buyer is that the seller sets at least a reserve price of v_H and the buyer expects a utility of zero. Analogously, the seller expects a utility of zero. Thus, any rationalizable action turns out to be a rational maximin action. □

Since in the underlying game of incomplete information there exists a continuum of equilibria, a continuum of actions is rationalizable, including extreme actions which prevent trade. As a result, the strategic uncertainty causes players to expect zero utility. Note that even actions are rationalizable which never lead to a positive utility and may lead to a negative utility. The following assumption excludes such extreme outcomes.[15]

[15] One could also assume "strategic sophistication" of both players as in Battigalli and

Assumption 1. *It is common knowledge that a buyer with valuation $v \in \{v_L, v_H\}$ never bids higher than v and a seller with costs $c \in \{c_L, c_H\}$ never sets a reserve price smaller than c.*

As a consequence, the adverse nature's actions space is not only restricted to rationalizable strategies but also to strategies compatible with Assumption 1.

As an additional result, I will investigate how the possibility to commit to certain bids or reserve prices changes the application of the rational maximin criterion. First, I will discuss *full commitment*, i.e., the possibility of a player to credibly commit to one single action. Formally, the action space of a player is a singleton (with the corresponding restrictions of the set of rationalizable actions and the adverse nature's action space). Assume that the buyer could credibly commit to a specific bid, then depending on the parameter constellation she would commit to bid $c_L + \epsilon$ or $c_H + \epsilon$ where ϵ is the smallest unit of money. Obviously, this commitment is almost not beneficial for the seller. The seller would like to commit herself to a reserve price $v_H - \epsilon$ or $v_L - \epsilon$. In other words, under full commitment both players prefer to be the only player with the possibility for full commitment.

Second, I will discuss *partial commitment*, i.e., the possibility of the buyer to commit to a minimum bid and the possibility of the seller to commit to a maximum reserve price. Formally, it is common knowledge that the action space of the buyer is bounded below and the action space of the seller is bounded above.

Proposition 6.

(i) Assume that the action space of the buyer A_b is given by (b^{min}, ∞) with

$$b^{min} = (1-p)\, v_H + \frac{p}{2\alpha} c_L$$

and $b^{min} < v_L$. Then a buyer with valuation v_H expects at least a utility of $\max\{0, (1-q)\,(v_H - 2\alpha v_L)\}$ *and her highest rationalizable bid is given by* $\min\{v_H, b^{max}\}$ *with*

$$b^{max} = \frac{q}{2\alpha} v_H + (1-q)\, v_L.$$

Siniscalchi (2003a). Under strategic sophistication a buyer who places a bid above c_L and a seller who sets a reserve price below v_H expect trade to take place with positive probability.

(iit) Assume that the action space of the seller A_s is given by $[0, r^{max})$ with

$$r^{max} = (1 - q)\, c_L + \frac{q}{2\alpha} v_H$$

and $r^{max} \geq c_H$. Then a seller with costs c_L expects at least a utility of $\max\{(1 - p)\,(2\alpha c_H - c_L)\}$ *and the lowest rationalizable reserve price is given by* $\max\{c_L, r^{min}\}$ *with*

$$r^{min} = \frac{p}{2\alpha} c_L + (1 - p)\, c_H.$$

If the buyer commits to bid at least b^{min}, the low-cost type of the seller will benefit form trading with both types of the buyer even if the seller expects that type v_H bids her own valuation. As a consequence, the highest rationalizable reserve price of type c_L becomes v_L. Thus, type v_H can expect a strictly positive utility when trading with type c_L which puts an upper bound on the highest rationalizable bid of type v_H, denoted by b^{max}. This, in turn, implies that a seller with costs c_H sets no higher reserve price than b^{max}. An analogous reasoning holds for the seller. Note that $b^{max} > r^{max}$ and $b^{min} > r^{min}$ from which follows that, in contrast to full commitment, every player benefits from the possibility of partial commitment and every player prefers the other player to partially commit.

I conclude this section by stating some insights.

Observation 3. *(i) The rational maximin belief of a player is not necessarily unique. However, given Assumption 1, the strategy as specified in (4), is the unique conjecture which is a rational maximin belief for every player and every type.*

(ii) As mentioned above, rational maximin strategies are not unique. They become unique under full commitment. In case of the buyer's partial commitment rational maximin strategies are given by $RM\,(c_L) = b^{min}$, $RM\,(c_H) = c_H$, $RM\,(v_L) = v_L$, and $RM\,(v_H) \in [0, v_H]$ if $v_H < b^{max}$, otherwise, $b\,(v_H) \in \{v_L, b^{max}\}$, .

(iii) If both buyer and seller apply the rational maximin criterion, the outcome is not necessarily efficient. The outcome is efficient under full commitment. Under partial commitment of the buyer the outcome

is inefficient if $v_H < b^{max}$. If $v_H > b^{max}$, the outcome may still be inefficient. In this case a buyer with valuation v_H is indifferent between b^{max} and v_L. If among two outcomes with the same expected utility, the buyer prefers the outcome where trade takes place with higher probability, the outcome under partial commitment of the buyer is efficient. An analogous reasoning holds for the seller.

7 Discussion

Minimax regret

In the first step of the rational maximin criterion the adverse nature's action space is restricted to rationalizable strategies. In the second step the adverse nature plays a zero-sum game against the player applying the criterion, i.e., the player applies the maximin criterion. Instead of the maximin criterion a player could apply the minimax regret criterion. Formally, a player would play a zero-sum game against an adverse nature and the utility of an action is given by its maximum regret. The analysis of this alternative criterion is beyond the scope of this paper. Moreover, the rational maximin criterion can be seen as the most cautious approach: the expected utility of a maximin action is guaranteed for sure since the set of possible conjectures is restricted by minimal assumptions and the maximin criterion guarantees that the minimal utility is maximized.

Cognitive complexity

Formally, the derivation of the set of rationalizable actions for an agent with a given type requires an infinite intersection of sets. However, the proofs show that there are techniques which make the computation tractable. One could argue that a sufficiently rational player can conduct the necessary calculations. But one could also argue that for some players these calculations may be too difficult. Therefore, similarly as in level-k models, one could define the concept of k-rationalizability. That is, a player i could know that her opponent can compute the set RS_j^k for all players j and for $k \in \mathbb{N}$, but cannot compute the sets $RS_j^{k'}$ for $k' > k$ (see Bernheim (1984)). Depending on the parameters, this knowledge can influence player i's rational maximin strategy.

Robustness

In addition to the rational maximin criterion, one could introduce an additional robustness criterion in the following sense: Does the maximin strategy of an agent change if the adverse nature deviates from her strategy to another strategy in an ϵ-neighborhood? If there is a change, does the strategy and the resulting expected utility change continuously?

As an example, consider a first-price auction under strategic uncertainty with a commonly known distribution function, two bidders and three valuations 0, θ and 1. Bidder A with valuation 1 has the rational maximin belief that bidder B with valuation 1 bids $\overline{b}_1$. Hence, bidder A with valuation 1 bids either $\overline{b}_\theta$ or zero. However, all bids in the interval $[0, \overline{b}_1]$ are rationalizable for a bidder with valuation 1. Hence, (if the bid grid is sufficiently fine) an ϵ-neighborhood of $\overline{b}_1$ and its intersection with the set of rationalizable actions contains bids lower than $\overline{b}_1$. If bidder A with valuation 1 has the subjective belief that bidder B with valuation 1 bids lower than $\overline{b}_1$, e.g. $\left(\overline{b}_1\right)^-$, then $\overline{b}_1$ becomes a best reply for bidder A with valuation 1. This constitutes a discontinuity in her best reply.

As a second example, consider a first-price auction under strategic uncertainty with two bidders and a commonly known common valuation v. To bid v is the highest rationalizable action for both bidders. Therefore, bidder A has the rational maximin belief that bidder B bids v. As a consequence, bidder 1 is indifferent between any bid in $[0, v]$. Assume that bidder 1 chooses the action v (or v^-). As any other bid, this leads to a utility of zero given the rational maximin belief that bidder B bids v. An ϵ-neighborhood of v and its intersection with the set of rationalizable actions contains only bids below v, e.g. it contains the bids v,v^- and $(v^-)^-$. The best replies to these bids are in an ϵ-neighborhood of v (or v^-) and the induced utilities are in an ϵ-neighborhood of zero. Hence, bidding v (or v^-) fulfills the robustness property that an ϵ-deviation of the rational maximin belief induces an ϵ-deviation of the best replies and expected utility.

Appendices

Proof of Lemma 1

Proof. Let $i \in \{1, \ldots, I\}$ be a player with valuation θ_i and for every $j \in \{1, \ldots, I\} \setminus \{i\}$ let β_j be a rationalizable strategy for player j. Let $a_i \in A_i$ be a best reply to β_{-i}, i.e., it holds that

$$a_i \in \underset{a_i' \in A_i}{\operatorname{argmax}} \; U_i\left(\theta_i, a_i', \beta_{-i}, F_{-i}\right).$$

It is to show action a_i is rationalizable. It is sufficient to show that a_i is an element of $RS_i^k\left(\theta_i\right)$ for every $k \geq 1$. The proof works by induction. It is true that action $a_i \in A_i$ is an element in $RS_i^1\left(\theta_i\right)$ since it holds by definition that $RS_i^1\left(\theta_i\right) = A_i$. Assume it is already shown that a_k is an element in $RS_i^k\left(\theta_i\right)$. Since the strategy profile β_{-i} is rationalizable, it holds for all $j \neq j$, for all $\theta_j \in \Theta_j$ and for all $a_j \in supp\left(\beta_j\left(\theta_j\right)\right)$ that $a_j \in RS_k^k\left(\theta_j\right)$. Since

$$a_i \in \underset{a_i' \in A_i}{\operatorname{argmax}} \; U_i\left(\theta_i, a_i', \beta_{-i}, F_{-i}\right),$$

it follows from the definition of rationalizable actions that $a_i \in RS_i^{k+1}\left(\theta_i\right)$ which completes the proof. □

Proof of Proposition 1

Proof. Let $\left(\beta_1, \ldots, \beta_I\right)$ together with $\hat{F}$ constitute a Bayes-Nash equilibrium. Let i be a player with valuation θ_i and a_i be an action such that $a_i \in supp\left(\beta_i\left(\theta_i\right)\right)$. It is to show that $a_i \in RS_i\left(\theta_i\right)$. I show by induction with respect to k that for every $j \in \{1, \ldots, I\}$, for every $k \geq 1$ and for all $\theta_j \in \Theta_j$ it holds that

$$a_j \in supp\left(\beta_j\left(\theta_j\right)\right) \Rightarrow a_j \in RS_j^k\left(\theta_j\right).$$

Then it follows that $a_j \in RS_j\left(\theta_j\right)$ and one can conclude that $a_i \in RS_i\left(\theta_i\right)$ because $a_i \in supp\left(\beta_i\left(\theta_i\right)\right)$. It holds for all $j \in \{1, \ldots, I\}$ that

$$a_j \in supp\left(\beta_j\left(\theta_j\right)\right) \Rightarrow a_j \in RS_j^1\left(\theta_j\right) \quad \text{for all } \theta_j \in \Theta_j$$

since $RS_j^1(\theta_j) = A_j$ by definition. Assume it is already shown that for all $j \in \{1, \ldots, I\}$ it holds that

$$a_j \in supp\left(\beta_j\left(\theta_j\right)\right) \Rightarrow a_j \in RS_j^k\left(\theta_j\right) \text{ for all } \theta_j \in \Theta_j.$$

It holds that for a player j with valuation θ_j the strategy profile β_{-j} fulfills the properties

(i) $a_l \in supp\left(\beta_l\left(\theta_l\right)\right) \Rightarrow a_l \in {RS_l}^k\left(\theta_l\right)$ for all $l \neq j$

(ii) $a_j \in supp\left(\beta_j\left(\theta_j\right)\right) \Rightarrow a_j \in \underset{a_j' \in A_j}{\operatorname{argmax}}\ U_j\left(\theta_j, a_j', \beta_{-j}, F_{-j}^j\right)$.

The first property follows from the induction hypothesis and the second property follows from the definition of a Bayes-Nash equilibrium. By definition of a rationalizable action, it follows that $\beta_j\left(\theta_j\right) \in RS_j^{k+1}$. Hence, it is shown that $a_j \in supp\left(\beta_j\left(\theta_j\right)\right) \Rightarrow a_j \in RS_j\left(\theta_j\right)$.

□

Proof of Lemma 2

Proof. Let $\theta_1, \ldots \theta_m$ be an ordered list of types in Θ with $\theta_1 = 0$ being the lowest type. I will show by induction that for every type $\theta_k \in \Theta$ it is not rationalizable to bid strictly above her valuation.

Induction start: $\theta_1 = 0$ The induction starts with type 0. In order to show the claim for type 0, I will use induction with respect to the bids starting with G. Bidding G can never be a best reply for a 0-type since she wins with positive probability when bidding G and hence gets a negative expected utility. Assume it has been already shown that all bids in the interval $[b, G]$ with $b > 0$ are not rationalizable for type 0. Then bidding b^- with $b > 0$ cannot be a best reply for a 0-type since she wins at least with probability $F(0)$ when bidding b^- and hence gets a negative expected utility. One can conclude that it is not rationalizable to bid strictly above 0 for a bidder with valuation 0.

Induction step: $\theta_k \rightarrow \theta_{k+1}$ Assume it has been already shown that for all types smaller or equal than θ_k it is not rationalizable to bid above her valuation. Since a positive mass of types does not bid higher than θ_k, it is never a best reply for a bidder with valuation θ_{k+1} to bid above her valuation since she would win at least with probability $F(\theta_k)$ and get a negative expected utility.

$\square$

Proof of Proposition 3

Proof. First, I show that $\overline{e}_L$ and $\frac{1}{4c_H}$ are rationalizable efforts for the low-cost and high-cost type respectively. I will show by induction that for every $k \geq 1$ it holds that $(0, \overline{e}_L] \subseteq RS^k(c_L)$ and $\left(0, \frac{1}{4c_H}\right] \subseteq RS^k(c_H)$. The induction start with $k = 1$ and the statement follows by definition as $RS^1(c_L) = RS^1(c_H) = \mathbb{R}^+$. Assume it has been shown that $(0, \overline{e}_L] \subseteq RS^k(c_L)$ and $\left(0, \frac{1}{4c_H}\right] \subseteq RS^k(c_H)$, then it is to show that $(0, \overline{e}_L] \subseteq RS^{k+1}(c_L)$ and $\left(0, \frac{1}{4c_H}\right] \subseteq RS^{k+1}(c_H)$. Since $\frac{1}{4c_H}$ is a best reply to both types exerting effort $\frac{1}{4c_H}$, it holds that

$$\frac{\frac{1}{4c_H}}{\left(\frac{1}{4c_H} + \frac{1}{4c_H}\right)^2} + \frac{\frac{1}{4c_H}}{\left(\frac{1}{4c_H} + \frac{1}{4c_H}\right)^2} = 2c_H$$

from which follows that $\frac{1}{4c_H} < \overline{e}_L$. The best-reply function in the case of complete information with type c_H is a bijection on $\left(0, \frac{1}{4c_H}\right]$ and since $\frac{1}{4c_H} < \overline{e}_L$, it follows that the efforts in the interval $\left(0, \frac{1}{4c}\right]$ are best replies to efforts in $RS^k(c_L)$ and $RS^k(c_H)$ from which follows that $\left(0, \frac{1}{4c_H}\right] \subseteq RS^{k+1}(c_H)$. Since by definition $\overline{e}_L$ is a best reply to the strategy where the low-cost type exerts effort $\overline{e}_L$ and the high-cost type exerts effort $\frac{1}{4c_H}$, it is a best reply to actions in $RS^k(c_L)$ and $RS^k(c_H)$. Recall that in the case of complete information with type c_L the maximum of the best-reply function is reached at $\frac{1}{4c_L}$ and strictly decreases if the opponent's efforts are decreasing. This implies that the optimal effort for type c_L decreases in the opponent's efforts if the opponent exerts efforts $\overline{e}_L$ and $\frac{1}{4c_H}$ (for low-cost and high-cost type). It follows that any effort in $(0, \overline{e}_L]$ can be induced by efforts in $(0, \overline{e}_L]$ and $\left(0, \frac{1}{4c_H}\right]$ and hence $(0, \overline{e}_L] \subseteq RS^{k+1}(c_L)$. This completes the induction step.

Second, I show that no higher efforts are rationalizable for types c_L and

c_H. Since the best-reply function in the case of complete information with type c_H is single-peaked at $\frac{1}{4c_H}$, it holds that no effort higher than $\frac{1}{4c_H}$ is rationalizable for type c_H. Let $\hat{e}_L$ and $\hat{e}_H$ be the efforts for the low-cost and high-cost type which induce the highest rationalizable effort for the low-cost type, denoted by $\overline{e}'_L$. Since $\hat{e}_L < \frac{1}{4c_L}$ and $\hat{e}_H < \frac{1}{4c_L}$, the best reply of a player is increasing in her opponent's efforts if her opponent exerts efforts $\hat{e}_L$ and $\hat{e}_H$. Thus, $\hat{e}_L$ and $\hat{e}_H$ have to be the highest rationalizable efforts for the low-cost and high-cost type. Therefore, $\overline{e}'_L$ is determined by the equation

$$\frac{\frac{1}{4c_H}}{\left(\frac{1}{4c_H} + \overline{e}'_L\right)^2} + \frac{\overline{e}'_L}{\left(\overline{e}'_L + \overline{e}_L\right)^2} = 2c_L$$

and it holds that $\overline{e}'_L = \overline{e}_L$. □

Proof of Proposition 4

Proof. As a preparation for the proof I first show the following lemma.

Lemma 5. *It holds that* $e^*_L > \frac{1}{4c_H}$.

Proof. To shorten notation, let $\alpha := \frac{c_L}{c_H}$. Then it is to show that

$$\frac{4\alpha + (1+\alpha)^2}{8c_L(1+\alpha)^2} > \frac{1}{4c_H}$$

$$\Leftrightarrow c_H\left(4\alpha + (1+\alpha)^2\right) > 2c_L(1+\alpha)^2$$

$$\Leftrightarrow (1+\alpha)^2(c_H - 2c_L) + 4\alpha c_H > 0$$

$$\Leftrightarrow \left(1 + \frac{2c_L}{c_H} + \frac{c_L^2}{c_H^2}\right)(c_H - 2c_L) + 4c_L > 0$$

$$\Leftrightarrow \left(c_H^2 + 2c_Lc_H + c_L^2\right)(c_H - 2c_L) + 4c_Lc_H^2 > 0$$

$$\Leftrightarrow c_H^3 - 2c_Lc_H^2 + 2c_Lc_H^2 - 4c_L^2c_H + c_L^2c_H - 2c_L^3 + 4c_Lc_H^2 > 0$$

$$\Leftrightarrow c_H^3 - 2c_L^3 - 3c_L^2c_H + 4c_Lc_H^2 > 0.$$

It holds that

$$c_H^3 - 2c_L^3 - 3c_L^2c_H + 4c_Lc_H^2 > c_H^3 - 2c_L^3 + c_Lc_H^2 > c_H^3 - 2c_L^3 + c_L^3 > 0.$$

$\square$

Consider the strategy where the low-cost type exerts effort e_L^* and the high-cost type exerts effort $\frac{1}{4c_H}$. Let $\tilde{e}_H$ denote the best reply of the high-cost type to this strategy. Since $\frac{1}{4c_H} < e_L^* < \overline{e}_L$, it holds that $\tilde{e}_H > RM\left(c_H\right)$. I will show that $e_H^* > \tilde{e}_H$ from which follows that $e_H^* > RM\left(c_H\right)$.

Recall that $e_H^* < \frac{1}{4c_H}$. Therefore, in order to prove that $e_H^* > \tilde{e}_H$, it is sufficient to show that if in equilibrium the high-cost type deviates to any $e_H^* + \epsilon$ for $0 < \epsilon < \frac{1}{4c_H} - e_H^*$, then the best reply of the high-cost type to this strategy is lower than e_H^*.

Let $0 < \epsilon < \frac{1}{4c_H} - e_H^*$ and let x be the best reply of the high-cost type to the strategy where the low-cost type exerts effort e_L^* and the high-cost type exerts effort $e_H^* + \epsilon$. It is left to show that $x < e_H^*$ Assume this is not the case. It holds that

$$\frac{e_L^*}{\left(e_L^* + e_H^*\right)^2} + \frac{1}{4e_H^*} = 2c_H$$

and

$$\frac{e_L^*}{\left(e_L^* + x\right)^2} + \frac{e_H^* + \epsilon}{\left(e_H^* + \epsilon + x\right)^2} = 2c_H$$

from which follows that

$$\frac{e_L^*}{\left(e_L^* + e_H^*\right)^2} + \frac{1}{4e_H^*} = \frac{e_L^*}{\left(e_L^* + x\right)^2} + \frac{e_H^* + \epsilon}{\left(e_H^* + \epsilon + x\right)^2}$$

$$\Leftrightarrow \frac{e_L^*}{\left(e_L^* + x\right)^2} - \frac{e_L^*}{\left(e_L^* + e_H^*\right)^2} = \frac{1}{4e_H^*} - \frac{e_H^* + \epsilon}{\left(e_H^* + \epsilon + x\right)^2}.$$

The RHS is strictly greater than zero since it it follow from $e_H^* \geq x$ that

$$\frac{1}{4e_H^*} - \frac{e_H^* + \epsilon}{\left(e_H^* + \epsilon + x\right)^2} \geq \frac{1}{4e_H^*} - \frac{e_H^* + \epsilon}{\left(e_H^* + \epsilon + e_H^*\right)^2}$$

and it holds that for all $z > 0$ that

$$\epsilon^2 > 0$$

$$\Leftrightarrow 4z^2 + 4z\epsilon + \epsilon^2 \geq 4z^2 + 4z\epsilon$$

$$\Leftrightarrow \left(2z + \epsilon\right)^2 \geq 4z\left(z + \epsilon\right)$$

$$\Leftrightarrow \frac{1}{4z} \geq \frac{z+\epsilon}{(2z+\epsilon)^2}$$

This implies that the LHS is also strictly greater than zero from which follows that $x < e_H^*$. Since this leads to a contradiction, I conclude that $x < e_H^*$ and hence $e_H^* > \tilde{e}_H$.

□

Proof of Proposition 6

Proof. Assume the buyer partially commits to b^{min}. If a seller with costs c_L sets a reserve price b^{min}, she obtains a utility of $2\alpha b^{min} - c_L$. The highest possible bid of a buyer with valuation v_H is v_H and therefore, the most profitable deviation profit of trading only with type v_H of the buyer is given by

$$(1-p)\left(2\alpha v_H - c_L\right).$$

It holds that

$$\begin{aligned} (1-p)\left(2\alpha v_H - c_L\right) &< \alpha b^{min} - c_L \\ \Leftrightarrow b^{min} &> (1-p)\, v_H + \frac{p}{2\alpha} c_L. \end{aligned} \tag{5}$$

Thus, if b^{min} fulfills condition (5t) and $b^{min} \leq v_L$, it is not rationalizable for the seller to set a reserve price above v_L since trading with both types of the buyer under the lowest possible price $2\alpha b^{min}$ is more profitable than trading only with the high-valuation buyer under highest possible price $2\alpha v_H$. As a consequence, it may be not rationalizable for the high-valuation buyer to bid v_H since bidding v_L ensures trade with the low-cost seller. The highest rationalizable bid of the high-valuation buyer, b^{max}, is determined by the equation

$$(1-q)\left(v_H - 2\alpha v_L\right) = q v_H + (1-q)\, v_H - 2\alpha b^{max}$$

from which follows that

$$b^{max} = \frac{q}{2\alpha} v_H + (1-q)\, v_L.$$

Thus, a buyer with valuation v_H obtains at least an expected utility of

$$v_H - 2\alpha b^{max} = (1-q)\left(v_H - 2\alpha v_L\right).$$

Similarly, if the seller sets a reserve price r^{max} such that

$$(1-q)\left(v_H - 2\alpha c_L\right) < v_H - 2\alpha r^{max}$$

$$\Leftrightarrow r^{max} < (1-q)\, c_L + \frac{q}{2\alpha} v_H$$

and $r^{max} \geq c_H$ then it is not rationalizable for a high-valuation buyer to bid below c_H. It follows that for the low-cost seller it is not rationalizable to set reserve prices below r^{min} which is determined by the equation

$$2\alpha r^{min} - c_L \geq (1-p)\left(2\alpha c_H - c_L\right)$$

$$\Leftrightarrow r^{min} = \frac{p}{2\alpha} c_L + (1-p)\, c_H.$$

Thus, a seller with costs c_L obtains at least an expected utility of

$$2\alpha r^{min} - c_L = (1-p)\left(2\alpha c_H - c_L\right).$$

□

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