Discussion Paper Series – CRC TR 224

Discussion Paper No. 214
Project B 01

Optimal Voting Mechanisms on Generalized Single-Peaked Domains

Tobias Rachidi [1]

June 2021
(First version: September 2020)

[1] Bonn Graduate School of Economics, E-Mail: tobias.rachidi@uni-bonn.de

Funding by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through CRC TR 224 is gratefully acknowledged.

Optimal Voting Mechanisms on Generalized Single-Peaked Domains*

Tobias Rachidi†

June 10, 2021

Abstract

This paper studies the design of voting mechanisms in a setting with more than two alternatives and voters who have generalized single-peaked preferences derived from median spaces as introduced in [Nehring and Puppe, 2007b]. This class of preferences is considerably larger than the well-known class of preferences that are single-peaked on a line. I characterize the voting rules that maximize the ex-ante utilitarian welfare among all social choice functions satisfying strategy-proofness, anonymity, and surjectivity. The optimal mechanism takes the form of voting by properties, that is, the social choice is determined through a collection of binary votes on subsets of alternatives involving qualified majority requirements that reflect the characteristics of these subsets of alternatives. This general optimality result is applied to the design of voting mechanisms for the provision of two costly public goods subject to the constraint that the provided level of one good is weakly higher than the provided level of the other good. *Keywords*: Voting; Generalized Single-Peaked Preferences; Mechanism Design

JEL Classification: D71, D72, D82

*I am grateful to my advisors Benny Moldovanu, Stephan Lauermann, and Jean-François Laslier for continued guidance and support. I also benefited from helpful discussions and comments by Pierre de Callataÿ (discussant), Felix Chopra, Deniz Kattwinkel, Patrick Lahr, Christina Luxen, Matías Núñez, Clemens Puppe, an editor, an anonymous referee as well as participants of the "BGSE Micro Workshop" (Bonn), the Summer School "Pluridisciplinary Analysis of Collective Decision Making" (Caen), the "Lunch Seminar Theory, Organisation and Markets" (PSE), the "EDP Jamboree 2019" (Louvain-la-Neuve), and the "4th CRC TR 224 Workshop for Young Researchers". Further, I thank the Paris School of Economics, where parts of this research were conducted, for its hospitality. Financial support from the Bonn Graduate School of Economics, the Deutsche Akademische Austauschdienst, the Deutsche Forschungsgemeinschaft through the Hausdorff Center for Mathematics, the Studienstiftung des deutschen Volkes, and support from the Deutsche Forschungsgemeinschaft through CRC TR 224 (Project B01) is gratefully acknowledged.

†Bonn Graduate School of Economics, E-Mail: tobias.rachidi@uni-bonn.de

1 Introduction

In this paper, I characterize the optimal utilitarian voting mechanisms, meaning, the voting rules that maximize the ex-ante utilitarian welfare, among all social choice functions satisfying strategy-proofness, anonymity, and surjectivity. The setting features more than two alternatives and the voters have generalized single-peaked preferences derived from median spaces as introduced in [Nehring and Puppe, 2007b], henceforth [NP, 2007b]. This class of preferences is much larger than the well-known class of preferences that are single-peaked on a line. For instance, the following collective decision-making problems are covered: Collective choice when preferences are single-peaked with respect to trees as introduced in [Demange, 1982], and voting on hypercubes, that is, voting on multiple binary decisions as studied in [Barberà et al., 1991]. As part of section 2 on the related literature, I discuss more comprehensively which kind of collective decision-making problems are captured by my analysis.

[NP, 2007b] extend previous work in strategy-proof social choice like the seminal contribution of [Moulin, 1980], who considers single-peaked preferences on a line, to generalized single-peaked domains. I build on [NP, 2007b]'s characterization of strategy-proof social choice functions. [Gershkov et al., 2017] study the stated optimality question for preferences which are single-peaked on a line.[1] For these preferences, they derive the utilitarian mechanism, and they show that, in this case, the optimal voting rule takes the form of a successive procedure with weakly decreasing thresholds that depend on the intensities of preferences. The present paper extends the work of [Gershkov et al., 2017] to a considerably larger class of preferences.

The motivation for studying optimal voting mechanisms on generalized single-peaked domains while deviating from single-peaked preferences on a line is twofold: On the one hand, from a practical point of view, [Benoit and Laver, 2006] analyze the dimensionality of policy spaces arising in reality and find that, in many countries, the diversity in the positions of political parties with regard to various policy issues cannot be accurately captured by a single dimension of political conflict (see chapter 5 of their book).[2] Therefore, the empirical evidence from [Benoit and Laver, 2006] suggests that there is need to deviate from single-peaked preferences on a line in order to better understand how to take collective decisions whenever the alternatives relate to more than one dimension of political conflict. Considering generalized single-peaked preferences makes it possible to account for the multidimensionality of politics in a number of ways. Furthermore, [Kleiner and Moldovanu, 2020] argue that in some real-world voting problems from the German as well as the British parliament preferences were single-peaked on a tree. On the other hand, from a theoretical perspective, it seems to be natural to move away from

[1] To be more precise, they assume that preferences are single-crossing and single-peaked on a line.

[2] Their comprehensive empirical analysis covering 47 countries is based on expert surveys that were mostly conducted in 2003. In terms of methods, they employ statistical techniques of data reduction.

single-peaked preferences on a line and to study optimal voting mechanisms for larger classes of preferences that, nevertheless, admit well-behaved strategy-proof social choice functions.

The characterization of optimal mechanisms for preference domains that are generalized single-peaked with respect to a median space essentially involves the following three assumptions: First, voters have private types that are distributed independently and identically across the voters. Second, the utility function that is common to all voters satisfies an additive separability condition, constituting a natural constraint in settings, where alternatives might be multidimensional. This condition is vacuously met in the special case of single-peaked preferences on trees. Third, I impose a constraint on the type distribution and the utility function that might be, very loosely speaking, interpreted as a concavity restriction on the preference intensities. This restriction is vacuously satisfied in the special case of hypercubes.

The utilitarian mechanism takes the form of voting by properties, that is, the social choice is determined through a collection of binary votes on subsets of alternatives and the involved qualified majority requirements reflect the characteristics of these subsets of alternatives. The characterization of optimal mechanisms for preference domains that are generalized single-peaked with respect to a median space constitutes the main result of this paper. To illustrate this finding, before introducing the general model, I discuss an application to the design of voting mechanisms for the provision of two costly public goods α and β subject to the constraint that the provided level of α is weakly higher than the provided level of β. For example, if α and β represent expansions of the rail and the road network respectively, this constraint might reflect the fight against climate change. Therefore, to get a more concrete idea how optimal mechanisms look like, I directly refer to section 3. Also, when developing the general result, I repeatedly revisit this application in order to illustrate the concepts and assumptions I employ in the general analysis in a less abstract setting.

The structure of this paper is as follows: In the following section 2, I discuss the related literature, and, in section 3, I present the public goods application. Next, in section 4, I introduce the general model, and, in section 5, I review the characterization of strategy-proof social choice functions from [NP, 2007b]. Then, in section 6, I present my general optimality finding. The following section 7 discusses the two special cases of trees and hypercubes. The final section 8 concludes. The proofs are contained in the Appendix.

2 Literature

The present paper relates to work on social choice and mechanism design and it contributes, specifically, to the literature on the evaluation of the utilitarian efficiency of voting rules. This literature starts with [Rae, 1969], who focuses on *binary decisions.*

More recent contributions that also consider binary decisions include [Nehring, 2004], [Schmitz and Tröger, 2012], and [Drexl and Kleiner, 2018].

When moving away from the binary setting and allowing for more than two alternatives, the *Gibbard-Satterthwaite-Theorem* ([Gibbard, 1973], [Satterthwaite, 1975]) implies that restrictions on the preference domain have to be imposed, since otherwise, dictatorship arises.[3]

[NP, 2007b] offer a characterization of strategy-proof social choice functions for all rich generalized single-peaked domains. Among many other preference structures, the unrestricted domain as well as domains that give rise to median spaces constitute generalized single-peaked domains.[4] Their results reveal that the latter preference domain admits a large class of well-behaved strategy-proof social choice functions, circumventing the Gibbard-Satterthwaite-Theorem. This is the reason why I consider generalized single-peaked domains derived from median spaces. Again, I make use of [NP, 2007b]'s characterization of strategy-proof social choice functions for these preference structures. Furthermore, [Nehring and Puppe, 2005] as well as [Nehring and Puppe, 2007a] also treat strategy-proof social choice on generalized single-peaked domains, albeit having each a somewhat different emphasis. The difference between these contributions and my work is that these authors characterize incentive-compatible mechanisms, whereas I maximize welfare over incentive-compatible mechanisms.

The preference domains from the literature discussed below are all instances of generalized single-peaked domains derived from median spaces (see [NP, 2007b]).[5] I refer to [NP, 2007b] for an overview and a classification of median spaces, revealing that, in addition to those from the literature discussed below, many more collective choice problems are instances of generalized single-peaked domains giving rise to median spaces. Subject to my assumptions on the preference distribution, my optimality analysis covers all these preference domains. In that way, I unify and generalize previous results in the mechanism design literature. To the best of my knowledge, the optimization over strategy-proof mechanisms on generalized single-peaked domains giving rise to median spaces while relying on the utilitarian principle is novel. This is the main contribution of this paper.

One strand of the literature investigates *hypercubes* or coupled binary decisions, meaning, voters face a collection of binary decisions. In terms of strategy-proof social choice, [Barberà et al., 1991] provide a characterization of strategy-proof and onto mechanisms when preferences are separable or additively separable across the binary issues. When it comes to mechanism design, [Jackson and Sonnenschein, 2007] offer a mechanism that is based

[3] In a setting with more than two alternatives, [Apesteguia et al., 2011] evaluate voting rules according to different normative standards. Since these authors assume voters to be non-strategic, they do not need to restrict the set of preferences.

[4] [NP, 2007b] generalize previous work by [Barberà et al., 1997].

[5] Related contributions that also allow for more than two alternatives, but do not fit in the subsequent classification include [Börgers and Postl, 2009], and [Kim, 2017].

on the idea of budgeting. For sufficiently many decisions, their mechanism is approximately Bayesian incentive-compatible as well as nearly ex-ante Pareto efficient. Other voting rules in the context of Bayesian mechanism design where voters report cardinal utility information include qualitative voting studied in [Hortala-Vallve, 2012] as well as storable votes due to [Casella, 2005]. In contrast to [Jackson and Sonnenschein, 2007], [Hortala-Vallve, 2010] considers finitely many decision problems as well as strategy-proof mechanisms. Allowing for random mechanisms, he finds that ex-ante Pareto efficiency cannot be attained and, moreover, in the presence of strategy-proofness, there is no unanimous mechanism which is sensitive to preference intensities.

Another branch of the literature considers preferences which are *single-peaked on a line*. [Moulin, 1980] characterizes in his seminal contribution peaks-only and strategy-proof social choice functions for the full domain of preferences which are single-peaked on a line. His elegant characterization involves min-max rules or generalized median mechanisms when restricting attention to anonymous social choice functions. Again, [Gershkov et al., 2017] characterize the utilitarian mechanism when preferences are single-crossing and single-peaked on a line. In contrast to their work, I allow for a much larger class of preferences, going beyond single-peaked preferences on a line. In terms of the proof argument for my optimality result, I expand on their insights, but the much larger class of preferences requires additional proof arguments as well as different assumptions. Further, similar to [Gershkov et al., 2017], [Gersbach, 2017] also emphasizes the importance of flexible majority rules. Moreover, [Kleiner and Moldovanu, 2017] analyze dynamic, binary, and sequential voting procedures. They identify conditions on the voting procedures under which the induced dynamic games possess an ex-post perfect equilibrium in which voters behave sincerely. Moreover, they illustrate their theoretical findings by means of several empirical case studies involving collective decisions from different parliaments.

Building on preferences which are single-peaked on a line, *products of lines*, the coupling of unidimensional decisions or, as [Barberà et al., 1993] put it, multidimensional single-peaked preferences have also received attention in the literature. Removing the peaks-only assumption in [Moulin, 1980], [Border and Jordan, 1983] as well as [Barberà et al., 1993] provide characterizations of strategy-proof social choice functions for the stated class of voting problems. Despite considering each somewhat different preferences, the main conclusion following from these contributions is that any strategy-proof social choice function is peaks-only and it can be decomposed into unidimensional functions such that each dimension is treated in a separate way. In other words, any strategy-proof social choice function is composed of a collection of the mechanisms that [Moulin, 1980] identified for the unidimensional case. Finally, regarding mechanism design, [Gershkov et al., 2019] consider a spatial voting environment, but they keep the voting procedure fixed in the sense that, essentially, the collective choice in each coordinate of the multidimensional setting is determined via simple majority voting. They show that the redefinition of the involved issues or, in other words, the rotation of the initial coordinate axes leads, generally, to improvements in terms of welfare.

While extending single-peaked preferences on a line in a somewhat different direction compared to products of lines, but maintaining the general idea of single-peakedness, [Demange, 1982] investigates preferences which are *single-peaked on trees*. She establishes that these domains ensure the existence of a Condorcet winner. However, when it comes to aggregation theory instead of voting, the majority relation need not be transitive. Moreover, [Kleiner and Moldovanu, 2020] study dynamic, binary, and sequential voting procedures in the context of single-peaked preferences on trees. They derive conditions on the voting procedures such that voting sincerely constitutes an ex-post perfect equilibrium and the Condorcet winner is implemented in this equilibrium. Also, again, they apply their theoretical findings to real-world voting problems from the German and the British parliament.

3 Public Goods Provision

The main purpose of this section is to illustrate the general optimality result presented in Theorem 2 below by means of an application to the design of voting mechanisms for the provision of two public goods subject to a constraint, but this application is also of interest in itself. Again, when developing the general optimality result subsequently, I repeatedly go back to this application in order to illustrate the concepts and assumptions I employ in the general analysis in a less abstract setting.

Suppose that there are two public goods α and β, and that, for each public good, there are three possible levels $\{1, 2, 3\}$. Further, assume that there is an exogenously given constraint imposing that the provided level of α has to be weakly higher than the provided level of β.[6] Again, for instance, if α and β represent expansions of the rail and the road network respectively, this constraint might reflect the fight against climate change. Therefore, the set of alternatives A amounts to

$$A := \{(k_\alpha, k_\beta) \in \{1, 2, 3\} \times \{1, 2, 3\} : k_\alpha \geq k_\beta\}.$$

Moreover, there is a finite set of voters $N := \{1, ..., n\}$ with $n \geq 2$. The subsequent specification of types and utilities suitably extends the linear utility model contained in [Gershkov et al., 2017] from one to two public goods. The voters' types are governed by the two-dimensional random variable $T := X \times Y$. The support of the type distribution

[6] Similar applications appear in [Barberà et al., 1997], [Nehring and Puppe, 2005], [Nehring and Puppe, 2007a], [Block, 2010], and [Block de Priego, 2014]. However, these authors are not concerned about welfare maximization, but they focus on characterizing strategy-proof social choice functions.

S is given by the right triangle

$$S := \{(x,y) \in \mathbb{R}^2 : l \leq x \leq u, l \leq y \leq u, y \leq x\}$$

for some $0 \leq l < u < \infty$. In particular, note that the set S is convex. Denote by G and g the cdf and density of the bivariate distribution of T and let G_X and g_X as well as G_Y and g_Y be the marginal cdfs and densities corresponding to the random variables X and Y respectively. Types are distributed independently and identically across voters, and each voter is privately informed about his or her type realization. Now, a voter having type realization $(x,y) \in S$ receives utility

$$u^{(k_\alpha,k_\beta)}(x,y) := G^{k_\alpha} \cdot x - c^{k_\alpha} + G^{k_\beta} \cdot y - c^{k_\beta}$$

from alternative $(k_\alpha, k_\beta) \in A$. The involved parameters satisfy $c^1 < c^2 < c^3$ and $0 \leq G^1 < G^2 < G^3$, and they are common knowledge. In words, utilities are additively separable across the two public goods, the realizations of X and Y capture the valuation of public good α and β respectively, the valuation for α is always weakly higher than the value for β, the function G^k with $k \in \{1,2,3\}$ translates public good level indices into utilities, and the function c^k with $k \in \{1,2,3\}$ represents the cost function. Take any public good $\gamma \in \{\alpha, \beta\}$ and consider two public good levels $k, m \in \{1,2,3\}$ with $k > m$: The cutoff

$$z^{m,k} := \frac{c^k - c^m}{G^k - G^m}$$

describes the valuation corresponding to the public good γ at which a voter is indifferent between providing level k and m of the good γ for any fixed level of the other public good. Note that these cutoffs are homogenous across the two public goods because the functions G^k and c^k are assumed to be homogenous across the two goods. Suppose that the cutoffs involving neighboring public good levels are ordered, that is, suppose that

$$z^{0,1} := l < z^{1,2} < z^{2,3} < u =: z^{3,4}.$$

This is a mild assumption on the involved parameters:[7] For example, it is satisfied if the function G^k is linear in k and the cost function c^k is convex in k. It implies that any alternative is the most preferred or peak alternative for some types. In particular, the most preferred alternative of a voter constitutes $(p_\alpha, p_\beta) \in A$ if and only if the type realization $(x,y) \in S$ satisfies $x \in [z^{(p_\alpha - 1, p_\alpha)}, z^{(p_\alpha, p_\alpha + 1)}]$ and $y \in [z^{(p_\beta - 1, p_\beta)}, z^{(p_\beta, p_\beta + 1)}]$.
Finally, following [Nehring and Puppe, 2007a] and [NP, 2007b], observe that any type realization induces an ordinal preference relation that is generalized single-peaked with respect to a median space. Specifically, the requirement of single-peakedness amounts

[7] [Gershkov et al., 2017] impose an analogous condition on the cutoffs.

here to the following condition: There exists an alternative $(p_\alpha, p_\beta) \in A$, which is the most preferred alternative, such that for all alternatives $(k_\alpha, k_\beta), (m_\alpha, m_\beta) \in A$ with $(k_\alpha, k_\beta) \neq (m_\alpha, m_\beta)$ it holds that whenever (k_α, k_β) lies on a shortest path in the graph shown in Figure 1 connecting (p_α, p_β) and (m_α, m_β), the voter must prefer (k_α, k_β) over (m_α, m_β). For instance, suppose that a voter's most preferred alternative is $(2, 2)$. Then,

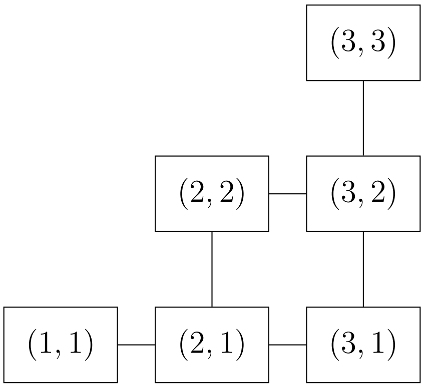

Figure 1: Public Goods Provision

single-peakedness requires, among other things, that this voter must prefer $(2, 1)$ and $(3, 2)$ over $(3, 1)$, but it does not impose whether $(2, 1)$ is preferred to $(3, 2)$ or the other way around. The specification of the support of the type distribution as well as the restriction on the cutoffs involving neighboring public good levels ensure that every type realization generates an ordinal preference relation that is generalized single-peaked in the described sense.

In the following, I present the direct mechanism that maximizes the utilitarian welfare among all strategy-proof, anonymous, and surjective mechanisms for the outlined setting.[8] In order to apply the general optimality result in Theorem 2 below, I impose three regularity assumptions on the type distribution. Specifically, assume that both marginal densities g_X and g_Y are log-concave and that $G_X \geq_{lr} G_Y$, where $\geq_{lr}$ denotes the likelihood ratio order. For instance, it can be verified that the three assumptions are met if the joint distribution G is the uniform distribution.[9]

The structure of the optimal mechanism can be described by means of four majority quotas $q_\alpha(1)$, $q_\alpha(2)$, $q_\beta(1)$, and $q_\beta(2)$, that is, four natural numbers weakly between 1 and n. Consider any public good $\gamma \in \{\alpha, \beta\}$ and public good level $k \in \{1, 2\}$: If there are at least $q_\gamma(k)$ voters with most preferred alternatives sharing the feature that the public

[8] There is an issue concerning the set of ordinal preferences generated by the utility representation introduced above: This set of ordinal preferences does not satisfy [NP, 2007b]'s richness condition on the preference domain. Therefore, the strategy-proof social choice functions they identify are strategy-proof for the outlined setting, but there might be other strategy-proof direct mechanisms in addition to those identified in their paper. However, [NP, 2007b]'s proof goes nevertheless through in the present setting, that is, there are no such other strategy-proof social choice functions. The argument for this claim is available on request from the author.

[9] Note that if the joint density g is log-concave, the marginal densities g_X and g_Y must be log-concave as well (see [Prékopa, 1973]).

good level of γ is weakly smaller than k, the social choice features the same property, that is, the provided level of γ is at most k. Otherwise, the provided level of γ is strictly larger than k. In other words, for every public good $\gamma \in \{\alpha, \beta\}$ and each level $k \in \{1, 2\}$, there is a binary vote between the following two subsets of alternatives: Alternatives sharing the feature that the public good level of γ is weakly smaller versus strictly larger than k. It follows from [Nehring and Puppe, 2007a] and [NP, 2007b] that, in the present setting, any strategy-proof, anonymous, and surjective social choice function takes this form subject to some constraints on the majority quotas. Now, under the regularity assumptions on the type distribution stated above, the described collection of binary votes on subsets of alternatives involving the majority quotas

$$q^*_\alpha(k) := \lceil \frac{n}{1 + \frac{\mathbb{E}[z^{k,k+1} - X | X \leq z^{k,k+1}]}{\mathbb{E}[X - z^{k,k+1} | X \geq z^{k,k+1}]}} \rceil$$

and

$$q^*_\beta(k) := \lceil \frac{n}{1 + \frac{\mathbb{E}[z^{k,k+1} - Y | Y \leq z^{k,k+1}]}{\mathbb{E}[Y - z^{k,k+1} | Y \geq z^{k,k+1}]}} \rceil,$$

where $k \in \{1, 2\}$, implements the welfare-maximizing mechanism among all social choice functions satisfying strategy-proofness, anonymity, and surjectivity.[10] The two main features of the optimal majority quotas are as follows: First, for both public goods, the associated quotas are decreasing in the public good level that determines the respective partition of the set of alternatives into two subsets, i.e., $q^*_\alpha(1) \geq q^*_\alpha(2)$ and $q^*_\beta(1) \geq q^*_\beta(2)$.[11] Second, the majority quotas corresponding to public good α are higher than the quotas linked to public good β, i.e., $q^*_\alpha(k) \geq q^*_\beta(k)$ for all $k \in \{1, 2\}$.[12] The designer faces a Bayesian inference problem, that is, he or she has to make inferences about the voters' preference intensities based on their vote choices in the described collection of binary votes. The optimal majority quotas that are shaped by ratios of preference intensities show how this inference problem is resolved. For concreteness, consider for example the the welfare-maximizing quota $q^*_\alpha(2)$: Rearranging the equation determining this quota while ignoring the integer problem yields

$$\frac{q^*_\alpha(2)}{n} \mathbb{E}[X | X \leq z^{2,3}] + \frac{n - q^*_\alpha(2)}{n} \mathbb{E}[X | X \geq z^{2,3}] = z^{2,3}.$$

Say that the designer is pivotal if there are exactly $q^*_\alpha(2)$ out of the n voters having most preferred alternatives that share the feature that the public good level of α is weakly

[10] In particular, this collection of binary votes represents a proper social choice function in the sense that it yields a unique alternative for all profiles of type realizations.

[11] This feature is also present in [Gershkov et al., 2017].

[12] Recall that the exogenously given constraint imposes that the provided level of α has to be weakly higher than the provided level of β.

smaller than 2, meaning, there are exactly $q^*_\alpha(2)$ voters with most preferred alternatives from the set $\{(1,1),(2,1),(2,2)\}$. Then, the quota $q^*_\alpha(2)$ is calibrated such that, conditional on being pivotal, the designer infers that the type component X governing the valuation for public good α equals the cutoff $z^{2,3}$, that is, the value at which a voter is indifferent between providing level 2 and 3 of α for any fixed level of β.[13] Furthermore, when rewriting the equation above once more, I obtain that, for all $k_\beta \in \{1,2\}$, it holds

$$\begin{aligned}&\frac{q^*_\alpha(2)}{n}\mathbb{E}[u^{(2,k_\beta)}(X,Y)|X \le z^{2,3}] + \frac{n-q^*_\alpha(2)}{n}\mathbb{E}[u^{(2,k_\beta)}(X,Y)|X \ge z^{2,3}]\\ =&\frac{q^*_\alpha(2)}{n}\mathbb{E}[u^{(3,k_\beta)}(X,Y)|X \le z^{2,3}] + \frac{n-q^*_\alpha(2)}{n}\mathbb{E}[u^{(3,k_\beta)}(X,Y)|X \ge z^{2,3}].\end{aligned}$$

In words, this equation means that, conditional on being pivotal, the designer is indifferent between any two alternatives such that the provided level of α is 2 versus 3, that is, it differs by exactly one, but the provided level of β is the same in both alternatives. In other words, the designer is indifferent between alternatives $(2,1)$ and $(3,1)$ as well as between $(2,2)$ and $(3,2)$. This characteristic of optimal quotas is not special to this public goods application, but it turns out that a generalization of it holds for all median spaces. Having presented the public goods application, in the following section, I introduce the general model.

4 Model

There is a finite set of voters $N := \{1, ..., n\}$ with $n \ge 2$ and a finite set of alternatives A with $|A| \ge 2$. Following [NP, 2007b], the set of alternatives is endowed with a property space structure. Elements of A are distinguished by *properties* which are described by $\mathcal{H} \subseteq \mathcal{P}(A)$, where $\mathcal{H} \neq \emptyset$, and $\mathcal{P}(A)$ denotes the power set of A. Each $H \in \mathcal{H}$ captures the property shared by all elements in $H \subseteq A$, but violated by all alternatives in $H^c := A \setminus H$. In other words, properties are subsets of the set of alternatives A. The set of properties $\mathcal{H}$ satisfies the regularity conditions

$$\begin{aligned}&H \in \mathcal{H} \Rightarrow H \neq \emptyset \text{ (non-triviality)},\\ &H \in \mathcal{H} \Rightarrow H^c \in \mathcal{H} \text{ (closedness under negation), and}\\ &\forall k, m \in A, k \neq m : \exists H \in \mathcal{H} : k \in H \wedge m \notin H \text{ (separation)}.\end{aligned}$$

Given some alternative $k \in A$, let $\mathcal{H}_k$ be the set of all properties shared by alternative k, meaning, define $\mathcal{H}_k := \{H \in \mathcal{H} : k \in H\}$. Due to separation, it holds that $\cap_{H \in \mathcal{H}_k} H = \{k\}$. Further, each pair (H, H^c) involving some property and its complement forms an

[13] This aspect which holds because utilities are affine in types appears also in [Gershkov et al., 2017]'s linear utility model.

issue, and the tuple $(A, \mathcal{H})$ is called *property space*. The property space $(A, \mathcal{H})$ induces a ternary relation on A, denoted by $B_{\mathcal{H}}$, in the following way: For all $(a, b, c) \in A \times A \times A$,

$$(a, b, c) \in B_{\mathcal{H}} :\Leftrightarrow [\forall H \in \mathcal{H} : \{a, c\} \subseteq H \Rightarrow b \in H].$$

The relation $B_{\mathcal{H}}$ is called *betweenness relation*. This means that some alternative b is between the alternatives a and c if and only if all properties that are jointly shared by a and c are also shared by b.
Moreover, I suppose that any property space constitutes a *median space* as introduced in [NP, 2007b].[14] This requires that the betweenness relation $B_{\mathcal{H}}$ satisfies the following constraint: For any $a, b, c \in A$, there exists some alternative $m = m(a, b, c) \in A$, called the median, such that

$$\{(a, m, b), (a, m, c), (b, m, c)\} \subseteq B_{\mathcal{H}}.$$

Take any set that is composed of three alternatives. The restriction of being a median space demands that there must be some alternative having the feature that it is between all pairs of alternatives that can be formed from the given set of three alternatives.
Based on these concepts, I introduce preferences. Following [NP, 2007b], an ordinal preference relation $\succ$ is said to be *generalized single-peaked* with respect to the underlying betweenness relation $B_{\mathcal{H}}$ if it satisfies the following condition: There exists some alternative $p \in A$ such that, for all $k, m \in A$ with $k \neq m$, it holds

$$(p, k, m) \in B_{\mathcal{H}} \Rightarrow k \succ m.$$

Intuitively, a generalized single-peaked preference relation is characterized by two main ingredients. On the one hand, the alternative p describes the peak of that preference relation. On the other hand, the constraint formalizing the generalized notion of single-peakedness requires that any alternative k distinct from m which is between the peak p and alternative m according to the betweenness relation $B_{\mathcal{H}}$ must be preferred to m. Let $\mathcal{P}_{\mathcal{H}}$ denote the set of all preference relations that are generalized single-peaked with respect to $B_{\mathcal{H}}$.

Public Goods Provision. Go back to the public goods application. Let $A_{Public\,Goods}$ be the set of alternatives for this application.[15] While following [Nehring and Puppe,

[14] Again, this assumption ensures that there is a rich class of non-degenerate incentive-compatible social choice functions.
[15] Recall that the set of alternatives is given by $\{(k_\alpha, k_\beta) \in \{1, 2, 3\} \times \{1, 2, 3\} : k_\alpha \geq k_\beta\}$.

2007a], consider, for all $k \in \{1,2\}$, the properties

$$H^{\alpha}_{\leq k} := \{(m_\alpha, m_\beta) \in A_{Public\,Goods} : m_\alpha \leq k\}$$
$$H^{\alpha}_{\geq k+1} := \{(m_\alpha, m_\beta) \in A_{Public\,Goods} : m_\alpha \geq k+1\}$$

as well as

$$H^{\beta}_{\leq k} := \{(m_\alpha, m_\beta) \in A_{Public\,Goods} : m_\beta \leq k\}$$
$$H^{\beta}_{\geq k+1} := \{(m_\alpha, m_\beta) \in A_{Public\,Goods} : m_\beta \geq k+1\}.$$

Denote by $\mathcal{H}_{Public\,Goods}$ the collection of these properties. Observe that the betweenness relation $B_{\mathcal{H}_{Public\,Goods}}$ induced by the property space $(A_{Public\,Goods}, \mathcal{H}_{Public\,Goods})$ satisfies the following condition: Alternative b is between alternatives a and c, meaning, $(a,b,c) \in B_{\mathcal{H}_{Public\,Goods}}$ if and only if b lies on a shortest path connecting a and c in the graph shown in Figure 1. Therefore, for this public goods application, the general definition of a generalized single-peaked preference relation introduced here exactly reduces to the definition based on the graphic notion of betweenness given in section 3. Moreover, it can be inferred from Figure 1 that the property space $(A_{Public\,Goods}, \mathcal{H}_{Public\,Goods})$ constitutes a median space.

Since I rely on the utilitarian principle as far as the objective criterion of the designer is concerned, I have to introduce a utility representation of ordinal preferences. Voters have types that are governed by the random variable T. Each voter is privately informed about his or her type realization. The distribution of T has full support on some non-empty set $S \neq \emptyset$. All subsequent expectations are taken with respect to this distribution. Throughout the paper, I suppose that types are distributed independently and identically across voters.

Assumption 1. *The types T are distributed independently and identically across voters.*

Now, $u^k(t)$ denotes the utility that a voter with type realization $t \in S$ receives if alternative $k \in A$ is implemented. I impose several constraints on the utility function and the type distribution. First, utilities are bounded, meaning, there exists some bound $B \in \mathbb{R}$ such that, for all type realizations $t \in S$ and for every alternative $k \in A$, $|u^k(t)| < B$. Second, I exclude indifferences, that is, for almost all type realizations $t \in S$ and for every pair of distinct alternatives $k, m \in A$ with $k \neq m$, it holds $u^k(t) \neq u^m(t)$. Third, of course, utilities must be consistent with generalized single-peakedness, that is, for almost all type realizations $t \in S$, there exists a generalized single-peaked preference relation $\succ \in \mathcal{P}_{\mathcal{H}}$ such that, for every pair of distinct alternatives $k, m \in A$ with $k \neq m$, it holds $k \succ m \Leftrightarrow u^k(t) > u^m(t)$. Fourth, I assume that the richness condition on the preference domain from [NP, 2007b] is satisfied. This means that the following two restrictions are met: First, for all $k, m \in A$ such that $\{k, m\} = \{l \in A : (k, l, m) \in B_{\mathcal{H}}\}$, there exists a set of type realizations $Z \subset S$ arising with positive probability $Pr(Z) > 0$ such that, for every element in this set $t \in Z$, it holds $u^k(t) > u^m(t) > u^l(t)$ for all $l \in A \setminus \{k, m\}$. Second, for all $p, k, m \in A$ such that $k \notin \{l \in A : (p, l, m) \in B_{\mathcal{H}}\}$, there exists a set of type realizations $Z \subset S$ arising with positive probability $Pr(Z) > 0$ such that, for every element in this set $t \in Z$, it holds $u^m(t) > u^k(t)$ and $u^p(t) > u^l(t)$ for all $l \in A \setminus \{p\}$.[16]

Finally, the designer maximizes the voter's ex-ante utilitarian welfare over all social choice functions that are strategy-proof, anonymous, and surjective. The timing is as follows:

1. The designer announces and commits to some strategy-proof, anonymous, and surjective direct mechanism.

2. The voters observe their type realizations and report them to the designer.

3. The designer implements an alternative according to the announced mechanism.

5 Incentive Compatibility

In this section, for completeness, I review the characterization of strategy-proof, anonymous, and surjective social choice functions for generalized single-peaked domains giving rise to median spaces due to [NP, 2007b].

First of all, I assume that the set of feasible mechanisms coincides with the set all possibly indirect deterministic mechanisms $\Gamma = (M, ..., M, f)$ inducing a game that admits a symmetric[17] dominant-strategy equilibrium, where M is the voters' finite message set and $f : M^n \to A$ is the outcome function. Also, I suppose that the mechanisms Γ are anonymous[18] and surjective.[19] Now, invoking a revelation principle in terms of payoffs due to [Jarman and Meisner, 2017] implies the following aspect: For each such anonymous and surjective mechanism Γ, there exists a direct mechanism $\Gamma' = (S, ..., S, f)$ that is dominant-strategy incentive-compatible, anonymous, and surjective, and the utilitarian welfare under Γ' is weakly higher than under Γ. In this sense, within the class of deterministic mechanisms, it is without loss to restrict attention to direct mechanisms.

However, the designer might be able to improve upon these deterministic mechanisms when allowing for stochastic mechanisms. There are two main reasons why I exclude

[16] The important point here is that there are no strategy-proof, anonymous, and surjective social choice functions apart from those identified in [NP, 2007b]'s characterization. In this sense, the utility representation of the public goods application is covered as well.

[17] Note that the voters are ex-ante identical.

[18] A mechanism Γ is anonymous if, for all $(m_1, ..., m_n) \in M^n$, $f(m_1, ..., m_n) = f(m_{\sigma(1)}, ..., m_{\sigma(n)})$, where σ is an arbitrary permutation of the set of voters N.

[19] A mechanism Γ is surjective if, for all $k \in A$, there exists $(m_1, ..., m_n) \in M^n$ such that $f(m_1, ..., m_n) = k$.

random mechanisms: First, random voting mechanisms are rarely used in practice.[20] Therefore, from a pragmatic point of view, it seems to be justified to focus on deterministic mechanisms. Second, to the best of my knowledge, it is not known how stochastic strategy-proof voting mechanisms look like when preferences are generalized single-peaked and, hence, there is a tractability issue when incorporating probabilistic mechanisms.[21] From now on, I restrict attention to deterministic direct mechanisms that are anonymous, surjective, and dominant-strategy incentive-compatible or, in other words, strategy-proof. A direct mechanism or, equivalently, a *social choice function* f is a mapping assigning to each type profile an alternative from the set A. In formal terms, this mapping amounts to $f : S^n \to A$. In the following, I recall some well-known properties of social choice functions.

Definition 1. *A social choice function f is strategy-proof if it holds, for all $i \in N$ and for all $t_i, t'_i \in S$ and $t_{-i} \in S^{n-1}$, that*

$$u^{f(t_i,t_{-i})}(t_i) \geq u^{f(t'_i,t_{-i})}(t_i).$$

In words, *strategy-proofness* requires that all voters have a weakly dominant strategy to truthfully reveal their types. Further, observe that strategy-proofness implies the following aspect: Consider any voter $i \in N$ and take two type realizations $t_i, t'_i \in S$ inducing the same ordinal preference relation. Then, a strategy-proof direct mechanism f must treat both types in the same way, that is, for any type realizations of the other voters $t_{-i} \in S^{n-1}$, it must hold that $f(t_i, t_{-i}) = f(t'_i, t_{-i})$.

Definition 2. *A social choice function f is anonymous if it holds, for all $(t_1, ..., t_n) \in S^n$, that $f(t_1, ..., t_n) = f(t_{\sigma(1)}, ..., t_{\sigma(n)})$, where σ is an arbitrary permutation of the set of voters N.*

Intuitively, *anonymity* imposes that mechanisms treat all voters equally. To put it differently, anonymity ensures that mechanisms respect the democratic principle of "one person, one vote".

Definition 3. *A social choice function f is surjective if, for all $k \in A$, there exists $(t_1, ..., t_n) \in S^n$ such that $f(t_1, ..., t_n) = k$.*

The requirement that social choice functions are *surjective* represents a mild condition ensuring that no alternative is a priori excluded from the set of outcomes.

[20] This observation may have several reasons: The designer might not have access to a credible randomization device. Moreover, in a stochastic mechanism, voters with the same type profile realization are not treated equally after the uncertainty is resolved. However, there is experimental evidence suggesting that people not only care about being treated equally before the uncertainty is resolved, but also about being treated in the same way after the resolution of the uncertainty (see the experimental papers [Brock et al., 2013] and [Cappelen et al., 2013]).

[21] Attempting to characterize these stochastic mechanisms is beyond the scope of this paper.

[NP, 2007b] show that strategy-proof and surjective social choice functions must be *peaks-only*, meaning, the outcome of any strategy-proof and surjective social choice function depends only on the voters' most preferred alternatives. Therefore, in the following, with abuse of notation, social choice functions are simply mappings $f : A^n \to A$ assigning to every profile of most preferred or peak alternatives $(p_1, ..., p_n) \in A^n$ some winning alternative from the set A.
In order to be able to state [NP, 2007b]'s characterization result, I need the following supplementary definitions from their paper. To begin with, introduce the notion of a *family of quotas* relative to some property space $(A, \mathcal{H})$.

Definition 4. *[NP, 2007b]*
Given some property space $(A, \mathcal{H})$, a family of quotas $\{q_H : H \in \mathcal{H}\}$ is a function that assigns an integer-valued quota $1 \leq q_H \leq n$ to each property $H \in \mathcal{H}$ such that, for all $H \in \mathcal{H}$, the associated quotas satisfy $q_H + q_{H^c} = n + 1$.

Take any property $H \in \mathcal{H}$. The associated absolute quota, threshold or majority requirement q_H describes the minimal number of votes that are needed in order to ensure that some alternative sharing property H is winning. Furthermore, the condition $q_H + q_{H^c} = n + 1$ reflects that whenever the quota q_H linked to property H is reached, the quota associated with the complementary property H^c cannot be attained, and vice versa. Moreover, exactly one of these two quotas is always achieved.
On the basis of the definition of families of quotas, consider the following class of functions which is termed anonymous *voting by properties*. These functions are central for the ensuing characterization result.

Definition 5. *[NP, 2007b]*
Given some property space $(A, \mathcal{H})$ and associated family of quotas $\{q_H : H \in \mathcal{H}\}$, voting by properties is the function $f_{\{q_H : H \in \mathcal{H}\}} : A^n \to \mathcal{P}(A)$ such that, for all profiles of peak alternatives $p = (p_1, ..., p_n) \in A^n$, it holds that

$$k \in f_{\{q_H : H \in \mathcal{H}\}}(p) :\Leftrightarrow [\forall H \in \mathcal{H}_k : |\{i \in N : p_i \in H\}| \geq q_H].$$

Intuitively, under voting by properties, the social choice is determined through a collection of binary votes on subsets of alternatives involving qualified majority requirements. In more detail, it works as follows: Take some family of quotas $\{q_H : H \in \mathcal{H}\}$. For any issue (H, H^c), it is collectively decided according to the quotas q_H and q_{H^c} whether the winning alternative is supposed to share property H or its complement H^c. These binary decisions yield a collection of properties that the winning alternative is supposed to share. However, it has to be ensured that this set of, loosely speaking, winning properties is consistent in the sense that the intersection of these properties is not empty, but it contains exactly one alternative which, then, constitutes the winning alternative. Thus,

in general, the considered mapping need not represent a proper social choice function. However, as the following result reveals, under some conditions on the family of quotas, the stated mapping forms a social choice function.
I state [NP, 2007b]'s characterization of strategy-proof, anonymous, and surjective social choice functions.

Theorem 1. *[NP, 2007b]*
A social choice function f is strategy-proof, anonymous, and surjective if and only if it is voting by properties $f_{\{q_H : H \in \mathcal{H}\}} : A^n \to A$ with a family of quotas $\{q_H : H \in \mathcal{H}\}$ such that, for all properties $H, K \in \mathcal{H}$, it holds

$$H \subseteq K \Rightarrow q_H \geq q_K.$$

Theorem 1 implies that, when searching for the optimal mechanism among all social choice functions satisfying strategy-proofness, anonymity, and surjectivity, it is sufficient to optimize over the set of quotas $\{q_H : H \in \mathcal{H}\}$ related to voting by properties while respecting the collection of inequalities stated in Theorem 1. I tackle this problem in the subsequent section.

Public Goods Provision. Before that, go again back to the public goods application. Recall that I described in section 3 strategy-proof, anonymous, and surjective social choice functions in terms of a collection of binary votes that are determined by the four majority quotas $q_\alpha(1)$, $q_\alpha(2)$, $q_\beta(1)$, and $q_\beta(2)$ subject to some constraints on these majority quotas that I did not specify there explicitly. Now, any such collection of binary votes coincides with a voting by properties mechanism, and the stated constraints are the restrictions from Theorem 1. To see this, for all $k \in \{1, 2\}$, set

$$q_{H^\alpha_{\leq k}} := q_\alpha(k), \text{ and } q_{H^\alpha_{\geq k+1}} := n + 1 - q_\alpha(k)$$

as well as

$$q_{H^\beta_{\leq k}} := q_\beta(k), \text{ and } q_{H^\beta_{\geq k+1}} := n + 1 - q_\beta(k).$$

6 Welfare Maximization

In this section, I characterize the welfare-maximizing mechanism among all social choice functions satisfying strategy-proofness, anonymity, and surjectivity, constituting the main result of this paper.
By Theorem 1, it is sufficient to find the optimal quotas related to voting by properties. Also, the existence of a solution is ensured since a bounded function is maximized over a finite set of elements. The structure of the proof of the main theorem below is as follows:

First, consider some optimal mechanism and derive necessary conditions for optimality by means of studying the implications of alterations of this optimal mechanism. Second, argue that these necessary conditions are also sufficient for optimality and conclude that they determine a unique optimal mechanism.[22]

When deriving the discussed necessary conditions for optimality, it turns out that I have to compare the welfare induced by the following two sets of alternatives: For every property $H \in \mathcal{H}$, define the sets of alternatives

$$A_H := H \cap [\cap_{\{M:M\subset H\}} M^c],$$

and

$$A_{H^c} := H^c \cap [\cap_{\{M:M\subset H^c\}} M^c].$$

Alternatives contained in the set A_H share property H, but these alternatives violate all properties that are subsets of H. Likewise, alternatives from the set A_{H^c} satisfy property H^c, but properties that are subsets of H^c are violated. In Lemma 1, I establish that the sets A_H and A_{H^c} have a particular tuple structure.

Lemma 1.
Consider any property $H \in \mathcal{H}$. The sets A_H and A_{H^c} satisfy $A_H \neq \emptyset$ and $A_{H^c} \neq \emptyset$. Moreover, all elements in both sets can be uniquely matched into tuples having the form (k, m) with $k \in A_{H^c}$ and $m \in A_H$ such that k and m are separated only by property H, meaning, $\{H\} = \{K \in \mathcal{H} : k \notin K \wedge m \in K\}$.

The proof of Lemma 1 employs a characterization of median spaces in terms of the involved properties instead of relying on the induced betweenness relation due to [NP, 2007b]. Let Z_H denote the set of tuples implied by Lemma 1. It is clear that $|A_H| = |A_{H^c}|$, but, in general, it does not hold that $|A_H| = |A_{H^c}| = 1$.

Public Goods Provision. Revisit again the public goods application. For instance, consider the property $H^{\beta}_{\leq 1}$. In this case, I have that $\{M : M \subset H^{\beta}_{\leq 1}\} = \{H^{\alpha}_{\leq 1}\}$. Hence, the set $A_{H^{\beta}_{\leq 1}}$ amounts to

$$A_{H^{\beta}_{\leq 1}} = \{(2, 1), (3, 1)\}.$$

Similarly, the set $A_{H^{\beta}_{\geq 2}}$ satisfies

$$A_{H^{\beta}_{\geq 2}} = \{(2, 2), (3, 2)\}.$$

[22] Conceptually, [Gershkov et al., 2017] employ a similar proof strategy.

In particular, it holds that $|A_{H^\beta_{\leq 1}}| = |A_{H^\beta_{\geq 2}}| \neq 1$. Moreover, the set of tuples $Z_{H^\beta_{\leq 1}}$ is given by

$$Z_{H^\beta_{\leq 1}} = \{((2,2),(2,1)),((3,2),(3,1))\}.$$

This is precisely the tuple structure established in Lemma 1: Alternatives $(2,2)$ and $(2,1)$ as well as $(3,2)$ and $(3,1)$ are each separated only by property $H^\beta_{\leq 1}$. Further, this tuple structure is unique for the following reason: When matching $(2,2)$ and $(3,1)$ as well as $(3,2)$ and $(2,1)$, the matched alternatives are each separated by more properties than just $H^\beta_{\leq 1}$, meaning, $\{H^\beta_{\leq 1}\} \subset \{K \in \mathcal{H} : (2,2) \notin K \wedge (3,1) \in K\}$ and $\{H^\beta_{\leq 1}\} \subset \{K \in \mathcal{H} : (3,2) \notin K \wedge (2,1) \in K\}$, violating the condition that the matched alternatives are each separated only by property $H^\beta_{\leq 1}$. More generally, for the public goods application, it can be verified that two alternatives form a tuple (k,m) with $k \in A_{H^c}$ and $m \in A_H$ such that k and m are separated only by property H if and only if, for one public good, the provided level is the same in k and m, and, for the other good, the levels differ by exactly one.

The tuple structure established in Lemma 1 implies that the comparison of the welfare generated by the sets A_H and A_{H^c} reduces to contrasting a collection of pairs of alternatives such that the elements within each pair are separated by one property only. Furthermore, in order to characterize the optimal quotas in a separable way, I have to make sure that the welfare gains and losses involved in the welfare comparison within the discussed tuples do not depend on the tuple under consideration, but that they are the same across all tuples. The purpose of Assumption 2 on the utility function is to ensure exactly that.

Assumption 2.
Consider any property $H \in \mathcal{H}$. For any two tuples of alternatives (k,m) and (k',m') with $k,k' \in A_{H^c}$ and $m,m' \in A_H$ such that k and m as well as k' and m' are each separated only by property H, that is, $\{H\} = \{K \in \mathcal{H} : k \notin K \wedge m \in K\} = \{K \in \mathcal{H} : k' \notin K \wedge m' \in K\}$, and, for all type realizations $t \in S$, the utility function satisfies

$$u^k(t) - u^m(t) = u^{k'}(t) - u^{m'}(t).$$

Following a related discussion in [NP, 2007b], Assumption 2 represents essentially an additive separability restriction on the utility function, making it a natural assumption in contexts, where alternatives might be multidimensional. Moreover, this assumption is vacuously met in the special case of trees that I discuss in section 7.

Public Goods Provision. Go back to the public goods application. Clearly, Assumption 2 is satisfied in the public goods application because the utility function is additively separable across the two public goods. For example, consider again the set of tuples $Z_{H^{\beta}_{\leq 1}} = \{((2,2),(2,1)),((3,2),(3,1))\}$. In this case, for all type realizations $(x,y) \in S$, it holds that

$$u^{(2,2)}(x,y) - u^{(2,1)}(x,y) = u^{(3,2)}(x,y) - u^{(3,1)}(x,y) = G^2 \cdot y - c^2 - G^1 \cdot y + c^1.$$

Finally, Assumption 3 takes care of the fact that the alterations of some optimal mechanism that build the basis for the derivation of necessary conditions for optimality need not be feasible due to the constraints on the family of quotas appearing in Theorem 1. Essentially, it implies that the discussed necessary conditions remain valid even if the considered alterations are not feasible.

Assumption 3.
Consider two arbitrary properties $H, K \in \mathcal{H}$ satisfying $H \subseteq K$. For any two tuples of alternatives (k,m) and (j,l) with $k \in A_{H^c}$ and $m \in A_H$ as well as $j \in A_{K^c}$ and $l \in A_K$ such that k and m are separated only by property H as well as j and l are separated only by property K, that is, $\{H\} = \{L \in \mathcal{H} : k \notin L \wedge m \in L\}$ and $\{K\} = \{L \in \mathcal{H} : j \notin L \wedge l \in L\}$, the following inequality holds:

$$\begin{aligned}\delta_H :=& \frac{\mathbb{E}[u^k - u^m | u^k > u^m]}{\mathbb{E}[u^k - u^m | u^k > u^m] + \mathbb{E}[u^m - u^k | u^m > u^k]} \\ \geq& \frac{\mathbb{E}[u^j - u^l | u^j > u^l]}{\mathbb{E}[u^j - u^l | u^j > u^l] + \mathbb{E}[u^l - u^j | u^l > u^j]} =: \delta_K.\end{aligned}$$

Note that in the presence of Assumption 2, for any two tuples of alternatives (k,m) and (k',m') with $k,k' \in A_{H^c}$ and $m,m' \in A_H$ such that k and m as well as k' and m' are each separated only by property H, it holds that

$$\begin{aligned}\delta_H =& \frac{\mathbb{E}[u^k - u^m | u^k > u^m]}{\mathbb{E}[u^k - u^m | u^k > u^m] + \mathbb{E}[u^m - u^k | u^m > u^k]} \\ =& \frac{\mathbb{E}[u^{k'} - u^{m'} | u^{k'} > u^{m'}]}{\mathbb{E}[u^{k'} - u^{m'} | u^{k'} > u^{m'}] + \mathbb{E}[u^{m'} - u^{k'} | u^{m'} > u^{k'}]}.\end{aligned}$$

Therefore, the notation δ_H is justified because δ_H does not depend on the considered tuple of alternatives that are separated only by property H.
Moreover, observe that the following requirement constitutes a sufficient condition for Assumption 3: For all $H, K \in \mathcal{H}$ such that $H \subseteq K$ or, equivalently, $K^c \subseteq H^c$, it holds that

$$\begin{aligned}&\mathbb{E}[u^m - u^k | u^m > u^k] \leq \mathbb{E}[u^l - u^j | u^l > u^j], \text{ and} \\ &\mathbb{E}[u^k - u^m | u^k > u^m] \geq \mathbb{E}[u^j - u^l | u^j > u^l],\end{aligned}$$

where $k \in A_{H^c}$ and $m \in A_H$ as well as $j \in A_{K^c}$ and $l \in A_K$ such that k and m are separated only by property H as well as j and l are separated only by property K. Taking into account a connected discussion in [NP, 2007b], very loosely speaking, Assumption 3 might be interpreted as a concavity restriction on the preference intensities in the following sense: In expectation, the utility decreases more when moving from alternative l to j in comparison with m to k as well as when moving from k to m in comparison with j to l. Further, this assumption is vacuously satisfied in the special case of hypercubes that I discuss in section 7.

Public Goods Provision. Consider again the public goods application. In this case, for all $k \in \{1, 2\}$, it can be verified that the considered ratios simplify to the following expressions:

$$\delta_{H^{\alpha}_{\leq k}} = \frac{1}{1 + \frac{\mathbb{E}[z^{k,k+1} - X | X \leq z^{k,k+1}]}{\mathbb{E}[X - z^{k,k+1} | X \geq z^{k,k+1}]}}, \text{ and } \delta_{H^{\alpha}_{\geq k+1}} = 1 - q_{H^{\alpha}_{\leq k}}$$

as well as

$$\delta_{H^{\beta}_{\leq k}} = \frac{1}{1 + \frac{\mathbb{E}[z^{k,k+1} - Y | Y \leq z^{k,k+1}]}{\mathbb{E}[Y - z^{k,k+1} | Y \geq z^{k,k+1}]}}, \text{ and } \delta_{H^{\beta}_{\geq k+1}} = 1 - q_{H^{\beta}_{\leq k}}.$$

Now, the interrelations between properties in the sense that one property is a subset of another property that are relevant for Assumption 3 are as follows:[23] First, for any public good $\gamma \in \{\alpha, \beta\}$, $H^{\gamma}_{\leq 1} \subset H^{\gamma}_{\leq 2}$. Hence, Assumption 3 requires that $\delta_{H^{\gamma}_{\leq 1}} \geq \delta_{H^{\gamma}_{\leq 2}}$. However, this aspect implied by the regularity condition imposed in section 3 that the marginal densities g_X and g_Y are log-concave.[24] Second, for any public good level $k \in \{1, 2\}$, $H^{\alpha}_{\leq k} \subset H^{\beta}_{\leq k}$. Thus, Assumption 3 demands that $\delta_{H^{\alpha}_{\leq k}} \geq \delta_{H^{\beta}_{\leq k}}$. In section 3, I assumed that $G_X \geq_{lr} G_Y$.[25] This condition is sufficient for $\delta_{H^{\alpha}_{\leq k}} \geq \delta_{H^{\beta}_{\leq k}}$.[26] Overall, the regularity conditions on the type distribution from section 3 ensure that Assumption 3 is satisfied in the public goods application. This suggests that, at least in the public goods application, Assumption 3 constitutes a rather mild constraint.

Having presented the required assumptions as well as some preliminary steps for the analysis, I state the main result of this paper, that is, I provide a characterization of the welfare-maximizing mechanism among all strategy-proof, anonymous, and surjective social choice functions.

[23] The conditions associated with all other interrelations of properties are automatically satisfied if the constraints related to the discussed relevant interrelations are met.

[24] To see this, recall that a random variable satisfies the decreasing mean residual life property as well as the increasing mean inactivity time property if its density is log-concave (see [Bagnoli and Bergstrom, 2005]).

[25] Again, the order $\geq_{lr}$ denotes the likelihood ratio order.

[26] The reason is that, if $G_X \geq_{lr} G_Y$, the same ordering holds in terms of the hazard as well as reversed hazard rate ordering, implying the claim (see [Shaked and Shanthikumar, 2007]).

Theorem 2.
Suppose that Assumptions 1, 2 and 3 hold.
The optimal mechanism among all strategy-proof, anonymous, and surjective social choice functions takes the form of voting by properties with quotas

$$q_H^* = \lceil n\delta_H \rceil \text{ for all } H \in \mathcal{H}.$$

While taking into account that mechanisms have to be dominant-strategy incentive-compatible, Theorem 2 characterizes the optimal utilitarian mechanism for generalized single-peaked domains derived from median spaces. In particular, Theorem 2 provides closed-form expressions for the welfare-maximizing quotas related to voting by properties. The intuition behind the optimal quotas $q_H^* = \lceil n\delta_H \rceil$ is as follows: Take any tuple of alternatives (k, m) with $k \in A_{H^c}$ and $m \in A_H$ such that k and m are separated only by property H. Now, first of all, observe that the quota q_H^* is shaped by the ratio of preference intensities

$$\frac{\mathbb{E}[u^m - u^k | u^m > u^k]}{\mathbb{E}[u^k - u^m | u^k > u^m]},$$

reflecting the utilitarian objective of the designer. Also, regarding comparative statics, the quota q_H^* decreases in the discussed ratio of preference intensities. For the purpose of a more detailed understanding, ignore the aspect that quotas must be integer-valued. Plugging in the term for δ_H and rearranging yields

$$\begin{aligned} &\frac{q_H^*}{n}\mathbb{E}[u^k | u^m > u^k] + \frac{n - q_H^*}{n}\mathbb{E}[u^k | u^k > u^m] \\ =&\frac{q_H^*}{n}\mathbb{E}[u^m | u^m > u^k] + \frac{n - q_H^*}{n}\mathbb{E}[u^m | u^k > u^m]. \end{aligned}$$

This expression shows how the designer's Bayesian inference problem is resolved: The optimal quota q_H^* is calibrated such that the designer is indifferent between implementing alternatives k and m conditional on being pivotal, that is, conditional on the event that exactly q_H^* out of the n voters prefer alternative m over alternative k. The latter event coincides with the event that there are exactly q_H^* voters whose most preferred alternatives share property H.[27] Consequently, the optimal quota q_H^* is set such that the designer is indifferent between any pair of alternatives separated only by property H conditional on being there exactly q_H^* voters with peaks from the set H.

Public Goods Provision. Revisit the public goods application, and recall the following two aspects: First, again, in the public goods application, two alternatives form a tuple (k, m) with $k \in A_{H^c}$ and $m \in A_H$ such that k and m are separated only by property H if

[27] This point follows from generalized single-peakedness (see [NP, 2007b]).

and only if, for one public good, the provided level is the same in k and m, and, for the other good, the levels in k and m differ by exactly one. Second, as discussed in section 3, the optimal quotas in the public goods application are calibrated in the following way: Conditional on being pivotal, the designer is indifferent between any two alternatives such that the provided level of one good is the same in both alternatives, but the levels of the other good differ by exactly one. This discussion shows how the indifference property of the optimal quotas in the public goods application described in section 3 generalizes to all median spaces.

Let me outline the proof of Theorem 2. Again, the proof builds on the corresponding proof in [Gershkov et al., 2017], but the much larger class of preferences requires additional arguments as well as different assumptions. In section 7, in the context of trees, I discuss how Theorem 2 extends the main result in [Gershkov et al., 2017].
To begin with, by Theorem 1, it is sufficient to optimize over the set of quotas related to voting by properties.[28] Furthermore, again, due to Theorem 1, for all $H, K \in \mathcal{H}$, the optimal quotas must satisfy

$$H \subseteq K \Rightarrow q_H \geq q_K.$$

Consider some property $H \in \mathcal{H}$ as well as the associated quota q^*_H which is supposed to be part of an optimal mechanism. To simplify the exposition, I divide the proof of Theorem 2 into two lemmata.

Lemma 2.
Suppose that Assumptions 1 and 2 hold. Consider any property $H \in \mathcal{H}$.
(i) If $H' \subset H \Rightarrow q^*_{H'} > q^*_H$ for all $H' \in \mathcal{H}$ such that $\nexists H'' \in \mathcal{H} : H' \subset H'' \subset H$, the inequality

$$q^*_H \geq n \cdot \delta_H$$

constitutes a necessary condition for optimality.
(ii) If $H \subset H' \Rightarrow q^*_H > q^*_{H'}$ for all $H' \in \mathcal{H}$ such that $\nexists H'' \in \mathcal{H} : H \subset H'' \subset H'$, any optimal mechanism meets the inequality

$$q^*_H \leq n \cdot \delta_H + 1.$$

Suppose that increasing q^*_H by 1 is feasible, meaning, this alteration does not violate the inequalities from Theorem 1. This change matters only if there are q^*_H voters having some peak from the set H and $n - q^*_H$ voters with peaks from the set H^c. In this case,

[28] Again, since a bounded function is maximized over a finite set of elements, the existence of a solution is ensured.

since $q^*_L \leq q^*_H$ for all $H \subset L$, the properties $\{L : H \subset L\}$ or, equivalently, $\{L^c : L \subset H^c\}$ are accepted whenever there are such properties. Additionally, since increasing q^*_H by 1 is feasible, I must have that $q^*_M > q^*_H$ for all $M \subset H$. Thus, the properties $\{M : M \subset H\}$ are rejected or, equivalently, the properties $\{M^c : M \subset H\}$ are winning whenever there are such properties. Putting these aspects together and using the introduced notation, if the quota is q^*_H, some element of the set $A_H \neq \emptyset$ is the winning alternative. However, if the quota amounts to $q^*_H + 1$, some element of the set $A_{H^c} \neq \emptyset$ is selected. Since q^*_H is part of an optimal mechanism, the modification of this quota should weakly decrease welfare. In other words, the expected welfare induced by alternatives from the set A_H must be weakly higher compared to the welfare generated by alternatives from the set A_{H^c}. This observation translates into a condition which is necessary for optimality whenever the considered change in the optimal quota q^*_H is feasible. Exploiting the tuple structure derived in Lemma 1, the comparison of the expected welfare induced by the two sets of alternatives reduces to contrasting a collection of tuples of alternatives such that the elements within each tuple are separated only by property H. Now, imposing Assumption 2 implies, as discussed above, that these within-tuple welfare comparisons are not sensitive to the tuple under consideration. This aspect simplifies the involved expressions and leads to the inequality appearing in part (i) of Lemma 2.

Studying the effect of a decrease of q^*_H by 1 yields via an analogous argument the inequality appearing in part (ii) of Lemma 2. This inequality is necessary for optimality as long as the considered decrease in the optimal quota q^*_H is feasible.

The second step of the proof of Theorem 2 is summarized in Lemma 3.

Lemma 3.

Suppose that Assumptions 1, 2 and 3 hold. Consider any properties $H', H \in \mathcal{H}$ such that $H' \subset H$ and $\nexists H'' \in \mathcal{H} : H' \subset H'' \subset H$.

If $q^*_{H'} = q^*_H$, any optimal mechanism nevertheless satisfies

$$q^*_H \geq n \cdot \delta_H$$

as well as

$$q^*_{H'} \leq n \cdot \delta_{H'} + 1.$$

The two alterations of the quota q^*_H that is part of an optimal mechanism considered above might not be feasible. Lemma 3 addresses this issue. Making use of Assumption 3, I show that the two inequalities derived in Lemma 2 still hold even if these alterations are not feasible.

Finally, it turns out that these inequalities are not only necessary, but also sufficient for optimality, and they determine the generically unique optimal mechanism featuring the quotas appearing in Theorem 2.

7 Applications

In this section, I apply the general characterization of welfare-maximizing mechanisms developed in Theorem 2 to the special cases of trees and hypercubes. In these settings Assumption 2 and Assumption 3 are vacuously met respectively. [NP, 2007b] identify trees and hypercubes as distinguished instances of median spaces.[29] The purpose of this section is to present these two instances of median spaces as in both settings one of the three assumptions in Theorem 2 is vacuously met.

7.1 Trees

To begin with, I consider the special case of single-peaked preferences on trees as introduced in [Demange, 1982]. Take any tree (A, E), that is, take any undirected graph that is connected and acyclic. The set of alternatives A coincides with the set of nodes and the set E captures the set of edges corresponding to the tree. In particular, the set E satisfies $E \subseteq \{V \in \mathcal{P}(A) : |V| = 2\}$. Following [Nehring and Puppe, 2007a], for any edge $V = \{b, c\} \in E$, define the two properties

$$H_{V,k} := \{a \in A : \text{“}a \text{ lies in direction of } k\text{”}\} \text{ and}$$
$$H_{V,m} := \{a \in A : \text{“}a \text{ lies in direction of } m\text{”}\}.$$[30]

Note that any property coincides with a set of nodes corresponding to a connected component of the underlying tree. Also, the properties of the form $(H_{V,k}, H_{V,m})$ constitute an issue. Let $\mathcal{H}_{Tree}$ denote the collection of all these properties. Further, observe that a preference relation is single-peaked with respect to the underlying tree as defined in [Demange, 1982] if and only if it is generalized single-peaked with respect to the betweenness relation $B_{\mathcal{H}_{Tree}}$. The former definition reads as follows: There exists an alternative $p \in A$, which is the most preferred or peak alternative, such that for all alternatives $k, m \in A$ with $k \neq m$ it holds that whenever k lies on the shortest path in the underlying tree connecting p and m, the voter must prefer k over m.

To illustrate this class of property spaces more concretely, take the simplest tree that is not a line: Suppose that there are four alternatives $\{1, 2, 3, 4\}$ and take the tree that is shown in Figure 2. In this case, the collection of properties $\mathcal{H}_{Tree}$ amounts to

[29] For any median space, [NP, 2007b] characterize the requirement of generalized single-peakedness in terms of a separability and a convexity condition, and they argue that, on trees and hypercubes, generalized single-peakedness reduces to convexity and separability respectively.

[30] More formally, following [Nehring and Puppe, 2007a], the property $H_{V,k}$ is composed of all alternatives $a \in A$ such that k lies on the shortest path from a to m in the tree (A, E). Similarly, $H_{V,m}$ comprises all alternatives $a \in A$ such that m lies on the shortest path from a to k in the tree (A, E).

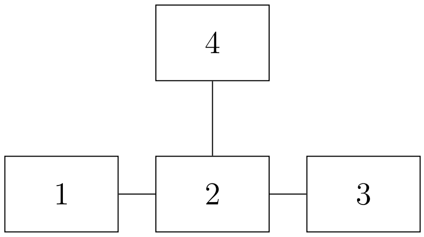

Figure 2: Tree Example

$$H_{\{2,4\},2} = \{1,2,3\}, H_{\{2,4\},4} = \{4\},$$
$$H_{\{1,2\},1} = \{1\}, H_{\{1,2\},2} = \{2,3,4\}, \text{ and}$$
$$H_{\{2,3\},2} = \{1,2,4\}, H_{\{2,3\},3} = \{3\}.$$

For instance, if some voter's most preferred alternative is 1, generalized single-peakedness requires here that alternative 2 is preferred over 3 and 4, but it does not impose whether 3 is preferred to 4 or the other way around.

Moreover, if preferences are single-peaked with respect to a tree (A, E), a voting by properties mechanism can be intuitively described as "voting by edges": Take any edge of the tree (A, E), cut this edge yielding two subsets of alternatives or, more precisely, two connected components of the tree. Then, perform a binary vote determining which of the two connected components is winning. This binary vote yields one winning connected component, that is, the social choice must be contained in the set of nodes associated with the connected component that is winning. These binary voting decisions are conducted for all edges yielding a collection of connected components that are winning. Eventually, the final outcome is given by the intersection of the sets of nodes linked to the connected components that are winning.[31]

Now, if preferences are single-peaked with respect to a tree (A, E), the following aspect follows from [NP, 2007b]: Take any edge $V = \{k, m\} \in E$ with $k, m \in A$. Any two alternatives form a tuple (j, l) with $j \in A_{H_{V,k}}$ and $l \in A_{H^c_{V,k}} = A_{H_{V,m}}$ such that j and l are separated only by property $H_{V,m}$ if and only if $j = k$ and $l = m$. This implies that, for any property $H \in \mathcal{H}_{Tree}$, the sets A_H and A_{H^c} considered in section 6 are singletons, i.e., $|A_H| = |A_{H^c}| = 1$. Consequently, Assumption 2 is vacuously met on trees, and I have the subsequent corollary of Theorem 2.

Corollary 1.

Consider the median space $(A, \mathcal{H}_{Tree})$, and suppose that Assumptions 1 and 3 are satisfied. The optimal mechanism among all strategy-proof, anonymous, and surjective social choice

[31]Of course, the quotas involved in the described binary votes on subsets of the set of alternatives have to satisfy the restrictions from Theorem 1.

functions takes the form of voting by properties with quotas

$$q_H^* = \lceil n\delta_H \rceil \text{ for all } H \in \mathcal{H}_{Tree}.$$

The general indifference property of the welfare-maximizing quotas discussed in section 6 reduces here to the following feature: Take any edge $V = \{k, m\} \in E$ with $k, m \in A$. Then, the corresponding optimal quotas $q^*_{H_{V,k}}$ and $q^*_{H_{V,m}}$ are calibrated such that, conditional on being pivotal, the designer is indifferent between the two graph neighbors k and m in the tree tree (A, E).
Now, let me discuss how Corollary 1 extends the main result in [Gershkov et al., 2017]. For concreteness, without loss of generality, suppose that the set of alternatives amounts to $A := \{1, ..., l\}$ with $l \geq 2$. Following [NP, 2007b], assume that, for all $1 \leq k < l$, the properties are given by

$$H_{\leq k} := \{m \in \{1, ..., l\} : m \leq k\} \text{ as well as}$$
$$H_{\geq k+1} := \{m \in \{1, ..., l\} : m \geq k+1\}.$$

Let $\mathcal{H}_{Line}$ denote the set of all these properties. Note that this collection of properties exactly coincides with $\mathcal{H}_{Tree}$ if the underlying tree (A, E) constitutes the line shown in Figure 3. In particular, a preference relation is generalized single-peaked with respect to

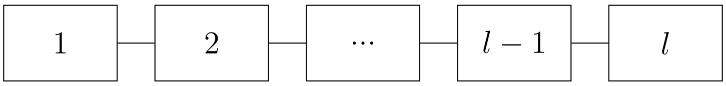

Figure 3: Line

the betweenness relation $B_{\mathcal{H}_{Line}}$ if and only if it is in the classical sense single-peaked on a line and, more precisely, it is single-peaked on a line with respect to the natural ordering $1 < 2 < ... < l - 1 < l$.
Specializing Corollary 1 to the case of single-peaked preferences on a line, I immediately obtain the following corollary.

Corollary 2. *[Gershkov et al., 2017]*
Consider the median space $(\{1, ..., l\}, \mathcal{H}_{Line})$, and suppose that Assumptions 1 and 3 are satisfied.
The optimal mechanism among all strategy-proof, anonymous, and surjective social choice functions takes the form of voting by properties with quotas

$$q_H^* = \lceil n\delta_H \rceil \text{ for all } H \in \mathcal{H}_{Line}.$$

Observe that Corollary 2 coincides with the main result in [Gershkov et al., 2017]: Assumption 1 is assumption A in their paper, and Assumption 3 reduces exactly to assumption B in their work.[32]

7.2 Hypercubes

Having treated collective choice when preferences are single-peaked on trees, I continue with the discussion of voting on hypercubes, that is, voting on multiple binary decisions as studied in [Barberà et al., 1991]. To start, assume that the set of alternatives A is given by $A := \{0,1\}^l$, where $l \geq 1$ is a natural number. This means that there are l binary decisions, each coordinate of an alternative corresponds to a binary decision, and, without loss of generality, each binary decision amounts either to 0 or 1. Following [NP, 2007b], suppose that, for all $1 \leq k \leq l$, the properties are given by

$$H_{0,k} := \{(m_1, ..., m_l) \in \{0,1\}^l : m_k = 0\}, \text{ and}$$
$$H_{1,k} := \{(m_1, ..., m_l) \in \{0,1\}^l : m_k = 1\}.$$

Let $\mathcal{H}_{Hypercube}$ denote the collection of these properties. Moreover, it follows from [NP, 2007b] that the requirement of generalized single-peakedness reduces here to the restriction of separable preferences imposed in [Barberà et al., 1991]. The latter requirement reads as follows: For any $1 \leq k \leq l$ and all sequences $m \in \{0,1\}^{k-1}$ and $m' \in \{0,1\}^{l-k}$, a voter prefers alternative $(0^{k-1}, 1, 0^{l-k})$ over $(0^{k-1}, 0, 0^{l-k})$ if and only if he or she prefers alternative $(m, 1, m')$ over $(m, 0, m')$.

Furthermore, it can be verified that, on hypercubes, there are no properties $H, K \in \mathcal{H}$ that are interrelated in the sense that $H \subseteq K$. Therefore, the restrictions from Theorem 1 as well as Assumption 3 are vacuously met.[33] Consequently, I obtain the following corollary of Theorem 2.[34]

Corollary 3.

Consider the median space $(\{0,1\}^l, \mathcal{H}_{Hypercube})$, and suppose that Assumptions 1 and 2 are satisfied.

The optimal mechanism among all strategy-proof, anonymous, and surjective social choice

[32]Unless there are at most three alternatives, i.e., unless $l \leq 3$, there is the following caveat: Because [Gershkov et al., 2017] assume that preferences are single-crossing and single-peaked, the set of ordinal preferences induced by their utility representation does not satisfy [NP, 2007b]'s richness condition on the preference domain. Hence, the strategy-proof social choice functions [NP, 2007b] identify are strategy-proof in [Gershkov et al., 2017]'s setting, but there might be more strategy-proof direct mechanisms. However, when combining results from [Moulin, 1980], [NP, 2007b], and [Saporiti, 2009], it can be inferred that also in [Gershkov et al., 2017]'s model there are no other strategy-proof social choice functions apart from those identified in [NP, 2007b]'s characterization.

[33]Similarly, in [Gershkov et al., 2017], their assumption B is vacuously satisfied if there are only two alternatives.

[34]Corollary 3 can also be obtained by combining, on the one hand, the results from [NP, 2007b] or [Barberà et al., 1991] and, on the other hand, the optimality findings for the two-alternatives case from, for example, [Nehring, 2004] or [Drexl and Kleiner, 2018].

functions takes the form of voting by properties with quotas

$$q_H^* = \lceil n\delta_H \rceil \text{ for all } H \in \mathcal{H}_{Hypercube}.$$

Here, any voting by properties social choice function amounts to performing qualified majority voting separately for each binary decision. Also, it follows from [NP, 2007b] that Assumption 2 is satisfied if and only if the voters' utilities are additively separable across the binary decisions, making it a natural assumption on hypercubes. Moreover, the general indifference property of the welfare-maximizing quotas from section 6 simplifies here to the following aspect: Take any $1 \leq k \leq l$. Then, the associated optimal quotas $q^*_{H_{0,k}}$ and $q^*_{H_{1,k}}$ are set such that, conditional on being pivotal, the designer is indifferent between any two alternatives that differ only with respect to the outcome in the k-th binary decision, that is, any two alternatives $(m, 1, m')$ and $(m, 0, m')$ with $m \in \{0, 1\}^{k-1}$ and $m' \in \{0, 1\}^{l-k}$.

8 Conclusion

In this paper, I offered a welfare analysis of voting rules. Specifically, I derived the optimal utilitarian mechanism among all strategy-proof, anonymous, and surjective social choice functions for generalized single-peaked domains giving rise to median spaces. The optimal mechanism takes the form of voting by properties, meaning, the social choice is determined through a collection of binary votes on subsets of alternatives involving qualified majority requirements that incorporate the characteristics of these subsets of alternatives. Consequently, on a qualitative level, my results emphasize the importance of flexible and qualified majority requirements for utilitarian welfare in voting on a broad scale. Moreover, my optimality analysis reveals that trees and hypercubes are distinguished instances of median spaces as in both settings one of the three assumptions of the general characterization of welfare-maximizing mechanisms is vacuously met.

Appendix

The proof of Lemma 1 employs a result from [NP, 2007b] that is stated as Lemma 1 below. In order to present this result, I need to introduce the notion of *critical families of properties* from their paper. These sets are collections of properties having the following characteristic.

Definition A.1. *[NP, 2007b]*
A set of properties $\mathcal{F} \subseteq \mathcal{H}$ is a critical family of properties if

$$\cap_{\bar{F}\in\mathcal{F}} \bar{F} = \emptyset \text{ and}$$
$$\forall F \in \mathcal{F} : \cap_{\bar{F}\in\mathcal{F}:\bar{F}\neq F} \bar{F} \neq \emptyset.$$

In words, a collection of properties constitutes a critical family of properties if the intersection of all involved properties is empty, but these properties have a non-empty intersection whenever an arbitrary single property of the collection is removed. Also, note that any critical family of properties involves at least two elements. Based on this definition, [NP, 2007b] obtain the following result about the size of critical families of properties in median spaces.

Lemma A.1. *[NP, 2007b]*
If $(A, \mathcal{H})$ constitutes a median space, all critical families of properties have length two.

Lemma 1 says that median spaces share the characteristic that there are no critical families of properties involving more than two properties.

Proof of Lemma 1.
Take any property $H \in \mathcal{H}$ and consider the related sets A_H and A_{H^c} as defined in the main text.
Concerning the first aspect, consider the set A_H. The argument for the set A_{H^c} is analogous. Towards a contradiction, suppose that $A_H = \emptyset$. If $\nexists M \in \mathcal{H} : M \subset H$, I have that $H \subseteq A_H$. Since $H \neq \emptyset$, it follows that $A_H \neq \emptyset$. If $\exists M \in \mathcal{H} : M \subset H$, $A_H = \emptyset$ implies $H \cap (\cap_{M\subset H} M^c) = \emptyset$. In other words, the collection of properties $\{H\} \cup \{M^c : M \subset H\}$ is not consistent. However, this means that there must be some subset of the set of these properties which constitutes a critical family of properties. If this critical family involves at least three elements, the desired contradiction is derived since, due to Lemma 1, all critical families have length two in a median space. In case this critical family involves only two properties, there are two possibilities. On the one hand, if H is part of this critical family, the other element must be some single property M^c such that $M \subset H$. However, the collection of these two properties cannot be inconsistent and, hence, not critical since the intersection of H and M^c must be non-empty. On the other hand, if H is not part of the critical family, this family must be composed of two properties from the

set $\{M^c : M \subset H\}$, but both of them are by definition supersets of H^c which means that they are consistent and, thus, not critical since $H^c \neq \emptyset$. Therefore, in the two possible cases, I derived the desired contradiction.
Regarding the second aspect, take any $k \in A_{H^c}$, and consider the following intersection of properties:

$$\mathcal{K}_k := (\cap_{\{K \in \mathcal{H}_k : K \neq H^c\}} K) \cap H.$$

Now, because of separation, the set $\mathcal{K}_k$ is either empty or it contains exactly one alternative, but it does not contain more than one alternative. I claim that $\mathcal{K}_k$ cannot be empty. Towards a contradiction, assume that $\mathcal{K}_k$ is empty. This means that the collection of properties $(\mathcal{H}_k \setminus \{H^c\}) \cup \{H\}$ is not consistent. To begin with, if $\mathcal{H}_k \setminus \{H^c\} = \emptyset$, the set of properties $(\mathcal{H}_k \setminus \{H^c\}) \cup \{H\}$ must be consistent since $H \neq \emptyset$. Thus, subsequently, assume that $\mathcal{H}_k \setminus \{H^c\}$ contains at least one property. $(\mathcal{H}_k \setminus \{H^c\}) \cup \{H\}$ being inconsistent implies that there must be some subset of this collection of properties which constitutes a critical family of properties. Since all property spaces are median spaces, due to Lemma 1, this critical family must involve exactly two properties. Again, there are two possibilities. On the one hand, if H is part of this critical family, the other element must be some single property $K \in \mathcal{H}_k \setminus \{H^c\}$ satisfying $K \subset H^c$. In particular, it must hold that $k \in K$. However, by definition of A_{H^c}, because of $K \subset H^c$, I have $k \in K^c$. This contradicts $k \in K$. On the other hand, if H is not part of the critical family, this family must be composed of two properties from the set $\mathcal{H}_k \setminus \{H^c\}$, but, by construction, the alternative k shares both of them which means that they are consistent and, thus, not critical. Hence, in both possible cases, I obtain a contradiction. Therefore, I infer that $\mathcal{K}_k$ is not empty, but it contains exactly one alternative. Denote this alternative by m. Now, by construction, k and m are separated only by property H. Further, I obtain that $m \in A_H$ for the following reason: If $\nexists M \in \mathcal{H} : M \subset H^c$, by definition of A_H, I have that $\mathcal{K}_k \cap A_H = \mathcal{K}_k \cap H = \mathcal{K}_k$. If $\exists M \in \mathcal{H} : M \subset H^c$, again by definition of A_H, it holds that

$$\mathcal{K}_k \cap A_H = \mathcal{K}_k \cap H \cap (\cap_{\{M : M \subset H^c\}} M^c) = \mathcal{K}_k \cap H \cap (\cap_{\{M : H \subset M^c\}} M^c) = \mathcal{K}_k.$$

Consequently, I conclude that there exists some $m \in A_H$ such that k and m are separated only by property H. Moreover, there cannot be another alternative $m' \in A_H$ with $m \neq m'$ such that k and m' are also separated only by property H since this would contradict separation. The argument for the other direction, meaning, starting with some $m \in A_H$ and showing that there is some unique $k \in A_{H^c}$ such that both alternatives are separated only by property H works in the same way. This establishes the claimed unique tuple structure. $\square$

Proof of Lemma 2.

Take any property $H \in \mathcal{H}$.

Assume that $H' \subset H \Rightarrow q^*_{H'} > q^*_H$ for all $H' \in \mathcal{H}$ such that $\nexists H'' \in \mathcal{H} : H' \subset H'' \subset H$.

Consider the quota q^*_H being part of an optimal mechanism and suppose that it is increased by 1, i.e. the quota linked to property H moves to $q^*_H + 1$. In particular, as long as $q^*_H \neq n$, the modified quota $q^*_H + 1$ is still feasible because $q^*_{H'} \geq q^*_H + 1 > q^*_H$ for all $H' \in \mathcal{H}$ such that $\nexists H'' \in \mathcal{H} : H' \subset H'' \subset H$.

This alteration matters only if there are q^*_H voters having some peak from the set H and $n - q^*_H$ voters with peaks from the set H^c. For simplicity, call this event "piv_H".

In this case, since $q^*_L \leq q^*_H$ for all $H \subset L$, the properties $\{L : H \subset L\}$ or, equivalently, $\{L^c : L \subset H^c\}$ are accepted whenever there are such properties. Additionally, $q^*_{H'} > q^*_H$ for all $H' \in \mathcal{H}$ such that $\nexists H'' \in \mathcal{H} : H' \subset H'' \subset H$ implies that $q^*_M > q^*_H$ for all $M \subset H$. Thus, the properties $\{M : M \subset H\}$ are rejected or, equivalently, the properties $\{M^c : M \subset H\}$ are winning whenever there are such properties. Putting these aspects together and using the notation introduced in the main text, if the quota is q^*_H, some element of the set $A_H \neq \emptyset$ is the winning alternative. However, if the quota amounts to $q^*_H + 1$, some element of the set $A_{H^c} \neq \emptyset$ is selected.

Therefore, for both quotas, employing Assumption 1, the expected welfare conditional on the event where the alteration of q^*_H matters, i.e., the expected welfare conditional on the event "piv_H", can be expressed in the following way: If the quota is q^*_H, the resulting welfare amounts to

$$\sum_{l \in A_H} Pr(l\, wins | piv_H) \cdot \{n \cdot \mathbb{E}[u^l(T) | piv_H \wedge l\, wins]\}.$$

In contrast, if the quota is $q^*_H + 1$, the induced welfare satisfies

$$\sum_{j \in A_{H^c}} Pr(j\, wins | piv_H) \cdot \{n \cdot \mathbb{E}[u^j(T) | piv_H \wedge j\, wins]\}.$$

Because q^*_H is part of an optimal mechanism, it must be that the former expression is weakly higher than the latter term. This necessary condition for optimality translates into the inequality

$$\sum_{l \in A_H} Pr(l\, wins | piv_H) \mathbb{E}[u^l(T) | piv_H \wedge l\, wins] \geq$$
$$\sum_{j \in A_{H^c}} Pr(j\, wins | piv_H) \mathbb{E}[u^j(T) | piv_H \wedge j\, wins].$$

Now, consider the tuple structure derived in Lemma 1 and, with abuse of notation, suppose that (j, l) constitutes a tuple of alternatives such that j and l are separated only by property H. This means that the events "$l\, wins \wedge piv_H$" and "$j\, wins \wedge piv_H$"

must coincide, meaning, they refer to the same set of type realizations. This is true for the following reason: The event "$j\ wins \wedge piv_H$" means that the properties $\mathcal{H}_j \setminus H^c$ are winning and the number of voters having peaks from the set H amounts to q_H^*. The event "$l\ wins \wedge piv_H$" means that the properties $\mathcal{H}_l \setminus H$ are winning and the number of voters having peaks from the set H is q_H^*. However, since j and l are separated only by property H, it holds that $\mathcal{H}_j \setminus H^c = \mathcal{H}_l \setminus H$. Therefore, the events "$j\ wins \wedge piv_H$" and "$l\ wins \wedge piv_H$" coincide. Call this event "$j/l\ win \wedge piv_H$". In particular, I have that

$$Pr(j/l\ win|piv_H) = Pr(l\ wins|piv_H) = Pr(j\ wins|piv_H).$$

Therefore, the inequality above can be rewritten as follows:

$$\sum_{(j,l)\in Z_H} Pr(j/l\ win|piv_H)\{\mathbb{E}[u^l(T) - u^j(T)|j/l\ win \wedge piv_H]\} \geq 0.$$

Now, take any pair of alternatives (k, m) with $k \in A_{H^c}$ and $m \in A_H$ such that k and m are separated only by property H. Due to Assumption 2, it holds that

$$u^k(t) - u^m(t) = u^l(t) - u^j(t)$$

for all tuples of alternatives $(j, l) \in Z_H$ and for all type realizations $t \in S$. Thus, the previous inequality can be written as

$$\sum_{(j,l)\in Z_H} Pr(j/l\ win|piv_H)\{\mathbb{E}[u^m(T) - u^k(T)|j/l\ win \wedge piv_H]\} \geq 0.$$

Next, by the law of total expectation, I obtain that

$$\mathbb{E}[u^m(T) - u^k(T)|piv_H] \geq 0.$$

Moreover, applying again the law of total expectations, this inequality can be written in the following way:

$$\begin{aligned} &Pr(\text{“}peak \in H\text{”}|piv_H)\mathbb{E}[u^m(T) - u^k(T)|\text{“}peak \in H\text{”} \wedge piv_H] \\ &+Pr(\text{“}peak \in H^c\text{”}|piv_H)\mathbb{E}[u^m(T) - u^k(T)|\text{“}peak \in H^c\text{”} \wedge piv_H] \geq 0, \end{aligned}$$

where "$peak \in H$" and "$peak \in H^c$" refer to the events that an arbitrary voter's most-preferred alternative or peak shares property H and H^c respectively. While using the definition of the event "piv_H", Assumption 1 implies that the probabilities involved in the inequality satisfy

$$Pr(\text{“}peak \in H\text{”}|piv_H) = \frac{q_H^*}{n} \text{ and}$$
$$Pr(\text{“}peak \in H^c\text{”}|piv_H) = \frac{n - q_H^*}{n}.$$

Also, Assumption 1 yields

$$\mathbb{E}[u^m(T) - u^k(T)|\text{“}peak \in H\text{”} \wedge piv_H] = \mathbb{E}[u^m(T) - u^k(T)|\text{“}peak \in H\text{”}] \text{ and}$$
$$\mathbb{E}[u^m(T) - u^k(T)|\text{“}peak \in H^c\text{”} \wedge piv_H] = \mathbb{E}[u^m(T) - u^k(T)|\text{“}peak \in H^c\text{”}].$$

Further, it follows from generalized single-peakedness that the events “$peak \in H$” and “$peak \in H^c$” are equivalent to the events “$u^m(T) > u^k(T)$” and “$u^k(T) > u^m(T)$” respectively (see Fact 2.1 in [NP, 2007b]). Taking these three aspects together, the inequality above simplifies to

$$\frac{q_H^*}{n}\mathbb{E}[u^m(T) - u^k(T)|u^m(T) > u^k(T)] + \frac{n - q_H^*}{n}\mathbb{E}[u^m(T) - u^k(T)|u^k(T) > u^m(T)] \geq 0.$$

Hence, rearranging yields

$$q_H^* \geq n \cdot \delta_H$$

while I use the notation introduced in the main text. In addition, if $q_H^* = n$, the derived inequality still holds since $\delta_H \in (0, 1)$. This establishes the first claim of the lemma.
Turning to the second point of the lemma, suppose that $H \subset H' \Rightarrow q_H^* > q_{H'}^*$ for all $H' \in \mathcal{H}$ such that $\nexists H'' \in \mathcal{H} : H \subset H'' \subset H'$.
Consider again the quota q_H^* related to an optimal mechanism and suppose that it is decreased by 1, i.e. the quota q_H^* moves to $q_H^* - 1$. In particular, the altered quota is still feasible as long as $q_H^* \neq 1$. This change matters only if there are $q_H^* - 1$ voters with peak alternatives from the set H and $n - q_H^* + 1$ voters having peaks that share property H^c. Following the steps employed in the reasoning above in an analogous way, it can be verified that the inequality

$$q_H^* \leq n \cdot \delta_H + 1$$

constitutes a necessary condition for optimality. Additionally, observe that the derived inequality also holds if $q_H^* = 1$ since $n \cdot \delta_H > 0$. □

Proof of Lemma 3.
Assume that there are properties $H', H \in \mathcal{H}$ with $H' \subset H$ as well as $\nexists H'' \in \mathcal{H} : H' \subset H'' \subset H$ and the quotas related to an optimal mechanism satisfy $q_{H'}^* = q_H^*$.

Define

$$\begin{aligned}\mathcal{Q} :=& \{K \in \mathcal{H} : [(K \subseteq H' \vee H \subseteq K) \wedge q_K^* = n] \text{ } \mathit{and} \\ & \nexists K' \in \mathcal{H} : [K \subset K' \wedge (K' \subseteq H' \vee H \subseteq K') \wedge q_{K'}^* = n]\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{R} :=& \{K \in \mathcal{H} : [(K \subseteq H' \vee H \subseteq K) \wedge q_K^* = 1] \text{ } \mathit{and} \\ & \nexists K' \in \mathcal{H} : [K' \subset K \wedge (K' \subseteq H' \vee H \subseteq K') \wedge q_{K'}^* = 1]\}.\end{aligned}$$

In the following, I perform a case distinction:
1) Suppose that $\mathcal{Q} \neq \emptyset$ and $\mathcal{R} \neq \emptyset$.
1a) $\exists \bar{Q} \in \mathcal{Q} : H \subseteq \bar{Q}$
By definition of $\mathcal{Q}$, it holds that $q_{\bar{Q}}^* = n$ and, because $H \subseteq \bar{Q}$, it follows that $q_H^* = n$. Thus, the inequality $q_H^* \geq n \cdot \delta_H$ is met.
Moreover, I obtain $H \subseteq Q$ for all $Q \in \mathcal{Q}$ since otherwise, $Q \subseteq H'$ or, equivalently, $Q \subset H$ which would imply $Q \notin \mathcal{Q}$ because $q_H^* = n$.
Take some $Q' \in \mathcal{Q}$ and consider the related set

$$\mathcal{S} := \{K \in \mathcal{H} : Q' \subset K \text{ } \mathit{and} \nexists K' \in \mathcal{H} : Q' \subset K' \subset K\}$$

of properties.
If $\mathcal{S} = \emptyset$, this means that there are no properties $Q'' \in \mathcal{H}$ such that $Q' \subset Q''$. Consequently, decreasing the quota $q_{Q'}^* = n$ by 1 is feasible and, thus, Lemma 2 implies that the inequality $q_{Q'}^* \leq n \cdot \delta_{Q'} + 1$ holds.
If $\mathcal{S} \neq \emptyset$, it must be that $q_S^* < q_{Q'}^*$ for all $S \in \mathcal{S}$. Suppose not, meaning, there exists some $S \in \mathcal{S}$ such that $q_S^* \geq q_{Q'}^*$. Since, by construction $Q' \subset S$, I obtain $q_S^* = q_{Q'}^* = n$. But, then, it holds that $H \subseteq Q' \subset S$ and $q_S^* = n$ and, thus, it follows that $Q' \notin \mathcal{Q}$ which is the desired contradiction.
Now, the aspect $q_S^* < q_{Q'}^*$ for all $S \in \mathcal{S}$ implies that decreasing the quota $q_{Q'}^*$ by 1 is feasible and, therefore, by Lemma 2, the inequality $q_{Q'}^* \leq n \cdot \delta_{Q'} + 1$ is met.
Hence, in both possible cases, the inequality $q_{Q'}^* \leq n \cdot \delta_{Q'} + 1$ is true. Furthermore,

$$q_{H'}^* = q_H^* = n = q_{Q'}^* \leq n \cdot \delta_{Q'} + 1 \leq n \cdot \delta_{H'} + 1$$

by Assumption 3 since $H' \subset H \subseteq Q'$. Thus, the inequality $q_{H'}^* \leq n \cdot \delta_{H'} + 1$ holds.

1b) $\exists \bar{R} \in \mathcal{R} : \bar{R} \subseteq H'$
By definition of $\mathcal{R}$, it holds $q_{\bar{R}}^* = 1$ and, since $\bar{R} \subseteq H'$, I obtain $q_{H'}^* = 1$. Therefore, the second inequality $q_{H'}^* \leq n \cdot \delta_{H'} + 1$ is true.

Furthermore, I obtain $R \subseteq H'$ for all $R \in \mathcal{R}$ because $H' \subset H \subseteq R$ would imply $R \notin \mathcal{R}$ since $q^*_{H'} = 1$.
Take some $R' \in \mathcal{R}$ and consider the related set

$$\mathcal{J} := \{K \in \mathcal{H} : K \subset R' \, and \, \nexists K' \in \mathcal{H} : K \subset K' \subset R'\}$$

of properties.
If $\mathcal{J} = \emptyset$, this means that there are no properties $R'' \in \mathcal{H}$ satisfying $R'' \subset R'$. Consequently, increasing the quota $q^*_{R'} = 1$ by 1 must be feasible yielding the inequality $q^*_{R'} \geq n \cdot \delta_{R'}$ because of Lemma 2.
If $\mathcal{J} \neq \emptyset$, it must be that $q^*_J > q^*_{R'}$ for all $J \in \mathcal{J}$. To see this point, suppose that the contrary is true, meaning, there exists some $J \in \mathcal{J}$ such that $q^*_J \leq q^*_{R'}$. Thus, because of $J \subset R'$, I obtain $q^*_J = q^*_{R'} = 1$. However, since $J \subset R' \subseteq H'$ and $q^*_J = 1$, the property R' cannot be part of the set $\mathcal{R}$ which contradicts $R' \in \mathcal{R}$.
Employing the aspect $q^*_J > q^*_{R'}$ for all $J \in \mathcal{J}$, I observe that increasing the quota $q^*_{R'}$ by 1 is feasible and, therefore, by Lemma 2, the inequality $q^*_{R'} \geq n \cdot \delta_{R'}$ holds.
Hence, in both possible scenarios, I obtain that the inequality $q^*_{R'} \geq n \cdot \delta_{R'}$ is satisfied. Consequently, since $R' \subseteq H' \subset H$, Assumption 3 implies

$$1 = q^*_H = q^*_{H'} = q^*_{R'} \geq n \cdot \delta_{R'} \geq n \cdot \delta_H.$$

Therefore, the inequality $q^*_H \geq n \cdot \delta_H$ is also true.

1c) $\forall \bar{Q} \in \mathcal{Q} : \bar{Q} \subseteq H'$ and $\forall \bar{R} \in \mathcal{R} : H \subseteq \bar{R}$
Define

$$\mathcal{O} = \{K \in \mathcal{H} : [K \subseteq H' \wedge q^*_K > q^*_H] \, and \, \nexists K' \in \mathcal{H} : [K \subset K' \subseteq H' \wedge q^*_{K'} > q^*_H]\}$$

and

$$\mathcal{P} = \{K \in \mathcal{H} : [H \subseteq K \wedge q^*_K < q^*_H] \, and \, \nexists K' \in \mathcal{H} : [H \subseteq K' \subset K \wedge q^*_{K'} < q^*_H]\}.$$

In particular, $\mathcal{O} \neq \emptyset$ as well as $\mathcal{P} \neq \emptyset$ since $\mathcal{Q} \neq \emptyset$ and $\mathcal{R} \neq \emptyset$.
Take some $O \in \mathcal{O}$. By construction, I have $q^*_L = q^*_H$ for all $L \in \mathcal{H}$ such that $O \subset L \subseteq H'$. Also, since $\mathcal{Q} \neq \emptyset$ and $\bar{Q} \subseteq H'$ for all $\bar{Q} \in \mathcal{Q}$, it must be that $q^*_H \neq n$.
Moreover, there exists some $L' \in \mathcal{H}$ such that $\nexists L'' \in \mathcal{H} : O \subset L'' \subset L' \subseteq H'$. Consider the set

$$\mathcal{I} := \{K \in \mathcal{H} : K \subset L' \, and \, \nexists K' \in \mathcal{H} : K \subset K' \subset L'\}$$

of properties. In particular, I have $\mathcal{I} \neq \emptyset$ because, by construction, $O \in \mathcal{I}$.
If $q^*_I > q^*_{L'}$ for all $I \in \mathcal{I}$, increasing $q_{L'}$ by 1 is feasible and, therefore, by Lemma 2, the inequality $q^*_{L'} \geq n \cdot \delta_{L'}$ holds.
If there exists $I' \in \mathcal{I}$ such that $q^*_{I'} \leq q^*_{L'}$, it follows that $q^*_{I'} = q^*_{L'} = q^*_H$ because $I' \subset L' \subseteq H'$. Now, employ the reasoning that I used to tackle L' and apply it to I'. Again, there are two possibilities: Either increasing $q^*_{I'}$ is feasible or there must be some property $I'' \in \mathcal{H}$ such that $I'' \subset I' \subseteq H'$ satisfying $q^*_{I''} = q^*_{L'} = q^*_H$. If necessary, since there are finitely many properties, repeat this argument for a finite number of times. This yields that there exist either some property $I''' \in \mathcal{H}$ with $I''' \subseteq H'$ such that increasing $q^*_{I'''}$ satisfying $q^*_{I'''} = q^*_{L'} = q^*_H$ is feasible or, otherwise, there must be some property $I'''' \in \mathcal{H}$ with $I'''' \subseteq H'$, $q^*_{I''''} = q^*_{L'} = q^*_H$ and $\nexists I''''' \in \mathcal{H} : I''''' \subset I''''$. However, concerning the latter case, increasing $q^*_{I''''}$ by 1 is feasible.
Therefore, in any scenario, there must be some $\tilde{I} \in \mathcal{H}$ with $\tilde{I} \subseteq L' \subseteq H' \subset H$ such that increasing $q_{\tilde{I}}$ by 1 is feasible and $q_{\tilde{I}}$ satisfies $q^*_{\tilde{I}} = q^*_{L'} = q^*_H$. Employing Lemma 2, this means that the inequality $q^*_{\tilde{I}} \geq n \cdot \delta_{\tilde{I}}$ is met. But, then, since $\tilde{I} \subset H$, Assumption 3 implies

$$q^*_H = q^*_{\tilde{I}} \geq n \cdot \delta_{\tilde{I}} \geq n \cdot \delta_H$$

and, thus, the inequality $q^*_H \geq n \cdot \delta_H$ is met.
Consider some arbitrary $P \in \mathcal{P}$. By construction, I have $q^*_M = q^*_{H'}$ for all $M \in \mathcal{H}$ such that $H \subseteq M \subset P$. Further, since $\mathcal{R} \neq \emptyset$ and $H \subseteq \bar{R}$ for all $\bar{R} \in \mathcal{R}$, it must be that $q^*_{H'} \neq 1$.
Additionally, there exists some $M' \in \mathcal{H}$ such that $\nexists M'' \in \mathcal{H} : H \subseteq M' \subset M'' \subset P$. Focus on the set

$$\mathcal{C} := \{K \in \mathcal{H} : M' \subset K \textit{ and } \nexists K' \in \mathcal{H} : M' \subset K' \subset K\}$$

of properties. In particular, I have $\mathcal{C} \neq \emptyset$ because, by construction, $P \in \mathcal{C}$.
If $q^*_C < q^*_{M'}$ for all $C \in \mathcal{C}$, decreasing $q_{M'}$ by 1 is feasible and, therefore, due to Lemma 2, the inequality $q^*_{M'} \leq n \cdot \delta_{M'} + 1$ holds.
If there exists $C' \in \mathcal{C}$ such that $q^*_{C'} \geq q^*_{M'}$, it follows that $q^*_{C'} = q^*_{M'} = q^*_{H'}$ because $H \subseteq M' \subset C'$. Now, employ the reasoning that I used to tackle M' and apply it to C'. Again, there are two possibilities: Either decreasing $q^*_{C'}$ is feasible or there must be some property $C'' \in \mathcal{H}$ such that $H \subseteq C' \subset C''$ satisfying $q^*_{C''} = q^*_{M'} = q^*_{H'}$. If necessary, since there are finitely many properties, repeat this argument for a finite number of times. This yields that there exist either some property $C''' \in \mathcal{H}$ with $H \subseteq C'''$ such that increasing $q^*_{C'''}$ satisfying $q^*_{C'''} = q^*_{M'} = q^*_{H'}$ is feasible or, otherwise, there must be some property $C'''' \in \mathcal{H}$ with $H \subseteq C''''$, $q^*_{C''''} = q^*_{M'} = q^*_{H'}$ and $\nexists C''''' \in \mathcal{H} : C'''' \subset C'''''$. However,

concerning the latter case, decreasing $q^*_{C''''}$ by 1 is feasible.
Therefore, in any scenario, there must be some $\tilde{C} \in \mathcal{H}$ with $H' \subset H \subseteq M' \subseteq \tilde{C}$ such that decreasing $q_{\tilde{C}}$ by 1 is feasible and $q_{\tilde{C}}$ satisfies $q^*_{\tilde{C}} = q^*_{M'} = q^*_{H'}$. Invoking Lemma 2, this means that the inequality $q^*_{\tilde{C}} \leq n \cdot \delta_{\tilde{C}} + 1$ is met. But, then, since $H' \subset \tilde{C}$, Assumption 3 implies

$$q^*_{H'} = q^*_{\tilde{C}} \leq n \cdot \delta_{\tilde{C}} + 1 \leq n \cdot \delta_{H'} + 1$$

and, thus, the inequality $q^*_{H'} \leq n \cdot \delta_{H'} + 1$ is met.
In conclusion, as desired, despite $q^*_{H'} = q^*_H$, both relevant inequalities are met at q^*_H.
2) If $\mathcal{Q} = \emptyset$ and $\mathcal{R} = \emptyset$, the argument from case 1c applies.
3) Suppose that $\mathcal{Q} \neq \emptyset$, but $\mathcal{R} = \emptyset$.
If $\exists \bar{Q} \in \mathcal{Q} : H \subseteq \bar{Q}$, the reasoning in case 1a yields the desired conclusion; in case $Q \subset H$ for all $Q \in \mathcal{Q}$, take the argument from case 1c.
4) Suppose that $\mathcal{R} \neq \emptyset$, but $\mathcal{Q} = \emptyset$.
In case $H \subset R$ for all $R \in \mathcal{R}$, replicate the steps in case 1c; if $\exists \bar{R} \in \mathcal{Q} : \bar{R} \subseteq H$, the argument from case 1b applies.
Taking all four cases together, this shows that the two relevant inequalities

$$q^*_H \geq n \cdot \delta_H \text{ and}$$
$$q^*_{H'} \leq n \cdot \delta_{H'} + 1$$

determining q^*_H as well as $q^*_{H'}$ hold despite $q^*_{H'} = q^*_H$. Therefore, overall, the claim in the lemma follows. □

Proof of Theorem 2.
It is sufficient to find the quotas related to voting by issues that are part of an optimal mechanism. The existence of a solution is ensured since a bounded function is optimized over a finite set of elements.

Recall, by Theorem 1, the optimal quotas must satisfy

$$H \subseteq K \Rightarrow q^*_H \geq q^*_K$$

for all $K', K \in \mathcal{H}$.
Consider some arbitrary property $H \in \mathcal{H}$ and the associated quota q^*_H being part of an optimal mechanism. Subsequently, I perform case distinctions.
1a) If $\forall H' \in \mathcal{H}$ with $H' \subset H$ and $\nexists H'' \in \mathcal{H} : H' \subset H'' \subset H$, it holds that $q^*_{H'} > q^*_H$, part (i) of Lemma 2 yields that the inequality $q^*_H \geq n \cdot \delta_H$ is met.
1b) If there is some $H' \in \mathcal{H}$ with $H' \subset H$ and $\nexists H'' \in \mathcal{H} : H' \subset H'' \subset H$ such that

$q^*_{H'} = q^*_H$, Lemma 3 implies that the inequality $q^*_H \geq n \cdot \delta_H$ holds.

Therefore, no matter the shape of the optimal mechanism, the inequality $q^*_H \geq n \cdot \delta_H$ constitutes a necessary condition for optimality.

2a) If $\forall H' \in \mathcal{H}$ with $H \subset H'$ and $\nexists H'' \in \mathcal{H} : H \subset H'' \subset H'$, it holds $q^*_H > q^*_{H'}$, part (ii) of Lemma 2 yields that the inequality $q^*_H \leq n \cdot \delta_H + 1$ is true.

2b) If there is some $H' \in \mathcal{H}$ with $H \subset H'$ and $\nexists H'' \in \mathcal{H} : H \subset H'' \subset H'$ such that $q^*_H = q^*_{H'}$, Lemma 3 implies that the inequality $q^*_H \leq n \cdot \delta_H + 1$ is satisfied.

Thus, no matter the shape of the optimal mechanism, the inequality $q^*_H \leq n \cdot \delta_H + 1$ is necessary for optimality.

Taking both inequalities together, since the quotas are integer-valued, the quotas q^*_H satisfying these inequalities are, generically, unique and they amount to $q^*_H = \lceil n\delta_H \rceil$ with $H \in \mathcal{H}$. Consequently, the equalities $q^*_H = \lceil n\delta_H \rceil$ with $H \in \mathcal{H}$ are necessary for optimality. Finally, it remains to be verified that these equalities are also sufficient for optimality. The quotas determined by these equalities are feasible in the sense that they constitute a family of quotas and that they meet the inequalities from Theorem 1. First, note that, for all $H \in \mathcal{H}$, since $0 < \delta_H < 1$, I have that $1 \leq q^*_H = \lceil n\delta_H \rceil \leq n$. Second, observe that, for any $H \in \mathcal{H}$, it holds that $q^*_H + q^*_{H^c} = n + 1$. Thus, the discussed quotas constitute a family of quotas. Also, it is immediate from Assumption 3 that these quotas satisfy the inequalities from Theorem 1. Moreover, the quotas determined by the equalities $q^*_H = \lceil n\delta_H \rceil$ with $H \in \mathcal{H}$ must be optimal because, again, there exists a solution and this solution has to meet these equalities. Consequently, the discussed equalities are also sufficient for optimality, and the theorem follows. □

References

Apesteguia, J., Ballester, M. A., and Ferrer, R. (2011). "On the Justice of Decision Rules". *Review of Economic Studies*, 78(1):1–16.

Bagnoli, M. and Bergstrom, T. (2005). "Log-concave probability and its applications". *Economic Theory*, 26(2):445–469.

Barberà, S., Gul, F., and Stacchetti, E. (1993). "Generalized Median Voter Schemes and Committees". *Journal of Economic Theory*, 61(2):262–289.

Barberà, S., Massó, J., and Neme, A. (1997). "Voting under Constraints". *Journal of Economic Theory*, 76(2):298–321.

Barberà, S., Sonnenschein, H., and Zhou, L. (1991). "Voting by Committees". *Econometrica*, 59(3):595–609.

Benoit, K. and Laver, M. (2006). *Party Policy in Modern Democracies.* Routledge.

Block, V. (2010). "Efficient and strategy-proof voting over connected coalitions: A possibility result". *Economics Letters*, 108(1):1–3.

Block de Priego, V. I. (2014). *Single-Peaked Preferences - Extensions, Empirics and Experimental Results.* PhD Thesis, https://publikationen.bibliothek.kit.edu/1000041208/3197881, accessed May 14, 2019.

Border, K. C. and Jordan, J. S. (1983). "Straightforward Elections, Unanimity and Phantom Voters". *Review of Economic Studies*, 50(1):153–170.

Börgers, T. and Postl, P. (2009). "Efficient compromising". *Journal of Economic Theory*, 144(5):2057–2076.

Brock, J. M., Lange, A., and Ozbay, E. Y. (2013). "Dictating the Risk: Experimental Evidence on Giving in Risky Environments". *American Economic Review*, 103(1):415–437.

Cappelen, A. W., Konow, J., Sørensen, E. Ø., and Tungodden, B. (2013). "Just Luck: An Experimental Study of Risk-Taking and Fairness". *American Economic Review*, 103(4):1398–1413.

Casella, A. (2005). "Storable votes". *Games and Economic Behavior*, 51(2):391–419.

Demange, G. (1982). "Single-peaked orders on a tree". *Mathematical Social Sciences*, 3(4):389–396.

Drexl, M. and Kleiner, A. (2018). "Why Voting? A Welfare Analysis". *American Economic Journal: Microeconomics*, 10(3):253–71.

Gersbach, H. (2017). "Flexible Majority Rules in democracyville: A guided tour". *Mathematical Social Sciences*, 85:37–43.

Gershkov, A., Moldovanu, B., and Shi, X. (2017). "Optimal Voting Rules". *Review of Economic Studies*, 84(2):688–717.

Gershkov, A., Moldovanu, B., and Shi, X. (2019). "Voting on multiple issues: What to put on the ballot?" *Theoretical Economics*, 14(2):555–596.

Gibbard, A. (1973). "Manipulation of Voting Schemes: A General Result". *Econometrica*, 41(4):587–601.

Hortala-Vallve, R. (2010). "Inefficiencies on linking decisions". *Social Choice and Welfare*, 34(3):471–486.

Hortala-Vallve, R. (2012). "Qualitative voting". *Journal of Theoretical Politics*, 24(4):526–554.

Jackson, M. O. and Sonnenschein, H. F. (2007). "Overcoming Incentive Constraints by Linking Decisions". *Econometrica*, 75(1):241–257.

Jarman, F. and Meisner, V. (2017). "Deterministic mechanisms, the revelation principle, and ex-post constraints". *Economics Letters*, 161:96–98.

Kim, S. (2017). "Ordinal versus cardinal voting rules: A mechanism design approach". *Games and Economic Behavior*, 104:350–371.

Kleiner, A. and Moldovanu, B. (2017). "Content-Based Agendas and Qualified Majorities in Sequential Voting". *American Economic Review*, 107(6):1477–1506.

Kleiner, A. and Moldovanu, B. (2020). "Voting Agendas and Preferences on Trees: Theory and Practice". *American Economic Journal: Microeconomics*. Forthcoming.

Moulin, H. (1980). "On strategy-proofness and single peakedness". *Public Choice*, 35(4):437–455.

Nehring, K. (2004). "The veil of public ignorance". *Journal of Economic Theory*, 119(2):247–270.

Nehring, K. and Puppe, C. (2005). "The Structure of Strategy-Proof Social Choice — Part II: Non-Dictatorship, Anonymity and Neutrality". Unpublished Manuscript, http://micro.econ.kit.edu/downloads/Puppe_Working_Papers_08_The_Structure_of_Strategy-Proof_Social_Choice._Part_II.pdf, accessed May 14, 2019.

Nehring, K. and Puppe, C. (2007a). “Efficient and Strategy-Proof Voting Rules: A Characterization”. *Games and Economic Behavior*, 59(1):132–153.

Nehring, K. and Puppe, C. (2007b). “The structure of strategy-proof social choice — Part I: General characterization and possibility results on median spaces”. *Journal of Economic Theory*, 135(1):269–305.

Prékopa, A. (1973). “On logarithmic concave measures and functions”. *Acta Scientiarum Mathematicarum*, 34:335–343.

Rae, D. W. (1969). “Decision-Rules and Individual Values in Constitutional Choice”. *American Political Science Review*, 63(1):40–56.

Saporiti, A. (2009). “Strategy-proofness and single-crossing”. *Theoretical Economics*, 4(2):127–163.

Satterthwaite, M. A. (1975). “Strategy-proofness and Arrow’s conditions: Existence and correspondence theorems for voting procedures and social welfare functions”. *Journal of Economic Theory*, 10(2):187–217.

Schmitz, P. W. and Tröger, T. (2012). “The (sub-)optimality of the majority rule”. *Games and Economic Behavior*, 74(2):651–665.

Shaked, M. and Shanthikumar, J. G. (2007). *Stochastic Orders*. Springer Science+Business Media, New York.