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Allocation With Correlated Information: Too Good to Be True

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Allocation with Correlated Information: Too good to be true *

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Abstract

A principal can allocate an indivisible good to an agent. The agent privately learns the value of the good while the principal privately learns the cost. Value and cost are correlated. The agent wants to have the good in any case. The principal wants to allocate whenever the value exceeds the cost. She cannot use monetary transfers to screen the agent.

I study how the principal utilizes her information in the optimal mechanism: when the correlation is negative, she bases her decision only on the costs, and when the correlation is positive, she screens the agent. To this end, she forgoes her best allocation opportunities: when the agent reports high valuations but her own costs are low. Under positive correlation, these realizations are unlikely; the principal will find them too good to be true. In contrast to standard results, this optimal mechanism may not allocate to a higher value agent with higher probability. I discuss applications to intra-firm allocations, task-delegation, and industry self-regulation.

Keywords: correlated information, mechanism design without transfers, bilateral trade, delegation;

JEL Codes: D61, D82, D86, L50

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1 Introduction

A principal (she) can allocate an indivisible good to an agent (he). She privately learns her costs c of allocating the good to him. He privately learns his valuation v for the good. She is benevolent and wants to allocate whenever v exceeds c. He does not incorporate the costs and wants the good in any case. In many applications of this setting, monetary transfers between the principal and the agent are not allowed. However, their information is often correlated. Examples include the following: the federal government decides whether to allocate a task to the local government, the management of a company decides whether to buy a capital good and allocate it to one of its departments, a regulator decides whether to approve a new product after she has investigated one of its features, and the producer of the good reports his assessment about the safety of the other features.

To my knowledge, this is the first study that analyzes the optimal mechanism in this bilateral trade setting with correlated information and without transfers.

As a benchmark, consider this setting when the designer could use monetary transfers to screen the agent. In this case, the efficient allocation could be implemented. The principal could charge her cost as the price for the good. Then, the agent would buy if and only if his valuation exceeds the cost.

In many situations, however, monetary transfers are not feasible. This can be for organizational reasons (companies and state agencies do not want to introduce internal budgets), for technological reasons (a website and its user might not have a payment channel), or for moral and legal reasons.

Moreover, it is natural that valuation and costs are not independent but correlated:

If the federal government delegates a task to the local government, her opportunity costs are given by the value she could have created in the task herself. In this example, the values which the agent (local government) and the principal (federal government) could create in the same task are positively correlated.

If the management buys the capital good from a third party, her costs are given by the price. This price could be a quality signal. Then, the price and the valuation for the good are positively correlated. If the regulator investigates one aspect of a new product and the producer investigates another, these aspects are likely positively correlated.

In my setting, the principal designs a mechanism that can be contingent on the realization of her cost. Her objective is to maximize the ex-ante expected efficiency of the allocation. When she screens the agent to allocate efficiently, she has to discourage an agent with a value below her costs to pretend to have a higher value. Without money, she can only use the allocation itself as an incentive. Thus, she must offer a low-value agent sufficiently high expected allocation probability to prevent him from mimicking an agent with a higher value. However, allocating to low-value agents distorts the expected efficiency. In the optimal mechanism, she has to square the inefficiencies that screening entails with the efficiency gains that the information about the agent's value permits.

Results. The revelation principle applies in this setting. Therefore, it is sufficient to study direct mechanisms. In a direct mechanism, the principal commits to a menu of allocation schedules, with one allocation schedule for each valuation. An allocation schedule specifies an allocation probability for any cost realization. The agent learns his valuation, forms expectations about the cost, and then reports his valuation to the principal. The principal learns the cost and allocates according to the schedule for the reported valuation. Because cost and valuation can be correlated, agents of different valuations form different expectations about the cost and evaluate allocation schedules differently. The agent must find it optimal to report truthfully. Therefore, he must expect the allocation schedule that corresponds to his true valuation to yield him the highest probability of allocation.

When cost and valuation are independent or negatively correlated, the optimal mechanism ignores the agent's report and bases the allocation decision only on the cost: the principal commits to implement the same allocation schedule after all valuation reports. She allocates if and only if the cost falls below a cutoff. The reason is that with negative correlation, low costs tend to occur with high valuations and vice versa. Therefore, basing the decision only on the costs is already quite efficient. Screening the agent would only slightly increase the expected efficiency.

When costs and valuations are positively correlated, the optimal mechanism might screen the agent. To illustrate this screening mechanism, suppose there are two possible valuations $v_L < v_H$. If the agent reports the low valuation v_L, the principal sticks to a cutoff allocation schedule: she only allocates if the cost remains below a cutoff, $c < \overline{c}_L$. If the agent claims to have high valuation v_H, the allocation schedule is characterized by two cost cutoffs $\underline{c}_H \leq \overline{c}_H$ that form an interval. If the cost is either too low or too high, the principal does not allocate the good. She allocates only after intermediate cost realization between the two cutoffs, $c \in (\underline{c}_H, \overline{c}_H)$. This means that the principal does not allocate when the gains from allocation $v - c$ are the highest. The form of this allocation schedule exploits the difference in beliefs of the two valuation types. Under positive correlation, an agent with v_L finds low costs likelier than an agent with v_H. When the principal chooses to not allocate after a high valuation report and low costs, she makes misreporting for the low type unattractive. She forgoes the most efficient allocation opportunity; however this allows her to allocate overall with a higher probability to v_H without giving the agent with v_L incentives to misreport.

With more than two valuations, the structure from the above example extends. I introduce a novel regularity assumption on the joint distribution. Under this assumption, the allocation schedule for all valuation reports has interval form. Furthermore, the intervals are ordered. For a higher valuation, both the upper and lower support of the allocation schedule exceed the upper and lower support of the allocation schedule of a lower valuation.

With correlated information, agents of different valuations hold different beliefs about the costs. Therefore, the interim expected allocation probabilities are insufficient to describe the mechanism and the standard approaches fail. In particular, the set of incentive compatible mechanisms cannot be easily characterized. Therefore, I directly characterize the optimal mechanism. First, I use the new regularity assumption to argue that the allocation schedules of the optimal mechanism are in interval form (Proposition 1). This assumption is similar (but weaker) to an assumption Jewitt (1988) uses in a moral hazard setting. I illustrate the role of this assumption by introducing a related problem. In this related problem, agents of different valuations share the same belief about the cost but differ in their risk preferences. The regularity assumption then guarantees that in the related problem, agents with higher valuation are more risk-averse. This leads to the interval form of the allocation schedules.

Next, I show that the allocation schedules are ordered (Proposition 2). When an agent evaluates an allocation schedule, he weighs the probability of receiving the good at a certain cost realization with his belief that this cost realization occurs. This evaluation is as if the agent would have a Bernoulli utility function with respect to costs that is equal to his belief and would evaluate a lottery that is given by the allocation schedule. I establish that these utility functions have single-crossing expectational differences. Finally, I use the resulting monotone comparative statics to show that the interval allocation schedules in the optimal mechanism are ordered. In contrast to standard results, this interval monotonicity does not imply that the interim allocation probabilities are monotone in the agent's valuation.

Applications. The characterization of the optimal mechanism has many interesting consequences for applications: if the principal buys the good from a third party to allocate it to an agent inside her organization, her demand schedule is no longer monotone. When the agent reports a high valuation, she will buy for intermediate prices but will not buy for high or low prices.

The optimal mechanism under positive correlation gives a rationale for inefficient governmental allocation of resources and tasks. This demands a high degree of commitment from the principal. It might be difficult for a public official to defend the decision to not allocate a task to a subordinate agency if (i) the subordinate agency predicts to be very successful in the task whereas (ii) the principal agency expects to perform poorly. This advocates for an intransparent allocation procedure, where the performance predictions do not become public record. In contrast, under negative correlation, the optimal mechanism can be transparently implemented: the principal does not collect any information from the agent. Given only the realization of the costs, her decision is efficient.

The possibility of using positive correlation for screening has interesting effects on the value of correlation. I demonstrate this with a numerical example, wherein the principal compares two joint distributions with equal marginal expectations for costs and valuation. The second distribution exhibits a higher degree of positive correlation than the first. As a benchmark, I consider the case without information asymmetry, where the principal can observe the agent's valuation. Here, she prefers the first distribution because the higher positive correlation lowers her expected value of the efficient allocation. However, if the valuation is the agent's private information, the efficient allocation is not implementable. Then, the principal prefers the second distribution. The higher degree of positive correlation allows her to screen the agent and to allocate more efficiently in the second-best solution.

Finally, I investigate whether a regulator can delegate parts of the certification process of a new product to the producer. I model this long-standing practice of industry self-regulation as follows. The regulator analyzes one aspect of the safety of the product and learns its value a_1; a second aspect a_2 is analyzed by the producer. The two aspects are positively correlated and jointly determine the safety of the product. To fit this setting into my model of valuation and cost $v - c$, I define the principal's cost as $c = -a_1$ and the agent's valuation as $v = a_2$. With this definition, cost and valuation are negatively correlated. In the optimal mechanism, the regulator bases her decision only on the aspect that she herself analyzes. In this model of industry self-regulation, delegation of an aspect implies ignoring it for the assessment of the safety of the product.

Related literature Myerson and Satterthwaite (1983) introduce the bilateral trade setting and show that with balanced transfers, the efficient allocation cannot be implemented. I deviate from their setting in three dimensions: (i) I let the principal (the seller in their terminology) design the mechanism. She can commit to make her decision contingent on her information. In contrast to Myerson and Satterthwaite, the optimal mechanism does not have to consider her incentive constraints. This also sets my model apart from informed principal problems (Myerson (1983); Maskin and Tirole (1990)). (ii) I allow cost and valuation to be correlated. (iii) I do not allow for monetary transfers. If monetary transfers are allowed and the information of the players is correlated, Cremer and McLean (1988); McAfee and Reny (1992); Riordan and Sappington (1988); Johnson et al. (1990); Neeman (2004) have demonstrated how and when a designer can exploit the differences in the beliefs of the players to implement any social choice function. I revisit some of their results in Section 3.1.

My model also relates to the literature on delegation started by Holmstrom (1977). To my knowledge, it is the first that studies delegation with two-sided information asymmetry. The interval structure of the optimal mechanism is similar to the interval delegation in Alonso and Matouschek (2008).

In settings without monetary transfers, the optimal mechanism often invents artificial money to screen agents. Hylland and Zeckhauser (1979) have agents trade allocation probabilities for indivisible goods; Jackson and Sonnenschein (2007) have voters trade the weights of their ballot in elections on different matters.

Bhargava et al. (2015) show how positively correlated beliefs among voters allow overcoming the impossibility of nondictatorial voting rules established by Gibbard (1973) and Satterthwaite (1975).

Guo and Hörner (2018) study a dynamic allocation problem between a benevolent principal and an agent. They use promises of future allocations as a means to incentivizes the agent to report truthfully.

Another strand in the literature on mechanism design studies allocation problems without money but allows for the costly verification of hidden information: Ben-Porath et al. (2014); Kattwinkel and Knoepfle (2019); Mylovanov and Zapechelnyuk (2017); Li (2017); Epitropou and Vohra (2019). Chakravarty and Kaplan (2013) study an allocation with costly signaling.

Gershkov et al. (2013); Manelli and Vincent (2010) analyze the equivalence between Bayesian optimal and ex-post incentive compatible implementation when monetary transfers are feasible. Ben-Porath et al. (2014); Erlanson and Kleiner (2019) and Kattwinkel and Knoepfle (2019) show that with costly verification and without monetary transfers—similar to the results in this paper—the optimal mechanism is ex-post incentive compatible if the values are independent or the correlation is negative. The screening mechanism I derive shows that in a setting without transfers and with positively correlated information, this equivalence can fail.

Fieseler, Kittsteiner, and Moldovanu (2003) study the problem of efficient trade with interdependent values. They find that only under negative interdependent values can balanced transfers implement efficiency. Like in my setting, in this case, the expected conflict is small: small values of one agent occur with high values of the others. Therefore, they can be mediated with money. In my setting, under negative correlation, screening offers only little efficiency improvement and the principal abstains from screening.

The regularity assumption that I introduce is weaker than a similar assumption that Jewitt (1988) uses to justify the first-order approach in a moral hazard problem. The intuition for why this assumption guarantees interval allocation schedules argues with Diamond and Stiglitz's (1974) notion of utility preserving spreads. More generally, this intuition sheds light on a connection between belief heterogeneity and the heterogeneity in risk preferences.

To prove that the allocation schedules are ordered, I use that the utility functions of the players have single crossing-differences for ordered lotteries. The difference from Kartik, Lee, and Rappoport (2019)'s result is that I need this property to hold only for ordered lotteries. The result I use is a consequence of the variation diminishing property of totally positive kernels (Schoenberg (1930); Motzkin (1936); Karlin (1968)).

2 Model

2.1 Setting

A principal (she) can allocate an indivisible good to an agent (he). Her allocation decision is denoted by $x \in \{0, 1\}$. If she mixes, $x \in [0, 1]$ denotes the probability of allocation. The allocation is costly for the principal. These costs are a random variable C taking values $c \in \mathbb{R}$. Of course, these costs can also be interpreted as opportunity costs. Then, C denotes the principal's valuation for the good. The realization of the costs is the principal's private information. The agent has a valuation for the good V, which takes values $v \in \mathbb{R}_{++}$. The realization of V is the agent's private information. Valuation V and cost C are jointly distributed according to a cdf $F(v, c)$. I assume that the supports of the valuation and the costs are both finite and denote them by $\mathcal{V}$ and $\mathcal{C}$, respectively. Let $f(v, c)$ denote the pdf. The assumption on the distribution of the costs is generalized in Section 4.2.

The principal's objective is to maximize social welfare; her Bernoulli utility function is given by,

$$w(v, c, x) = x \cdot (v - c).$$

The agent does not bear the costs; his utility reads as follows:

$$u(v, c, x) = x \cdot v.$$

because $v > 0$, he always prefers receiving the good. The agent's outside option is zero so that he is always willing to participate in the mechanism.

The correlation between the agent's information V and the principal's information C is captured by the following likelihood ratios. For all $v' < v'' \in \mathcal{V}$:

$$\frac{f(v', c)}{f(v'', c)}. \tag{1}$$

I distinguish three cases.

1. Negative affiliation: the likelihood ratios (1) are strictly increasing in c. The higher the costs, the more likely is a low valuation by the agent.
2. Independence: the likelihood ratios (1) are constant in c. In this case, the costs are not informative about the valuation.
3. Positive affiliation: the likelihood ratios (1) are strictly decreasing in c. The

higher the costs, the more likely is a high valuation.

In the remainder I will use the term positive correlation for positive affiliation and negative correlation for negative affiliation.

2.2 Mechanism

The principal announces and commits to a mechanism before she learns the costs. The realization of the costs is contractible. The agent learns his valuation (but not the costs) and then plays a Bayesian best response.

Formally, a mechanism is given by a message set M and an allocation function $x : M \times \mathcal{C} \to [0, 1]$ that specifies an allocation probability for any pair of message and cost realization. The revelation principle applies so that it suffices to have valuations as messages, $M = \mathcal{V}$ and to study mechanisms where the incentives are such that the agent reports his type truthfully.

Given a valuation report v, a direct mechanism specifies an allocation probability for all cost realization $c \in \mathcal{C}$. Denote this vector

$$x(v) = (x(v, c))_{c \in \mathcal{C}}$$

and term it as v's allocation schedule.

2.3 Agent's problem

The agent takes a mechanism x as given. He does not know the costs of the good, but forms expectations about it based on the realization of his valuation v. If he reports truthfully, he faces the random allocation lottery $x(v, C)$. If he reports $\hat{v}$, he faces a different lottery $x(\hat{v}, C)$. Therefore, the Bayesian incentive constraints read as follows:

$$\forall v, \hat{v} \in \mathcal{V}: \qquad v \cdot \mathbb{E}\left[x(v, C) \mid V = v\,\right] \;\; \geq \;\; v \cdot \mathbb{E}\left[x(\hat{v}, C) \mid V = v\,\right]. \qquad (2)$$

Every type derives strictly positive utility from the good ($v > 0$ for all $v \in \mathcal{V}$); it follows that the intensity of type v's preferences can be eliminated from the incentive constraint: the agent maximizes his expected allocation probability.

$$IC(v,\hat{v}) = \sum_{c\in\mathcal{C}} f(v,c)\,[x(v,c) - x(\hat{v},c)] \;\geq\; 0. \tag{3}$$

This shows that the setting is equivalent to a setting where the valuation v does not enter the utility of the agent: he has utility 1 if he receives the good and 0 otherwise. In this setting, the valuation v is the value of the principal. Formally, in this equivalent setting, the principal's utility function stays the same and the agent's utility is given by

$$\overline{u}(x,v,c) = x.$$

Moreover, note that the expected allocation probability at a certain misreport is not independent of the true valuation type, as different valuation types have different conditional beliefs over the distribution of C. The interim expectations are therefore insufficient to describe the mechanism.

2.4 Principal's problem

The principal designs a mechanism that maximizes social welfare. If she could observe the valuation of the agent, she would allocate the good efficiently: that is, if and only if $v - c \geq 0$. However, because she only observes the costs, she must incentivize the agent to report his valuation. Her problem can be stated as the following linear program:

$$\max_{0\leq \boldsymbol{x}\leq 1} \mathbb{E}\left[\, x(V,C)\cdot(V-C)\,\right] \qquad \text{s.t. } \forall(v,\hat{v})\in\mathcal{V}\times\mathcal{V}\colon\; IC(v,\hat{v})\geq 0. \tag{4}$$

3 Optimal mechanism

3.1 If monetary transfers were feasible

If the principal could, in addition to the allocation decision $x\in[0,1]$, set a monetary transfer $t\in\mathbb{R}$ that would enter the agent's utility additively,

$$\hat{u}(v,c,x,t) = u(v,c,x) - t = x\cdot v - t,$$

the efficient allocation could be achieved: the principal can offer the good at price c to the agent and completely align the agent's interest in the mechanism with hers. To achieve efficiency, it is necessary that the agent's valuation and the principal's valuation for the good coincide. In (3), it is shown that without transfers, it is not relevant whether v enters the agent's utility. This is not the case if transfers are feasible. If the agent's utility is not affected by v and is just 1 if he receives the good and 0 otherwise, selling the good at price c does not induce the efficient allocation.

However, with transfers and correlation, there is another distinct way to achieve efficiency. This method exploits the differences of the beliefs that agents of different valuations v hold about the cost realization. It requires that the beliefs identify the corresponding valuations. Therefore, it must not be the case that there exists a type v' and $\lambda(v) \geq 0$ for all $v \in \mathcal{V} - \{v'\}$ such that,

$$f(c|v') = \sum_{v \in \mathcal{V}-\{v'\}} \lambda(v) \cdot f(c|v) \quad \text{for all } c \in \mathcal{C}. \tag{5}$$

This rank condition[1] ensures that the belief that an agent holds is a sufficient statistic for the valuation v. If costs and valuation are independent, this condition is not fulfilled.

Theorem 1 (Cremer and McLean (1988))
If monetary transfers are feasible and the spanning condition is met, for any $x(v, c)$ there exists $t(v, c)$ implementing x with $\mathbb{E}[t(v, C)] = 0$ for all $v \in \mathcal{V}$.

The proof is an adaption of Cremer and McLean (1988)'s proof for the setting studied here: if the rank condition is met, the principal offers the agent a menu of lotteries whose payments depend on the realizations of the costs. For each valuation, there is one lottery. An agent with this valuation expects the corresponding lottery to pay out 0 and expects all other lotteries' payout to be lower than some bound $b < 0$. This bound b can be chosen uniformly for all valuations and can be set arbitrarily low. By setting b arbitrarily low, the mechanism gives the agent an arbitrarily high incentive to choose the lottery that corresponds to his valuation and thereby reveal it, irrespective of the allocation rule that the principal implements with this information. This mechanism also works if v does not enter into the agent's utility.

[1] An equivalent formulation of this condition is that the set of beliefs for all types are the extreme points of their convex hull.

3.2 Independence and negative correlation

In the case of independence ((1), case 2), the costs do not convey any information about the agent's value. In turn, agents of all valuations share the same belief about the costs. The principal cannot use the difference in the agent's belief to distinguish between them. As a result, optimally, there is no meaningful communication between the principal and the agent.

Ignorant mechanism A mechanism that ignores the agent's report,

$$x(c, v') = x(c, v'') \text{ for all } v', v'' \in \mathcal{V},$$

is ex-post incentive compatible in the sense that an agent who would learn the cost realization before reporting to the mechanism is still incentivized to report truthfully. In practice, such a mechanism can be implemented by a procedure that does not ask the agent for his valuation at all. The optimal ignorant mechanism is given[2] by

$$x(c, v) = 1 \text{ if and only if } \mathbb{E}[V \mid C = c] \geq c.$$

More surprisingly, there is also no ground for communication when the correlation is negative.

Theorem 2
If costs and valuation are negatively correlated ((1), case 1) or independent ((1), case 2), then it is optimal for the principal to offer an ignorant mechanism. In these cases, the optimal (ignorant) mechanism is given by a simple cutoff rule:

$$\bar{c} = \min\{c \in \mathcal{C} \mid \mathbb{E}[V \mid C = c] \geq c\}.$$

and

$$x(c, v) = 1 \textit{ if and only if } c \leq \bar{c}.$$

The proof (in the appendix) introduces a relaxed problem: it disregards all incentive constraints for the agent to understate his valuation. For a fixed valuation report v' and two cost realizations $c' < c''$, there always exists a modification of the

[2] The optimal ignorant mechanism is unique except for the allocation probability after cost realizations c with $\mathbb{E}[V \mid C = c] = c$. I assume that in this case, the principal allocates the good.

allocation schedule for v', $x(v') \to \tilde{x}(v')$, that decreases the allocation probability after the high cost realization c'', $x(v, c'') \downarrow$, and in turn increases the probability after the lower cost realization c', $x(v, c') \uparrow$, such that

$$\mathbb{E}[\tilde{x}(v', C) \,|\, V = v'] = \mathbb{E}[x(v', C) \,|\, V = v']$$

and for all $v < v'$

$$\mathbb{E}[\tilde{x}(v', C) \,|\, V = v] \leq \mathbb{E}[x(v', C) \,|\, V = v].$$

The existence of this modification follows from the negative correlation (negative affiliation): an agent with lower valuation v puts relatively more likelihood on the realizations of high cost than on the realizations of low compared with an agent with a higher valuation v'. By shifting mass from high costs to low costs, one can keep the high valuation type indifferent while harming a lower type. The principal strictly prefers this modification because on average she has to bear lower costs while the overall probability of allocation to this valuation type remains constant.

Under negative correlation, the interest of the principal and the higher valuation type v' are aligned (put mass on low costs) and are distinct from the interest of a lower type v (put mass on high cost realization).

In the optimal mechanism, any allocation schedule must be in cutoff form. Otherwise it could be improved by a modification of the above form. Finally, the proof shows that these cutoff allocation schedule are optimally identical for all valuations. Hence, the optimal (Bayesian) mechanism is ignorant.

3.3 Positive correlation

Under positive correlation, the optimal mechanism offers different allocation schedules and the agents of different valuations sort themselves by their choice.

Regularity Assumption. To study the optimal mechanisms under positive correlation, I introduce a new regularity assumption on the joint distribution of cost and valuation. Under positive correlation (positive affiliation) the likelihood ratios (1) are strictly decreasing in c. The regularity assumption demands that these ratios are convex decreasing. The next definition formalizes the convexity of a function on a discrete set.

Definition 1. A function l on $\mathcal{C}$ is strictly convex if for all $c' < c''$ and for all $\alpha \in (0,1)$ with $\alpha \cdot c' + (1-\alpha) \cdot c'' \in \mathcal{C}$, $l(\alpha \cdot c' + (1-\alpha) \cdot c'') < \alpha \cdot l(c') + (1-\alpha) \cdot l(c'')$.

If the set is convex,this definition coincides with the usual definition of convexity. The regularity assumption reads as follows.

$$\text{For all } v' < v'' \in \mathcal{V}, \frac{f(v',c)}{f(v'',c)} \text{ is strictly convex in } c. \tag{6}$$

Jewitt (1988) introduces a similar condition in a moral hazard setting. He assumes that the increasing function that maps c to

$$\frac{f(v'',c)}{f(v',c)} = \frac{1}{\frac{f(v',c)}{f(v'',c)}}$$

is concave. These conditions are very similar; in general, they demand that the extent of the interference about the valuation decreases with increasing observed costs. However, the condition used here is weaker than the condition in Jewitt (1988).

Corollary 1
If a function $g : \mathbb{R} \to \mathbb{R}_{++}$ is increasing and concave, then the function $\frac{1}{g}$ is decreasing and convex. The reverse implication might not be true.

The proof is in the appendix. The regularity assumption is met by many distributions. I list some of them in Section 4.2.

Next, the optimal mechanism is characterized.

Relaxation. Again, it is sufficient to consider a relaxed problem, where all incentive constraints that prevent the agent from understating his type are disregarded and only the upward incentive constraints,

$$IC(v, \hat{v}) \geq 0 \text{ for } v < \hat{v} \in \mathcal{V},$$

are considered.

3.3.1 Plateau mechanisms

The first characteristic of the optimal mechanism concerns the form of the allocation schedules that the principal offers to the agents with different values.

Definition 2 (Single plateau)**.** Let x be a mechanism and $v \in \mathcal{V}$ be a valuation. The allocation schedule $x(v)$ has a single plateau if for all $c' < c'' < c''' \in \mathcal{C}$ it holds,

$$x(v, c') > 0 \text{ and } x(v, c''') > 0 \Rightarrow x(v, c'') = 1.$$

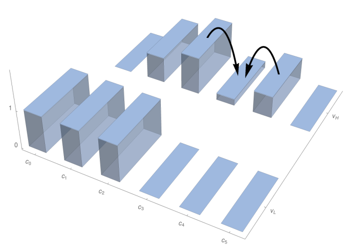
A mechanism exhibits single plateaus if for all $v \in \mathcal{V}$, the allocation schedule $x(v)$ has a single plateau.

The support of an allocation schedule with a single plateau is always an interval in $\mathcal{C}$: it is given by $\mathcal{C} \cap [\underline{c}, \overline{c}]$ for some $\underline{c}, \overline{c} \in \mathcal{C}$. The allocation probability in the interior of this interval is always 1 and outside the interval is 0. Allocation probabilities different from $\{0, 1\}$ are only possible on the boundary of its support. In Section 4.2, the case of continuously distributed costs is analyzed. In this case the mechanism is deterministic: the allocation probability is either 0 or 1.

The allocation schedules studied in Section 3.2 allocate whenever the cost is below a cutoff. These cutoff schedules also have a single plateau. But the plateau is not interior. Intuitively, an allocation schedule with an interior single plateau allocates the good for intermediate costs with certainty and for extreme costs with zero probability.

Proposition 1
If costs and valuations are positively correlated ((1), case 3) and the distribution meets the regularity assumption (6), then the optimal solution to the relaxed problem has single plateaus.

If an allocation schedule has a single plateau, then the following modification is not feasible: for some triple $c' < c'' < c''' \in \mathcal{C}$, decrease $x(v, c') \downarrow$ and $x(v, c''') \downarrow$ while increasing $x(v, c'') \uparrow$. To show that the optimal mechanism has single plateaus, it suffices to show that there always exists such a modification that respects the relaxed incentive constraints while increasing the principal's expected payoff.



These shifts resemble the shifts of the payoffs that are used to study risk aversion (Diamond and Stiglitz (1974)). The formal proof of the Lemma uses a version of Farkas' Lemma and is relegated to the appendix.

Instead, I present a related problem which makes the connection to risk preferences transparent and illustrates how the regularity assumption leads to single plateau allocation schedules.

Intuition: a related problem with homogeneous beliefs. Suppose there are two valuations $\{v_L, v_H\}$ with $v_H - c \geq 0$ for all $c \in \mathcal{C}$. Let $f(v, c)$ be a twice differentiable interpolation of f for all $c \notin \mathcal{C}$. Before describing the related problem, I want to emphasize three important features of the setting that is studied in this paper. (i) Agents of different valuations hold different beliefs about the costs. (ii) The agent does not bear the costs of the allocation. (iii) The problem is equivalent to a setting where all the agents of different valuations have the same utility function.

The related problem contrasts the original setting in these three respects: (i) Agents of different valuations and the principal share a common belief about the costs. (ii) The agent bears the costs of the allocation. (iii) Agents with different valuations and the principal differ in their risk preferences with respect to the costs.

Suppose that the following related setting exists: costs are distributed according to a pdf g. Again, $g(c)$ denotes a twice differenteiable interpolation of g for all $c \notin \mathcal{C}$. There exist twice differentiable and increasing utility functions $\tilde{u}_L(-c), \tilde{u}_H(-c), \tilde{w}(-c)$ for the agent with low valuation v_L, the agent with high valuation v_H and the principal, respectively. $u_H(-c)$ describes the utility an agent with valuation v_H derives from receiving the good if the cost realization is c. The utilities are such that the preferences over allocation schedules for v_H coincide with the preferences in the original problem, formally:

$$\tilde{u}_H(-c) \cdot g(c) = f(v_H, c), \tag{7}$$

$$\tilde{u}_L(-c) \cdot g(c) = f(v_L, c), \tag{8}$$

$$\tilde{w}(-c) \cdot g(c) = f(v_H, c) \cdot (v_H - c). \tag{9}$$

It follows from this equation, that the optimal mechanism in the original setting and in the related problem coincide. Take any allocation schedule for v_H as given. For

$c' < c'' < c''' \in \mathcal{C}$, consider the following modification of the mechanism:

$$\text{decreases } x(v_H, c') \downarrow \text{and } x(v_H, c''') \downarrow, \text{ and increase } x(v_H, c'') \uparrow, \tag{10}$$

such that the utility for an agent with valuation v_H remains constant. When does such a modification simultaneously increase the principal's expected utility and decrease the expected utility of a low valuation agent?

Diamond and Stiglitz (1974) show that this is the case for all such modifications if a player with utility $\tilde{w}$ is more risk averse than a player with $\tilde{u}_H$ and a player with $\tilde{u}_H$ is more risk averse than a player with $\tilde{u}_L$, each in the sense of Arrow-Pratt:

$$\tilde{w} \geq_{AP} \tilde{u}_H \geq_{AP} \tilde{u}_L.$$

This is equivalent (see for example Jewitt (1989)) to,

$$\frac{\tilde{u}'_H(-c)}{\tilde{w}'(-c)} \text{ and } \frac{\tilde{u}'_L(-c)}{\tilde{u}'_H(-c)} \text{ are decreasing in } c. \tag{11}$$

In the appendix, I show that (7)–(9) and (11) can only hold together if

$$\frac{\partial^2}{\partial^2 c}\left(\frac{f(v_L, c)}{f(v_H, c)}\right) \geq 0.$$

In the related problem, the players' heterogeneity of beliefs is transformed into heterogeneity of their risk preferences. The regularity assumption ensures that these risk preferences can be ordered in a way that makes the modification of the form (10) simultaneously incentive compatible and worthwhile for the principal. In the optimal mechanism this modifications must not be feasible. Therefore, it must have single plateaus.

3.3.2 Plateau-monotone mechanisms

Because $\mathcal{V}$ is ordered, one can number its elements.

$$\mathcal{V} = \{v_0, v_1, \ldots, v_{m-1}\}.$$

To state the second characteristic of the optimal mechanism—a form of monotonicity—one needs to introduce a partial order on the set of allocation schedules $\{x(v) \mid v \in \mathcal{V}\}$.

Definition 3. Let x be a mechanism. Define the partial order $\succeq$ on $\{x(v) \mid v \in \mathcal{V}\}$ such that for $v', v'' \in \mathcal{V}$

$$x(v'') \succeq x(v') :\Leftrightarrow \quad \forall c' < c'' \in \mathcal{C} \colon x(v'', c') \geq x(v', c') \Rightarrow x(v'', c'') \geq x(v', c''),$$
$$x(v', c'') \geq x(v'', c'') \Rightarrow x(v', c') \geq x(v'', c').$$

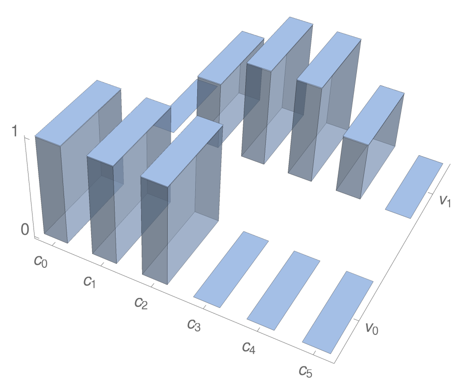
Or equivalently,

$$x(v'') \succeq x(v') :\Leftrightarrow \quad c \mapsto x(v'', c) - x(v', c) \text{ is single-crossing from below.}$$

Define the strict order $x(v'') \succ x(v')$ if $x(v'') \succeq x(v')$ and $x(v'') \neq x(v')$.
A mechanism is monotone if

$$v'' > v' \Rightarrow x(v'') \succeq x(v')$$

or equivalently, a mechanism is monotone if $x(\cdot, \cdot)$ has single-crossing differences from below.



Proposition 2
If costs and valuations are positively correlated ((1), case 3) and the distribution meets the regularity assumption (6), then the optimal mechanism in the relaxed problem is plateau-monotone.

Fix two valuations $v'' > v'$ and consider a plateau-monotone mechanism. Denote the upper and lower bounds of the support of the allocation schedule for v'' by $\overline{c}''$ and $\underline{c}''$, respectively. Denote the bounds for v' respectively as $\overline{c}'$ and $\underline{c}'$. Then,

$$\overline{c}'' \geq \overline{c}' \text{ and } \underline{c}'' \geq \underline{c}'.$$

The plateau of the higher valuation type is shifted to the right.

Proof sketch. The formal proof is in the appendix.
Step 1: It is established, that if x is an optimal mechanism in the relaxed problem, then the partial order $\succeq$ is total on $\{x(v) \,|\, v \in \mathcal{V}\}$. For any pair of $v', v'' \in \mathcal{V}$ either $x(v') \succeq x(v'')$ or $x(v'') \succeq x(v')$. Step 2: The following result about the variation diminishing property of totally positive functions is used:

Lemma (Schoenberg (1930); Karlin (1968))
If a real function function $K\colon \mathcal{V} \times \mathcal{C} \to \mathbb{R}$ is strictly totally positive of order 2 (STP_2) and there are allocation schedules: $x(\tilde{v}) \succ x(v)$, then for any $v' < v'' \in \mathcal{V}$ it holds

$$\sum_{c \in \mathcal{C}} K(v', c) \cdot (x(\tilde{v}, c) - x(v, c)) \geq 0 \;\Rightarrow\; \sum_{c \in \mathcal{C}} K(v'', c) \cdot (x(\tilde{v}, c) - x(v, c)) > 0,$$
$$\sum_{c \in \mathcal{C}} K(v'', c) \cdot (x(\tilde{v}, c) - x(v, c)) \leq 0 \;\Rightarrow\; \sum_{c \in \mathcal{C}} K(v', c) \cdot (x(\tilde{v}, c) - x(v, c)) < 0.$$

Or equivalently, $v' \mapsto \sum_{c \in \mathcal{C}} K(v', c) \cdot (x(\tilde{v}, c) - x(v, c))$ crosses zero at most once and then from below.[3]

When the players compare two allocation schedules $x(\tilde{v}) \succeq x(v)$, they form expectations about the costs C and then evaluate the difference between these two schedules $x(\tilde{v}, C) - x(v, C)$. The above variation diminishing results ensure that the ranking over the schedules is monotone in the valuation of the agents: setting $K(v, c) = f(v, c)$ yields, for example, that if an agent of valuation v' ranks $x(\tilde{v})$ over $x(v)$ then an agent with a higher valuation $v'' > v'$ ranks these allocation schedules in the same way.

An analogue monotone comparative static result follows for the principal. Again, suppose that $x(\tilde{v}) \succeq x(v)$. Setting $K(v, c) = f(v, c) \cdot (v - c)$ yields that if assigning the allocation $x(\tilde{v})$ to an agent with valuation v' yields her a higher expected payoff than $x(v)$ the same must hold true for an agent with a higher valuation $v'' > v'$.

A key step,is to establish that the assumptions of Karlin's lemma is met. The pdf $f(v, c)$ is strictly affiliated and strictly positive and therefore strictly totally positive (STP_2). Also, restricted on $\{(v, c) \in \mathcal{V} \times C \,|\, v > c\}$, the function $(v, c) \mapsto v - c$ is STP_2 and so is the product$f(v, c) \cdot (v - c)$. The proof in the appendix shows that all relevant comparisons between allocations schedules take place in this restricted set.

[3] A touching of zero is counted as a crossing.

Step 3: Suppose for this proof sketch, that the only relevant incentive constraints are the local-upward incentive constraints of the form: $IC(v_{k-1}, v_k) \geq 0$.

If an optimal mechanism was not plateau-monotonic, there would be some k such that $x(v_{k-1}) \succ x(v_k)$. Consider the following modification: offer v_k the allocation schedule $x(v_{k-1})$. An agent with v_{k-1} ranks schedule $x(v_{k-1})$ over $x(v_k)$. Because $x(v_{k-1}) \succeq x(v_k)$, an agent with type v_k ranks them in the same way (step 2). The modification would therefore not violate any of the local-upward incentive constraints. Hence, this modification cannot be optimal for the principal:

$$\mathbb{E}[(v_k - C) \cdot x(v_{k-1}, C) \,|\, V = v_k] \leq E[(v_k - C) \cdot x(v_k, C) \,|\, V = v_k].$$

This implies (step 2) that

$$\mathbb{E}[(v_k - C) \cdot x(v_{k-1}, C) \,|\, V = v_{k-1}] < E[(v_k - C) \cdot x(v_k, C) \,|\, V = v_{k-1}].$$

However, then the mechanism can not have been optimal in the first place: modifying it such that it offers v_{k-1} the allocation schedule $x(v_k)$ would be a strict improvement for the principal and would not violate the incentive constraints. □

Theorem 3

If costs and valuations are positively correlated ((1), case 3) and the distribution meets the regularity assumption (6), then the optimal mechanism has single plateaus and is plateau-monotone. It fulfills all local-upward incentive constraints with equality, for all $0 < k < m$:

$$\sum_{c \in \mathcal{C}} f(v_{k-1}, c) \cdot [x(v_{k-1}, c) - x(v_k, c)] = 0.$$

In the relaxed problem, the principal is not restrained by the incentive constraints that prevent the agent from misrepresenting his valuation as the lowest v_0. Therefore, it cannot be the case that the principal allocates the good to the lowest type at some cost c'' but does not allocate with certainty at some lower costs $c' < c''$. If this were the case, she could profitably shift the allocation probability from c'' to c', keeping the lowest type indifferent. This rules out any interior single plateau as the allocation schedule for the lowest valuation.

Corollary 2

Under the assumptions of Theorem 3, the allocation schedule for the lowest valuation

v_0 *is always in cutoff form.*

$$x(v_0, c) = \begin{cases} 1, & c < \tilde{c} \\ 0, & c > \tilde{c} \end{cases}.$$

3.4 Monotonicity

Proposition 2 establishes that the optimal allocation schedules are plateau ordered. Of course, the ignorant cutoff mechanisms that are optimal under negative correlation are also ordered in this sense. However, in general, this monotonicity does not imply that a higher valuation type receives the good with higher probability. Again, there is a difference between positive and negative correlation. Under negative correlation, the expected allocation probabilities in the optimal ignorant mechanism are increasing in the agent's valuation. That is because, the function $x(c) = \begin{cases} 1, & c \leq \bar{c} \\ 0, & \text{otherwise} \end{cases}$ is decreasing on $\mathcal{C}$. By negative affiliation, $\mathbb{E}[x(C) \,|\, V = v]$ is therefore increasing in v. If the optimal mechanism under positive correlation is also ignorant, it must have the cutoff from. [4] However, in this case, the expected allocation probabilities are decreasing in the agent's valuation because under positive correlation $\mathbb{E}[x(C) \,|\, V = v]$ is decreasing in v.

Example 4 shows that the interim expected allocation probabilities can be decreasing under positive affiliation, even if the optimal mechanism screens.

This difference with respect to standard results steams from the fact that the agents of different valuations hold different beliefs. For $v'' > v'$ by incentive compatibility,

$$\mathbb{E}[x(v'', C) \,|\, V = v''] \geq \mathbb{E}[x(v', C) \,|\, V = v'']$$
$$\mathbb{E}[x(v', C) \,|\, V = v'] \geq \mathbb{E}[x(v'', C) \,|\, V = v'],$$

but because v' and v'' do not share the same belief, one cannot compare $\mathbb{E}[x(v'', C) \,|\, V = v'']$ and $\mathbb{E}[x(v', C) \,|\, V = v']$ with the above equations.

[4] Otherwise, if the ignorant allocation schedule was interior, it could be shifted to lower costs, keeping the expected allocation value constant but decreasing the expected allocation costs

4 Extensions

4.1 Discrete costs

In Section 2.1, I assumed that the support of the costs is finite. However, the proofs in the appendix allow the support to be countably infinite. Therefore, the case of many discrete cost distributions is covered.

Example 1. Suppose that valuations are finite: $\mathcal{V} = \{v_0, v_1, \ldots, v_{m-1}\}$.

1. Suppose that conditional on the valuation $V = v_k$, the cost C is binomally distributed with equal number-of-trials parameter n and success probabilities p_k. Formally, $f(C = i|V = v_k) = \binom{n}{i} \cdot p_k^i \cdot (1 - p_k)^{n-i}$. The distribution is strictly positively affiliated if and only if $p_0 < p_1 < \cdots < p_{m-1}$. Then, for all $j < k$ the decreasing likelihood ratio, $\frac{f(C=i|V=v_j)}{f(C=i|V=v_k)}$ is also convex in i. Therefore, the regularity assumption (6) is fulfilled.
2. Suppose that conditional on the valuation $V = v_k$, the cost C is geometrically distributed with failure probabilities p_k. Formally, $f(C = i|V = v_k) = (1 - p_k)^i \cdot p_k$. The joint distribution is strictly positively affiliated if and only if $p_0 > p_1 > \cdots > p_{m-1}$. Then, the regularity assumption is also fulfilled.
3. Suppose that conditional on the valuation $V = v_k$, the cost C has a Poison distribution with parameter λ_k. Formally, $f(C = i|V = v_k) = \frac{\lambda_k^i e^{-\lambda_k}}{i!}$. The joint distribution is strictly positively affiliated if and only if $\lambda_0 < \lambda_1 < \cdots < \lambda_{m-1}$. Then, the regularity assumption is also fulfilled.

4.2 Continuous costs

Here, I assume that the costs are continuously distributed on an interval while the valuations remain finitely distributed. Formally, there is a measurable positive function $f(v, c)$ such that for all $B \subset \mathbb{R}$,

$$\mathbb{P}(V = v, C \in B) = \int_B f(v, c)\, \mathrm{d}c.$$

I assume that the cost distribution has the same interval as the support for all v, i.e. there exists an interval $\mathcal{C} \subset \mathbb{R}$ such that for all $v \in \mathcal{V}$:

$$\mathcal{C} = \{c \in \mathbb{R} | f(c, v) > 0\}.$$

Furthermore $c \mapsto f(v,c)$ is assumed to be continuous on $\mathcal{C}$ for all $v \in \mathcal{V}$.

An allocation schedule for a valuation report v is given by a measurable function $x(v) : \mathcal{C} \to [0,1]$, and a mechanism specifies an allocation schedule for all valuations $v \in \mathcal{V}$. I distinguish the same three cases for the likelihood ratios with $v'' > v'$

$$\frac{f(c,v')}{f(c,v'')}.$$

Of course, manipulations on a set $B \subset \mathcal{C}$ of measure zero, neither affect the incentive constraints nor the principal's expected payoff. Mechanisms that differ only on a zero measure subset of $\mathcal{C}$ are equivalent. The regularity assumption (6) remains the same.

Theorem 4 *1. If costs and valuations are independent or negatively correlated, the optimal mechanism is equivalent to an ignorant cutoff mechanism of the form*

$$x(v,c) = \begin{cases} 1, & c \le c^* \\ 0, & c > c^* \end{cases},$$

for some $\bar{c} \in \mathcal{C}$.

2. If costs and valuations are positively correlated and the regularity assumption (6) is met, the optimal mechanism is equivalent to a plateau-monotone mechanism of the form

$$x(v,c) = \begin{cases} 1, & \underline{c}(v) \le c \le \bar{c}(v) \\ 0, & \textit{otherwise} \end{cases},$$

with increasing functions $\underline{c}(v) \le \bar{c}(v)$.

If the costs are continuous, there is always an optimal mechanism that is deterministic.

Example 2. If the distribution of cost and valuation is of an exponential family, i.e., $f(c|v) = h(c) \cdot e^{\eta(v) \cdot T(c) - A(v)}$, then it is strictly positive affiliated if $T'(c) \cdot (\eta(v') - \eta(v'')) < 0$ for all c and $v' < v''$. The regularity assumption (6) is met if $(T'(c) \cdot (\eta(v') - \eta(v''))^2 > T''(c) \cdot (\eta(v') - \eta(v''))$ for all c and $v' < v''$. This yields that the regularity assumption is met if cost and valuation are jointly normally distributed and have positive correlation.[5] Also, if conditionally on $V = v_k$ the cost is exponentially distributed with parameter λ_k, then the distribution is positively affiliated if and only if $\lambda_0 > \lambda_1 > \cdots > \lambda_{m-1}$. In this case, the regularity assumption is also met.

Optimal ignorant mechanism. Under negative correlation or independence, the optimal ignorant mechanism was a simple cutoff rule. Despite the regularity assumption, the optimal ignorant mechanisms under positive affiliation can be quite irregular. For sets $C', C'' \subset \mathcal{C}$ define $C' \leq C''$ if for all $c' \in C'$ and $c'' \in C''$: $c' \leq c''$. An ignorant mechanism $x(v,c) = x(c)$ has a hole if there exist sets $C' \leq C'' \leq C''' \subset \mathcal{C}$ of positive measure such that $x(c) > 0$ for all $c \in \mathcal{C}' \cup C'''$ and $x(c) = 0$ for all $c \in C''$. The holes of a mechanism are counted in the following way:

$$\sup\{k \in \mathbb{N} \,|\, \exists C_1 \leq B_1 \leq C_2 \leq B_2 \leq \cdots \leq C_{k+1} \colon c \in C_i \colon x(c) = 0 \& c \in B_i \colon x(c) > 0\}.$$

Lemma 1

For any $k \in \mathbb{N} \cup \{\infty\}$, there exists a positively correlated (Item 3) joint distribution that meets the regularity assumption such that the optimal ignorant mechanism $x(c)$ has k holes.

Under negative correlation, $\mathbb{E}[V|C = c]$ is decreasing in c. Therefore, $\mathbb{E}[V|C = c] - c$ is also decreasing, and the optimal ignorant mechanism is always in the cutoff form. Under positive correlation, $\mathbb{E}[V|C = c]$ is increasing and therefore $\mathbb{E}[V|C = c] - c$ can cross zero infinitely many times. The proof (in the appendix) of Lemma 1 constructs such an example.

An optimal mechanism is in plateau form and therefore has no holes. The regularity assumption separately guarantees the plateau form of all single allocation schedules. However, the ignorant mechanism consists of one allocation schedule for all v. The separate regularity of the single optimal schedules does not aggregate to the regularity of the single optimal schedule for the ignorant mechanism. Finally if the optimal ignorant mechanism is irregular, it can be strictly improved, by an non-ignorant mechanism.

Corollary 3

If the optimal ignorant mechanism under positive affiliation has a hole, then the optimal mechanism is not ignorant.

[5] Of course this paper does not cover this case, since I assume that valuation types are finite

4.3 Binary values

When $\mathcal{V} = \{v_L < v_H\}$ and the costs are continuously distributed on an interval $\mathcal{C} = [a, b)$ with $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{\infty\}$, the optimal mechanism can be pinned down. With binary values, there is only one likelihood ratio of interest:

$$l(c) = \frac{f(v_L, c)}{f(v_H, c)}.$$

As a convex function, l is almost surely differentiable.

Proposition 3
When costs and valuations are positively affiliated and $[v_L, v_H] \subset \mathcal{C}$, then either:

1. *The optimal mechanism is ignorant and of a cutoff form:*

$$x(v, c) = x(c) = \begin{cases} 1, & c \leq \bar{c} \\ 0, & \textit{otherwise} \end{cases}.$$

In this case, $\bar{c} \in \mathcal{C}$ is the unique solution to

$$c \in [v_L, v_H]: \quad \frac{f(v_L, c)}{f(v_H, c)} = \frac{v_H - c}{c - v_L}. \tag{12}$$

2. *The optimal mechanism is not ignorant and of the following form:*

$$x(v_L, c) = \begin{cases} 1, & c \leq \bar{c}_L \\ 0, & \textit{otherwise} \end{cases}, \quad x(v_H, c) = \begin{cases} 1, & \underline{c}_H \leq c \leq \bar{c}_H \\ 0, & \textit{otherwise} \end{cases}.$$

In this case, $a < \underline{c}_H \leq \bar{c}_H$ and $\bar{c}_L < \bar{c}_H$ are the unique solution to:

$$l(\underline{c}_H) = \frac{v_H - \underline{c}_H}{\bar{c}_L - v_L} \tag{13}$$

$$l(\bar{c}_H) = \frac{v_H - \bar{c}_H}{\bar{c}_L - v_L} \tag{14}$$

$$F(\bar{c}_L | V = v_L) = F(\bar{c}_H | V = v_L) - F_L(\underline{c}_H | V = v_L). \tag{15}$$

Remark 5. The system (12) always has a solution. If it is not unique then the optimal ignorant mechanism has holes and by Corollary 3, the optimal mechanism is not ignorant. Then case 2 applies. Only if the solution is unique the optimal mechanism can be ignorant.

The system (13)–(15) has at most one solution. If it has no solution then the optimal mechanism is ignorant. If both (12) and (13)–(15) have a unique solution, then they are the two candidates for the optimal mechanism. It cannot be the case that simultaneously (12) is not unique and (13)–(15) do not have a solution.

When monetary transfers are feasible, any form of correlation can be exploited. Without transfers, negative correlation is not used to screen the agent. The next result shows that under positive correlation, the degree of correlation must exceed a certain threshold at the lowest cost.

Lemma 2
Suppose that l' is continuous at a. If

$$\frac{-l'(a)}{l(a)} < \frac{1}{v_H},$$

then the optimal mechanism is ignorant.

Example 3. If $f(c|V = v_\omega) = \lambda_\omega e^{-\lambda_\omega \cdot c}$ for $\omega \in \{L, H\}$ and $\lambda_L > \lambda_H$, the condition from Lemma 2 translates to $\lambda_L - \lambda_H < \frac{1}{v_H}$.

The characterizations of this section can be used to efficiently calculate numerical examples:

Example 4. $f(c, V = v_\omega) = \frac{1}{2} \cdot \lambda_\omega e^{-\lambda_\omega \cdot c}$ for $\omega \in \{L, H\}$ with $\lambda_L = 1, \lambda_H = 1/5$ and $v_H = 5, v_L = 0$. The optimal mechanism screens. It is given by $\bar{c}_L = 1.22$ and $(\underline{c}_H, \bar{c}_H) = (0.34, 4.88)$. The optimal ignorant mechanism has: $\bar{c} = 4.32$. It is worth noticing that the lower type has a higher expected allocation probability: $\mathbb{E}[x(v_L, C) \,|\, V = v_L] = .71 > \mathbb{E}[x(v_H, C) \,|\, V = v_H] = 0.56$.

4.4 General allocation values

The principal's value from the allocation can be generalized from $v - c$ to

$$w(x, v, c) = x \cdot z(v, c)$$

with z increasing in v and decreasing in c. Under negative correlation or independence the same results go through.

Theorem 6

If costs and valuations are negatively correlated ((1), case 1) or independent ((1), case 2), then it is optimal for the principal to offer an ignorant mechanism. In these cases, the optimal (ignorant) mechanism is given by a simple cutoff rule:

$$\bar{c} = \min\{c \in \mathcal{C} \mid \mathbb{E}[z(V,c) \,|\, C = c] \geq c\}.$$

and

$$x(c,v) = 1 \textit{ if and only if } c \leq \bar{c}.$$

For the case of positive correlation, the regularity assumption generalizes to the following: for all $v' < v'' \in \mathcal{V}$ and all $c' < c'' < c''' \in \mathcal{C}$,

$$\frac{\frac{f(v',c'')}{f(v'',c')} - \frac{f(v',c')}{f(v'',c'')}}{\frac{f(v',c'')}{f(v'',c'')} - \frac{f(v',c''')}{f(v'',c''')}} \geq \frac{z(v'',c') - z(v'',c'')}{z(v'',c'') - z(v'',c''')}. \tag{16}$$

It demands that the decreasing likelihood ratio (on the left hand side) must be more convex than the allocation value of the principal (on the right hand side). This regularity assumption ensures the plateau form of the optimal mechanism in the relaxed problem. To establish that the optimal mechanism is plateau-monotone, it must be the case that z is log-supermodular on $\{(v,c) \,|\, z(v,c) > 0\}$, formally for all $v'' > v'$ and $c'' > c'$,

$$\text{if } z(v',c'') > 0: \qquad z(v'',c'') \cdot z(v',c') > z(v'',c') \cdot z(v',c'').$$

Theorem 7

Under positive correlation ((1), case 3), if the generalized regularity assumption is fulfilled and z is log-supermodular where it is positive, then the optimal mechanism has single plateaus and is monotone. It fulfills all local-upward incentive constraints with equality.

5 Applications

5.1 Intra-firm allocation

A new computer model is about to enter the market. The management of a company has to decide whether it should buy the new computer. Before the market opens, the research department privately learns the value that the computer could generate for the company. The management wants to buy whenever the value exceeds the purchasing price; the research department wants to have the computer in any case. Before the competitive market price realizes, the management can communicate with the research department and commit to a demand schedule in the upcoming competitive market. Can the management use future market information to counter the internal information asymmetry?

Suppose that the correlation between market price and valuation stems from an unobservable quality factor $q \in [0,1]$ with strictly positive density $h(q)$. At $t = 0$, the quality realizes and the research department learns the value for the firm. The value is either low or high: $v \in \{v_L < v_H\}$. Let $f(v \mid q)$ denote the probability that valuation is $v \in \{v_L, v_H\}$ if the quality level is q. Suppose that $l(q) = \frac{f(v_L \mid q)}{f(v_H \mid q)}$ is twice differentiable and strictly decreasing. High quality indicates higher values. At $t = 1$, a competitive market for the computer opens. The management learns the rational expectation equilibrium price and can buy the computer at this price. The market is abstractly modeled as an equilibrium price function, which defines for any quality level q a price $p(q)$. Suppose that the price is strictly increasing in the quality. Defining the costs as $c = p(q)$ puts this in the positive correlation case. The regularity assumption then translates to $l(p^{-1}(p))$ being convex; this is the case if

$$-\frac{l''}{l'} > -\frac{p''}{p'}. \tag{17}$$

By assumption, $l' < 0$ and $p' > 0$. This condition is most easily fulfilled when the price is a convex increasing function of the quality or if the price is linear and l is convex. The price must react faster to an increase in quality than the value for the firm.

If the condition is met and the optimal mechanism involves screening, the management's demand schedule in the market is not monotone. Suppose the research department reports that the value of the computer is high. Then, the management will buy for intermediate prices and will not buy if the price is too low or too high.

5.2 Task delegation

The state wants to open a new hospital in a city. She has to decide if she wants to operate the hospital on her own or wants to delegate the operation to the city. Both the state and the city privately know what benefit for the society they would create as operators of the hospital. Denote the state's value by v_P and the city's value by v_A. These values are positively correlated: whenever the state can create a high value, it is more likely that the city could also create a high value. The state wants to delegate the operation to whoever creates the highest social value. The city wants to have control over the hospital in any case. If the state decides to delegate to the city, she forgoes the social value that she could create. This opportunity cost $c = -v_P$ represents her costs of allocation, whereas the social value that can the city can create is the value of the allocation: $v = v_A$. v and c are positively correlated, therefore the optimal mechanism might screen. If there is any screening then $x(\sup \mathcal{V}, \inf \mathcal{C}) < 1$. The state has to commit that she will not delegate when the city reports the highest possible value while she predicts the lowest possible value. Such a decision might be difficult to publicly defend. This advocates for an intransparent procedure, where the reported values are not part of the public record.

The next numerical example illustrates the effect of screening on the value of correlation.

Numerical example: value of correlation. Suppose that the state can build the hospital in one of two cities. After she builds the hospital, her and the city's operation value realize and she has to decide who operates it.

Both cities are with probability 1/2 good operators of the hospital and would create a social value of $v_H^1 = v_H^2 = 5$ and with probability 1/2 bad operators creating zero value. The states ability differs in the two cities. In the first city, her value is exponentially distributed with parameter $\lambda_H^1 = 1/4$ if the city itself is a good operator and with parameter $\lambda_L^1 = 1/2$ when the city is a bad operator. In the second city, the respective parameters are $\lambda_H^1 = 1/5$ and $\lambda_L^1 = 1$. The unconditional expectation of the state's operation value is the same in both cities. However, the degree of positive correlation is higher in city 2.

Without asymmetric information, the principal would prefer to build the hospital in city 1. If she could observe the realization of the city's operation value, she would implement the efficient allocation. This would yield her 2.146 for city 1 and 1.839 for city 2. With more positive correlation, if one of the two values is low, it is likelier that the other value is also low. In this sense, negative correlation serves as an insurance against low values.

The state's choice changes when she takes the asymmetric information into account. Now she prefers to build the hospital in city 2. The optimal mechanism in city 1 is ignorant. It allocates whenever $v_P > 2.376$. This yields the state 0.546. The extent of the positive correlation is not sufficient to screen out the city's type. In city 2, the positive correlation is sufficiently strong. The optimal mechanism screens the agent: When the city reports v_L^2, the state lets her operate the hospital whenever $v_P < 1.222$. when the city reports v_H^2, the principal lets her operate the hospital if $v_p \in (0.338261, 4.87646)$. This yields her $0.5887 > 0.546$.

5.3 Self-regulation

A firm seeks a regulator's approval for a new product. There are two aspects, $a_1 and a_2$, that are positively correlated and jointly determine the probability that the product is not faulty $p(a_1, a_2)$. The probability increases in both aspects. An example could be the approval of a new airplane model. If the regulator approves and the product turns out to not be faulty a social benefit $B > 0$ is created. If the regulator approves and the product is faulty, a social loss $L > 0$ occurs. The expected social benefit from approving reads as follows.

$$p(a_1, a_2) \cdot B - (1 - p(a_1, a_2)) \cdot L.$$

If the product is not approved, the effect on social welfare is independent of whether the product is faulty or not and normalized to zero.

It is a long-standing practice that regulators delegate parts of the certification process to the producers of the product. In the case of airplanes, the Search Results Web result with site links Federal Aviation Administration (FAA) fostered this development with their "Organizational Designation Authorization" program in 2005.[6] The extend of the delegation is substantial; the Transportation Departments Inspec-

[6]see Federal Aviation Administration (2005).

tor General (2005) reported, "One aircraft manufacturer approved about 90 percent of the design decisions for all of its own aircraft."

The regulators argue that the delegation increases the efficiency of the process whilst ensuring its safety.[7]

To analyze this claim, suppose that the regulator delegates the certification of the first aspect to the producer and bases her decision on the reports from the producer and her own investigation of the second aspect. Setting

$$w(x, v, c) = x \cdot (p(v, -c) \cdot B - (1 - p(v, -c)) \cdot L)$$

as the principal's objective translates the positive correlation between the aspects, a_1 and a_2, into negative correlation between valuation $v = a_1$ and costs $c = -a_1$. The optimal mechanism therefore ignores the producer's report about the first aspect and bases the decision solely on the regulator's own findings. Delegation of an aspect implies that it is ignored it for the assessment of the safety of the product.

6 Conclusion

I study the bilateral trade setting with correlated information when monetary transfers are not feasible and characterize the welfare maximizing mechanism. This mechanism uses positive correlation to screen the agent; whereas under negative correlation, the optimal mechanism does not elicit the agent's valuation. My characterization of the optimal mechanism has interesting consequences for applications. Screening makes it necessary to forgo the highest gains from allocation. Also, the optimal mechanism may not allocate to higher valuation-types with higher probability. I introduce a novel regularity assumption that ensures an interval form of the optimal mechanism. The possibility of using positive correlation for screening opens interesting new directions of future research. My characterization is the first step toward analyzing a richer setting in which the principal endogenously chooses the degree of positive correlation.

[7] See, for example, the testimony of the FAA's acting administrator Daniel K. Elwell (2019) in a senate hearing on the Boeing 737 Max crashes.

A Proofs

All the proofs in this section are for the more general case of allocation values of the form $w(x, v, c) = x \cdot z(v, c)$ with

- $z(v, c)$ is increasing in v and decreasing in c.
- $z(v, c)$ is log-supermodular on $\{(v, c) \mid z(v, c) > 0\}$.

This generalization is introduced in Section 4.4. The allocation value $z(v, c) = v - c$ is of this form, since

$$\partial_v \partial_c \log(v - c) = \frac{1}{(v - c)^2} \geq 0.$$

As $\mathcal{V}, \mathcal{C}$ are assumed to be discrete and ordered one can number its elements:

$$\mathcal{V} = \{v_0, v_1, \ldots, v_{m-1}\}, \quad \mathcal{C} = \{c_0, c_1, \ldots, c_{n-1}\}.$$

The support of $\mathcal{C}$ can be countably infinite, then $n = \infty$.

Notation For two vectors $a, b \in \mathbb{R}^n$ let $a \cdot b$ denote their standard inner product.[8] Let $a \circ b$ denote the vector of their element-wise products.

If there were monetary transfers

Independence and negative correlation

Proof of Theorem 2.

Take any incentive compatible mechanism $\boldsymbol{x}$. If it is not ignorant and in cutoff form I construct a modification of this mechanism that yields the principal a higher expected utility while keeping the incentive constraints satisfied.

step 0: (Relaxation)

In a relaxation of the problem the designer maximizes the same objective but disregards the incentive constraints for the agent not to report a lower type. The only incentive constraints which a solution of the relaxed problem has to respect are:

$$\forall v < \hat{v} \colon IC(v, \hat{v})$$

[8] If $n = 1$ than this denotes the standard product of two real numbers.

step 1: For all v there exists a cutoff $c(v)$ such that $x(v,c) = \begin{cases} 1, & c < c(v) \\ 0, & c > c(v) \end{cases}$.
Suppose there exists v'' and $c' < c''$ with $x(v'', c') < 1$ and $x(v'', c'') > 0$. Consider the following modification of the mechanism:

- allocate with probability $x(v'', c') + \mathrm{d}x(v'', c')$ after (v'', c')
- and with $x(v'', c'') + \mathrm{d}x(v'', c'')$ after (v'', c''),

with

$$\mathrm{d}x(v'', c'') = -\frac{f(v'', c')}{f(v'', c'')} \cdot \mathrm{d}x(v'', c')$$

and $\mathrm{d}x(v'', c') > 0$ small enough such that: $x(v'', c') + \mathrm{d}x(v'', c') \leq 1$ and $x(v'', c'') + \mathrm{d}x(v'', c') \geq 0$. It follows that

$$f(v'', c') \cdot \mathrm{d}x(v'', c') + f(v'', c'') \cdot \mathrm{d}x(v'', c'') = 0,$$

i.e. the probability of allocation for type v' (if he reports truthfully) remains the same in the modified mechanism. Since the allocation probability for all reports $v''' > v''$ were not modified, v'' has no new incentives to report a higher type. The principal's expected value increases:

$$f(v'', c') \cdot \mathrm{d}x(v'', c') \cdot (-c') + f(v'', c'') \cdot \mathrm{d}x(v'', c'') \cdot (-c'') = f(v'', c') \cdot (c'' - c') > 0.$$

Furthermore for any $v' \leq v''$,

$$f(v', c') \cdot \mathrm{d}x(v'', c') + f(v', c'') \cdot \mathrm{d}x(v'', c'') = \mathrm{d}x(v'', c') \cdot f(v', c'') \cdot \left(\frac{f(v', c')}{f(v', c'')} - \frac{f(v'', c')}{f(v'', c'')} \right) \leq 0,$$

i.e. for lower types misreporting their type as v'' becomes less attractive. Since the original mechanism was assumed to be incentive compatible in the relaxed problem, it follows that the modified mechanism remains compatible to the relaxed problem. We set $c(v)$ such that[9]

$$x(v, c) = \begin{cases} 1, & c < \lfloor c(v) \rfloor \\ c(v) - \lfloor c(v) \rfloor, & c = \lfloor c(v) \rfloor \\ 0, & c > c(v) \end{cases}.$$

[9] If $c(v) = 3.7$ the corresponding allocation is given by $x(v, \cdot) = (1, 1, 1, 0.7, 0, \cdots)$. If $c(v) = 3$ the allocation is $x(v, \cdot) = (1, 1, 0, 0, \cdots)$

step 2: There exists $x : \mathcal{C} \to [0,1]$ with $x(v,c) = x(c)$ for all v.
First, it can be ruled out that there exists $v' < v''$ such that $c(v') < c(v'')$ since this would violated $IC(v', v'')$. Suppose next, there exists $v' < v''$ such that $c(v') > c(v'')$. Define

$$v' = \sup\{v \in \mathcal{V} \,|\, \exists v'' > c\colon c(v) > c(v'')\}, \tag{18}$$

$$v'' = \inf\{v \in \mathcal{V} \,|\, v > v', c(v) < c(v')\}. \tag{19}$$

Suppose there was $v \in \mathcal{V}$ with $v' < v < v''$. From $v < v''$ it follows that $c(v) \geq c(v')$ and from $v > v'$ that $c(v) \leq c(v'')$. Together this yields a contradiction to the assumption that $c(v') > c(v'')$. Therefore v' and v'' must be successors. This entails that all of v' incentive constraints in the relaxed problem must slack: $c(v') > c(v'') \geq c(v''')$ for all $v''' \geq v''$. If the mechanism is optimal in the relaxed problem, it must be the case that $w(v', \lfloor c(v') \rfloor) \geq 0$. As $w(\cdot,\cdot)$ is increasing in the first component and decreasing in the second, it follows that $w(v'', c) \geq 0$ for all $c \geq \lfloor c(v) \rfloor$.
step 3: By step 2 we can assume that the optimal mechanism is ignorant. Since any ignorant mechanism is incentive compatible, the optimal ignorant mechanism is a solution to the relaxed and to the original problem. □

Positive correlation

Proof of Corollary 1.
Suppose $g(c)$ is concave. For all $c' < c''$ and for all rational $\alpha \in (0,1) \cap \mathbb{Q}$ we have

$$\frac{1}{g(\alpha \cdot c' + (1-\alpha) \cdot c'')} \leq \frac{1}{\alpha \cdot g(c') + (1-\alpha) \cdot g(c'')}$$

Since α was assumed to .be rational there are natural numbers $j < n \in \mathbb{N}$ such that $\alpha = \frac{j}{n}$. But then it follows that

$$\frac{n}{j \cdot g(c') + (n-j) \cdot g(c'')} \leq \frac{j}{n} \cdot \frac{1}{g(c')} + \frac{n-j}{n} \cdot \frac{1}{g(c'')},$$

since the arithmetic mean exceeds the harmonic mean. For $\alpha \in (0,1)$ not rational, it can be expressed as limit of rational αs, and the result follows as a limit. Note that as g is assumed to be concave it must be continuous on all interior points of its support. Since $\alpha \in (0,1)$ the point $\alpha \cdot c' + (1-\alpha)c''$ is always in the interior of the

support.
To see that the reverse might not be true consider the function $c \mapsto 2 - \sqrt{c}$ on $(0, 1)$ it is strictly convex but $c \mapsto \frac{1}{2-\sqrt{c}}$ is not concave on $(0, 1)$. □

The next result gives an equivalent characterization fo convexity for a decreasing function. It both applies when the function is defined on a interval and when the function is defined on a discrete set and convexity is discrete convexity. (See Definition 1)

Corollary 4
A real decreasing function g defined on some set $I \subset \mathbb{R}$ is strictly convex iff $\forall c' < c'' < c''' \in I$:

$$(g(c') - g(c'')) \cdot (c'' - c') > (g(c'') - g(c''')) \cdot (c''' - c'').$$

Proof. For $\alpha = \frac{c'''-c''}{c'''-c'}$ it holds that $c'' = \alpha \cdot c' + (1 - \alpha) \cdot c'''$. Plugging this into the definitions yields the result. □

This corollary shows that if $z(v, c) = v - c$ as in the main body, the generalized formulation of the regularity assumption (16) coincides with convexity of the liklihood ratios. Now, we are all set for the proof of Theorem 3. The proof proceeds in several steps. First I show that all the allocation schedules of the optimal mechanism in the relaxed problems have single plateaus.

Proof of Proposition 1.
Suppose the allocation schedule for v_j, $x(v_j)$, was not having a single plateau. This means that there exists $c' < c'' < c'''$ with $x(v_j, c') > 0, x(v_j, c'') = 0$ and $x(v_j, c''') > 0$. The remainder shows that under the regularity assumption there exists a modification of the mechanism, given by

$$\tilde{x}(v, c) = \begin{cases} x(v, c) + dx(v, c), & \text{if } v = v_j \text{ and } c \in \{c', c'', c'''\} \\ x(v, c), & \text{otherwise} \end{cases}, \tag{20}$$

with $dx(v_j, c') < 0, dx(v_j, c''') < 0$ and $dx(v_j, c'') > 0$. This proof uses a form of Farkas' Lemma (Farkas' Alternative) which I restate here:

Lemma (Farkas')
Suppose that $A \in \mathbb{R}^{k,l}, b \in \mathbb{R}^l$ then exactly one of these two alternatives is true:

1. $\exists p \in \mathbb{R}^l$ *with*

- $Ap \geq 0.$
- $p \cdot b < 0.$
- $p > 0 :\Leftrightarrow p \geq 0 \wedge p \neq 0.$

2. $\exists y \in \mathbb{R}^k$ *with*

- $y'A \leq b.$
- $y \geq 0.$

A proof for this version of Farkas' Lemma can be found in Gyula Farkas' original paper, Farkas (1902). The above formulation is taken from Border (2013), Corollary 11. Setting

$$A = \begin{pmatrix} f(v_0, c') & -f(v_0, c'') & f(v_0, c''') \\ \vdots & -\vdots & \vdots \\ f(v_{j-1}, c') & -f(v_{j-1}, c'') & f(v_{j-1}, c''') \\ -f(v_j, c') & f(v_j, c'') & -f(v_j, c''') \end{pmatrix} \in \mathbb{R}^{j+1\times 3}, \quad b = \begin{pmatrix} z(v', c') \cdot f(v', c') \\ -z(v', c'') \cdot f(v', c'') \\ z(v', c''') \cdot f(v', c''') \end{pmatrix}$$

establishes equivalence between alternative 1 and the existence of a feasible, strictly profitable deviation of the form (20), which is then given by

$$\begin{pmatrix} dx(v_j, c') \\ dx(v_j, c') \\ dx(v_j, c') \end{pmatrix} := \begin{pmatrix} -p_1 \\ p_2 \\ -p_3 \end{pmatrix}.$$

The last entry of $Ap \geq 0$ guaranties that all incentive constraints, $IC(v_j, v'') \geq 0$ for $v' \geq v_j$ are fulfilled in the modified mechanism. The other entries ensure that $IC(v', v_j) \geq 0$ for all $v' \leq v_j$. No other incentive constraints in the relaxed are affected by the modification. Therefore the modified mechanism is also incentive compatible. Furthermore, $0 > b \cdot p =$ ensures strict profitability.

To conclude the proof one needs only to rule out alternative 2 under the regularity assumption. Suppose alternative 2 was true, then $\exists y \in \mathbb{R}_+^{j+1}$ such that $y'A \leq b$. This implies, that

$$y_j + z(v_j, c') \geq \sum_{i=0}^{j-1} \frac{f(v_i, c')}{f(v_j, c')} y_i \tag{21}$$

$$y_j + z(v_j, c'') \leq \sum_{i=0}^{j-1} \frac{f(v_i, c'')}{f(v_j, c'')} y_i \tag{22}$$

$$y_j + z(v_j, c''') \geq \sum_{i=0}^{j-1} \frac{f(v_i, c''')}{f(v_j, c''')} y_i \tag{23}$$

(21) − (22) yields:

$$z(v_j, c') - z(v_j, c'') \geq \sum_{i=0}^{j-1} y_i \cdot \left(\frac{f(v_i, c'')}{f(v_j, c')} - \frac{f(v_i, c')}{f(v_j, c'')} \right) \tag{24}$$

(23) − (22) yields:

$$z(v_j, c''') - z(v_j, c'') \geq \sum_{i=0}^{j-1} y_i \cdot \left(\frac{f(v_i, c''')}{f(v_j, c''')} - \frac{f(v_i, c'')}{f(v_j, c'')} \right) \tag{25}$$

$$\Leftrightarrow z(v_j, c'') - z(v_j, c''') \leq \sum_{i=0}^{j-1} y_i \cdot \left(\frac{f(v_i, c'')}{f(v_j, c'')} - \frac{f(v_i, c''')}{f(v_j, c''')} \right) \tag{26}$$

Note that by affiliation of the distribution and since $z(v_j, c)$ is decreasing c, all terms of the sum in (24) and (26) are positive.
Therefore (24) / (26) yields:

$$\frac{z(v_j, c') - z(v_j, c'')}{z(v_j, c'') - z(v_j, c''')} \geq \frac{\sum_{i=0}^{j-1} y_i \cdot \left(\frac{f(v_i, c'')}{f(v_j, c')} - \frac{f(v_i, c')}{f(v_j, c'')} \right)}{\sum_{i=0}^{j-1} y_i \cdot \left(\frac{f(v_i, c'')}{f(v_j, c'')} - \frac{f(v_i, c''')}{f(v_j, c''')} \right)} \tag{27}$$

$$\Leftrightarrow \sum_{i=0}^{j-1} y_i \cdot (z(v_j, c') - z(v_j, c'')) \cdot \left(\frac{f(v_i, c'')}{f(v_j, c'')} - \frac{f(v_i, c''')}{f(v_j, c''')} \right) \tag{28}$$

$$\geq \sum_{i=0}^{j-1} y_i \cdot (z(v_j, c'') - z(v_j, c''') \cdot \left(\frac{f(v_i, c'')}{f(v_j, c')} - \frac{f(v_i, c')}{f(v_j, c'')} \right) \tag{29}$$

But by the regularity assumption and since $y_i \geq 0$ it holds $\forall i \in \{1, \ldots, j-1\}$:

$$y_i \cdot (z(v_j, c') - z(v_j, c'')) \cdot \left(\frac{f(v_i, c'')}{f(v_j, c'')} - \frac{f(v_i, c''')}{f(v_j, c''')} \right) \tag{30}$$

$$\leq y_i \cdot (z(v_j, c'') - z(v_j, c''') \cdot \left(\frac{f(v_i, c'')}{f(v_j, c')} - \frac{f(v_i, c')}{f(v_j, c'')} \right) \tag{31}$$

Finally suppose that $\forall i \in \{1, \ldots j-1\} : y_i = 0$. Then (26) would imply that, $z(v_j, c'') - z(v_j, c''') \leq 0$. Contradiction since $z(v_j, c)$ is assumed to be strictly decreasing!
So we can assume that the above equation (31) holds strictly for at most one $i \in \{1, \ldots j-1\}$ contradicting therefore (29). Thus, alternative 2 can be ruled out.

□

To proof monotonicity the following Lemma is used:

Lemma 3. Suppose plateau allocation rules $x'', x' : \mathcal{C} \to [0,1]$ are strictly ordered: $x'' \succ x'$. If there is $v \in \mathcal{V}$ with

$$f(v) \cdot (x'' - x') = 0$$

then it must hold that

$$f(v) \circ w(v,c) \cdot (x'' - x') < 0$$

Proof. Define

$$C'' = \{c \in \mathcal{C} \,|\, x''(c) - x'(c) > 0\}, \quad C' = \{c \in \mathcal{C} \,|\, x''(c) - x'(c) < 0.$$

As the allocation rules are in plateau form an ordered there exists $c^* \in \mathcal{C}$ with

$$\forall c'' \in C'' \, \forall c' \in \mathbf{C}' \colon c' \leq c^* \leq c''.$$

As $w(v, \cdot)$ is strictly increasing it follows that

$$\sum_{cinC''} f(v,c) \cdot w(v,c) \cdot (x''(c) - x'(c)) \leq w(v,c^*) \cdot \left(\sum_{c \in C''} f(v,c) \cdot (x''(c) - x'(c)) \right)$$
$$= w(v,c^*) \cdot \left(\sum_{c \in C'} f(v,c) \cdot (x''(c) - x'(c)) \right) \leq \sum_{c \in C'} f(v,c) \cdot w(v,c) \cdot (x''(c) - x'(c)).$$

As $x'' \succ x'$ at least one of the two inequalities must be strict. □

I also use the following result about totally positive kernels and adapt its formulation to my setting:

Lemma 4 (Schoenberg (1930); Karlin (1968))**.** If a real function function $K\colon \mathcal{V}\times\mathcal{C} \to \mathbb{R}$ is strictly totally positive of order 2 (STP_2) and there are allocation schedules: $x(\tilde{v}) \succ x(v)$ then for any $v' < v'' \in \mathcal{V}$ it holds

$$\sum_{c\in\mathcal{C}} K(v',c)\cdot(x(\tilde{v},c)-x(v,c)) \geq 0 \;\Rightarrow\; \sum_{c\in\mathcal{C}} K(v'',c)\cdot(x(\tilde{v},c)-x(v,c)) > 0,$$
$$\sum_{c\in\mathcal{C}} K(v'',c)\cdot(x(\tilde{v},c)-x(v,c)) \leq 0 \;\Rightarrow\; \sum_{c\in\mathcal{C}} K(v',c)\cdot(x(\tilde{v},c)-x(v,c)) < 0.$$

Or equivalently, $v' \mapsto \sum_{c\in\mathcal{C}} K(v',c)\cdot(x(\tilde{v},c)-x(v,c))$ crosses zero at most once and then from below.[10]

A proof can be found in Karlin (1968). Setting $K(v',c) = f(v',c)$ or $K(v',c' = f(v',c)\cdot z(v,c)$

Proof of Proposition 2.
Let x be a solution to the relaxed problem. With *Proposition* 1 we can assume that the mechanism is in plateau form. Let

$$k = \min\{0 < i < m \,|\, x(v_i) \succeq x(v_{i-1})\}$$

If $k = 1$, the mechanism is plateau monotone. Otherwise we have $\neg[x(v_k) \succeq x(v_{k-1})]$. This means

$$\neg[\underline{c}(v_k) \geq \underline{c}(v_{k-1}) \text{ and } \overline{c}(v_k) \geq \overline{c}(v_{k-1})].$$

One has to to distinguish three cases:

1. $\underline{c}(v_k) \leq \underline{c}(v_{k-1})$ and $\overline{c}(v_k) \geq \overline{c}(v_{k-1})$ with at least one inequality strict. This case can be ruled out immediately, since if this were the case, then

   $$f(v_{k-1})\cdot x(v_{k-1}) < f(v_{k-1})\cdot x(v_k),$$

   which would be a violation of $IC(v_{k-1}, v_k)$.
2. $\underline{c}(v_k) > \underline{c}(v_{k-1})$ and $\overline{c}(v_k) \leq \overline{c}(v_{k-1})$ with at least one inequality strict.
   For all $j < k$ it holds,

   $$f(v_j)\cdot x(v_j) \geq f(v_j)\cdot x(_{k-1}) > f(v_j)\cdot x(v_k).$$

[10] A touching of zero is counted as a crossing.

Since $j \leq k-1$, the first inequality follows from incentive compatibility. The second strict inequality follows from the initial assumption about x_k and x_{k-1}. As a consequence, all incentive constraints $IC(v_j, v_{,k})$ for $j < k$ slack. This could have only been optimal if for all $c \leq \lfloor \bar{c}_{k-1} \rfloor$ it was the case that $w(v_k, c) \leq 0$. But then as w is strictly increasing in v it follows that $w(v_{k-1}, v) < 0$ for all $c \leq \lfloor \bar{c}_{k-1} \rfloor$. But this is a contradiction since, for all $l \geq k$ it holds:

$$f(v_{k-1}) \cdot x(v_k - 1) > f(v_{k-1}) \cdot x(v_k) \geq f(v_{k-1}) \cdot x(v_l).$$

The first inequality is again by the initial assumption and the second follows since $x_l \succeq x_k$. This means that all incentive constraints $IC(v_{k-1}, v_l)$ for $l > k-1$ slack. So the choice of x_{k-1} could not have been optimal in the first place.

3. $\underline{c}(v_k) \leq \underline{c}(v_{k-1})$ and $\bar{c}(v_k) \geq \bar{c}(v_{k-1})$ with at least one inequality strict.
step 1: *Define the upper end of the support of $x(v_{k-1})$ as $\bar{c}$, i.e. $\bar{c} = \lceil \bar{c}(v_{k-1}) - 1 \rceil$. In this step I show that: $w(v_{k-1}, \bar{c}) > 0$.*
First suppose that $w(v_{k-1},) \leq 0$. If

$$f(v_{k-1}) \cdot x(v_{k-1}) = f(v_{k-1}) \cdot x(v_k)$$

then allocating the good after v_{k-1} according to allocation rule $x(v_k)$ instead of $x(v_{k-1})$ does not generate new profitable deviations. But since $x(v_k) \succ x(v_{k-1}$ it would strictly improve the principals expected utility (Lemma 3). We can therefore assume that $IC(v_{k-1}, v_k) > 0$. But since for all $l \geq k$

$$f(v_k) \cdot x(v_k) \geq f(v_k) \cdot x(v_l) \Rightarrow f(v_{k-1}) \cdot x(v_k) \geq f(v_{k-1}) \cdot x(v_l),$$

it follows that $IC(v_{k-1}, v_j) > 0$. Since there are only finitely many types there is $\varepsilon > 0$ such that $IC(v_{k-1}, v_l) \geq \varepsilon$,i.e. v_{k-1} incentive constraints slack uniformly. This result directly in a contradiction if $w(v_{k-1}, \bar{c}) < 0$ since then $x(v_{k-1}, \lceil \bar{c}(v_{k-1}) - 1 \rceil)$ could be lowered to increase the principals expected payoff. If $w(v_{k-1}, \bar{c}) = 0$ then $x(v_{k-1}, \bar{c})$ could still be lowered until either $IC(v_{k-1}, v_k)$ binds or if the incentives still slack with $x(v_{k-1}, \bar{c}) = 0$ one could proceed and lower $x(v, \bar{c}-)$. In the former case, one could now improve the principals payoff by allocating the good after v_{k-1} according to allocation rule $x(v_k)$ instead of the modified $x(v_{k-1})$.

step 2: Consider the following modification of the mechanism: after v_k allocate according to $x(v_{k-1})$ instead of $x(v_k)$. This change does not introduce new profitable deviations since,

$$f(v_{k-1}) \cdot x(v_{k-1}) \geq f(v_{k-1}) \cdot x(v_k) \Rightarrow f(v_k) \cdot x(v_{k-1}) \geq f(v_k) \cdot x(v_k).$$

Therefore, this modification cannot yield a higher expected value for the principal:

$$f(v_k) \circ w(v_k) \cdot x(v_{k-1}) \leq f(v_k) \circ w(v_k) \cdot x(v_k)$$

It follows by Lemma 4 and since $w(v_{k-1}, \overline{c}) > 0$ that

$$f(v_{k-1}) \circ z(v_{k-1}) \cdot x(v_{k-1}) < f(v_{k-1}) \circ z(v_{k-1}) \cdot x(v_k)$$

For any $\alpha \in (0,1)$ it would be strictly profitable to modify the mechanism in the following way: after v_{k-1} allocate according to $\alpha x(v_k) + (1-\alpha)x(v_{k-1})$ instead of $x(v_{k-1})$. It must therefore be the case that any such modification would violate the incentive constraints. This can only be the case if there is a binding incentive constraint for v_{k-1}, i.e. there exists $l \geq k$ such that

$$f(v_{k-1}) \cdot x(v_{k-1}) = f(v_{k-1}) \cdot x(v_l).$$

and $x(v_l) \succ x(v_k)$. Otherwise — if $x(v_l) = x(v_k)$ — there would be no incentive violation in the modified mechanism.
As $IC(v_{k-1}, v_k) \geq 0$, it follows that

$$f(v_{k-1}) \cdot x(v_l) \geq f(v_{k-1}) \cdot x(v_k).$$

But since $x(v_l) \succ x(v_k)$ it follows that $l > k$ and

$$f(v_k) \cdot x(v_l) > f(v_k) \cdot x(v_k),$$

a contradiction to $IC(v_k, v_l) \geq 0$.

□

Lemma 5. In the optimal solution to the relaxed problem all local incentive constraints are binding. That is for all $0 < k < m$ $IC(v_{k-1}, v_k) = 0$, i.e.

$$f(v_{k-1}) \cdot x(v_{k-1}) = f(v_{k-1}) \cdot x(v_k - 1).$$

Proof. Suppose there was some k with $IC(v_{k-1}, v_k) > 0$. It cannot be the case that $x_k = x_{k-1}$. Therefore $x_k \succ x_{k-1}$. By Lemma 4 it follows for all $j < k$ that

$$f(v_j) \cdot x(v_{k-1}) > f(v_j) \cdot x(v_k)$$

For all $j < k$ the relaxed incentive compatibility implies

$$f(v_j) \cdot x(v_j) \geq f(v_j) \cdot x(v_k),$$

it follows that for all $j < k$ the incentive constraints slack, $IC(v_j, v_k) > 0$. As $x(v_k) \succ x(v_j)$ and $f(v_{k-1}) \cdot x(v_{k-1}) > f(v_{k-1}) \cdot x(v_k)$, it must be the case that $x(v_k, c_0) < 1$. Choose $u = \max\{i : 0 < i < n, x(v_k, c_i) > 0\}$. Since all incentive constraints slack one can freely shift mass from $x(v_k, c_u) \downarrow$ to $x(v_k, c_0) \uparrow$ at a rate that keeps v_k's expected allocation probability constant,

$$dx(v_k, c_0) \cdot f(v_k, c_0) = -dx(v_k, c_u) \cdot f(v_k, c_u),$$

by Lemma 3 this modification would strictly improve the principal's expected utility. Contradiction.

□

Lemma 6. The plateau-monotonic solution to the relaxed problem is a solution to the original problem.

Proof. Let x be the plateau-monotonic solution to the relaxed problem. Suppose that x was not a solution to the original problem. This means that a high type has incentive to misrepresent himself as a lower type. Set

$$k = \min\{l \in \mathbb{N} \,|\, 0 < l < m, \exists j < l \text{ with } f(v_l) \cdot x(v_l) < f(v_l) \cdot x(v_j)\}.$$

It follows that $x(v_k) \succ x(v_j)$. By Lemma 5 it holds that,

$$f(v_{k-1}) \cdot x(v_k) = x(v_{k-1}).$$

By Lemma 4 an since $x(v_k) \succeq x(v_{k-1})$ (monotonicity) it follows that

$$f(v_{k-1}) \cdot x(v_k) = x(v_{k-1}).$$

One can deduce that $k-1 > j$. Therefore, since x is monotone: $x(v_{k-1}) \succeq x(v_j)$.

k was chosen minimally, therefore:

$$f(v_{k-1}) \cdot x(v_{k-1}) \geq f(v_{k-1}) \cdot x(v_j).$$

But then again by Lemma 4 it follows that,

$$.f(v_k) \cdot x(v_{k-1}) \geq f(v_k) \cdot x(v_j).$$

Contradicting, the initial assumption that $IC(V_K, v_j) < 0$. □

Taking all these results together proofs Theorem 3.

Continuous Costs

Lemma 7 (Lemma 1)**.** For any $k \in \mathbb{N} \cup \{\infty\}$ there exists a positively affiliated joint distribution that fulfills the regularity assumption (R) such that the optimal ignorant mechanism $x(c)$ has k holes.

Proof. Suppose that there are only two valuations $v_L < v_H$. The optimal ignorant mechanism reads:

$$x(c) = \begin{cases} 1, & \mathbb{E}[V \mid C = c] > c \\ 0, & \mathbb{E}[V \mid C = c] < c \end{cases}.$$

For $c \in (v_L, v_H)$:

$$\mathbb{E}[V \mid C = c] > c \Leftrightarrow \frac{f(v_L, c)}{f(v_H, c)} < \frac{v_H - c}{c - v_L}.$$

Setting $v_H = 3, v_L = 0$ and

$$f(v_H, c) = \begin{cases} K, & c \in (1,2) \\ 0, & \text{otherwise} \end{cases}, f(v_L, c) = \begin{cases} K \cdot \left(\frac{v_H - c}{c - v_L} + \varepsilon \cdot \sin\left(\frac{1}{c-1}\right)(c-1)^4\right), & c \in (1,2) \\ 0, & \text{otherwise} \end{cases}$$

$\frac{v_H - c}{c - v_L}$ is a strictly convex decreasing function. For $v_H = 3, v_L = 0$ $\frac{v_H - c}{c - v_L}$, $-\left(\frac{v_H - c}{c - v_L}\right)'$, and

$\left(\frac{v_H - c}{c - v_L}\right)''$ are all bounded from below by a strictly positive bound on the whole support $(1,2)$. Since $|\sin\left(\frac{1}{c-1}\right)(c-1)^4|$, $|\left(\sin\left(\frac{1}{c-1}\right)(c-1)^4\right)'|$ and $|\left(\sin\left(\frac{1}{c-1}\right)(c-1)^4\right)''|$ are bounded from above on $(1,2)$ there exists $\varepsilon > 0$ such that for all $K > 0$ $f(v_L, c) > 0$ and $\frac{f(v_L,c)}{f(v_H,c)}$ is decreasing convex. Since $\int_1^2 \sin\left(\frac{1}{c-1}\right)(c-1)^4 \, \mathrm{d}c$ exists and is bounded $K > 0$ can be chosen such that $f(v, c)$ is a density. $\square$

Proof of Theorem 4.

The proof essentially replicates all the arguments from the finite realization case. First we show that it is in plateau form, which corresponds to intervals in this new setting:

Let λ denote the Lebesgue measure. Let C', C'', C''' be measurable subset of $\mathcal{C}$ with positive measure that are ordered in the following sense: $\forall (c', c'', c''') \in C' \times C'' \times C'''$: $c' \leq c'' \leq c'''$. The sets are assumed to be distinct on a positive measure: $\lambda(C' \Delta C'') > 0, \lambda(C'' \Delta C''') > 0$ and $\lambda(C' \Delta C''') > 0$, where Δ denotes the symmetric differences of sets.

Suppose that x is an optimal mechanism and there exists $v' \in \mathcal{V}$ with:

$$\int_{C'} x(v, c) \, \mathrm{d}c > 0, \int_{C''} (1 - x(v, c)) \, \mathrm{d}c > 0, \int_{C'''} x(v, c) \, \mathrm{d}c > 0$$

It is without loss to assume that $x(v', c) > 0$ for all $c \in C' \cup C'''$ and $x(v', c) < 1$ for all $c \in C''$ and that $c' < c'' < c'''$. Otherwise use $C' - C'' - C''' - \{c \in \mathcal{C} \mid x(v', c) = 0\}$ instead of C' and corresponding subsets of C'' and C'''.

Since for any measurable $C \subset \mathcal{C}$ the function $b \mapsto \lambda(C \cap [-b, b])$, is continuous, there exists $\tilde{C}' \subset C'$, $\tilde{C}'' \subset C''$, and $\tilde{C}''' \subset C'''$ with $\lambda(\tilde{C}') = \lambda(\tilde{C}'') = \lambda(\tilde{C}''') > 0$. There exist measure preserving bijections $\phi \colon \tilde{C}' \to \tilde{C}''$ and $\psi \colon \tilde{C}' \to \tilde{C}''$ (see for example Alós-Ferrer (1999) , Lemma 3). For all $c' \in \tilde{C}'$ we can apply **?** to $c' < \phi(c') < \psi(c'')$. This gives for any c' that the set of feasible, incentive compatible and strictly profitable deviations $dx(v', c')$, $d(v', c'')$, $dx(v', c'')$ is not empty. One needs to argue why this correspondence exhibits a measurable selection. The usual measurable selection theorems are formulated for closed correspondences. But since the deviations here must be strictly profitable the definition of the correspondence includes a strict inequality. To get around this, consider the correspondence that maps c to the set of deviations which are feasible, incentive compatible and profitable or neutral for the principal. This does not involve strict inequalities therefore it is compact valued. It is also weakly measurable, denote it by Γ.

From above we know that the most profitable deviation in $\Gamma(c')$ is a strictly profitable deviation. Also, the function which evaluates the profitability of a deviation $(c', x_1, x_2, x_3) \mapsto f(v', c') \cdot x_1 + f(v', \phi(c')) \cdot x_2 + f(v', \psi(c')) \cdot x_3$ is measurable in c' and continuous in (x_1, x_2, x_3). Hence, it is a Caratheodory function.

With this we can apply a measurable maximum theorem (Aliprantis and Border (2006), Theorem 18.19) and get a measurable selection of the most profitable deviation, which is strictly profitable. Denote it by $\gamma(c') = (\gamma_1(c'), \gamma_2(c'), \gamma_3(c'))$. With this we can construct a new measurable mechanism $\tilde{x}$, which has $\tilde{x}(v', c') = x(v', c) + \gamma_1(c')$, $x(v', \phi(c')) = x(v', \phi(c')) + \gamma_2(c')$, and $x(v', \phi(c')) = x(v', \phi(c')) + \gamma_3(c')$ for all $c' \in \tilde{C}'$ and equals x elsewhere. This new mechanism has a strictly higher payoff for the principal on a set with positive measure. Therefore the original mechanism can not have been optimal.

It follows that the optimal mechanism to the relaxed problem can be characterized by two functions: $\underline{c}, \overline{c} : \mathcal{V} \to \mathcal{C}$ with $\forall v \in \mathcal{V}$: $\underline{c}(v) \leq \overline{c}(v)$. Define an analogous partial order on the set of allocation schedules in plateau form:

$$x(v'') \succeq x(v') \Leftrightarrow \underline{c}(v'') \geq \underline{c}(v') \text{ and } \overline{c}(v'') \geq \overline{c}(v').$$

The variation diminishing property also holds for kernels that are continuous in c:

Lemma (Schoenberg (1930); Karlin (1968))
If a real function function $K : \mathcal{V} \times \mathcal{C} \to \mathbb{R}$ is strictly totally positive of order 2 (STP_2) and there are allocation schedules: $x(\tilde{v}) \succ x(v)$ then for any $v' < v'' \in \mathcal{V}$ it holds

$$\int_{\mathcal{C}} K(v', c) \cdot (x(\tilde{v}, c) - x(v, c)) \, \mathrm{d}c \geq 0 \;\Rightarrow\; \int_{\mathcal{C}} K(v'', c) \cdot (x(\tilde{v}, c) - x(v, c)) \, \mathrm{d}c > 0,$$

$$\int_{\mathcal{C}} K(v'', c) \cdot (x(\tilde{v}, c) - x(v, c)) \, \mathrm{d}c \leq 0 \;\Rightarrow\; \int_{\mathcal{C}} K(v', c) \cdot (x(\tilde{v}, c) - x(v, c)) \, \mathrm{d}c < 0.$$

The cost-continuous analog of Lemma 3 directly follows, if in its proof the sums are replaced by integrals.

With these results in hand, all the other steps of the proof with finite realizations can be replicated analogously. □

Binary valuation

Proof of Proposition 3.

First, since $v_H \in (a,b)$, there always exists $d > a$ such that the ignorant mechanism $x(c) = \begin{cases} 1, & c \in [a,d) \\ 0, & c \in [d,b) \end{cases}$ dominates the mechanism that never allocates. Similarly, the ignorant mechanism $x(c) = \begin{cases} 1, & c \in [a,v_H) \\ 0, & c \in [v_H,b) \end{cases}$ dominates the mechanism that always allocates.

Suppose first, that the optimal mechanism x screens, i.e. $\bar{c}_L < \bar{c}_H$ and $\underline{c}_H > a$. I consider two feasible infinitesimal deviations that keep the incentives intact. Then it must be the case for each of them that they do not increase the principal's expected utility:

1. Change $\bar{c}_L$ and $\underline{c}_H$ simultaneously such that

$$f(v_L,\bar{c}_L) \cdot d\bar{c}_L = f(v_L,\underline{c}_H) \cdot d\underline{c}_H \tag{32}$$

   The principal's expected utility changes by

$$f(v_H,\underline{c}) \cdot (v_H - \underline{c}_H) \cdot d\underline{c}_H + f(v_L,\bar{c}_L) \cdot (v_L - \bar{c}_L) d\bar{c}_L.$$

   Plugging in (32) yields:

$$f(v_L,\bar{c}_L) \cdot \left(\frac{f(v_H,\underline{c}_H)}{f(v_L,\underline{c}_H)} \cdot (v_H - \underline{c}_H) + (v_L - \bar{c}_H) \right) \cdot d\bar{c}_L$$

   To rule out any strictly profitable modification of the mechanism it must be the case:

$$\frac{f(v_H,\underline{c}_H)}{f(v_L,\underline{c}_H)} \cdot (v_H - \underline{c}_H) + (v_L - \bar{c}_H) = 0 \Leftrightarrow \frac{v_H - \underline{c}_H}{\bar{c}_H - v_L} = \frac{f(v_L,\underline{c}_H)}{f(v_H,\underline{c}_H)} = l(\underline{c}_H). \tag{33}$$

2. Change $\bar{c}_L$ and $\bar{c}_H$ simultaneously such that

$$f(v_L,\bar{c}_L) \cdot d\bar{c}_L = f(v_L,\bar{c}_H) \cdot d\bar{c}_H. \tag{34}$$

   Again, the principal's expected utility cannot be strictly improved if,

$$\frac{v_H - \bar{c}_H}{\bar{c}_L - v_L} = \frac{f(v_L,\bar{c}_H)}{f(v_H,\bar{c}_H)}. \tag{35}$$

From Theorem 3 we know that $IC(v_L, v_H) = 0$. This gives a third equation

$$F(\overline{c}_L|v_L) = F(\overline{c}_h|v_L) - F(\underline{c}_H|v_L). \tag{36}$$

These three equations ((33),(35),(36)) uniquely pin down the optimal mechanism with screening, since the optimum must fulfill them and I will show that they have at most one solution.

For a a fix $\overline{c}_L$ consider the function $c \mapsto \frac{v_H - c}{\overline{c}_L - v_L}$ on $c \in [v_L, v_H]$. This linear function can be described as a straight line that hits zero at $c = v_H$. For higher $\overline{c}_L$ its slope gets less negative. This linear function intersects at most twice with the convex decreasing function $l(c)$. denote—if existent—the lower intersection point by $\underline{c}_H(\overline{c}_L)$ and the higher intersection point by $\overline{c}_H(\overline{c}_L)$. By the convexity of l it follows that for $\overline{c}_H(\overline{c}_L) - \underline{c}_H(\overline{c}_L)$ is decreasing in $\overline{c}_L$. Therefore

$$F(\overline{c}_L|v_L) - (F(\overline{c}_H(\overline{c}_L)|v_L) - F(\underline{c}_H(\overline{c}_L)|v_L))$$

is increasing in $\overline{c}_L$ It follows that there is at most one $\overline{c}_L$ solving the equations (33), (35), (36) simultaneously.

Suppose now that the optimal mechanism is ignorant. Then the optimal cutoff $\overline{c} \in [v_l, v_H]$ must be locally optimal:

$$0 = f(v_H, \overline{c}) \cdot (v_H - \overline{c}) \cdot d\overline{c} + f(v_L, \overline{c}) \cdot (v_L - \overline{c}) \cdot d\overline{c} \Leftrightarrow \frac{v_H - \overline{c}}{\overline{c} - v_L} = l(\overline{c})$$

If this equation is not unique then we know by Corollary 3 that the optimal mechanism is not ignorant. Then it must be uniquely characterized by the equations of the screening case. □

Proof of Lemma 2.

Under which circumstances can we rule out that the system (33), (35) and (36) has a solution. One of this instances is given, if the tangent at a of l crosses zero before v_H. Since then no linear function crossing zero at v_H can twice intersect with l. Formally this is the case if:

$$l(a) + l'(a) \cdot v_H < 0 \Leftrightarrow v_H >$$

□

B Related Problem

$$\begin{aligned} \tilde{u}_H(y) \cdot g(y) = f(v', -y) = f(v', c) \\ \tilde{u}_L(y) \cdot g(y) = f(v, -y) = f(v, c) \\ 0 \leq \left(\frac{\tilde{u}'_L(y)}{u'_H(y)} \right)' = (-l'(-y) \cdot \xi(x) \tilde{u}_L(y)) \cdot g(y) = f(v, -y) \\ \tilde{w}(y) \cdot g(y) = f(v', -y) \cdot (v' + y) \end{aligned}$$

It follows:

$$\begin{aligned} \frac{\tilde{u}_L(x)}{\tilde{u}_H(x)} = \frac{f(v, -x)}{f(v', -x)} =: l(x) \\ \frac{\tilde{w}(x)}{\tilde{u}_H(x)} = x. \end{aligned}$$

From that we get,

$$\tilde{u}'_L(y) = -l'(y) \cdot \tilde{u}_L(y) + l(y) \cdot \tilde{u}_L(y) \tag{37}$$

$$\tilde{w}'(y) = \tilde{u}_H(y) + y \cdot \tilde{u}'_H(y) \tag{38}$$

To have the principal with utility $\tilde{w}$ to be more risk avers than an agent with utility $\tilde{u}_H$ we need to have,

$$0 \geq \left(\frac{\tilde{w}'(y)}{\tilde{u}'_H(y)} \right)' = \left(\underbrace{\frac{\tilde{u}_H(y)}{\tilde{u}'_H(y)}}_{=:\xi(y)} + y \right)' = \xi'(y) + 1$$

To also have an agent with utility function $\tilde{u}_H$ be more risk averse than an agent with utility function $\tilde{u}_L$ it must be the case, that

$$0 \leq \left(\frac{\tilde{u}'_L(y)}{u'_H(y)} \right)' = (-l'(-y) \cdot \xi(y) + l(-y))' = l''(-y) \cdot \xi(y) - l'(-y) \cdot \xi'(y) - l'(y)$$

putting both conditions together yields:

$$l''(y) \cdot \xi(y) \geq l'(-x) \cdot (\xi'(y) + 1) \leq 0$$

Assuming that $\tilde{u}_H, u'_H \geq 0$ and $l' \leq 0$ we conclude,

$$l''(-y) \geq 0.$$

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