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Optimal Testing and Social Distancing of Individuals With Private Health Signals

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Abstract

We consider individuals who are privately informed about the probability of being infected by a potentially dangerous disease. Depending on its private health signal, an individual may assign a positive or negative value to getting tested for the disease. Individuals dislike social distancing. The government has scarce testing capacities and scarce resources for enforcing social-distance keeping. We solve the government's problem of setting up an optimal testing-and-social-distancing schedule, taking into account that individuals may lie about their private health signal. Rather than modelling the infection dynamics, we take a snapshot view, that is, we ask what should be done at a particular point in time to curb the current spread of the disease while taking the current well-being of the individuals into account as well. If testing capacities are sufficiently scarce, then it can be optimal to test only a randomly selected fraction of those who want to be tested, and require maximal social distancing precisely for those individuals who wanted a test and ended up not belonging to the tested fraction.

1 Introduction

Consider an individual who believes that she may have been infected by a virus that is potentially dangerous. Initial symptoms caused by the virus infection can be specific to the particular virus, but also may be rather unspecific. For example, a person who may have been infected by Covid 19 may experience some rather unspecific respiratory problems, but may also experience some more specific symptoms like a loss of her sense of smell. Another typical situation is that the individual has no symptoms at all, but knows that she has been in contact with a possibly infected person. Such an individual faces a dilemma. If she quickly undergoes a test for the disease, then she can expect an early treatment after a positive test result which can be quite beneficial to her health. On the other hand, if she decides to not undergo the test at this point in time, then she avoids the hassle of traveling

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to and spending time at a test facility, and she avoids the immediate personal quarantine that will result in case her test result is positive.

The government cannot simply advise all individuals with sufficiently severe symptoms or with contacts to possibly infected persons to undergo tests or to remain socially distant, because any individual is, to a large extent, *privately informed* about its health status, that is, an individual can downplay or exaggerate its symptoms when communicating with medical personal, and the individual can conceal or falsely claim recent contacts with possibly infected persons.

In this paper, we analyze how a government should optimally design a testing-and-social-distancing schedule in such a situation. We assume that the government is concerned about both the current well-being of its citizens and curbing the spread of the disease. Any individual that is infected and is not quarantined causes a negative externality on the population of individuals because it may infect others. On the other hand, given that test capacities may be scarce, spending a test unit on any particular individual has an opportunity cost, and in addition there may be a surveillance cost of making sure that the individual keeps any required social distancing or her quarantine.

In contrast to a large fraction of the literature, we do not model the dynamics of the infectious disease, but take a snapshot view. That is, fixing a particular point in time with a particular state of diffusion of the disease, we ask what should a government do in order to curb the current spread of the disease while taking the current well-being of its citizens into account as well. In this sense, our analysis is partial. Nevertheless, it provides very clear insights. These insights should be useful as a building block for a dynamic model that takes the incentive effects into account that we describe.

The most basic insight is that individuals who believe it to be sufficiently unlikely that they are affected, in expectation, lose out from being tested because the small chance of having the disease and thus getting a beneficial early treatment is outweighed by the hassle of undergoing a test. On the other hand, individuals who feel sufficiently ill, in expectation, gain from being tested because the expected benefit from an early treatment outweighs the hassle cost of undergoing the test. Formally, there exists a number p^* such that any individual who assigns a probability larger than p^* to the event of being infected expects to gains from undergoing the test, while any individual who believes that she has the disease with a probability smaller than p^* believes that she loses out from undergoing the test.

We call the individual's personal probability assessment of being infected her *type*. We assume that each individual is privately informed about her type.[1] Thus, when communicating with medical personal, she can claim to be of a smaller type than her true type (i.e., downplay her symptoms or conceal her contacts with possibly infected persons) and, alternatively,

[1]We are not assuming that individuals consciously do calculations with probabilities. Nevertheless, thinking in terms of degrees of likelihood is common sense and is relevant to many aspects of life beginning with the weather forecast. While psychological research has identified many biases in decision-making under uncertainty (e.g., Gigerenzer (2008), Kahneman (2011)), it is fair to say that the individuals in our model face quite simple decision problems, given the optimal testing-and-social-distancing schedules that we propose.

claim to be of a larger type (i.e., exaggerate her symptoms or claim to have been in contact with a possibly infected person). The population consists of individuals whose types are drawn independently from a given interval of probabilities. The lowest feasible type (which may be equal to 0 or strictly larger than 0) feels quite healthy, while the highest feasible type (which may be equal to 1 or strictly below 1) feels rather ill.

The government sets up a testing-and-social-distancing schedule. That is, it specifies, for each type of individual, a probability of being tested and a degree of social distancing, a real number between 0 and 1, where 1 should be interpreted as quarantine. We take a mechanism-design approach, that is, we allow the government to optimize over all schedules that are mathematically feasible. The optimal schedule turns out to be simple and intuitive.

If individuals could not misrepresent their types, computing the optimal schedule would be straightforward. According to this so-called *first-best solution*, there would exist a threshold type such that all types above the threshold type are tested and all types below the threshold type are not tested; the level of the threshold type depends on the parameters of the environment such as the government's cost of a test, its cost of enforcing any social-distancing requirements, and the weight it puts on an individual's current utility relative to the weight put on curbing the current spread of the disease. Depending on the parameters, it can also be optimal to require maximal social distancing (i.e., quarantine) for a range of types below the threshold type of the testing schedule. That is, it can be optimal to quarantine some individuals rightaway, without testing them first.

A simple, but very important, observation is that, with the exception of some special cases, the first-best solution is not implementable because individuals of certain types have an incentive to downplay or exaggerate their types. For concreteness, suppose that the parameters are such that the threshold type of the testing schedule is below p^*. Then there exists a range of types that are above the threshold type (and thus are supposed to get tested) and are below p^* (and thus lose out from being tested). Any individual with such a type has an incentive to misrepresent her type by claiming a type below the threshold type so that she avoids being tested. Similarly, if the threshold type of the testing schedule is above p^*, individuals of certain types have an incentive to claim a higher type than their true type. Further incentive problems occur if some, but not all, of the untested individuals are to be put in quarantine. These incentive problems prevent the first-best solution from being implementable.

The main contribution of the paper is to solve the government's problem of setting up an optimal testing and social-distancing schedule, while taking the individuals' incentive constraints into account. That is, extending methods from mechanism-design theory, we compute an optimal schedule, taking it as given that any individual is free to lie about her type. The resulting solution is called the second-best optimal schedule.

The second-best optimal schedule generally looks rather different from the first-best schedule—the fact that each individual is privately informed about their health status has a tremendous impact on the nature of the optimal schedule.

We distinguish four different cases concerning the possible nature of the second-best solution. First of all, it can be optimal for the government to test nobody (at the considered point in time) and quarantine everybody; testing nobody should be interpreted here as saving any available testing capacity for a different population of individuals or for use at a different point in time. This extreme solution is incentive compatible because no individual is even asked about their type—the same regulation is enforced on everyone. An example of a situation in which this is the solution is when the government's overwhelming concern is curbing the current spread of the disease, while it can enforce any social-distancing requirements at negligible cost. Let us turn to the other three cases.

Secondly, it can be optimal for the government to not regulate anything. We call this the null mechanism. Here, the government lets any individual decide freely whether or not they want to be tested. No social distancing is required for those who decide not to get tested. As a result, all individuals with types above p^* will be tested, and all individuals with types below p^* will not be tested. We summarize this case by saying that the marginally tested type, denoted $\check{p}$, is equal to p^*.

Third, it can be optimal to set up a testing-and-social-distancing schedule such that the marginally tested type, $\check{p}$, is smaller than p^*. In other words, there is now an interval of types, from $\check{p}$ to p^*, who are supposed to get tested although they lose out from being tested. The trick to make individuals with these types reveal themselves so they can be tested, is to require some social distancing for any individual who decides to remain untested. This lowers the payoff from not getting tested so that, if the level of social distancing is chosen right, individuals of type $\check{p}$ become indifferent between being tested and not being tested. Incentive compatibility is then satisfied. This form of the testing-and-social-distancing schedule explains why it can be optimal to require some social distancing even for those individuals who are quite sure to not be infected.

The remaining, fourth, possibility for the optimum is that the marginally tested type $\check{p}$ is higher than p^*. Now there is an interval of types, from p^* to $\check{p}$, who are supposed to not get tested although they gain from being tested. A solution like this can be optimal if test capacities are rather scarce, that is, if the opportunity cost of a test is very high. How are individuals with types in between p^* and $\check{p}$ prevented from snatching a test by claiming a higher type than their true type? The optimal solution is to introduce probabilistic testing. Only a randomly selected fraction of the individuals who claim to have types above $\check{p}$ are tested. For any individual who claims a type above $\check{p}$, if the randomization implies that this individual does not belong to the tested fraction, maximal social distancing is required. Each individual now faces a gamble if she claims a type above $\check{p}$: on the one hand, this allows her to grab a test with some probability, but, on the other hand, it sends her in quarantine for sure if (through the randomization) she ends up not getting tested. Higher types are more willing than lower types to take such a gamble because for them the test is more valuable, while the hassle of being put in quarantine for those who do not get a test is type-independent.[2]

All four solution cases have in common that at most a binary informa-

[2] As for a concrete application example, imagine this schedule to be used for the group of individuals who arrive at an airport on a given day if tests are too scarce to test everybody.

tion is required from each individual: the individual is never asked anything beyond the information whether it feels relatively healthy (type below $\check{p}$) or rather ill (type above $\check{p}$).

In the paper, we also provide some detailed results and examples concerning which of the four solution cases is second-best optimal, depending on the exogeneous parameters of the model, and what are the properties of the optimal testing threshold $\check{p}$.

Literature

There is a huge epidemological literature that analyzes the dynamics of infectious diseases.[3] A small subset of this literature investigates behavioral aspects, that is, models individual choices of possibly infected individuals (see the survey by Klein, Laxminarayan, Smith, and Gilligan (2007)). The behavioral aspect that is modelled most frequently is that each individual may choose their level of contacts with other, possibly infected, individuals. A reason for government intervention may arise because of the negative externalities of any contact (see, e.g., Kremer (1996), considering the HIV/AIDS epidemic, and Fenichel, Castillo-Chavez, Ceddia, Chowell, Parra, Hickling, Holloway, Horan, Morin, Perrings, et al. (2011)). By modeling the individual choice of undergoing a test, rather than the choice of the level of contacts, we differ sharply from this literature.

An important building block of our model is the assumption that individuals are heterogenous with respect to the individual probability assessments of being infected. Such heterogenous assessments have been modeled by Gong (2015) and, similarly, Paula, Shapira, and Todd (2014), who show empirically for the HIV/AIDS epidemic that these probability assessments are behaviorally relevant. They find that the individual belief has a strong impact on the individual level of contacts. Heterogenous individual health-status beliefs are also modelled in Brotherhood, Kircher, Santos, and Tertilt (2020), who introduce a state of "fever".

Chen (2006) considers a disease for which, in contrast to the disease we consider, a vaccination is available; each individual chooses whether to get vaccinated, which incurs a personal cost. Due to the incentive effects of a vaccination, its overall welfare effect can be ambiguous.

Caplin and Eliaz (2003), in a static model, combine individual choices of being tested and contact choices that are conditional on a certificate of the test result. Fear of a positive test result is introduced as a psychological bias, and the optimal certification policy of the government is determined.

Testing and social distancing as a design problem of the government has been considered in the literature. Berger, Herkenhoff, and Mongey (2020) recognize the importance of testing asymptomatic individuals and applying conditional quarantine. However, in their model the individuals cannot choose anything, but the government's policy is applied mechanically to all individuals depending on their health states. Acemoglu, Makhdoumi, Malekian, and Ozdaglar (2020) distinguish agents with high and low values of social contacts who choose their level of contacts in a network. Different types of

[3] See von Thadden (2020) for an adaptation to the epidemiological specifics of the current Covid-19 pandemic.

agents can react differently to policies, and optimal policies are generally type dependent.

Brotherhood, Kircher, Santos, and Tertilt (2020) assume that individuals choose their hours of work, domestic leisure, and leisure outside the house; this choice implies, in particular, a level of contacts with possibly infected persons.[4] The main assumption of the paper is that individuals are heterogeneous with respect to a payoff-relevant observable characteristic, age. The government can, and should, condition its testing and social-distancing policies on this characteristic.[5]

Another strand of the literature models the government's optimal testing and social distancing policy as a control problem across the evolution of the disease (Piguillem and Shi (2020), Kruse and Strack (2020)).

On a technical level, finding the second-best solution in our model is a mechanism-design problem in which an individual's required degree of social distancing acts as a quasi-money that steers every individual's incentives to reveal her type. From the individual's point of view, getting tested is like receiving a good that may have a positive or negative value for the individual. The technical challenge of solving the government's problem mainly arises from the fact that the probability of becoming quarantined is restricted between 0 and 1, thus restricting the amount of quasi-money that can be paid by any individual.[6]

2 Model

We consider an individual who is uncertain about whether or not she is infected by a given disease. At time 0, the individual possesses a private signal, her type $p \in [\underline{p}, \overline{p}]$, that describes the individual's personal probability assessment that she is infected, given her current symptoms and recent contacts with other people, where $0 \leq \underline{p} \leq \overline{p} \leq 1$. In most applications, it is reasonable to assume that no individual can be absolutely sure to have the disease (i.e., $\overline{p} < 1$), and also never be sure to be healthy (i.e., $\underline{p} > 0$), but our model also encompasses environments in which absolutely certain individuals may exist (i.e., $\underline{p} = 0$ and/or $\overline{p} = 1$). Although our model considers a single individual, it is instructive to imagine a population of individuals out of which the

[4] Similarly, Jones, Philippon, and Venkateswaran (2020) introduce a choice of shopping time and working time, inducing a level of social contacts.

[5] One way to capture age in our model would be by recognizing that the benefit from early treatment is larger for old people than for young people, giving rise to a smaller threshold probability p^* for old people, and a different second-best optimal schedule in an old population compared to a young population. In order to capture a mixed-age poulation in our model, we would need to define a welfare objective that puts positive weights on the welfare of different age groups and recognizes the possibility of cross-infections.

[6] Again on a technical level, our setup may be seen as a case of mechanism design with costly state-verification (see Ben-Porath, Dekel, and Lipman (2014)). In this literature, a designer commits to verifying states and implementing outcomes conditional on agents' reports when agents have private information related to these states. In our setting, the government is able to verify an individual's health state by testing for the infection, but she is not able to verify the agent's type. The verification of the health state carries a cost not only to the government, but is also costly to the individual.

considered individual is a representative member.[7]

We assume that, across the population, individuals' probability assessements are not systematically wrong, that is, for all p, among all individuals who think that they are infected with probability p, the expected fraction p is in fact infected.[8] Let F denote the c.d.f. for the distribution of types p in the population of individuals. We assume that F has a density f that is strictly positive on the open interval $(\underline{p}, \overline{p})$.

At a point in time after time 0, the individual may develop clearer symptoms, at which point the individual may regret not having learned about its infection at time 0, which would have opened the opportunity to begin an early treatment. Our analysis focusses on time 0.

At time 0, the individual may be tested for the illness. The test perfectly reveals the health state. Getting tested is a hassle for the individual; let $c^t > 0$ denote the individual cost of having the test done. On the other hand, if the test comes out positive, then an early treatment can be started; let $b > 0$ denote the individual's anticipated benefit of being treated early.[9]

Social distancing (starting at time 0) may be enforced on the individual. The degree of social distancing is a real number between 0 and 1. We use the term quarantine to indicate the maximal social-distancing level, 1. For the purposes of the model, any social-distancing level may be thought of as a probability of being put into quarantine.

Being in quarantine is unpleasant; the cost for the individual is denoted $c^q > 0$. We assume that being quarantined is more unpleasant than being tested:

$$c^q \quad > \quad c^t. \tag{1}$$

A quarantined individual cannot spread the disease. Putting an infected individual into quarantine yields a social benefit of $b^q > 0$. Putting a non-infected individual into quarantine has no benefit.

We assume that every positively tested individual will be quarantined, and no negatively tested individual will be quarantined. Thus, an individual of any type p expects to get quarantined with probability p after a test, implying that the individual's *expected value of being tested* is given by

$$v(p) \quad = \quad p(b - c^q) - c^t. \tag{2}$$

[7]What defines a population depends on the application. A population may be large, such as the group of citizens in a jurisdiction, or more confined, such as the group of individuals who arrive at an airport on a given day.

[8]A model variation in which agents are systematically too pessimistic or optimistic, or have s-shaped probability distortions as in prospect theory (see Kahneman (2011) for an introduction), may also be considered. A psychological bias towards overestimating the probability of being infected may be considered a plausible model variation in a population with a small rate of infections when the illness nevertheless draws a lot of public attention.

[9]An interesting model extension would distinguish individuals with low b (young, no preexisting illnesses) and high b (old, preexisting illnesses). The social welfare function would then depend on a convex combination of both groups' expected utilities, with the weights depending on the groups' relative sizes in the population. The interaction between the groups would arise from the possibility of cross-group infections.

We assume that the benefit of the early treatment is so high that the expected value of getting tested is positive for the highest type, that is,

$$\overline{p}(b - c^q) \quad > \quad c^t.$$

The less likely an individual deems itself infected, the smaller the expected value of getting tested; an individual who is sure to be healthy will have a negative value because $v(0) = -c^t < 0$. We assume that the lowest type is so close to 0 that its expected value is negative,

$$\underline{p}(b - c^q) \quad < \quad c^t.$$

The inequalities above mean that there is enough heterogeneity in the population so that a conflict exists between those who, in the absence of any other incentives, would like to get tested and those who do not want to get tested. Let

$$p^* \quad = \quad c^t/(b - c^q) \tag{3}$$

denote the indifferent type, that is, $v(p^*) = 0$. By the assumptions above, $\underline{p} < p^* < \overline{p}$.

The government's goal is to set up a rule for determining who gets tested and what level of social distancing will be required.

Even before introducing the government's welfare function, it is pretty clear that the optimal testing-and-social-distancing rule will, in general, be type-dependent. For all p, let $m(p)$ denote the probability that an individual of type p is tested, and let $q(p)$ denote the required degree of social distancing for such an individual, conditional on the event that the individual does not get tested. Naturally,

$$0 \leq m(p) \leq 1 \quad \text{for all } p. \tag{4}$$

Similarly, recalling that any degree of social distancing is interpreted as a probability of getting quarantined,

$$0 \leq q(p) \leq 1 \quad \text{for all } p. \tag{5}$$

The pair of functions (m, q) defines the government's rule or (direct) mechanism.

The main difficulty for the government is that, given any individual's personal cost of getting tested and cost of getting quarantined, individuals may lie about their personal health signal. Due to the revelation principle, there is no loss of generality in restricting attention to mechanisms (m, q) that are *incentive compatible*, that is, direct mechanisms in which no individual can gain from making a false claim about her type. In order to spell out this condition, let

$$U(\hat{p}, p) \quad = \quad v(p)m(\hat{p}) - c^q(1 - m(\hat{p}))q(\hat{p}) \tag{6}$$

denote the current expected utility of an individual of any type p who pretends to be of some type $\hat{p}$. The incentive-compability condition requires that

$$U(p, p) \geq U(\hat{p}, p) \quad \text{for all } \hat{p} \text{ and } p. \tag{7}$$

Our model does not attempt to capture the dynamic aspects of the spread of the disease. Rather, we are interested in the problem which mechanism is optimal at the given current point in time. Thus, we take it is given that the goverment is concerned about two things: first, the current expected utility of an (average) individual, which should be kept high; second, the probability that any given individual spreads the disease, which should be kept low.

An individual can spread the disease if and only if it is infected and is not quarantined. Let $b^q > 0$ denote the social benefit of quarantining an infected individual. The *expected quarantining benefit* that is achieved by the goverment's rule with respect to type p is equal to

$$b^q(m(p)p + (1 - m(p))pq(p)).$$

This is because, in case the individual is tested (probability $m(p)$), the benefit b^q occurs if and only if the individual is infected (probability p) because the test is perfect; if, however, the individual is not tested (probability $1-m(p)$), then being infected (probability p) and gettting quarantined (probability $q(p)$) are stochastically independent events, so that the benefit b^q only occurs with probability $pq(p)$.

Denoting the government's welfare weight on the individual's utility by $w^1 > 0$,[10] the government's welfare objective is given by

$$\begin{aligned} W &= E_{p\sim F}\left[w^1U(p,p) + b^q p\,(m(p) + (1 - m(p))q(p))\right. \\ &\qquad \left. - c^{gt}m(p) - c^{gq}(1 - m(p))q(p)\right], \end{aligned} \tag{8}$$

where $c^{gt} > 0$ denotes the government's production cost of a test, and $c^{gq} \geq 0$ denotes the government's surveillance cost of enforcing the quarantine of an untested person.[11]

Interpreting c^{gt} as an opportunity cost, we can view c^{gt} as a measure of the current scarcity of test medication units or test facilities, that is, the higher c^{gt} the higher is the cost of using a test unit for any particular individual. In this view, c^{gt} is the government's value of saving a test unit for a different point in time or of using it for an individual outside the considered population of individuals.

The cost c^{gq} can be interpreted as a measure of the availability of surveillance and enforcement infrastructure. Furthermore, c^{gq} can be seen as measuring the lack of social norms towards voluntary quarantine keeping in the considered population of individuals.

Since we do not set up a dynamic model, we cannot determine the optimal value of w^1.[12] Instead we identify structural properties of the goverment's current welfare-maximizing rule, taking w^1 as a given parameter.

[10] From a theoretical point of view, w^1 is a redundant parameter. The relative weight of the individual utility in the welfare objective could also be captured by scaling the parameters b, c^q, and c^t appropriately. We keep the parameter w^1 for convenience.

[11] For simplicity, we assume that there is no cost of enforcing the quarantine of a positively tested person. While such a cost could be easily incorporated into our model, it is reasonable to assume that a positively tested person will keep its quarantine voluntarily, or the medical facility into which the person is transferred will enforce the quarantine without incurring significant extra costs beyond the cost of caring for and treating the individual.

[12] In a dynamic model, because the fraction of infected individuals in the population becomes a variable, the definition of an agent's current expected utility must be altered such that she

The government's goal is to solve the following (second-best) welfare-maximization problem:

$$\max_{m(\cdot),q(\cdot)} W \quad \text{s.t. } (4), \quad (5), \quad (7),$$

where the expected utilities that occur in (7) are computed via (6).

First best

As a benchmark,[13] we now discuss the rule the government would use if it could directly observe the individual's type and thus could set up any rule without relying on the individuals' type reports. Such an omniscient and omnipotent government would solve the following first-best problem:

$$\max_{m(\cdot),q(\cdot)} W \quad \text{s.t. } (4), \quad (5).$$

To solve this problem, we replace the expression (6) for $U(p,p)$ in W and rearrange terms,

$$\begin{aligned} W &= E_{p\sim F}\big[w^1\left(v(p)m(p) - c^q(1-m(p))q(p)\right) + b^q p\left(m(p) + (1-m(p))q(p)\right) \\ &\qquad - c^{gt}m(p) - c^{gq}(1-m(p))q(p)\big] \\ &= E_{p\sim F}\left[C(p)m(p) + D(p)(1-m(p))q(p)\right], \end{aligned} \tag{9}$$

where we use the shortcuts

$$C(p) = w^1 v(p) + b^q p - c^{gt}, \tag{10}$$
$$D(p) = -w^1 c^q + b^q p - c^{gq}. \tag{11}$$

Note that both C and D are linear and strictly increasing functions of p, and C is steeper than D because $w^1 > 0$.

Using (9), the welfare-maximizing value of $m(p)$ and $q(p)$ can be determined separately for each p. Denoting by

$$\underline{p}^q = \frac{w^1 c^q + c^{gq}}{b^q}$$

the number such that $D(\underline{p}^q) = 0$, constraint (5) together with (9) shows that an optimal quarantining schedule is given by[14]

$$q^{**}(p) = \begin{cases} 1 & \text{if } p \geq \underline{p}^q, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Using this schedule, the welfare can be expressed as a function of the testing schedule m:

$$W = E_{p\sim F}\left[C(p)m(p) + \max\{0, D(p)\}(1-m(p))\right].$$

obtains a benefit from being healthy. The testing-and-social-distancing rule would be adapted dynamically. The welfare-maximizing value of w^1 would depend on the impact of the current spread of the disease on the discounted expected utility of forward-looking agents.

[13] This section can be skipped on first reading.

[14] It is possible that $\underline{p}^q < \underline{p}$, in which case everybody should be quarantined, or $\underline{p}^q \geq \overline{p}$, in which case nobody should be quarantined.

Thus, by constraint (4), an optimal testing schedule is given by

$$m^{**}(p) \quad = \quad \begin{cases} 1 & \text{if } C(p) - \max\{0, D(p)\} \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

In order to achieve a more explicit form for the optimal testing schedule, we distinguish two cases. Define $\underline{p}^t$ such that $C(\underline{p}^t) = 0$, that is,

$$\underline{p}^t \quad = \quad \frac{w^1 c^t + c^{gt}}{w^1(b - c^q) + b^q}.$$

Suppose first that $C(\underline{p}^q) \geq 0$, that is,

$$\underline{p}^t \quad \leq \quad \underline{p}^q. \tag{13}$$

In this case, $D(p) \leq 0$ for all $p \leq \underline{p}^t$, implying $m^{**}(p) = 0$. For all $p \in (\underline{p}^t, \underline{p}^q]$, we have $C(p) > 0$ and $D(p) \leq 0$, implying $m^{**}(p) = 1$. For all $p > \underline{p}^q$, it is also true that $m^{**}(p) = 1$ because

$$C(p) - D(p) > C(\underline{p}^q) - D(\underline{p}^q) = C(\underline{p}^q) \geq 0,$$

where the first inequality follows from the fact that C is steeper than D.

Summarizing the insights so far, we have seen that, if condition (13) holds, then

$$m^{**}(p) \quad = \quad \begin{cases} 1 & \text{if } p \geq \underline{p}^t, \\ 0 & \text{otherwise.} \end{cases}$$

Note that, under condition (13), the optimal quarantining required by (12) never comes to play: due to (13), all types that are optimally quarantined if they are not tested are tested anyway.

Secondly, consider the case in which (13) does not hold, that is, $\underline{p}^q < \underline{p}^t$.

In this case, $C(p) < 0$ for all $p \leq \underline{p}^q$, implying $m^{**}(p) = 0$. For all $p > \underline{p}^q$, we have $D(p) > 0$, implying

$$C(p) - \max\{0, D(p)\} = C(p) - D(p).$$

Let $\underline{p}^{qt}$ be such that $C(\underline{p}^{qt}) - D(\underline{p}^{qt}) = 0$, that is,

$$\underline{p}^{qt} \quad = \quad \frac{c^{gt} - c^{gq} - w^1(c^q - c^t)}{w^1(b - c^q)}.$$

Because $C(\underline{p}^q) - D(\underline{p}^q) = C(\underline{p}^q) < 0$ and C is steeper than D, we have

$$\underline{p}^{qt} \quad > \quad \underline{p}^q,$$

and thus

$$m^{**}(p) \quad = \quad \begin{cases} 1 & \text{if } p \geq \underline{p}^{qt}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, in the case where (13) does not hold, the range of types that are tested is smaller than the range of types that are quarantined. In other words, there is a range of intermediate types that are quarantined rightaway, without a test being applied; only high types are tested. We summarize.

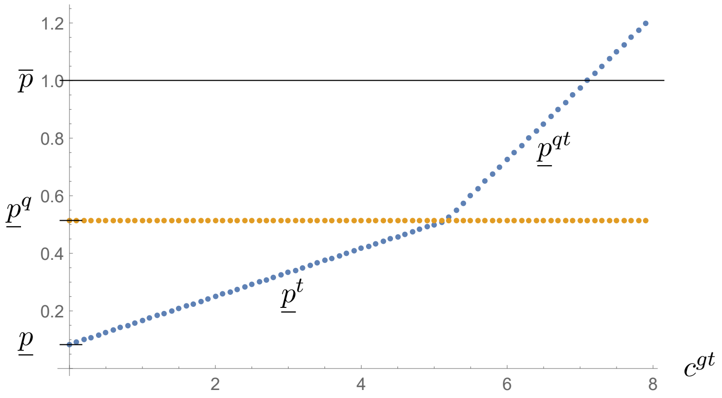

Figure 1: Example of a government's first-best optimal rule as a function of the cost of a test unit, c^{gt}. The blue curve given by the value of $\underline{p}^t$ or, resp., $\underline{p}^{qt}$, indicates the marginally tested type. The orange line indicates the marginally quarantined type. For this diagram, it is assumed that $w^1 = 1$, F is the uniform distribution on the interval $[\underline{p}, \overline{p}] = [0.1, 1]$, $b = 8$, $c^q = 4$, $c^t = 1$, $b^q = 8$, $c^{gq} = 0.1$. The computations were performed using Mathematica 12.

Proposition 1. *A first-best best testing-and-quarantining schedule is given as follows. If condition (13) holds, then an individual is tested if and only if her type is at least* $\underline{p}^t$*; no social distancing is required for untested individuals. If condition (13) fails, then an individual is tested if and only if her type is at least* $\underline{p}^{qt}$*; there is a nonempty interval of types—the types in* $[\underline{p}^q, \underline{p}^{qt})$*—such that an individual with such a type is quarantined rightaway, without being tested; no social distancing is required for individuals with types in* $[\underline{p}, \underline{p}^q)$*.*

We illustrate the first-best via a discussion of the comparative statics with respect to the cost of testing, c^{gt}.

If test capacities are abundant (i.e., $c^{gt} \approx 0$), then (13) is satisfied because

$$\underline{p}^t \approx \frac{w^1 c^t}{w^1(b - c^q) + b^q} < \frac{w^1 c^q}{w^1(b - c^q) + b^q} < \frac{w^1 c^q + c^{gq}}{b^q} = \underline{p}^q.$$

All individuals with types above this threshold are tested, and no social distancing is required for untested types. Note that $\underline{p}^t < \underline{p}$ if $\underline{p} > 0$ and b^q is sufficiently large. That is, unless some individuals are almost certainly healthy (i.e., $\underline{p} = 0$), all individuals will be tested if the public benefit b^q is sufficiently large.

As c^{gt} increases, the marginally tested type $\underline{p}^t$ increases, so that fewer and fewer individuals are tested. There exists $\overline{c^{gt}}$ such that $\underline{p}^t = \underline{p}^q$. If test capacities become even scarcer, (13) fails. The set of tested types shrinks ever more as c^{gt} increases further, but the quarantining threshold $\underline{p}^q$ remains constant.

At some point, test capacity is so scarce that $\underline{p}^{qt} \geq \overline{p}$. Then nobody is tested anymore, but quarantining of individuals with types in the interval $(\underline{p}^q, \overline{p}]$ persists; depending on the parameters, it can be optimal to quarantine nobody (if $\underline{p}^q > \overline{p}$) or everybody (if $\underline{p}^q \leq \underline{p}$).

Figure 1 provides an illustration. It shows an example of the marginally tested type and the marginally quarantined type as functions of the testing cost c^{gt}, keeping the other parameters fixed.

A simple, but very important, observation is that, with the exception of extreme cases or non-generic cases, the first-best solution is not incentive compatible. For concreteness, suppose that condition (13) is safisfied and that the parameters are such that individuals with high types should be tested, whereas individuals with sufficiently low types should not be not tested, that is,

$$\underline{p} < \underline{p}^t < \overline{p}.$$

According to (6), in the first-best solution, the payoff of an individual with type p who announces a type $\hat{p}$ is given by

$$U(\hat{p}, p) \quad = \quad \begin{cases} v(p) & \text{if } \hat{p} > \underline{p}^t, \\ 0 & \text{if } \hat{p} \leq \underline{p}^t. \end{cases}$$

If $\underline{p}^t < p^*$, then any individual with a type $p \in (\underline{p}^t, p^*)$ can improve her payoff by announcing a type $\hat{p} < \underline{p}^t$ because

$$U(p, p) = v(p) < 0 = U(\hat{p}, p).$$

By such a misrepresentation of the personal health signal, the individual avoids an unwanted test.

Similarly, if $\underline{p}^t > p^*$, then any individual with a type $p \in (p^*, \underline{p}^t)$ can improve her payoff by announcing a type $\hat{p} > \underline{p}^t$. By such a misrepresentation of the personal health signal, the individual snatches an undeserved test.

Only in the non-generic case where the parameters happen to be such that $\underline{p}^t = p^*$, it is true that the first-best solution is incentive compatible.

The described misrepresentation of the personal health signal is not possible if all types are tested (i.e., $\underline{p}^t \leq \underline{p}$) or no type is tested (i.e., $\underline{p}^t > \overline{p}$). We should mention that incentive compatibility can be violated even if the parameters are such that no type is tested. This violation happens if the first-best solution requires that some, but not all, individuals are quarantined; i.e., $\underline{p} < \underline{p}^q < \overline{p}$. Incentive compatibility fails for individuals with types above $\underline{p}^q$; they are supposed to be quarantined, but can avoid this by claiming that their type is below $\underline{p}^q$.

Our conclusion that the first best typically fails to be incentive compatible makes clear that, in order to achieve its welfare goal, the government must take the individual's incentive compatibiliy conditions into account. Our results will show that this approach leads to a rather different solution of the government's problem.

3 Results

In this section, we provide a general solution of the government's problem, we consider a number of important special cases, and we discuss the role of certain exogeneous parameters, that is, we consider comparative statics.

Proposition 2 is our fundamental result that describes how the government optimally combines testing and social distancing to resolve the tradeoff between maximizing the individual's current expected utility and curbing the spread of the disease, while taking the incentive-compatibility conditions into account.

Proposition 2. *It is either optimal for the government to test nobody and quarantine everybody, or the government's problem has a solution* (m^*, q^*) *that takes the following form. There exists* $\check{p} \in [\underline{p}, \overline{p}]$ *such that, for all types* p*, the optimal testing schedule is*

$$m^*(p) \quad = \quad \begin{cases} 0 & \text{if } p < \check{p}, \\ \check{m} & \text{if } p \geq \check{p}, \end{cases}$$

where

$$\check{m} \quad = \quad \frac{c^q}{c^q + \max\{v(\check{p}), 0\}}. \tag{14}$$

If $\check{p} \leq p^*$*, then the optimal quarantining probability is*

$$q^*(p) \quad = \quad \frac{-v(\check{p})}{c^q} \quad \text{for all } p < \check{p}.$$

If $\check{p} > p^*$*, then the optimal quarantining probability is*

$$q^*(p) \quad = \quad 0 \quad \text{for all } p < \check{p},$$

and

$$q^*(p) \quad = \quad 1 \quad \text{for all } p \geq \check{p}.$$

It should come as no surprise that it can be optimal for the government to test nobody and quarantine everybody (at the considered point in time); testing nobody should be interpreted here as saving any available testing capacity for a different population of individuals or for use at another point in time. This extreme solution applies, for example, if the government puts almost zero weight on the current expected utility of the individual (i.e., $w^1 \approx 0$), and has almost zero cost of surveillance of the quarantine (i.e., $c^{gq} \approx 0$), so that, essentially, its only concern is the spread of the disease (cf. Corollary 2). The other possible form of the solution described in Proposition 2 is more interesting.

In the described optimum, there exists a fixed testing probability $\check{m}$ and a marginal type, $\check{p}$, such that an individual may be tested only if she claims to be feeling sufficiently ill (i.e., be at or above the marginal type); all individuals with types above the marginal type are tested with the same probability. Accordingly, only two different quarantining probabilities are applied in any given solution: one probability for those who claim to be of the marginal type or higher, and one for those who claim to be below. Thus, the mechanism only makes use of a binary information: in essence, it asks whether the individual feels relatively healthy (type below $\check{p}$) or rather ill (type at or above $\check{p}$).

The quarantining probability $q^*(p)$ is designed such that, for each type p, an individual of type p is willing to reveal her type truthfully to the mechanism. In other words, no individual has an incentive to lie about their personal health signal.

The simplest solution possibility is that the marginal type $\check{p} = p^*$, that is, $v(\check{p}) = 0$. This solution features $\check{m} = 1$, that is, the types above p^* are tested for sure, whereas the types below p^* are not tested at all. This is exactly what the individual would like to happen in the absence of any additional incentives. Accordingly, no social distancing is required in the optimum (i.e., $q^*(p) = 0$).

Another possibility for the optimum is that $\check{p} < p^*$, that is, the marginal type has a negative test value, $v(\check{p}) < 0$. In this case, it is still optimal to use the testing probability $\check{m} = 1$, that is, all those who feel relatively ill are tested for sure. But now there is an interval of types, from $\check{p}$ to p^*, who are supposed to get tested, but would refuse so in the absence of additional incentives. In order to make individuals with these types reveal themselves so they can be tested, some social distancing (i.e., $q^*(p) > 0$) is enforced for individuals who remain untested. This lowers each individual's payoff from not getting tested. Thus, an individual of type p^*, who would otherwise be indifferent, now strictly prefers to be tested, and so do the types in between $\check{p}$ and p^*. For any individual with a type $p \geq \check{p}$, the value of $q^*(p)$ is irrelevant because individuals with such types p are tested for sure; in the proposition, $q^*(p)$ is specified only for the types $p < \check{p}$.

Note that, within the cases with $\check{p} < p^*$, an extreme possibility is that $\check{p} = \underline{p}$, that is, everybody is tested; the individual will be quarantined if and only if the test is positive.

The remaining possibility for the optimum is that $\check{p} > p^*$, that is, the marginal type has a positive test value, $v(\check{p}) > 0$. In this case, it is optimal to use a testing probability $\check{m} < 1$, that is, even those who feel rather ill are not tested for sure. Applying a testing probability $\check{m}$ below 1 may be interpreted as randomly selecting a fraction $\check{m}$ from the group of individuals who claim to be relatively ill and test only those. In the absence of additional incentives via quarantining, individuals with types in between p^* and $\check{p}$ would pretend to be rather sick in order to snatch a test. In order to prevent this, the testing is probabilistic, that is, only a randomly selected fraction $\check{m}$ of the individuals who claim to have types above $\check{p}$ are tested, and each individual of such a type that does not belong to the tested fraction is put in quarantine for sure. Each individual now faces a gamble if she volunteers to get tested: with some probability she is then *not* tested and is *still* put in quarantine, whereas she would not have been put in quarantine had she not volunteered. Individuals with higher types are more willing than those with lower types to take such a gamble because for them the test is more valuable, while the hassle of being put in quarantine for those who do not get a test is type-independent.

Note that one possibility is that $\check{p} = \overline{p}$. Such a solution is essentially equivalent to no-testing-no-quarantining (strictly speaking, the highest type, $\overline{p}$, is tested with a positive probability, but this exact type occurs with probability 0, and the government may as well not test this type).

What is missing from Proposition 2 is the characterization (in terms of the exogeneous parameters of the model) of the cases in which it is optimal to test nobody and quarantine everybody, and, concerning the other cases, a characterization of the optimal threshold $\check{p}$. This gap will be closed via Proposition 3. Some auxiliary functions must be specified. These functions are defined via the exogeneous parameters of the model. For all types p, define

$$B(p) = \left(-(b-c^q)w^1 p - c^{gt} - \frac{c^{gq}}{c^q}v(p)\right)F(p) + \left((b-c^q)w^1 + b^q + \frac{b^q}{c^q}v(p)\right)E_{p'\sim F}[p'|p'\leq p]F(p). \quad (15)$$

For all $\lambda \geq 0$ and all types p, define

$$A^\lambda(p) = B(p) + \mathbf{1}_{p>p^*}\cdot v(p)(\lambda - \lambda^*), \quad (16)$$

where

$$\lambda^* = -w^1 + \frac{b^q E_{p'\sim F}[p'] - c^{gq}}{c^q}.$$

For all $\lambda \geq 0$, define

$$\alpha^\lambda = -\min_p A^\lambda(p) + A^0(\overline{p}) - \lambda c^q. \quad (17)$$

The following lemma implies that the function $\lambda \mapsto \alpha^\lambda$ is strictly decreasing on $[0,\infty)$, and, by the intermediate-value theorem, intersects the horizontal axis. Thus, there exists a unique $\check{\lambda}$ such that $\alpha^{\check{\lambda}} = 0$. The proof is straightforward and is relegated to the Appendix.

Lemma 1. *The function $\lambda \mapsto \alpha^\lambda$ is Lipschitz continuous. Its derivative satisfies the inequalities $-v(\overline{p}) - c^q \leq d\alpha^\lambda/d\lambda \leq -c^q$. Moreover, $\alpha^0 \geq 0$, and $\alpha^\lambda \leq 0$ for all sufficiently large λ.*

The following result determines which of the solutions that are described in Proposition 2 applies. Proposition 3 not only characterizes the optimal solution, but also provides a computational path to solving the government's problem for any parameter constellation.

Proposition 3. *Let $\check{\lambda} \geq 0$ be such that $\alpha^{\check{\lambda}} = 0$. If $\check{\lambda} \leq \lambda^*$, then the government's problem has a solution such that nobody is tested and everybody is quarantined.*

Alternatively, suppose that $\check{\lambda} \geq \lambda^*$. Let $\check{p}$ be a minimizer of $A^{\check{\lambda}}$. Then $\check{p}$ yields a solution for the government's problem as described in Proposition 2.

The proof of Proposition 2 and Proposition 3 is relegated to Section 4. Examples will be provided below.

Concerning the form of the solution of the government's problem, the most fundamental distinction occurs between four different categories of solutions (cf. the explanations below Proposition 2): first, *no-testing-always-quarantining*, second, a mechanism that *sets up testing incentives* (i.e., $\check{p} < p^*$), third, *testing disincentives* (i.e., $\check{p} > p^*$), and fourth, the "null" mechanism that is characterized by the *absence of testing incentives or disincentives*, that is, the individual behavior remains unregulated (i.e., $\check{p} = p^*$).

Proposition 4 provides conditions on the model parameters that can be verified directly in order to check which of the four possibilities applies in any particular environment. In order to formulate these conditions, additional notation is needed. Let

$$\underline{B} = \min_{p\leq p^*} B(p),$$

$$\bar{l} \quad = \quad \frac{1}{c^q}(A^0(\overline{p}) - \underline{B}), \quad \text{and } \overline{\lambda} = \max\{0, \bar{l}\}.$$

For any $\lambda \geq 0$, define

$$\underline{A}^\lambda \quad = \quad \min_{p \geq p^*} A^\lambda(p).$$

Also note that, if $\lambda^* \geq 0$ and (17) is evaluated at $\lambda = \lambda^*$, then the definition simplifies to

$$\alpha^{\lambda^*} \quad = \quad -\min_p B(p) + B(\overline{p}) - (v(\overline{p}) + c^q)\lambda^*. \tag{18}$$

The conditions provided in Proposition 4 refer to the four numbers λ^*, α^{λ^*}, $\underline{B}$, and $\underline{A}^{\overline{\lambda}}$. Computing these numbers is relatively easy: λ^* is defined directly in terms of the exogeneous model parameters, and to compute each of the other three numbers, a single one-dimensional minimization problem must be solved. Thus, computing the four numbers is easier than fully computing the function $\lambda \mapsto \alpha^\lambda$, which would be required to apply Proposition 3 directly. On the other hand, Proposition 3 fully specifies a solution to the government's problem, whereas Proposition 4 mainly serves to characterize four different categories of solution possibilities; in particular, in Proposition 4 the exact value of $\check{p}$ is not specified in the case in which testing disincentives are optimal (i.e., $\check{p} > p^*$).

Note that, while this is not explicit from the statement, Proposition 4 allows to distinguish cases in which setting up testing incentives is optimal (i.e., $\check{p} < p^*$) from cases in which the null mechanism is optimal (i.e., $\check{p} = p^*$): if the restriction of B to the interval $[\underline{p}, p^*]$ is minimized at p^*, then the null mechanism is optimal; if it is minimized at a point below p^*, setting up testing incentives is optimal.

Proposition 4. *If $\lambda^* \geq 0$ and $\alpha^{\lambda^*} \leq 0$, then the government's problem has a solution such that nobody is tested and everybody is quarantined.*

Alternatively, suppose that $\lambda^* < 0$, or $\lambda^* \geq 0$ and $\alpha^{\lambda^*} > 0$. Then there exists a solution to the government's problem with marginal type $\check{p}$ as defined in Proposition 3 such that the following holds.

If $\underline{B} \leq \underline{A}^{\overline{\lambda}}$, then $\check{p} \in \arg\min_{p \leq p^*} B(p)$.

If $\underline{B} > \underline{A}^{\overline{\lambda}}$, then $\check{p} > p^*$.

Here is a sketch of the proof. The condition for the optimality of no-testing-always-quarantining means that the strictly decreasing function $\lambda \mapsto \alpha^\lambda$ has already dipped below the horizontal axis when it reaches the point $\lambda = \lambda^*$. Thus, it intersects the horizontal axis to the left of the point λ^*, which corresponds to the condition on $\check{\lambda}$ given in Proposition 3. To understand where the other conditions arise, suppose for simplicity that $\bar{l} \geq 0$, that is, $\overline{\lambda} = \bar{l}$. Then

$$0 \quad = \quad -\underline{B} + A^0(\overline{p}) - \overline{\lambda} c^q,$$

that is, $\overline{\lambda}$ is the number at which we would have $\alpha^{\overline{\lambda}} = 0$ if the minimizer $\check{p}$ of $A^{\overline{\lambda}}$ belonged to $[\underline{p}, p^*]$, that is, if $\underline{B} \leq \underline{A}^{\overline{\lambda}}$. In this case, by Proposition 3, the government's problem has a solution with $\check{\lambda} = \overline{\lambda}$ and thus $\check{p} \leq p^*$. Similar

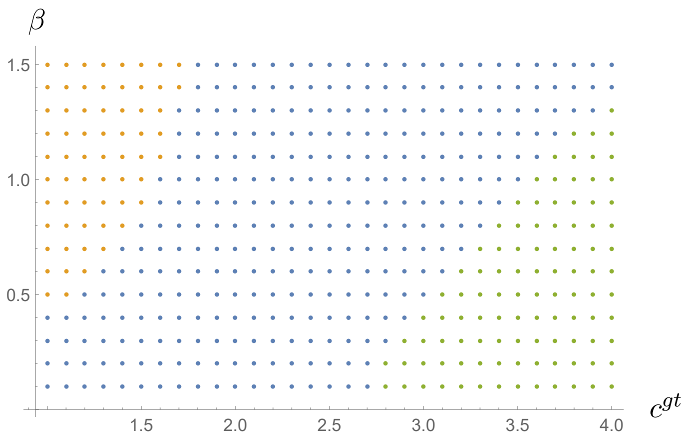

Figure 2: Examples of parameter constellations such that setting up testing incentives is optimal (orange dots), leaving behavior unregulated is optimal (blue dots), and setting up testing disincentives is optimal (green dots). For this diagram, it is assumed that $w^1 = 1$, $\overline{p} = 1$, $\underline{p} = 0.1$, $F(p) = ((p - \underline{p})/(\overline{p} - \underline{p}))^\beta$, $b = 8$, $c^q = 4$, $c^t = 1$, $b^q = 8$, $c^{gq} = 0.1$. Consequently, $p^* = 0.25$. Each orange dot represents a pair (c^{gt}, β) such that $\check{p} < p^*$; each blue dot represents a pair (c^{gt}, β) such that $\check{p} = p^*$; each green dot represents a pair (c^{gt}, β) such that $\check{p} > p^*$. The computations were performed using Mathematica 12.

arguments imply that, if the opposite inequality $\underline{B} > \underline{A}^{\overline{\lambda}}$ holds, then $\check{\lambda} > \overline{\lambda}$ and the minimizer $\check{p}$ of $A^{\check{\lambda}}$ cannot belong to $[\underline{p}, p^*]$, that is, $\check{p} > p^*$. The details of the proof of Proposition 4 are relegated to the Appendix.

Figure 2 illustrates Proposition 4. Fixing all other parameters, we consider testing costs that range from 1 to 4, and type distributions that are indexed by a parameter β that ranges from 0.1 to 1.5. Specifically, we assume that the lowest type $\underline{p} = 0.1$, the highest type $\overline{p} = 1$, and consider the distribution

$$F(p) = \left(\frac{p - \underline{p}}{\overline{p} - \underline{p}}\right)^\beta.$$

This specification is meant to capture that, the larger β, the more ill is the population, that is, the more probability mass is shifted to higher types.

Each blue dot in Figure 2 represents a pair (c^{gt}, β) such that no regulation is optimal (i.e., $\check{p} = p^*$); each orange dot represents a pair (c^{gt}, β) such that it is optimal to test more than in the absence of regulation, that is, setting up testing incentives is optimal (i.e., $\check{p} < p^*$); each blue dot represents a pair (c^{gt}, β) such that it is optimal to test less than in the absence of regulation, that is, setting up testing disincentives is optimal (i.e., $\check{p} > p^*$). Naturally, the more ill is the population (i.e., the higher is β), the larger is the range of testing-cost levels c^{gt} such that setting up testing incentives is optimal, and the smaller is the range of testing-cost parameters such that setting up testing disincentives is optimal.

In order to provide more insight into the nature of the solution of the government's problem, we now present a number of special cases.

Special cases and comparative statics

First of all, we have the following benchmark:

Corollary 1. *If, for given values of the other parameters, the public cost and benefit parameters c^{gt}, c^{gq}, and b^q, are sufficiently close to 0, then $\check{p} = p^*$, that is, it is optimal to leave the individual behavior unregulated.*

The benchmark considered here is a situation in which the spread of the disease is considered almost irrelevant, so that the individual's current expected utility is essentially the only important design aspect. The government's motivation for any regulation is the externality of the individual's behavior concerning the spread of the disease. If this motivation ceases to be relevant, the optimum is the null contract.

To obtain a heuristic argument towards the proof, consider the hypothetical limit case $c^{gt} = c^{gq} = b^q = 0$. By definition of λ^*, we have $\lambda^* = -w^1 < 0$. Hence, by Proposition 3 there exists a solution with a marginal type $\check{p}$.

Also note that (15) implies that

$$B(p) \quad = \quad (b - c^q)w^1 E_{p' \sim F}[\mathbf{1}_{p' \leq p} \cdot (-p + p')],$$

which has the slope

$$B'(p) \quad = \quad -(b - c^q)w^1 F(p) \ < \ 0,$$

implying that B is strictly decreasing.

On the other hand, (16) together with $\lambda^* = -w^1$ implies that, for all $p > p^*$ and all $\lambda \geq 0$,

$$A^\lambda(p) \quad = \quad B(p) + v(p)(\lambda + w^1),$$

which has the slope

$$(A^\lambda)'(p) \quad = \quad B'(p) + (b - c^q)(\lambda + w^1) \ \geq \ (b - c^q)w^1(1 - F(p)) \ > \ 0,$$

implying that A^λ is strictly increasing for all $p \geq p^*$.

We conclude that A^λ, and in particular $A^{\check{\lambda}}$, is minimized at p^*, showing that $\check{p} = p^*$ by Proposition 3. The arguments above are easily extended to the case in which the parameters c^{gt}, c^{gq}, and b^q are not exactly equal to 0, but are sufficiently close to 0; the details are omitted.

Note that, according to the first-best solution described in Proposition 1, if the public cost and benefit parameters c^{gt}, c^{gq}, and b^q are close to 0, but not exactly equal to 0, then, generically, $\underline{p}^t \neq p^*$. In these cases, the first-best solution differs from the null mechanism while the null mechanism is still second-best optimal, as shown in Corollary 1. In other words, while the omniscient and omnipotent government's optimal rule reacts with stipulating some regulation to even the slighest concern about public costs and benefits, a government that must take the incentive constraints into account optimally sticks to the null mechanism if public costs and benefits are small.

Next we consider the opposite extreme case where the government does not care much about the individual's current expected utility, that is, w^1 is close to 0 and, in addition, the quarantine can be enforced at almost zero cost.

Corollary 2. *If, for given values of the other parameters, w^1 and c^{gq} are sufficiently close to 0, then testing nobody and quarantining everybody is optimal.*

Again, we obtain a heuristic argument towards the proof by considering the hypothetical limit case where $w^1 = 0$ and $c^{gq} = 0$. By definition,

$$\lambda^* \quad = \quad \frac{b^q}{c^q} E[p'] > 0$$

and

$$B(p) \quad = \quad \left(-c^{gt}\right) F(p) + \left(b^q + \frac{b^q}{c^q} v(p)\right) E[p'|p' \leq p] F(p). \tag{19}$$

In particular,

$$B(\overline{p}) = -c^{gt} + \left(b^q + \frac{b^q}{c^q} v(\overline{p})\right) E[p'] = -c^{gt} + (c^q + v(\overline{p}))\lambda^*.$$

Thus, using (18),

$$\alpha^{\lambda^*} = -\min_p B(p) - c^{gt} = -\min_p (B(p) + c^{gt}). \tag{20}$$

Note also that $v(p) \geq -c^t \geq -c^q$, implying $v(p)/c^q + 1 \geq 0$. Hence,

$$b^q + \frac{b^q}{c^q} v(p) \quad \geq \quad 0. \tag{21}$$

This together with (19) implies that $B(p) + c^{gt} \geq 0$. Thus, (20) implies that $\alpha^{\lambda^*} \leq 0$. Thus, testing nobody and quarantining everybody is optimal by Proposition 4. Combining the above arguments with appropriate continuity arguments leads to the proof of Corollary 2; the details are relegated to the Appendix.

Next we consider comparative statics with respect to the government's cost of testing the individual, c^{gt}. We distinguish the case in which the current individual utility is relatively important (i.e., w^1 above a threshold) and the opposite case where curbing the spread of the disease is considered relatively more important (i.e., w^1 below the threshold). In the first case, the marginally tested type $\check{p}$ is increasing in the cost c^{gt} until a cost level $\overline{c}^{gt}$ is reached at which no-testing-*no*-quarantining is optimal; this remains the solution at all higher cost levels.

In the remaining oppositive case where the current individual utility is less important (i.e., w^1 below the threshold), the marginally tested type is increasing in the cost c^{gt} until a cost level $\overline{c}^{gt}$ is reached at which no-testing-*always*-quarantining is optimal; this remains the solution at all higher cost levels.

Corollary 3. *(Comparative statics with respect to c^{gt}, keeping the other parameters fixed.)*

Consider the case $w^1 \geq \frac{b^q E_{p' \sim F}[p'] - c^{gq}}{c^q}$. Then there exists a marginally tested type $\check{p}$. Moreover, choosing either the minimal or the maximal $\check{p}$ in case of multiplicity, $\check{p}$ is weakly increasing in c^{gt}, and $\check{p} \to \overline{p}$ as $c^{gt} \to \infty$.

Consider the case $w^1 < \frac{b^q E_{p' \sim F}[p'] - c^{gq}}{c^q}$. Then there exists a threshold $\overline{c}^{gt}$ such that, for all $c^{gt} < \overline{c}^{gt}$, the marginally tested type $\check{p}$ (choose the minimal or maximal $\check{p}$ in case of multiplicity) is weakly increasing in c^{gt}; no-testing-always-quarantining is optimal for all $c^{gt} \geq \overline{c}^{gt}$.

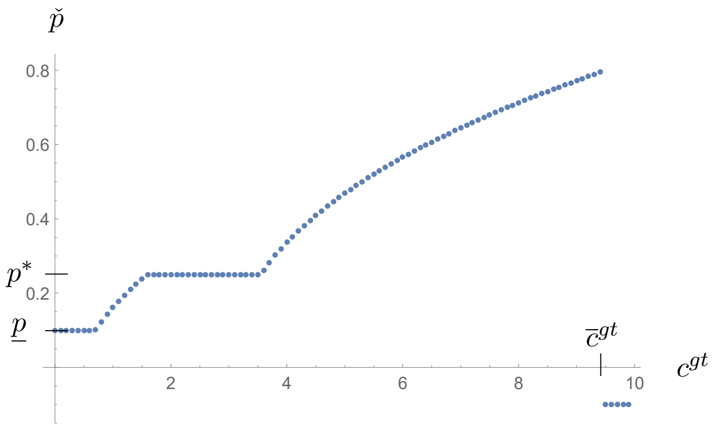

Figure 3: Example of a government's second-best optimal marginally tested type, $\check{p}$, as a function of the government's cost of a test unit, c^{gt}. The cases in which $c^{gt} > \overline{c}^{gt}$, where no-testing-always-quarantining is optimal, are represented via a negative value of $\check{p}$. For this diagram, it is assumed that $w^1 = 1$, F is the uniform distribution on the interval $[\underline{p}, \overline{p}] = [0.1, 1]$, $b = 8$, $c^q = 4$, $c^t = 1$, $b^q = 8$, $c^{gq} = 0.1$. The computations were performed using Mathematica 12.

The proof of Corollary 3 is relegated to the Appendix. Figure 3 illustrates the case $w^1 < (b^q E_{p' \sim F}[p'] - c^{gq})/c^q$ in Corollary 3. In this example, $p^* = 0.25$. If the testing cost c^{gt} is small, it is optimal to test everybody, that is $\check{p} = \underline{p}$. Then there is a range of testing costs in which it is optimal to set up testing incentives, but not everybody is tested, that is, $\underline{p} < \check{p} < p^*$. This is followed by a range of testing costs such that the null mechanism is optimal. If the testing cost is even higher, it becomes optimal to provide ever stronger testing disincentives. At the point $\overline{c}^{gt} \approx 9.4$, testing capacity is is so scarce that no-testing-always-quarantine is optimal if the cost is even higher.

Note that the second-best solution is strikingly different from of the first-best solution at the same parameters that was illustrated in Figure 1. At low testing costs (c^{gt} below ≈ 5.1), the first best relies on testing some types without extra quarantining of untested types; such a solution violates incentive compatibility and thus is not feasible; in the second best, type-revelation incentives are provided via social-distancing of untested types or randomized testing, and this also changes the optimal testing schedule relative to the first best.

At very high testing costs (c^{gt} above ≈ 7.1), since testing is now very expensive, tests are not applied at all in the first-best solution—the first best then relies entirely on quarantining of individuals with sufficiently high types, which again violates incentive compatibility and thus is not feasible; in the second best, some tests are still applied, in an incentive-compatible way, until the testing cost is so high that the government gives up on testing and resorts to quarantining everybody.

Figure 4 illustrates the case $w^1 > (b^q E_{p' \sim F}[p'] - c^{gq})/c^q$ in Corollary 3. This case is reached because now we assume that $\underline{p} = 0.0001$—some types are almost sure to not be infected. These types remain untested even if the testing cost c^{gt} is very close to zero, implying that $\check{p}$ is strictly increasing essentially from the start. As in the previous example, there is a range of testing costs such that the null mechanism is optimal, and if testing cost are even higher, it becomes optimal to provide ever stronger testing disincentives.

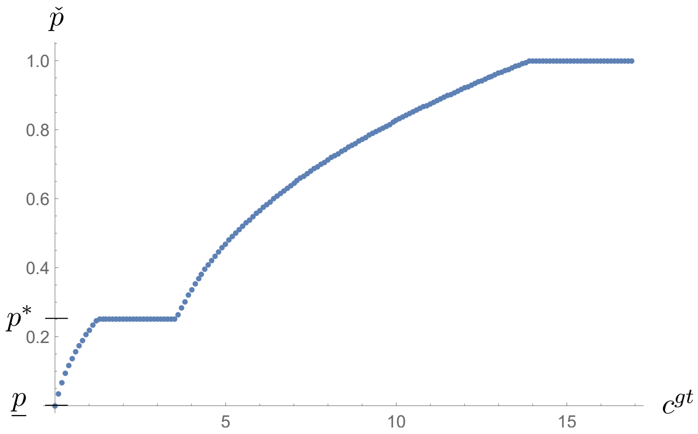

Figure 4: Example of the government's second-best optimal marginally tested type, $\check{p}$, as a function of the government's cost of a test unit, c^{gt}. If the testing cost is sufficiently high, then $\check{p} = \overline{p}$, that is, no-testing-no-quarantining is optimal. For this diagram, it is assumed that $w^1 = 1$, F is the uniform distribution on the interval $[\underline{p}, \overline{p}] = [0.0001, 1]$, $b = 8$, $c^q = 4$, $c^t = 1$, $b^q = 8$, $c^{gq} = 0.1$. The computations were performed using Mathematica 12.

In contrast to the previous example, however, no-testing-no-quarantining is optimal if testing capacities are sufficiently scarce.

4 Proof of Proposition 2 and Proposition 3

As a first step, we rewrite the government's problem as a convex maximization problem over testing schedules $m(\cdot)$. As a second step, we show that the solution $m^*(\cdot)$ described in Proposition 2 and Proposition 3 satisfies the (Lagrangian first-order) sufficient conditions for solving the problem as rewritten in the first step. As a third step, we show that the optimal quarantining schedule $q^*(\cdot)$ described in Proposition 2 and Proposition 3 is a consequence of the optimal testing schedule $m^*(\cdot)$.

Step 1: rewriting the government's problem

Using standard techniques from mechanism design (see, e.g., Börgers (2015), Chapter 3), we have the following result.

Lemma 2. *A rule (m, q) is incentive compatible if and only if*

$$U(p,p) = (b - c^q) \int_{\underline{p}}^{p} m(p')dp' + U(\underline{p}, \underline{p}) \quad \textit{for all } p, \tag{22}$$

and

$$m(p) \leq m(p') \quad \textit{for all } p < p' \tag{23}$$

The first condition (22) is an envelope or integrability condition that yields a "revenue-equivalence" result: the testing schedule $m(\cdot)$ determines the individual's expected utility as a function of the type, up to the constant $U(\underline{p}, \underline{p})$. Thus, by equating the integrability condition with the individual utility expression (6), we get, for each type p, a condition for the quarantine probability $q(p)$ such that the integrability condition is satisfied:

$$v(p)m(p) - c^q(1 - m(p))q(p) \quad = \quad (b - c^q) \int_{\underline{p}}^{p} m(p')dp' + U(\underline{p}, \underline{p}).$$

Rearranging, we can express the joint probability of remaining untested and being quarantined:

$$(1-m(p))q(p) = \underbrace{\frac{1}{c^q}\left(v(p)m(p)-(b-c^q)\int_{\underline{p}}^{p} m(p')dp' - U(\underline{p},\underline{p})\right)}_{\equiv\ \psi^{U(\underline{p},\underline{p}),m}(p)}. \tag{24}$$

Now consider a testing schedule $m(\cdot)$ that satisfies the monotonicity condition (23) and the probability condition (4).

We would like to characterize the set of $m(\cdot)$ such that (m,q) is incentive compatible for some quarantining schedule $q(\cdot)$. Given some $m(\cdot)$, the question is then whether or not there exists $q(\cdot)$ that satisfies the probability condition (5) together with the equation (24), where (setting $\hat{p}=p=\underline{p}$ in (6))

$$U(\underline{p},\underline{p}) = v(\underline{p})m(\underline{p}) - c^q(1-m(\underline{p}))q(\underline{p}).$$

Note that this last equation already follows from (24) as applied with $p=\underline{p}$. Thus, we can treat the number $U(\underline{p},\underline{p})$ as a variable that can take any value that satisfies (24).

Multiplying (5) with $1-m(q)$, we obtain the essentially equivalent condition

$$0 \le (1-m(p))q(p) \le 1-m(p) \quad \text{for all } p, \tag{25}$$

(Note that this condition, in contrast to (5), leaves $q(p)$ undetermined if $m(p)=1$; this change, however, is inessential because the quarantining probability $q(p)$ is irrelevant for an individual who is tested for sure.)

Plugging (24) into (25), we obtain a condition on $m(\cdot)$ that is necessary and sufficient for the existence of a $q(\cdot)$ such that (m,q) is incentive compatible:

$$0 \le \psi^{U(\underline{p},\underline{p}),m}(p) \le 1-m(p) \quad \text{for all } p, \tag{26}$$

The next step is to express the welfare W as a function of the testing schedule $m(\cdot)$ and the number $U(\underline{p},\underline{p})$. This is achieved by plugging into (8) the expressions obtained in (22) and (24), giving

$$\begin{aligned} W &= E_{p\sim F}\Bigg[w^1\left((b-c^q)\int_{\underline{p}}^{p} m(p')dp' + U(\underline{p},\underline{p})\right) \\ &+ b^q p\left(m(p)+\psi^{U(\underline{p},\underline{p}),m}(p)\right) - c^{gt}m(p) - c^{gq}\psi^{U(\underline{p},\underline{p}),m}(p)\Bigg]. \end{aligned} \tag{27}$$

The government's goal is to solve the following problem:

$$\max_{U(\underline{p},\underline{p}),\ m(\cdot)} W \quad \text{s.t.} \quad (4),\ (23),\ (26)$$

The left condition in (26) is satisfied for all p if and only if it is satisfied for the p that minimizes the function $\psi^{U(\underline{p},\underline{p}),m}(p)$. The minimizer is $p=p^*$; to see this, consider any $p \neq p^*$ and note that[15]

$$\begin{aligned}
&\left(\psi^{U(\underline{p},\underline{p}),m}(p) - \psi^{U(\underline{p},\underline{p}),m}(p^*)\right) c^q \\
&= v(p)m(p) - (b - c^q) \int_{p^*}^{p} m(p')dp' \\
&= (b - c^q)(p - p^*)m(p) - (b - c^q) \int_{p^*}^{p} m(p')dp'.
\end{aligned}$$

Due to (23), the last integral is bounded above by $(p - p^*)m(p)$, showing that $\psi(p) \geq \psi(p^*)$.

Thus we can replace the left condition in (26) by the simpler condition $0 \leq \psi^{U(\underline{p},\underline{p}),m}(p^*)$ or, equivalently, using (24), by the condition

$$U(\underline{p},\underline{p}) \quad \leq \quad -(b - c^q) \int_{\underline{p}}^{p^*} m(p')dp'. \tag{28}$$

The right condition in (26) is satisfied for all p if and only if it is satisfied for the p that maximizes the function $\psi^{U(\underline{p},\underline{p}),m}(p) + m(p)$. The maximizer is $p = \overline{p}$; to see this, consider any $p < \overline{p}$ and note that

$$\begin{aligned}
&\left(\psi^{U(\underline{p},\underline{p}),m}(\overline{p}) + m(\overline{p}) - \psi^{U(\underline{p},\underline{p}),m}(p) - m(p)\right) c^q \\
&= (v(\overline{p}) + c^q)m(\overline{p}) - (v(p) + c^q)m(p) - (b - c^q) \underbrace{\int_{p}^{\overline{p}} m(p')dp'}_{\leq (\overline{p}-p)m(\overline{p}) \text{ by (23)}} .
\end{aligned}$$

Due to (1), $v(p) + c^q > 0$. Thus, again using that $m(p) \leq m(\overline{p})$ from (23), we can continue the above equation via the estimation

$$\begin{aligned}
&\geq (v(\overline{p}) + c^q)m(\overline{p}) - (v(p) + c^q)m(\overline{p}) - (b - c^q)(\overline{p} - p)m(\overline{p}) \\
&= (v(\overline{p}) - v(p))m(\overline{p}) - (b - c^q)(\overline{p} - p)m(\overline{p}) \\
&= 0,
\end{aligned}$$

where the last equality relies on the definition of v in (2).

Thus, we can replace the right condition in (26) by the simpler condition $\psi^{U(\underline{p},\underline{p}),m}(\overline{p}) \leq 1 - m(\overline{p})$ or, equivalently, using (24), by the condition

$$(v(\overline{p}) + c^q)m(\overline{p}) - (b - c^q) \int_{\underline{p}}^{\overline{p}} m(p')dp' - c^q \quad \leq \quad U(\underline{p},\underline{p}). \tag{29}$$

At this point, it is useful to take stock: we have replaced the condition (26), which is required for all p, by two one-dimensional conditions: (28) provides an upper bound for $U(\underline{p},\underline{p})$, and (29) provides a lower bound for $U(\underline{p},\underline{p})$.

The first step towards solving the government's problem is to eliminate the variable $U(\underline{p},\underline{p})$.

According to (24) and (27), the government's objective W is linear with respect to $U(\underline{p},\underline{p})$, with slope

$$w^1 - \frac{b^q E_{p\sim F}[p] - c^{gq}}{c^q} \quad = \quad -\lambda^*.$$

[15] By convention, an integral $\int_{p^*}^{p} \ldots$ with $p^* > p$ is equal to $-\int_{p}^{p^*} \ldots$.

In the following computations, we have to distinguish two cases, depending on the sign of λ^*. Suppose first that

$$\lambda^* \quad \leq \quad 0. \tag{30}$$

Then W is weakly increasing in $U(\underline{p}, \underline{p})$. Thus, there exists an optimal $U(\underline{p}, \underline{p})$ that hits the upper bound provided by (28), that is,

$$U(\underline{p}, \underline{p}) \quad = \quad -(b - c^q) \int_{\underline{p}}^{p^*} m(p')dp'. \tag{31}$$

Plugging (31) into (29), we have

$$(v(\overline{p}) + c^q)m(\overline{p}) - (b - c^q) \int_{\underline{p}}^{\overline{p}} m(p')dp' - c^q \quad \leq \quad -(b - c^q) \int_{\underline{p}}^{p^*} m(p')dp'.$$

This (one-dimensional) condition replaces (26) in the government's optimization. Rearranging, we obtain the equivalent condition

$$(v(\overline{p}) + c^q)m(\overline{p}) - (b - c^q) \int_{p^*}^{\overline{p}} m(p')dp' \quad \leq \quad c^q. \tag{32}$$

Next we rewrite the welfare W by plugging into (27) the expression for $U(\underline{p}, \underline{p})$ that was obtained in (31):

$$\begin{aligned} W \quad = \quad & E_{p \sim F} \left[w^1 (b - c^q) \int_{p^*}^{p} m(p')dp' \right. \\ & \left. + (b^q p - c^{gt})m(p) + (b^q p - c^{gq})\psi^{U(\underline{p},\underline{p}),m}(p) \right]. \end{aligned}$$

Similarly, $U(\underline{p}, \underline{p})$ can be substituted on the right-hand side of (24), yielding

$$\psi^{U(\underline{p},\underline{p}),m}(p) \quad = \quad \frac{1}{c^q} \left(v(p)m(p) - (b - c^q) \int_{p^*}^{p} m(p')dp' \right). \tag{33}$$

In summary, we obtain the expression

$$\begin{aligned} W \quad = \quad & E_{p \sim F} \left[w^1 (b - c^q) \int_{p^*}^{p} m(p')dp' \right. \\ & \left. + (b^q p - c^{gt})m(p) + (b^q p - c^{gq}) \frac{1}{c^q} \left(v(p)m(p) - (b - c^q) \int_{p^*}^{p} m(p')dp' \right) \right] \\ = \quad & E_{p \sim F} \left[\left(w^1 (b - c^q) - \frac{b^q p - c^{gq}}{c^q} (b - c^q) \right) \int_{p^*}^{p} m(p')dp' \right] \\ & + E_{p \sim F} \left[\left(b^q p - c^{gt} + \frac{b^q p - c^{gq}}{c^q} v(p) \right) m(p) \right] \\ = \quad & - \int_{\underline{p}}^{\overline{p}} \int_{p^*}^{p} (b - c^q)\kappa(p)m(p')dp'dp \ + \ \int_{\underline{p}}^{\overline{p}} L(p)m(p)dp, \end{aligned} \tag{34}$$

where we have used the auxiliary functions

$$\kappa(p) = \left(-w^1 + \frac{b^q p - c^{gq}}{c^q}\right) f(p) \tag{35}$$

$$\text{and } L(p) = \left(b^q p - c^{gt} + \frac{b^q p - c^{gq}}{c^q} v(p)\right) f(p). \tag{36}$$

The first of the two terms in (34) can be rewritten into a more useful form. To do this, we split it into two integrals:

$$-\int_{\underline{p}}^{\overline{p}} \int_{p^*}^{p} m(p')\kappa(p)dp'dp = \int_{\underline{p}}^{p^*} \int_{p}^{p^*} m(p')\kappa(p)dp'dp - \int_{p^*}^{\overline{p}} \int_{p^*}^{p} m(p')\kappa(p)dp'dp.$$

Each of these double integrals can be simplified via changing the order of integration.

$$\int_{\underline{p}}^{p^*} \int_{p}^{p^*} m(p')\kappa(p)dp'dp = \int_{\underline{p}}^{p^*} \int_{\underline{p}}^{p'} m(p')\kappa(p)dpdp' = \int_{\underline{p}}^{p^*} K(p')m(p')dp',$$

where we have used the auxiliary function

$$K(p) = \int_{\underline{p}}^{p} \kappa(p')dp' \tag{37}$$

Similarly, the second integral can be written as

$$-\int_{p^*}^{\overline{p}} \int_{p^*}^{p} m(p')\kappa(p)dp'dp = -\int_{p^*}^{\overline{p}} \int_{p'}^{\overline{p}} m(p')\kappa(p)dpdp' = \int_{p^*}^{\overline{p}} (K(p') - K(\overline{p}))m(p')dp'.$$

Summing up,

$$-\int_{\underline{p}}^{\overline{p}} \int_{p^*}^{p} m(p')\kappa(p)dp'dp = \int_{\underline{p}}^{\overline{p}} \left(K(p') - \mathbf{1}_{p \geq p^*} \cdot K(\overline{p})\right) m(p')dp'.$$

Note that

$$K(\overline{p}) = \lambda^*.$$

Thus, (34) has been simplified as

$$W = \int_{\underline{p}}^{\overline{p}} \left((b - c^q)K(p) + L(p) - \mathbf{1}_{p \geq p^*} \cdot (b - c^q)\lambda^*\right) m(p)dp. \tag{38}$$

So far we have achieved the following reformulation of the government's problem

$$(\text{case } \lambda^* \leq 0) \qquad \max_{m(\cdot)} (38) \qquad \text{s.t.} \quad (4), \quad (23), \quad (32).$$

We can use, e.g., the space $PC[\underline{p}, \overline{p}]$ of piecewise continuous and right-continuous functions for the testing schedules $m(\cdot)$; this is a linear vector space. The constraints (23) and (4) define a convex subset Ω of $PC[\underline{p}, \overline{p}]$. Then the government's problem can be written as

$$\text{(case } \lambda^* \leq 0\text{)} \qquad \max_{m(\cdot)\in\Omega} (38)$$
$$\text{s.t.} \qquad (32).$$

Now suppose that

$$\lambda^* \geq 0. \tag{39}$$

Then the government's objective W is weakly decreasing in $U(\underline{p}, \underline{p})$. Hence, it is optimal to choose $U(\underline{p}, \underline{p})$ such that it hits the lower bound provided by (29), that is,

$$U(\underline{p}, \underline{p}) = (v(\overline{p}) + c^q)m(\overline{p}) - (b - c^q)\int_{\underline{p}}^{\overline{p}} m(p')dp' - c^q. \tag{40}$$

Plugging (40) into (28) yields the same constraint (32) that we obtained when we plugged (31) into (29).

Recall that the (one-dimensional) condition (32) replaces (26) in the government's optimization.

Next we rewrite the welfare W by plugging into (27) the expression for $U(\underline{p}, \underline{p})$ that was obtained in (40):

$$W = E_{p\sim F}\left[w^1\left(-(b - c^q)\int_p^{\overline{p}} m(p')dp' + (v(\overline{p}) + c^q)m(\overline{p}) - c^q\right)\right.$$
$$\left. + b^q p\left(m(p) + \psi^{U(\underline{p},\underline{p}),m}(p)\right) - c^{gt}m(p) - c^{gq}\psi^{U(\underline{p},\underline{p}),m}(p)\right].$$

Similarly, $U(\underline{p}, \underline{p})$ can be substituted on the right-hand side of (24), yielding

$$\psi^{U(\underline{p},\underline{p}),m}(p) = \frac{1}{c^q}\left(v(p)m(p) + (b - c^q)\int_p^{\overline{p}} m(p')dp' - (v(\overline{p}) + c^q)m(\overline{p})\right) + 1. \tag{41}$$

In summary, we obtain the expression

$$
\begin{aligned}
W &= E_{p\sim F}\Bigg[w^1\left(-(b-c^q)\int_p^{\overline{p}} m(p')dp' + (v(\overline{p})+c^q)m(\overline{p}) - c^q\right)\\
&\quad +(b^q p - c^{gt})m(p)\\
&\quad +\frac{b^q p - c^{gq}}{c^q}\left(v(p)m(p) + (b-c^q)\int_p^{\overline{p}} m(p')dp' - (v(\overline{p})+c^q)m(\overline{p})\right)\\
&\quad + b^q p - c^{gq}\Bigg].\\
&= E_{p\sim F}\Bigg[\left(-w^1 + \frac{b^q p - c^{gq}}{c^q}\right)(b-c^q)\int_p^{\overline{p}} m(p')dp'\\
&\quad + \left(b^q p - c^{gt} + \frac{b^q p - c^{gq}}{c^q}v(p)\right)m(p)\\
&\quad + \left(w^1 - \frac{b^q p - c^{gq}}{c^q}\right)(v(\overline{p})+c^q)m(\overline{p})\\
&\quad - w^1 c^q + b^q p - c^{gq}\Bigg]\\
&= \int_{\underline{p}}^{\overline{p}}\Bigg(\kappa(p)(b-c^q)\int_p^{\overline{p}} m(p')dp' \;+\; L(p)m(p)\\
&\qquad - \kappa(p)(v(\overline{p})+c^q)m(\overline{p}) + \kappa(p)c^q\Bigg)\,dp, \qquad (42)
\end{aligned}
$$

where we have used the functions defined in (35) and (36).

The first of the four terms in (42) can be rewritten into a more useful form. The double integral can be simplified via changing the order of integration.

$$
\int_{\underline{p}}^{\overline{p}}\int_p^{\overline{p}} m(p')\kappa(p)dp'dp = \int_{\underline{p}}^{\overline{p}}\int_{\underline{p}}^{p'} m(p')\kappa(p)dpdp' = \int_{\underline{p}}^{\overline{p}} K(p')m(p')dp',
$$

where we have used the function K defined in (37). Thus, (42) can be written as

$$
W = \int_{\underline{p}}^{\overline{p}} \left((b-c^q)K(p) + L(p)\right)m(p)dp - K(\overline{p})\left((v(\overline{p})+c^q)m(\overline{p}) - c^q\right). \qquad (43)
$$

So far we have achieved the following reformulation of the government's problem:

$$
\begin{aligned}
&(\text{case } \lambda^* \geq 0) \qquad && \max_{m(\cdot)}\ (43)\\
&\qquad\qquad \text{s.t.} && (4),\quad (23),\quad (32).
\end{aligned}
$$

Analogously to the earlier case $\lambda^* \leq 0$, the government's problem can also be written as

$$\text{(case } \lambda^* \geq 0) \qquad \max_{m(\cdot)\in\Omega} (43)$$
$$\text{s.t.} \quad (32).$$

Step 2: solving the rewritten problem

We will now show that the solution m^* described in Proposition 2 and Proposition 3 solves the government's problem as reformulated in *Step 1*.

As in Step 1, we distinguish two cases depending on the sign of λ^*. Suppose first that $\lambda^* \leq 0$.

Consider the reformulated problem from *Step 1* (case $\lambda^* \leq 0$). The following two Lagrangian conditions are sufficient for a solution (see, e.g., Luenberger (1968), Chapter 8). First, there exists a number $\lambda \geq 0$ ("Lagrange multiplier") such that $m^*(\cdot)$ solves the problem

$$\max_{m(\cdot)\in\Omega} \int_{\underline{p}}^{\overline{p}} \left((b-c^q)K(p) + L(p) - \mathbf{1}_{p\geq p^*} \cdot (b-c^q)\lambda^*\right) m(p)dp$$
$$- \lambda\left((v(\overline{p}) + c^q)m(\overline{p}) - (b-c^q)\int_{p^*}^{\overline{p}} m(p')dp'\right). \qquad (44)$$

Second, (32) is satisfied with equality at $m = m^*$.

In order to show that m^* satisfies these conditions, we begin by rewriting the objective of the Lagrangian problem (44):

$$W^\lambda = \int_{\underline{p}}^{\overline{p}} \overbrace{\left((b-c^q)K(p) + L(p) + \mathbf{1}_{p\geq p^*} \cdot (b-c^q)(\lambda - \lambda^*)\right)}^{\equiv a^\lambda(p)} m(p)dp$$
$$-\lambda(v(\overline{p}) + c^q)m(\overline{p}). \qquad (45)$$

In order to rewrite W^λ, we introduce additional notation. For any type p, define the conditional expectations

$$\eta(p) = E_{p'\sim F}[p'|p' \leq p],$$
$$\eta_2(p) = E_{p'\sim F}[(p')^2|p' \leq p].$$

Thus, using integration by parts,

$$\int_{\underline{p}}^{p} F(p')dp' = -\int_{\underline{p}}^{p} p'f(p')dp' + pF(p) = (p - \eta(p))F(p). \qquad (46)$$

Similarly,

$$\int_{\underline{p}}^{p}\int_{\underline{p}}^{p'} f(p'')dp''dp' = -\int_{\underline{p}}^{p} (p')^2 f(p')dp' + p\int_{\underline{p}}^{p} f(p'')dp'' \qquad (47)$$
$$= (p\eta(p) - \eta_2(p))F(p). \qquad (48)$$

Using (35) and (37),

$$K(p) = -\left(w^1 + \frac{c^{gq}}{c^q}\right)F(p) + \frac{b^q}{c^q}\int_{\underline{p}}^{p} p'f(p')dp'.$$

Thus, using (46) and (48),

$$\int_{\underline{p}}^{p} K(p')dp' = -\left(w^1 + \frac{c^{gq}}{c^q}\right)(p - \eta(p))F(p) + \frac{b^q}{c^q}(p\eta(p) - \eta_2(p))F(p).$$

Using the definition (36),

$$\begin{aligned} L(p) &= \left(-c^{gt} + \frac{c^{gq}}{c^q}c^t\right) f(p) + \left(b^q - \frac{b^q}{c^q}c^t - \frac{c^{gq}}{c^q}(b - c^q)\right) pf(p) \\ &\quad + \frac{b^q}{c^q}(b - c^q)p^2 f(p). \end{aligned}$$

Thus,

$$\begin{aligned} \int_{\underline{p}}^{p} L(p')dp' &= \left(-c^{gt} + \frac{c^{gq}}{c^q}c^t\right) F(p) + \left(b^q - \frac{b^q}{c^q}c^t - \frac{c^{gq}}{c^q}(b - c^q)\right) \eta(p)F(p) \\ &\quad + \frac{b^q}{c^q}(b - c^q)\eta_2(p)F(p). \end{aligned} \tag{49}$$

Combining the derived expressions,

$$\begin{aligned} \int_{\underline{p}}^{p} \left((b - c^q)K(p') + L(p')\right) dp' &= \left(-(b - c^q)\left(w^1 + \frac{c^{gq}}{c^q}\right) p - c^{gt} + \frac{c^{gq}}{c^q}c^t\right) F(p) \\ &\quad + \left((b - c^q)\left(w^1 + \frac{b^q}{c^q}p\right) + b^q - \frac{b^q}{c^q}c^t\right) \eta(p)F(p) \\ &= \left(-(b - c^q)w^1 p - c^{gt} - \frac{c^{gq}}{c^q}v(p)\right) F(p) \\ &\quad + \left((b - c^q)w^1 + b^q + \frac{b^q}{c^q}v(p)\right) \eta(p)F(p) \\ &= B(p). \end{aligned}$$

where we have used the definition (15). Thus, by definition of the function a^λ,

$$\begin{aligned} \int_{\underline{p}}^{p} a^\lambda(p')dp' &= \int_{\underline{p}}^{p} \left((b - c^q)K(p') + L(p') + \mathbf{1}_{p' \geq p^*} \cdot (b - c^q)(\lambda - \lambda^*)\right) dp' \\ &= B(p) + \mathbf{1}_{p \geq p^*} \cdot \underbrace{(b - c^q)(p - p^*)}_{=v(p) \text{ by } (3)} (\lambda - \lambda^*) \\ &= A^\lambda(p), \end{aligned}$$

where the last equality follows from (16).

With this in mind, we apply integration by parts to the right-hand side of (45), yielding

$$W^\lambda = -\int_{\underline{p}}^{\overline{p}} A^\lambda(p)dm(p) + A^\lambda(\overline{p})m(\overline{p}) - \lambda(v(\overline{p}) + c^q)m(\overline{p}),$$

where m is interpreted as a c.d.f..

Note that, using the definition (16),

$$A^{\lambda}(\overline{p}) \quad = \quad B(\overline{p}) + v(\overline{p})(\lambda - \lambda^*).$$

Thus, we obtain the simplified formula

$$\begin{aligned} W^{\lambda} &= -\int_{\underline{p}}^{\overline{p}} A^{\lambda}(p)dm(p) + (B(\overline{p}) - v(\overline{p})\lambda^* - \lambda c^q)\, m(\overline{p}) \\ &= -\int_{\underline{p}}^{\overline{p}} A^{\lambda}(p)dm(p) + \left(A^0(\overline{p}) - \lambda c^q\right) m(\overline{p}). \end{aligned} \tag{50}$$

Now consider specifically the Lagrange multiplier $\lambda = \check{\lambda}$ from Proposition 3. Fixing any $m(\overline{p})$ $(0 \leq m(\overline{p}) \leq 1)$, $W^{\check{\lambda}}$ is maximized if m puts all of the mass $m(\overline{p})$ on a point $\check{p}$ where $A^{\check{\lambda}}$ is minimized, that is,

$$m(p) \quad = \quad \begin{cases} 0 & \text{if } p < \check{p}, \\ m(\overline{p}) & \text{if } p \geq \check{p}. \end{cases}$$

Given such an m, the value of the Lagrangian can be written as

$$\begin{aligned} W^{\check{\lambda}} &= \left(-\min_p A^{\check{\lambda}}(p) + A^0(\overline{p}) - \check{\lambda} c^q\right) m(\overline{p}) \\ &= \alpha^{\check{\lambda}} m(\overline{p}) \\ &= 0, \end{aligned}$$

because $\alpha^{\check{\lambda}} = 0$ according to the assumption in Proposition 3.

In particular, m^* as described in Proposition 2 maximizes $W^{\check{\lambda}}$. Thus, the first of the two Lagrangian conditions is satisfied.

It remains to verify the second condition, that the constraint (32) is satisfied with equality.

Suppose that $\check{p} \leq p^*$. Then $m^*(\overline{p}) = \check{m} = 1$ according to the formula given for $\check{m}$ in Proposition 2. Thus, (32) is satisfied with equality because

$$\begin{aligned} &(v(\overline{p}) + c^q)m^*(\overline{p}) - (b - c^q)\int_{p^*}^{\overline{p}} m^*(p')dp' \\ &= (v(\overline{p}) + c^q)\check{m} - (b - c^q)(\overline{p} - p^*)\check{m} \\ &= c^q \check{m} \\ &= c^q, \end{aligned}$$

where we have used the definitions of $v(\overline{p})$ and p^*.

Now suppose that $\check{p} > p^*$. Then at $m(\overline{p}) = 1$ the left-hand side of (32) would be strictly larger than c^q. Thus, there exists $\check{m} < 1$ such that, at $m(\overline{p}) = \check{m}$, (32) is satisfied with equality. It is straighforward to check that the formula for $\check{m}$ given in Proposition 2 yields the required value.

Now suppose that $\lambda^* \geq 0$.

Consider the reformulated problem from *Step 1* (case $\lambda^* \geq 0$). The following three Lagrangian conditions are sufficient for a solution (see, e.g., Luenberger (1968), Chapter 8). First, there exists a number $\lambda_2 \geq 0$ ("Lagrange multiplier") such that $m^*(\cdot)$ solves the problem

$$\max_{m(\cdot)\in\Omega} \int_{\underline{p}}^{\overline{p}} \left((b-c^q)K(p)+L(p)\right) m(p)dp - \lambda^* \left((v(\overline{p})+c^q)m(\overline{p})-c^q\right)$$
$$- \lambda_2 \left((v(\overline{p})+c^q)m(\overline{p}) - (b-c^q)\int_{p^*}^{\overline{p}} m(p')dp' \right). \tag{51}$$

Second, (32) is satisfied at $m = m^*$. Third, if (32) is satisfied with strict inequality at $m = m^*$, then $\lambda_2 = 0$.

In order to show that m^* satisfies these conditions, we begin by rewriting the objective of the Lagrangian problem (51):

$$\begin{aligned} &= \int_{\underline{p}}^{\overline{p}} \left((b-c^q)K(p)+L(p)+\lambda_2(b-c^q)\mathbf{1}_{p\geq p^*}\right) m(p)dp - (\lambda_2+\lambda^*)(v(\overline{p})+c^q)m(\overline{p}) + \lambda^* c^q \\ &= W^{\lambda_2+\lambda^*} + \lambda^* c^q, \end{aligned} \tag{52}$$

where the last equality is immediate from a comparison with (45). Note that the term $\lambda^* c^q$ is constant and thus can be dropped from the maximization problem.

First we consider the case $\check{\lambda} \leq \lambda^*$. Fix the Lagrange multiplier $\lambda_2 = 0$. Then,

$$\alpha^{\lambda_2+\lambda^*} \leq 0 \tag{53}$$

because α is a decreasing function.

Fixing any $m(\overline{p})$ ($0 \leq m(\overline{p}) \leq 1$) and applying (50) with $\lambda = \lambda_2 + \lambda^*$, we see that the objective of the Lagrangian problem is maximized if m puts all of the mass $m(\overline{p})$ on a point $\check{p}$ where $A^{\lambda_2+\lambda^*}$ is minimized, that is,

$$m(p) = \begin{cases} 0 & \text{if } p < \check{p}, \\ m(\overline{p}) & \text{if } p \geq \check{p}. \end{cases}$$

Given such an m, the value of the Lagrangian can be written as

$$W^{\lambda_2+\lambda^*} = \alpha^{\lambda_2+\lambda^*} m(\overline{p}) + \lambda^* c^q,$$

and, due to (53), this expression is maximized by setting $m(\overline{p}) = 0$. That is, no testing is optimal. The constraint (32) is obviously satisfied.

Now suppose that $\check{\lambda} \geq \lambda^*$. Then, we consider the Lagrange multiplier $\lambda_2 = \check{\lambda} - \lambda^*$. Using the fact that the Lagrangian can be written in the form (52), the rest of the proof is as in the case $\lambda^* \leq 0$ that was treated above.

Step 3: optimal quarantining schedule

As in *Step 1* and in *Step 2*, we distinguish two cases depending on the sign of λ^*. Suppose first that $\lambda^* \leq 0$.

Consider first the case $\check{p} \leq p^*$. Then $\check{m} = 1$. Using (31),

$$U(\underline{p}, \underline{p}) = -(b-c^q)\int_{\check{p}}^{p^*} \check{m} dp' = -(b-c^q)(p^* - \check{p}) = v(\check{p}).$$

Thus, for all $p < \check{p}$, (24) with $m = m^*$ implies that

$$q^*(p) = \frac{1}{c^q}\left(-U(\underline{p},\underline{p})\right) = \frac{-v(\check{p})}{c^q},$$

as was to be shown.

Now consider the case $\check{p} > p^*$. Then $\check{m} < 1$. Using (31),

$$U(\underline{p},\underline{p}) = 0.$$

Thus, for all $p < \check{p}$, (24) with $m = m^*$ implies that $q^*(p) = 0$, as was to be shown.

For all $p \geq \check{p}$, (24) with $m = m^*$ implies that

$$\begin{aligned}(1-\check{m})q^*(p) &= \frac{1}{c^q}\left(v(p)\check{m} - (b-c^q)\int_{\check{p}}^{p}\check{m}dp' - 0\right)\\ &= \frac{1}{c^q}\left(v(p) - (b-c^q)(p-\check{p})\right)\check{m}\\ &= \frac{1}{c^q}v(\check{p})\check{m}.\end{aligned}$$

Dividing both sides by $1-\check{m}$ yields the formula

$$q^*(p) = v(\check{p})\frac{\check{m}}{(1-\check{m})c^q} \quad \text{for all } p \geq \check{p}.$$

Plugging into the right-hand side the formula (14), we obtain the desired conclusion $q^*(p) = 1$.

Now suppose that $\lambda^* \geq 0$.

Consider first the case that testing nobody is optimal, $m(p) = 0$ for all p. Then (40) implies

$$U(\underline{p},\underline{p}) = -c^q.$$

Thus, (24) implies that $q(p) = 1$ for all p, that is, everybody will be quarantined, as was to be shown.

Now consider the remaining possibility for the optimum, that is, the case with a marginally tested type $\check{p}$.

Using (40),

$$\begin{aligned}U(\underline{p},\underline{p}) &= (v(\overline{p}) + c^q)\check{m} - (b-c^q)(\overline{p}-\check{p})\check{m} - c^q\\ &= (v(\overline{p}) + c^q)\check{m} - (v(\overline{p}) - v(\check{p}))\check{m} - c^q\\ &= (v(\check{p}) + c^q)\check{m} - c^q\end{aligned}$$

Thus, for all $p < \check{p}$, (24) with $m = m^*$ implies that

$$\begin{aligned}q^*(p) &= \frac{1}{c^q}\left(-U(\underline{p},\underline{p})\right) = \frac{-v(\check{p})}{c^q}\\ &= -\frac{v(\check{p}) + c^q}{c^q}\check{m} + 1. \qquad (54)\end{aligned}$$

In particular, if $\check{p} \leq p^*$ so that $\check{m} = 1$, then

$$q^*(p) = -\frac{v(\check{p})}{c^q},$$

as was to be shown.

Finally, suppose that $\check{p} > p^*$. Then (14) implies that

$$\check{m} \quad = \quad \frac{c^q}{c^q + v(\check{p})}.$$

Plugging this into the formula (54) yields that, for all $p < \check{p}$,

$$q^*(p) = -\frac{v(\check{p}) + c^q}{c^q}\check{m} + 1 = 0,$$

as was to be shown.

For all $p \geq \check{p}$, (24) with $m = m^*$ implies that

$$\begin{aligned}(1-\check{m})q^*(p) &= \frac{1}{c^q}\left(v(p)\check{m} - (b - c^q)\int_{\check{p}}^{p} \check{m}dp' - ((v(\check{p}) + c^q)\check{m} - c^q)\right) \\ &= \frac{1}{c^q}\left(-(c^q\check{m} - c^q)\right) \\ &= 1 - \check{m}.\end{aligned}$$

Dividing both sides by $1 - \check{m}$ yields that

$$q^*(p) \quad = \quad 1,$$

as was to be shown. This completes the proof of Proposition 2 and Proposition 3.

5 Conclusion

The timing of individuals when deciding to get tested for an infectious disease can be crucial. Getting tested at an early stage when the symptoms are still ambiguous can be very helpful towards curbing the spread of the disease. We emphasize the government's role in providing such incentives via putting a testing-and-social-distancing schedule in place that takes the individuals' private health signals into account.

6 Appendix

Proof of Lemma 1. Because $A^\lambda(p)$ is strictly increasing in λ if $p > p^*$ and is independent of λ if $p \leq p^*$, the expression $\min_p A^\lambda(p)$ is weakly increasing in λ, showing that α^λ is weakly decreasing in λ.

To show that $\alpha \mapsto \alpha^\lambda$ is Lipschitz continuous, it remains to verify that there exists a number $\overline{L} > 0$ such that, for all $\lambda_2 > \lambda_1$,

$$\alpha^{\lambda_2} - \alpha^{\lambda_1} \quad \geq \quad -\overline{L}(\lambda_2 - \lambda_1). \tag{55}$$

To see this, let p_1 denote a minimizer of A^{λ_1}. Then $\min_p A^{\lambda_2}(p) \leq A^{\lambda_2}(p_1)$, implying

$$\alpha^{\lambda_2} - \alpha^{\lambda_1} \geq -A^{\lambda_2}(p_1) + A^{\lambda_1}(p_1) - (\lambda_2 - \lambda_1)c^q = -\mathbf{1}_{p_1 > p^*} \cdot v(p_1)(\lambda_2 - \lambda_1) - c^q(\lambda_2 - \lambda_1),$$

so that Lipschitz continuity is satisfied with $\overline{L} = v(\overline{p}) + c^q$.

By Lipschitz continuity, the derivative $d\alpha^\lambda/d\lambda$ exists almost everywhere. Using the envelope theorem (Milgrom and Segal (2002)), and letting p^λ denote a minimizer of A^λ,

$$\frac{d\alpha^\lambda}{d\lambda} = -\frac{d}{d\lambda}\min_p A^\lambda - c^q = -\mathbf{1}_{p^\lambda > p^*} \cdot v(p^\lambda) - c^q,$$

from which the inequalities stated in the lemma are immediate.

Note that $\alpha^0 \geq 0$ from (17).

If we choose λ larger than $-\frac{1}{b-c^q}\min_{p>p^*} dB/dp$, then A^λ is strictly increasing on the interval $(p^*, \overline{p}]$, showing that any minimizer of A^λ belongs to the interval $[\underline{p}, p^*]$. For all p in this interval, we have $A^\lambda(p) = B(p)$. Thus, for all sufficiently large λ,

$$\alpha^\lambda \quad = \quad -\min_{p \leq p^*} B(p) + A^0(\overline{p}) - \lambda c^q,$$

showing that $\alpha^\lambda < 0$ if λ is sufficiently large. □

Proof of Proposition 4. Suppose that $\lambda^* \geq 0$ and $\alpha^{\lambda^*} \leq 0$. By Lemma 1, there exists $\check{\lambda} \leq \lambda^*$ such that $\alpha^{\check{\lambda}} = 0$. Thus, Proposition 3 implies that no-testing-always-quarantining solves the government's problem.

Now suppose that $\lambda^* < 0$, or $\lambda^* \geq 0$ and $\alpha^{\lambda^*} > 0$. By Lemma 1, there exists $\check{\lambda} \geq \max\{0, \lambda^*\}$ such that $\alpha^{\check{\lambda}} = 0$. Thus, Proposition 3 implies that the government's problem has a solution with a threshold $\check{p}$. Choose $\check{p}$ minimal if multiple solutions exist.

Note that, for all $\lambda \geq 0$ and all $p \leq p^*$, $A^\lambda(p) = B(p)$. Thus,

$$\min_{p \leq p^*} A^\lambda(p) = \underline{B}.$$

Consider first the case $\underline{B} \leq \underline{A}^{\overline{\lambda}}$. Thus,

$$\underline{B} = \min_p A^{\overline{\lambda}}(p) \leq A^{\overline{\lambda}}(\overline{p}).$$

This implies $\overline{l} \geq 0$ because otherwise we would have $\overline{\lambda} = 0$, implying $\underline{B} \leq A^0(\overline{p})$ by the inequality above, implying $\overline{l} \geq 0$ by the definition of $\overline{l}$.

Thus, $\overline{\lambda} = \overline{l}$.

Suppose that $\check{\lambda} < \overline{\lambda}$. Then $\underline{A}^{\check{\lambda}} \leq \underline{A}^{\overline{\lambda}}$, implying

$$\alpha^{\check{\lambda}} = -\min\{\underline{A}^{\check{\lambda}}, \underline{B}\} + A^0(\overline{p}) - \check{\lambda}c^q > -\min\{\underline{A}^{\overline{\lambda}}, \underline{B}\} + A^0(\overline{p}) - \overline{\lambda}c^q = -\underline{B} + A^0(\overline{p}) - \overline{l}c^q = 0,$$

contradicting the definition in Proposition 3.

Thus, $\check{\lambda} \geq \overline{\lambda}$. In the case $\underline{B} < \underline{A}^{\overline{\lambda}}$, we cannot have a solution with $\check{p} > p^*$ because this would imply

$$A^{\check{\lambda}}(\check{p}) \geq A^{\overline{\lambda}}(\check{p}) \geq \underline{A}^{\overline{\lambda}} > \underline{B},$$

contradicting the fact that $\check{p}$ minimizes $A^{\check{\lambda}}$ on the interval $[\underline{p}, \overline{p}]$.

Similarly, in the case $\underline{B} = \underline{A}^{\overline{\lambda}}$ and $\check{\lambda} > \overline{\lambda}$, we cannot have a solution with $\check{p} > p^*$ because this would imply

$$A^{\check{\lambda}}(\check{p}) > A^{\overline{\lambda}}(\check{p}) \geq \underline{A}^{\overline{\lambda}} = \underline{B},$$

again contradicting the fact that $\check{p}$ minimizes $A^{\check{\lambda}}$ on the interval $[\underline{p}, \overline{p}]$.

In the case $\underline{B} = \underline{A}^{\overline{\lambda}}$ and $\check{\lambda} = \overline{\lambda}$, the function $A^{\check{\lambda}}$ has a minimizer that is $\leq p^*$, showing that $\check{p} \leq p^*$, as claimed.

Now consider the case $\underline{B} > \underline{A}^{\overline{\lambda}}$. This implies

$$\min\{\underline{A}^{\overline{\lambda}}, \underline{B}\} < \underline{B}.$$

Suppose first that $\bar{l} \geq 0$. Then $\overline{\lambda} = \bar{l}$, implying

$$\alpha^{\overline{\lambda}} = -\min\{\underline{A}^{\overline{\lambda}}, \underline{B}\} + A^0(\overline{p}) - \overline{\lambda} c^q > -\underline{B} + A^0(\overline{p}) - \bar{l} c^q = 0,$$

Thus, $\check{\lambda} > \overline{\lambda}$ because α^{λ} is decreasing.

Suppose that $\check{p} \leq p^*$. This would imply $\underline{B} \leq \underline{A}^{\check{\lambda}}$, thus

$$\alpha^{\check{\lambda}} = -\underline{B} + A^0(\overline{p}) - \check{\lambda} c^q < -\underline{B} + A^0(\overline{p}) - \bar{l} c^q = 0,$$

contradicting the definition of $\check{\lambda}$.

Finally, consider the case $\bar{l} < 0$, that is, $A^0(\overline{p}) - \underline{B} < 0$. Suppose that $\check{p} \leq p^*$. This would imply $\underline{B} \leq \underline{A}^{\check{\lambda}}$, thus

$$\alpha^{\check{\lambda}} = -\underline{B} + A^0(\overline{p}) - \check{\lambda} c^q \leq -\underline{B} + A^0(\overline{p}) < 0,$$

contradicting the definition of $\check{\lambda}$. □

Proof of Corollary 2. By Proposition 4, it is sufficient to show that $\lambda^* \geq 0$ and $\alpha^{\lambda^*} \leq 0$.

In the following, we view λ^*, B, and α^{λ^*} as functions of w^1 and c^{gq}. Accordingly, we use the notation $\lambda^* = \lambda^{w^1, c^{gq}}$, $B = B^{w^1, c^{gq}}$, and $\alpha^{\lambda^*} = \alpha^{w^1, c^{gq}}$.

Note that all these quantities are continuous in w^1 and c^{gq}, where B is endowed with the max-norm for continuous functions on $[\underline{p}, \overline{p}]$, and the continuity of $(w^1, c^{gq}) \mapsto \alpha^{w^1, c^{gq}}$ follows from Berge's maximum theorem.

Thus, it is sufficient to show that $\lambda^{0,0} > 0$ and $\alpha^{0,0} < 0$.

By definition of λ^*,

$$\lambda^{0,0} = \frac{b^q}{c^q} E[p'] > 0.$$

Note that, for all p,

$$B^{0,0}(p) = \left(-c^{gt}\right) F(p) + \left(b^q + \frac{b^q}{c^q} v(p)\right) E[p'|p' \leq p] F(p).$$

Using the same arguments as in the heuristics provided below the statement of Corollary 2 (cf. (20)),

$$\alpha^{0,0} = -\min_p (B^{0,0}(p) + c^{gt}). \tag{56}$$

Fix $\epsilon > 0$ such that

$$\epsilon \quad < \quad \max\left\{c^{gt}(1-F(p^*)), b^q E[p'|p' \leq p^*]F(p^*)\right\}.$$

Using the definition of $B^{0,0}$ and (21),

$$B^{0,0}(p) + c^{gt} \quad \geq \quad c^{gt}(1-F(p)) \quad \text{for all } p \in [\underline{p}, \overline{p}].$$

Thus, if $p \leq p^*$, then

$$B^{0,0}(p) + c^{gt} \geq c^{gt}(1-F(p^*)) > \epsilon.$$

If $p \geq p^*$, then

$$B^{0,0}(p) + c^{gt} \geq b^q E[p'|p' \leq p^*]F(p^*) > \epsilon.$$

Thus, (56) implies that $\alpha^{0,0} \leq -\epsilon < 0$, as was to be shown. □

Proof of Corollary 3. We indicate the dependence of A^λ on c^{gt} by using the notation $A^\lambda_{c^{gt}}$. Similarly, we will use the notation $\alpha^\lambda_{c^{gt}}$.

For any $\lambda \geq 0$ and any $c^{gt} > 0$, let $p^\lambda_{c^{gt}}$ denote the smallest minimizer of $A^\lambda_{c^{gt}}(p)$; the proof will be identical if we select the largest minimizer for all λ and all c^{gt}. Recalling the definition (16), the envelope theorem (Milgrom and Segal (2002)) yields that the function $c^{gt} \mapsto \min_p A^\lambda_{c^{gt}}(p)$ is Lipschitz continuous and its derivative is, for Lebesgue-almost every c^{gt}, given by

$$\frac{d}{dc^{gt}} \min_p A^\lambda_{c^{gt}}(p) = \frac{\partial A^\lambda_{c^{gt}}}{\partial c^{gt}}(p^\lambda_{c^{gt}}) = -F(p^\lambda_{c^{gt}}).$$

Similarly,

$$\frac{\partial}{\partial c^{gt}} A^0_{c^{gt}}(\overline{p}) = -1.$$

Thus, using (17),

$$\frac{\partial}{\partial c^{gt}} \alpha^\lambda_{c^{gt}} = F(p^\lambda_{c^{gt}}) - 1. \tag{57}$$

For any c^{gt}, let $\check{\lambda}(c^{gt})$ denote the unique point λ where $\alpha^\lambda_{c^{gt}} = 0$ (cf. Lemma 1).

By (57), $\partial\alpha^\lambda_{c^{gt}}/\partial c^{gt} \leq 0$. Together with the fact that $\alpha^\lambda_{c^{gt}}$ is strictly decreasing in λ (cf. Lemma 1), this implies that the function $c^{gt} \mapsto \check{\lambda}(c^{gt})$ is weakly decreasing. Next we show that this function is Lipschitz continuous, implying that its derivative exists almost everywhere.

Consider any two cost levels $c^{gt}_1 < c^{gt}_2$. Then

$$\begin{aligned}
0 &= \alpha^{\check{\lambda}(c^{gt}_2)}_{c^{gt}_2} - \alpha^{\check{\lambda}(c^{gt}_1)}_{c^{gt}_1} \\
&= \alpha^{\check{\lambda}(c^{gt}_2)}_{c^{gt}_2} - \alpha^{\check{\lambda}(c^{gt}_2)}_{c^{gt}_1} - \left(\alpha^{\check{\lambda}(c^{gt}_1)}_{c^{gt}_1} - \alpha^{\check{\lambda}(c^{gt}_2)}_{c^{gt}_1}\right) \\
&= \int_{c^{gt}_1}^{c^{gt}_2} \frac{\partial}{\partial c^{gt}} \alpha^{\check{\lambda}(c^{gt}_2)}_{c^{gt}} dc^{gt} - \int_{\check{\lambda}(c^{gt}_2)}^{\check{\lambda}(c^{gt}_1)} \frac{\partial \alpha^\lambda_{c^{gt}_1}}{\partial \lambda} d\lambda.
\end{aligned} \tag{58}$$

Thus, using (57) and the estimate $-d\alpha^\lambda/d\lambda \geq c^q$ from Lemma 1,

$$0 \quad \geq \quad (-1)(c^{gt}_2 - c^{gt}_1) + \left(\check{\lambda}(c^{gt}_1) - \check{\lambda}(c^{gt}_2)\right) c^q,$$

implying that

$$\check{\lambda}(c_1^{gt}) - \check{\lambda}(c_2^{gt}) \quad \leq \quad \frac{1}{c^q}(c_2^{gt} - c_1^{gt}).$$

This completes the proof that the function $c^{gt} \mapsto \check{\lambda}(c^{gt})$ is Lipschitz continuous. Because the function is also weakly decreasing,

$$\check{\lambda}'(c^{gt}) \quad \leq \quad 0 \quad \text{for Lebesgue-almost every } c^{gt}.$$

Using (16), for all p,

$$\frac{d}{dc^{gt}} A_{c^{gt}}^{\check{\lambda}(c^{gt})}(p) \quad = \quad -F(p) + \mathbf{1}_{p>p^*} v(p) \check{\lambda}'(c^{gt}).$$

Thus, for all p_1, p_2 with $p_2 > p_1$, and all c_1^{gt}, c_2^{gt} with $c_2^{gt} > c_1^{gt}$,

$$\begin{aligned}
& A_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})}(p_2) - A_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})}(p_1) - \Big(A_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})}(p_2) - A_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})}(p_1)\Big) \\
&= A_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})}(p_2) - A_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})}(p_2) - \Big(A_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})}(p_1) - A_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})}(p_1)\Big) \\
&= -\underbrace{(F(p_2) - F(p_1))}_{>0}\underbrace{(c_2^{gt} - c_1^{gt})}_{>0} + \underbrace{(\mathbf{1}_{p_2>p^*} v(p_2) - \mathbf{1}_{p_1>p^*} v(p_1))}_{\geq 0}\underbrace{\big(\check{\lambda}(c_2^{gt}) - \check{\lambda}(c_1^{gt})\big)}_{\leq 0} \\
&< 0. \qquad (59)
\end{aligned}$$

Recall that

$$p_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})} \quad \in \quad \arg\min_p A_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})}(p)$$

and

$$p_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})} \quad \in \quad \arg\min_p A_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})}(p).$$

Thus, for all $p < p_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})}$,

$$A_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})}(p_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})}) - A_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})}(p) \quad \leq \quad 0.$$

Applying (59) with $p_2 = p_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})}$ and $p_1 = p$, we conclude that

$$A_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})}(p_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})}) - A_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})}(p) \quad < \quad 0.$$

Thus,

$$p \quad \notin \quad \arg\min_p A_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})}(p),$$

implying that[16]

$$p_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})} \quad \geq \quad p_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})}.$$

[16]For a general background of this type of monotone-comparative-statics argument, see Milgrom and Shannon (1994)).

Thus, the marginal-type function $c^{gt} \mapsto p_{c^{gt}}^{\check{\lambda}(c^{gt})}$ is weakly increasing. An analogous argument shows that, for all $\lambda \geq 0$, the marginal-type function $c^{gt} \mapsto p_{c^{gt}}^{\lambda}$ is weakly increasing, implying that $p_{c^{gt}}^{\check{\lambda}(c_2^{gt})} \leq p_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})}$ for all $c^{gt} \leq c_2^{gt}$.

Thus, for any two cost levels $c_1^{gt} < c_2^{gt}$, (57) implies that

$$\begin{aligned}\int_{c_1^{gt}}^{c_2^{gt}} \frac{\partial}{\partial c^{gt}} \alpha_{c^{gt}}^{\check{\lambda}(c_2^{gt})} dc^{gt} &= -\int_{c_1^{gt}}^{c_2^{gt}} \left(1 - F\left(p_{c^{gt}}^{\check{\lambda}(c_2^{gt})}\right)\right) dc^{gt} \\ &\leq -(c_2^{gt} - c_1^{gt}) \left(1 - F\left(p_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})}\right)\right).\end{aligned}$$

On the other hand, the estimate $-d\alpha^{\lambda}/d\lambda \leq c^q + v(\overline{p})$ from Lemma 1 implies that

$$-\int_{\check{\lambda}(c_2^{gt})}^{\check{\lambda}(c_1^{gt})} \frac{\partial \alpha_{c_1^{gt}}^{\lambda}}{\partial \lambda} d\lambda \leq \left(\check{\lambda}(c_1^{gt}) - \check{\lambda}(c_2^{gt})\right)(c^q + v(\overline{p})).$$

In summary, (58) implies that

$$0 \leq -(c_2^{gt} - c_1^{gt}) \left(1 - F\left(p_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})}\right)\right) + \left(\check{\lambda}(c_1^{gt}) - \check{\lambda}(c_2^{gt})\right)(c^q + v(\overline{p})).$$

Rearranging this yields the inequality

$$\frac{\check{\lambda}(c_1^{gt}) - \check{\lambda}(c_2^{gt})}{c_2^{gt} - c_1^{gt}} \geq \frac{1}{c^q + v(\overline{p})} \left(1 - F\left(p_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})}\right)\right).$$

Taking the limit $c_1^{gt} \to c_2^{gt}$ yields that

$$-\check{\lambda}'(c_2^{gt}) \geq \frac{1}{c^q + v(\overline{p})} \left(1 - F\left(p_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})}\right)\right). \tag{60}$$

This implies

$$\lim_{c^{gt} \to \infty} p_{c^{gt}}^{\check{\lambda}(c^{gt})} = \overline{p}$$

because otherwise the derivative (60) is bounded away from zero for all c^{gt}, contradicting the fact that $\check{\lambda}(c^{gt}) \geq 0$.

Next, we show that

$$\lim_{c^{gt} \to \infty} \check{\lambda}(c^{gt}) = 0. \tag{61}$$

Otherwise there exists a sequence $(c_n^{gt})_{n=1,2,\ldots}$ with $c_n^{gt} \to \infty$ and a number $\epsilon > 0$ such that $\check{\lambda}(c_n^{gt}) > \epsilon$ for all n. Then (17) implies that

$$\begin{aligned}&\limsup_n \alpha_{c_n^{gt}}^{\check{\lambda}(c_n^{gt})} \\ &\leq \limsup_n \left(-A_{c_n^{gt}}^{\check{\lambda}(c_n^{gt})}(p_{c_n^{gt}}^{\check{\lambda}(c_n^{gt})}) + A_{c_n^{gt}}^{0}(\overline{p})\right) - \liminf_n \check{\lambda}(c_n^{gt}) c^q \\ &\leq \limsup_n \left(-A_{c_n^{gt}}^{\check{\lambda}(c_n^{gt})}(\overline{p}) + A_{c_n^{gt}}^{0}(\overline{p})\right) + \limsup_n \left(A_{c_n^{gt}}^{\check{\lambda}(c_n^{gt})}(\overline{p}) - A_{c_n^{gt}}^{\check{\lambda}(c_n^{gt})}(p_{c_n^{gt}}^{\check{\lambda}(c_n^{gt})})\right) - \liminf_n \check{\lambda}(c_n^{gt}) c^q \\ &\leq -v(\overline{p})\epsilon + 0 - \epsilon c^q < 0,\end{aligned}$$

contradicting the optimality condition $\alpha_{c_n^{gt}}^{\check{\lambda}(c_n^{gt})} = 0$ from Proposition 3.

The condition $w^1 \geq \frac{b^q E_{p' \sim F}[p'] - c^{gq}}{c^q}$ is equivalent to the condition $\lambda^* \leq 0$.

The opposite condition $w^1 < \frac{b^q E_{p' \sim F}[p'] - c^{gq}}{c^q}$ means that $\lambda^* > 0$. From (61), there exists $\bar{c}^{gt}$ be such that $\check{\lambda}(\bar{c}^{gt}) = \lambda^*$.

Setting $\check{p} = p_{c^{gt}}^{\check{\lambda}(c^{gt})}$, the desired claims hold by Proposition 3. $\square$

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