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The Multiple-Volunteers Principle

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The multiple-volunteers principle*

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Abstract

We consider mechanisms for assigning an unpleasant task among a group of agents with heterogenous abilities. We emphasize threshold rules: every agent decides whether or not to "volunteer"; if the number of volunteers exceeds a threshold number, the task is assigned to a random volunteer; if the number is below the threshold, the task is assigned to a random non-volunteer. We show that any non-extreme threshold rule allows for a symmetric equilibrium in which every ability type is strictly better off than in a random assignment. This holds for arbitrarily high costs of performing the task.

Within the class of binary-action mechanisms, some threshold rule is utilitarian optimal.

The first-best can be approximated arbitrarily closely with a threshold rule as the group size tends to infinity; that is, there exist threshold numbers such that with probability arbitrarily close to 1 the task is performed by an agent with an ability arbitrarily close to the highest possible ability. The optimal threshold number goes to infinity as the group size tends to infinity.

1 Introduction

Volunteering, according to Wilson (2000), is "any activity in which time is given freely to benefit another person, group or cause." Implicit here is the assumption that there is little or no remuneration for the activity. Volunteering plays an important role in many different areas of any modern economy. It concerns services as diverse as chairing a university department, engaging in environmental activities such as bird counting, teaching the host

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country's language to a refugee, and providing company for terminally ill patients in hospices.[1] While many volunteering activities are informal, in the UK the economic value of formal volunteering alone is estimated at £39 billion, according to the report by Low, Butt, Ellis, and Davis Smith (2007).

For many tasks, it is crucial to not just find any volunteer, but to find a well-qualified volunteer. As humans, we are heterogenous with respect to our abilities, whether the task is to lead a department, to spot a specimen of a rare species in the field, to work with a refugee, or to talk to a dying person. If a highly able person volunteers, everybody benefits.

Our paper focusses on the economic problem of assigning a task to the most able person in a given group. No remuneration is possible. The task cannot be delayed or avoided: one of the group members must perform the task. Each agent is privately informed about her ability, which is defined as the benefit that accrues to everybody if she performs the task. There is a free-rider problem because performing the task is costly.

A simple task-assignment rule that naturally comes to mind consists in asking every agent whether or not she "volunteers". We imagine all agents being asked simultaneously; if at least one agent volunteers, the task is assigned randomly among the volunteering agents; if no agent volunteers, the task is assigned randomly among all agents. This *any-volunteer rule* can, however, lead to rather poor volunteering incentives. In particular, if performing the task is sufficiently costly, then the any-volunteer rule leads, in any symmetric equilibrium, to a purely-random assignment because nobody will volunteer.

In this paper, we present alternative task-assignment rules. The seed for our construction can be found in the writings of Thomas Schelling (Schelling, 2006). As a proposal in passing, he casts the idea of "volunteering if 20 others do likewise" (p. 95). We may call this idea the multiple-volunteers principle. Schelling's half-sentence immediately raises many questions. What happens if the threshold number of 20 is not reached? Does the multiple-volunteers principle lead to a welfare improvement relative to the any-volunteer rule? Can we use the multiple-volunteers idea to construct a mechanism that is optimal in some sense? Which threshold number should be set? Our paper elaborates on these questions.

We consider task assignment rules that are set by a social planner. Thus, Schelling's advice to declare one's conditional willingness to volunteer is recast as follows. The rule allows each agent a choice between two actions that we call "volunteering" and "not volunteering". If at least i^* (e.g., $i^* = 20$) agents volunteer, then all volunteers participate in a uniform lottery that determines the service provider. However, because the task cannot be avoided, a fully specified task-assignment rule must go beyond Schelling's advice: it must also specify who performs the task if the threshold number of i^* volunteers is not reached. We stipulate that the task is then assigned randomly among the non-volunteers by a uniform lottery. We generalize this construction slightly: we allow that, if the number of volunteers is equal to the threshold i^*, then, rather than assigning the task to a volunteer for sure, a lottery may be used to decide whether the task is assigned to a volunteer or a non-volunteer. We

[1] The German association of hospices reports that most of the 120.000 individuals working under their roof are volunteers who are not remunerated, see https://www.dhpv.de/themen_hospiz-palliativ_ehrenamt.html.

call any such mechanism a *threshold rule*.

Our main results advertise the class of threshold rules, by demonstrating three properties. First, in any threshold rule that is non-extreme in a sense that will be defined,[2] there exists an equilibrium such that every ability type, including every non-volunteering type, is strictly better off than under a purely-random assignment of the task. Thus, every type—even those with very low abilities and those with very high abilities—has a strict incentive to participate if the default is a purely-random assignment of the task. This property holds for arbitrarily high costs of performing the task. Thus, the existence of an improvement over the purely-random assignment is a detail-free conclusion.

Secondly, we show that, given the utilitarian welfare criterion, some threshold rule is optimal among all binary-action rules. In other words, in order to outperform the class of threshold rules, more complicated mechanisms with at least three actions would be needed.

Third, the first-best can be approximated arbitrarily closely via an appropriate sequence of threshold numbers as the group size tends to infinity. That is, in a large population a threshold rule (in particular, a binary-action rule) is always good enough.

While the utilitarian-optimal threshold number tends to infinity as the population size tends to infinity, considerable welfare improvements are often achieved already with very small threshold numbers. Consider, for instance, a large population in which the average ability is equal to 1, the highest possible ability is equal to 5, and the individual cost of taking on the task is equal to 3. The following equilibrium outcomes can be computed. In equilibrium, only agents with abilities close to 5 will volunteer, implying that the expected ability of a volunteer is close to 5, and the expected ability of a non-volunteer is close to 1. If a single volunteer is required (i.e., the any-volunteer rule is used), then the task will be assigned to a volunteer with probability 45%; if two volunteers are required, the task will be assigned to a volunteer with probability 69%; if ten volunteers are required, the task will be assigned to a volunteer with probability 89%.

Related literature

The allocation problem considered in our paper is a (very) special case of a social-choice setting with informational and allocative externalities (Jehiel and Moldovanu, 2001). However, in contrast to the focus in that literature, we consider here mechanisms without monetary transfers and with only two actions, while maintaining a continuum of types. In the absence of these restrictions, that is, with quasilinear preferences, arbitrary monetary transfers, and arbitrary action spaces, the first best could be obtained in our setting by simply asking all agents for their ability types, assigning the task to the highest type, and reimbursing the cost of performing the task, which is identical across agents.[3]

[2]The set of non-extreme threshold rules includes any rule with a threshold number i^* from 2 to $n-2$, where n is the group size.

[3]The fact that the efficient allocation can thus be implemented renders our setting “non-generic” from the point of view of Jehiel and Moldovanu.

The earlier game-theoretic literature on volunteering assumes that agents have identical abilities, but are heterogenous with respect to the opportunity cost of providing the public good (i.e., performing the task) or, equivalently, the personal benefit from consuming the public good. Moreover, in the literature it is usually allowed that the group of agents may fail to provide the public good (i.e., the task can be avoided). Such a setting resembles the classic public-good provision problem with private values (Clarke, 1971; Groves, 1973), except that no monetary transfers are feasible. In a private-values setting, incentives for volunteering arise from the threat that the public good is not provided *at all* rather than, as in our setting, the threat that the public good is provided in *low quality*.

Rather than taking a mechanism-design approach, the volunteering literature has focussed on two particular binary-action game rules. The *coordinated volunteer's game* assigns the task randomly among the volunteers if at least one person volunteers, and otherwise avoids the task altogether. The *uncoordinated volunteer's game* is different in the sense that not just one, but all volunteers pay the cost of performing the task. Olson (2009, first edition: 1965) conjectured that if a volunteer's game is played in a large population, then the probability that a volunteer is found will be smaller than in a small population. The first equilibrium analysis of the (uncoordinated) volunteer's game is due to Diekmann (1985). The subsequent literature has evaluated Olsen's conjecture in various settings (Makris, 2009; Bergstrom, 2017; Nöldeke and Peña, 2020).

In between our paper's assumption that the public good must be provided and the opposite assumption that it can be avoided lies the possibility that the provision can be delayed, leading to discounted costs and benefits. The possibility of delay naturally leads to a war-of-attrition game in which each agent waits, or engages in some other costly search process, until someone agrees to provide the service. Within the heterogenous-cost setting, such a game has been analyzed by Bliss and Nalebuff (1984).[4] In equilibrium, it is often the "right" person who volunteers first, e.g., the one who has the lowest cost of providing the service, but it can also be the one who has the highest cost of waiting, and substantial waiting costs may have to be incurred before a volunteer is found.

2 Model

A task of public interest needs to be allocated among a group of agents $1, \ldots, n$, where $n \geq 2$. Each agent is privately informed about her *ability* at performing the task. Each agent's ability type is independently distributed on an interval $[\underline{\theta}, \overline{\theta}]$ according to some strictly increasing and continuous cumulative distribution function F.

In a *binary mechanism*, each player chooses between two actions, denoted by "Y" and "N". Assuming anonymity, a binary mechanism is characterized

[4]See also Bilodeau and Slivinski (1996) for a related model with complete information. See Klemperer and Bulow (1999) for a general approach to war-or-attrition games, and see LaCasse, Ponsati, and Barham (2002) and Sahuguet (2006) for more special extensions.

by a list

$$p_1, \ldots, p_{n-1}.$$

For all $j = 1, \ldots, n-1$, the number p_j denotes the probability that the task is assigned to a randomly selected Y-player; with probability $1 - p_j$, it is assigned to a randomly selected N-player. If the number of Y-players is 0 or n, the task is assigned randomly among all agents, that is, each agent gets assigned the task with probability $1/n$.

The *purely-random-assignment* mechanism $(p_1, \ldots, p_{n-1})$ is given by $p_j = j/n$ for all j. These probabilities imply that the task is always assigned with equal probability to any agent, independently of the agents' strategies.

Given any mechanism $(p_1, \ldots, p_{n-1})$, a symmetric strategy profile is characterized by a function σ that determines the strategy for each agent, where $\sigma(\theta)$ denotes the probability that type $\theta \in [\underline{\theta}, \overline{\theta}]$ chooses Y.

The expected utility $U_a(\sigma, \theta)$ of any type θ taking action a, who anticipates that the other agents will use the strategy σ, is denoted $U_a(\sigma, \theta)$.

The function σ is an *equilibrium* if the following implications hold for all θ:

$$\begin{aligned}
&\text{if } \sigma(\theta) > 0 \text{ then } U_Y(\sigma, \theta) - U_N(\sigma, \theta) \geq 0, \\
&\text{if } \sigma(\theta) < 1 \text{ then } U_Y(\sigma, \theta) - U_N(\sigma, \theta) \leq 0.
\end{aligned}$$

Mechanism-equilibrium combinations $(p_1, \ldots, p_{n-1}, \sigma)$ and $(p'_1, \ldots, p'_{n-1}, \sigma')$ are *equivalent* if each type obtains the same expected utility in both combinations.

Notation: selection-probability functions

Before continuing with the model description, we introduce four auxiliary functions, h_Y, h_N, q_Y and q_N, and discuss their basic properties. These functions will play a fundamental role throughout the paper. We call them selection-probability functions.

Taking the point of view of an agent who has chosen an action (Y or N), the functions q_Y and q_N describe the probability of personally getting assigned the task, and the functions h_Y and h_N describe the probability that anyone in the set of agents who take a particular action gets assigned the task. For the most part, in our computations, binomial sums will remain hidden behind the selection-probability functions.

The argument of the selection-probability functions is the ex-ante probability that a given agent chooses Y,

$$y = \int \sigma(\theta) \mathrm{d}F(\theta). \tag{1}$$

For any $y \in [0, 1]$, the probability that anyone of the Y-playing agents is selected, conditional on the event that a given agent plays Y, is denoted

$$h_Y(y) = \sum_{j=0}^{n-1} B_y^{n-1}(j) p_{j+1}. \tag{2}$$

Here, y denotes every other (i.e., not the given) agent's probability of playing Y. Using the binomial distribution, $B_y^{n-1}(j) = \binom{n-1}{j}(1-y)^{n-1-j}y^j$ denotes[5] the probability that, from the point of view of the given agent, j other agents choose Y. We also use the notation $p_n = 1$.

The probability that anyone of the Y-playing agents is selected, conditional on the event that the given agent plays N, is denoted

$$h_N(y) \quad = \quad \sum_{j=0}^{n-1} B_y^{n-1}(j)p_j, \tag{3}$$

where we use the notation $p_0 = 0$.

The probability that the given agent is selected if she chooses action $a = Y, N$ is denoted $q_a(y)$; i.e.,

$$q_Y(y) \quad = \quad \sum_{j=0}^{n-1} B_y^{n-1}(j)\frac{p_{j+1}}{j+1} \tag{4}$$

and

$$q_N(y) \quad = \quad \sum_{j=0}^{n-1} B_y^{n-1}(j)\frac{1-p_j}{n-j}. \tag{5}$$

Often we will omit the argument y from h_Y, h_N, q_Y and q_N.

For an illustration of the selection-probability functions in a special case, consider the purely-random-assignment rule. Here, a computation that applies standard properties of the binomial distribution to the definitions (2)–(5) shows that

$$h_Y(y) = \frac{1+y(n-1)}{n}, \quad h_N(y) = \frac{y(n-1)}{n}, \tag{6}$$
$$q_Y(y) = \frac{1}{n}, \quad q_N(y) = \frac{1}{n}.$$

To understand the numerators in the formulas for h_Y and h_N, note that, from the point of view of a given agent, the expected number of other Y-players is equal to $y(n-1)$; by playing Y, the agent adds in herself (1+).

Next we establish several useful algebraic relations between the selection-probability functions. These relations hold independently of the underlying mechanism.[6] A particularly simple formula is available for expressing q_Y and q_N in terms of h_Y and h_N. To see this, suppose that all agents play Y with probability y. Then the probability that the task is assigned to a Y-player can be expressed in the form $yh_Y + (1-y)h_N$. Alternatively, the same probability can be expressed in the form nyq_Y because every Y-playing agent is selected with the same probability:

$$nyq_Y \quad = \quad yh_Y + (1-y)h_N. \tag{7}$$

[5]We use the convention that $0^0 = 1$.

[6]Note that the Bernstein polynomials $y \mapsto B_y^{n-1}(j)$ ($j = 0, 1, \ldots, n-1$) form a basis of the vector space of polynomials of degree at most $n-1$. Thus, from each of the four selection-probability functions, the underlying mechanism $(p_1, \ldots, p_{n-1})$ can be recovered, implying that each of the four functions determines the other three functions.

Similarly, the probability that an N-player is selected is given by

$$n(1-y)q_N = y(1-h_Y)+(1-y)(1-h_N). \tag{8}$$

Adding up the equations (7) and (8) confirms the ex-ante probability that any given agent is selected:

$$yq_Y+(1-y)q_N = \frac{1}{n}. \tag{9}$$

We state one other relation between the selection-probability functions; it refers to the derivatives of q_Y and q_N. The proof, which relies on standard properties of Bernstein polynomials, is relegated to the Appendix.

Lemma 1. *Consider any mechanism and any* $0<y<1$. *Then*

$$q_Y'(y) = \frac{h_Y-h_N-q_Y}{y}, \tag{10}$$

$$q_N'(y) = \frac{h_N-h_Y+q_N}{1-y}. \tag{11}$$

To interpret (10), take the point of view of an agent who considers switching from playing N to playing Y. Here, $h_Y-h_N-q_Y$ equals the change in the probability that a Y-player other than herself gets assigned the task. As long as this change is positive, it holds that $q_Y'(y)>0$, that is, an increase of y increases the probability that the agent herself gets assigned the task if she plays Y; similarly if the change is negative. An analogous interpretation applies to (11).

Expected utilities

We assume the following preferences. Suppose the task is performed by an agent of ability θ. Then every agent obtains the benefit θ. In addition, the performing agent bears a cost $c\geq 0$, where c is commonly known and independent of the identity of the agent. Agents are expected-utility maximizers.

Consider a strategy σ and y defined via (1). Towards computing equilibria, it is crucial to evaluate an agent's expected-utility gain from playing Y versus playing N, assuming that all other agents use the strategy σ. We will establish a convenient expression for this utility gain. To this end, we express the agents' expected-utility functions in terms of the selection-probability functions and conditional expected abilities. The conditional expected ability of an agent who chooses Y is denoted

$$E_Y = \frac{1}{y}\int \sigma(\theta)\theta \mathrm{d}F(\theta) \quad \text{if } y>0.$$

In other words, E_Y is the expected benefit that accrues to every agent if the task is assigned to a Y-player.

The conditional expected ability of an agent who chooses N is denoted

$$E_N = \frac{1}{1-y}\int (1-\sigma(\theta))\theta \mathrm{d}F(\theta) \quad \text{if } y<1.$$

That is, E_N is the expected benefit that accrues to every agent if the task is assigned to an N-player.

Using this notation, the agents' expected-utility functions are

$$U_Y(\sigma,\theta) = (h_Y - q_Y)E_Y + q_Y \cdot (\theta - c) + (1 - h_Y)E_N, \tag{12}$$
$$U_N(\sigma,\theta) = (1 - h_N - q_N)E_N + q_N \cdot (\theta - c) + h_N E_Y. \tag{13}$$

The interpretation of these expressions is straightforward. Consider the expected utility (12) from playing Y: the first term captures the payoff that arises from the event that the task is performed by a Y-player other than the agent herself, which happens with probability $h_Y - q_Y$; the second term captures the event that the agent is selected herself, which happens with probability q_Y, yielding the utility $\theta - c$; the third term captures the payoff that arises from the event that the task is performed by an N-player. The interpretation of the expression (13) for the expected utility from playing N is analogous.

Combining the expressions (12) and (13) and cancelling terms, the utility gain from playing Y versus playing N is

$$\begin{aligned} U_Y(\sigma,\theta) - U_N(\sigma,\theta) &= (q_Y - q_N)(\theta - c) \\ &\quad + (h_Y - h_N - q_Y)E_Y + (h_N - h_Y + q_N)E_N. \end{aligned} \tag{14}$$

The three terms on the right-hand side reflect that an agent's choice of action affects three probabilities: to be selected herself (first term), the probability that a Y-player other than herself is selected (second term), and the probability that an N-player other than herself is selected (third term).

The purely-random assignment is a natural benchmark for our analysis. In the purely-random-assignment rule, every strategy is an equilibrium. To see this formally, note that, from (6), the right-hand side of (14) equals 0 for all σ. Moreover, all equilibria are equivalent: using (12) and the law of iterated expectations (that is, $yE_y + (1 - y)E_N = E[\theta]$),

$$U_a(\sigma,\theta) = (1 - \frac{1}{n})E[\theta] + \frac{1}{n}(\theta - c) \quad \text{for all } \sigma, \text{ all } \theta, \text{ and } a = Y, N. \tag{15}$$

These purely-random-assignment payoffs in fact obtain not only if the random-assignment rule is used. These payoffs obtain whenever the mechanism and equilibrium are such that an agent's probability of getting selected is independent of her action.[7]

Remark 1. *Any mechanism-equilibrium combination $(p_1, \ldots, p_{n-1}, \sigma)$ such that $q_Y(y) = 1/n = q_N(y)$ (where y is given by (1)) is equivalent to a purely-random assignment.*

[7] The equilibrium assumption in Remark 1 is indispensible. If a (possibly non-equilibrium) strategy leads to a y with the property $q_Y(y) = \frac{1}{n}$ and $q_N(y) = \frac{1}{n}$, it does *not* follow that $h_Y(y)$ and $h_N(y)$ are given as in the case of a purely-random assignment. For example, if $n = 3$, $(p_1, p_2) = (0, 1)$, and $y = 1/2$, then $q_Y(y) = \frac{1}{n} = q_N(y)$ and $h_Y(y) = 3/4$, whereas we would obtain $h_Y(y) = 2/3$ from using the pure-random-assignment rule.

Here is a sketch of the proof. The conclusion is straightforward if the mechanism-equilibrium combination is such that all types prefer Y to N or vice versa, or the conditional expected quality of the task is the same across the two actions. Suppose now that some type is indifferent between the actions N and Y, and the conditional expected quality of the task is not the same across the two actions. Consider an agent who changes her action from N to Y. By assumption, this change has no impact on the probability of getting selected. Thus, the change increases the probability that a Y-player other than herself is selected by the same amount as it decreases the probability that an N-player other than herself is selected. Then a type who is indifferent between the actions can exist only if the change of the agent's action does not actually increase (or decrease) the probability that Y-player other than herself is selected. Thus, any type's expected utility is as in a purely-random assignment. The formal proof is relegated to the Appendix.

Threshold equilibria

We now introduce a special class of equilibria, threshold equilibria. Our analysis will focus on this class. We show in Lemma 2 that this focus is without loss of generality.

A strategy σ has the *threshold form* if there exists $\hat{\theta} \in [\underline{\theta}, \overline{\theta}]$ such that $\sigma(\theta) = 0$ for all $\theta < \hat{\theta}$ and $\sigma(\theta) = 1$ for all $\theta > \hat{\theta}$. Ignoring probability-0 events, any strategy in threshold form is characterized by the playing-Y-probability $y = 1 - F(\hat{\theta})$. Whenever we deal with a threshold strategy y, we will use the notation

$$E_Y(y) = E[\theta \mid \theta \geq F^{-1}(1-y)] \quad \text{and} \quad E_N(y) = E[\theta \mid \theta \leq F^{-1}(1-y)]$$

for the expected ability of a Y-player and an N-player, respectively; we define the continuous extensions $E_Y(0) = \overline{\theta}$ and $E_N(1) = \underline{\theta}$.

Similarly, we will use the notation $U_a(y, \theta)$ for the expected payoff of type θ from taking action $a = Y, N$ if all others use the strategy y. Given any threshold strategy y,

$$E_Y(y) \quad > \quad E_N(y). \tag{16}$$

The strategies $y = 0$ and $y = 1$ imply that one action is chosen with probability 1, so that the purely-random-assignment payoffs obtain. In any equilibrium y in which both actions are chosen with positive probability (i.e., $0 < y < 1$), the type $\hat{\theta} = F^{-1}(1-y)$ is indifferent between the two actions, that is,

$$\Delta(y) = 0, \quad \text{where} \quad \Delta(y) = U_Y(y, F^{-1}(1-y)) - U_N(y, F^{-1}(1-y)). \tag{17}$$

Moreover, using (14) and the equilibrium condition, if $0 < y < 1$ then an agent's switch from the action N to the action Y cannot reduce the probability that she gets selected,

$$q_Y(y) \quad \geq \quad q_N(y). \tag{18}$$

Lemma 2 shows that focussing on threshold equilibria is without loss of generality, and the properties (17) and (18) can be maintained even if $y = 0$ or $y = 1$.

Given the property (18), from now on we interpret the action Y as "volunteering" and the action N as "non-volunteering".

Lemma 2. *For any mechanism-equilibrium combination, there exists an equivalent mechanism-threshold-equilibrium combination,* $(p_1, \ldots, p_{n-1}, y)$*, such that the properties (17) and (18) hold.*

The intuition is that we can construct an equivalent mechanism-equilibrium combination by switching the labels of the actions Y and N. The formal proof is relegated to the Appendix.

For later use, we establish a simple property of those threshold equilibria in which volunteering actually increases the probability of getting selected: in such an equilibrium it cannot be true that all types volunteer.[8]

Remark 2. *Any threshold equilibrium y with $q_Y(y) > q_N(y)$ satisfies $y < 1$.*

The intuition behind this result is simple: if an agent expects that with probability $y = 1$ somebody else will volunteer, then by volunteering herself she will reduce the expected ability of the selected agent if she herself is endowed with the lowest ability $\underline{\theta}$ or an ability close to that. The formal proof is relegated to the Appendix.

Because the inequality (18) and its strict version will occur frequently in the subsequent analysis, it is useful to note that these inequalities can be expressed in an alternative form if the strategy is such that some types do not volunteer. The proof is straightforward from (9).

Remark 3. *Consider any threshold strategy $y < 1$. Then the inequality (18) holds if and only if $q_Y(y) \geq 1/n$. The inequality $q_Y(y) > q_N(y)$ holds if and only if $q_Y(y) > 1/n$.*

The planner's (binary-second-best) problem

We consider the utilitarian welfare objective. Given our focus on symmetric equilibria, this objective is equivalent to maximizing any agent's ex-ante expected utility. Because the task cannot be avoided, each agent pays the cost c/n in any mechanism-equilibrium combination. Thus, the planner's objective boils down to assigning the task such that the expected ability of the selected agent is maximized.

Without loss of generality, we restrict the allowed mechanism-equilibrium combinations in line with the result of Lemma 2. Given any strategy y, a volunteer is selected with probability nyq_Y and a non-volunteer is selected with probability $n(1-y)q_N$. Thus, the expected ability of the selected agent is

$$\mathcal{E} \quad = \quad nyq_Y E_Y + n(1-y)q_N E_N. \tag{19}$$

[8] The assumption $q_Y(y) > q_N(y)$ in Remark 2 cannot be replaced by the weaker condition (18); this condition would leave open the possibility of a purely-random-assignment rule in which $y = 1$ is in fact an equilibrium.

Hence, the planner's (binary-second-best) problem is to

$$\max_{p_1 \cdots, p_{n-1}, y} \mathcal{E}$$

$$\begin{aligned} \text{s.t.} \quad & 0 \leq p_j \leq 1 \qquad (j = 1, \ldots, n-1), \\ & 0 \leq y \leq 1, \\ & \Delta(y) = 0, \\ & q_Y(y) - q_N(y) \geq 0. \end{aligned}$$

Using (9), we can alternatively write the objective purely in terms of q_Y, as

$$\mathcal{E} \quad = \quad nyq_Y(E_Y - E_N) + E_N. \tag{20}$$

We will solve this problem in Section 3.2.

The binary-first-best problem

In this section, we solve, as a benchmark, the problem of a planner who is not restricted by equilibrium constraints. Secondly, we show how the cost of performing the task creates a conflict between the solution to the first-best problem and the equilibrium condition. Finally, we characterize the large-population limit of the binary-first-best solution.

The planner's *binary-first-best problem* is as follows:

$$\max_{p_1 \cdots, p_{n-1}, y} \mathcal{E}$$

$$\begin{aligned} \text{s.t.} \quad & 0 \leq p_j \leq 1 \qquad (j = 1, \ldots, n-1), \\ & 0 \leq y \leq 1. \end{aligned}$$

The interpretation is that, by setting any y, the planner has the power to make the types in $[F^{-1}(1-y), \overline{\theta}]$ play Y and to make the types in $[\underline{\theta}, F^{-1}(1-y)]$ play N.

The binary-first-best problem maintains the restriction to binary mechanisms and threshold strategies. The standard first best, in contrast, is defined without these restrictions. The solution to the *standard* first-best problem is to always assign the task to the agent with the highest ability among all agents. Given that a continuum of types exist, this solution can obviously not be reached exactly with a binary mechanism.

The mechanism

$$(p_1 \ldots, p_{n-1}) \quad = \quad (1, \ldots, 1)$$

is called the *any-volunteer rule.*

Proposition 1. *The solution to the binary-first-best problem involves using the any-volunteer rule. Denoting by* y^{b*} *the volunteering rate in a solution, we have* $0 < y^{b*} < 1$ *and* $d\mathcal{E}/dy|_{y=y^{b*}} = 0$.

Proof. Given any $0 < y < 1$, (16) together with (20) shows that the optimal mechanism maximizes $q_Y(y)$. Thus, from (4) the any-volunteer rule is the unique optimal mechanism if $0 < y < 1$. Moreover, the any-volunteer rule is *an* optimal mechanism if $y = 0$ or $y = 1$. Consequently, a binary-first-best optimal $y = y^{b*}$ is found by solving the problem

$$\max_y \mathcal{E} \ \text{ s.t. } \ 0 \leq y \leq 1,$$

where $\mathcal{E}$ is evaluated specifically for the case of the any-volunteer rule $(p_1 \ldots, p_{n-1}) = (1, \ldots, 1)$.

In the case of the any-volunteer rule, (2) and (3) imply that, for all y,

$$h_Y(y) = 1 \quad \text{and} \quad h_N(y) = 1 - (1-y)^{n-1}. \tag{21}$$

Thus, (7) implies that

$$q_Y(y) = \frac{1}{ny}(1 - (1-y)^n) \quad \text{if } y > 0. \tag{22}$$

Also, (5) implies that

$$q_N(y) = \frac{1}{n} B_y^{n-1}(0) = \frac{1}{n}(1-y)^{n-1}. \tag{23}$$

Thus, using (19), in the case of the any-volunteer mechanism,

$$\mathcal{E} = (1 - (1-y)^n) E_Y + (1-y)^n E_N.$$

The first-order effect of increasing y is

$$\frac{d\mathcal{E}}{dy} = n(1-y)^{n-1}(E_Y - E_N) + (1 - (1-y)^n)\frac{dE_Y}{dy} + (1-y)^n \frac{dE_N}{dy},$$

where

$$\frac{dE_Y}{dy} = \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{y}\int_{F^{-1}(1-y)}^{\overline{\theta}} \theta \mathrm{d}F(\theta)\right) = \frac{F^{-1}(1-y) - E_Y}{y} \tag{24}$$

and

$$\frac{dE_N}{dy} = \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{1-y}\int_{\underline{\theta}}^{F^{-1}(1-y)} \theta \mathrm{d}F(\theta)\right) = \frac{E_N - F^{-1}(1-y)}{1-y}. \tag{25}$$

Clearly, any binary-first-best y^{b*} satisfies $0 < y^{b*} < 1$ because otherwise the purely-random assignment would obtain. To confirm, one can verify that $d\mathcal{E}/dy|_{y=1} = \underline{\theta} - E[\theta] < 0$ and $d\mathcal{E}/dy|_{y=0} = (n-1)(\overline{\theta} - E[\theta]) > 0$. This completes the proof of Proposition 1. □

In the proof above we have evaluated the welfare effect of marginally increasing the volunteering rate y when the any-volunteer rule is used. If a general binary-action rule is used, there is a surprisingly simple and useful formula that connects this welfare effect to the marginal type's utility gain from playing Y versus playing N. This formula, stated in Lemma 3, captures how the conflict between the planner's welfare goal and an agent's equilibrium condition depends on the cost of performing the task.

Lemma 3. *Consider any mechanism and any (not necessarily equilibrium) threshold strategy y. Then*

$$\frac{d\mathcal{E}}{dy} = n\Delta(y) + n(q_Y(y) - q_N(y))c.$$

Proof. Applying the product differentiation rule to (19), we find

$$\begin{aligned}\frac{1}{n}\frac{d\mathcal{E}}{dy} &= q_Y E_Y - q_N E_N + y\frac{dq_Y}{dy}E_Y + (1-y)\frac{dq_N}{dy}E_N \\ &\quad + yq_Y\frac{dE_Y}{dy} + (1-y)q_N\frac{dE_N}{dy}. \end{aligned} \tag{26}$$

Plugging (24), (25), (10), and (11) into (26), we get

$$\frac{1}{n}\frac{d\mathcal{E}}{dy} = (E_Y - E_N)(h_Y - h_N) + q_N E_N - q_Y E_Y + F^{-1}(1-y)(q_Y - q_N).$$

Using (14) and the definition of $\Delta(y)$ in (17), the claimed formula follows. □

An immediate implication of Lemma 3 is that, if the cost of performing is positive ($c > 0$) and an agent's task-assignment probability is not independent of her action (i.e., $q_Y(y) > q_N(y)$), then at any equilibrium volunteering rate y (i.e., $\Delta(y) = 0$), the welfare is strictly increasing in the volunteering rate. This free-rider problem vanishes if the volunteering cost is equal to 0.

Corollary 1. *Assume $c = 0$. Then the any-volunteer rule, together with any binary-first-best volunteering probability, solves the planner's binary-second-best problem.*

Proof. If $c = 0$, then any binary-first-best volunteering probability $y = y^{b*}$ is an equilibrium in the any-volunteer rule. This is because $\frac{d\mathcal{E}}{dy}|_{y=y^{b*}} = 0$ by binary-first-best optimality, so that Lemma 3 implies $\Delta(y^{b*}) = 0$. Moreover, as shown above, any solution to the binary-first-best problem involves the any-volunteer rule because $0 < y^{b*} < 1$. □

The binary-first-best expected ability of the selected agent is

$$\mathcal{E}^{b*} = ny^{b*}q_y(y^{b*})(E_Y(y^{b*}) - E_N(y^{b*})) + E_N(y^{b*}).$$

Remark 4 states that, in the binary first-best in a large population, the individual volunteering probability tends to 0, the probability that at least one agent volunteers tends to 1, and the expected ability of the selected agent tends to the highest possible ability. This follows from the fact that in a large population, an agent with an ability close to the highest possible ability exists with a probability close to 1. A detailed proof can be found in the Appendix.

Remark 4. *As $n \to \infty$, $y^{b*} \to 0$, $ny^{b*}q_Y(y^{b*}) \to 1$, and $\mathcal{E}^{b*} \to \overline{\theta}$.*

An immediate implication of Remark 4 is that the standard first-best is approximated by the binary first best if the population is large.

Threshold rules

A mechanism $(p_1, \ldots, p_{n-1})$ is called a *threshold rule* if there exists a number i^* $(1 \leq i^* \leq n-1)$ such that $p_j = 1$ for all $j > i^*$ and $p_j = 0$ for all $j < i^*$. Our main results will concern threshold rules.

The any-volunteer rule is a threshold rule; set $i^* = 1$ and $p_{i^*} = 1$. More generally, a threshold rule captures the idea of what we call the multiple-volunteers principle. Each agent anticipates that playing Y puts her in a lottery box together with the other Y-playing agents if altogether more than i^* players play Y, and releases her from the task if altogether fewer than i^* players play Y. If the threshold number i^* is reached exactly, the decision whether or not she will be in the lottery box may itself be randomized (via the probability p_{i^*}). Stipulations are analogous if the agent plays N. If the number of other agents who play Y equals $i^* - 1$ or i^*, then the agent can be pivotal, that is, her own action choice can have an impact on whether the task is assigned via a lottery among the Y-players or via a lottery among the N-players.

From a given agent's point of view, the pivotality of her action choice may be measured in terms of the difference between the selection probabilities defined in (2) and (3). When applied to a threshold rule, this difference simplifies to

$$h_Y(y) - h_N(y) \quad = \quad B_y^{n-1}(i^*)(1 - p_{i^*}) + B_y^{n-1}(i^* - 1)p_{i^*}. \qquad (27)$$

This difference will play an important in our analysis. In particular, a very useful property is its quasiconcavity: as the volunteering rate increases, the pivotality first increases and then decreases. More precisely, the following holds.

Lemma 4. *If $n = 2$, then the threshold rule with $i^* = 1$ and $p^{i^*} = 1/2$ satisfies $h_Y(y) - h_N(y) = 1/2$ for all $y \in [0, 1]$.*

For any other threshold rule if $n = 2$, and for any threshold rule if $n \geq 3$,

$$\begin{aligned} &\exists y^{*m} \in [0,1]\ \forall y \in (0,1): \qquad (28)\\ &\quad (h_Y - h_N)'(y) > 0 \text{ if } y < y^{*m}, \text{ and } (h_Y - h_N)'(y) < 0 \text{ if } y > y^{*m}. \end{aligned}$$

Note that formula (28) is immediate from standard properties of binomial probabilities if $p_{i^*} = 1$ or $p_{i^*} = 0$. The complete proof, in which we also consider the "mixed" cases where $0 < p_{i^*} < 1$, and the special case $n = 2$, is relegated to the Appendix.

For later reference, we restate the other two selection-probability functions as specialized for a threshold rule:

$$q_Y(y) \quad = \quad \sum_{j=i^*}^{n-1} B_y^{n-1}(j)\frac{1}{j+1} + B_y^{n-1}(i^* - 1)p_{i^*}\frac{1}{i^*} \qquad (29)$$

and

$$q_N(y) \quad = \quad \sum_{j=0}^{i^*-1} B_y^{n-1}(j)\frac{1}{n-j} + B_y^{n-1}(i^*)(1 - p_{i^*})\frac{1}{n - i^*}. \qquad (30)$$

3 Results

3.1 Improvement over the purely-random assignment

In this section, we define non-extreme threshold rules and show that any such rule always has an equilibrium such that every type of agent is strictly better off than in a purely-random assignment (Proposition 3). The any-volunteer rule, in general, does not have this strict-improvement property (Proposition 4).

We begin by showing that any binary mechanism provides a weak improvement compared to a purely-random assignment, and formulate conditions for a strict improvement (Proposition 2). The weak-improvement property justifies our formulation of the designer's problem without a participation constraint: if upon rejection of the planner's rule a purely-random assignment obtains, all types find it weakly optimal to participate in the rule.

Proposition 2. *Consider any mechanism-threshold-equilibrium combination* $(p_1 \ldots, p_{n-1}, y)$. *Then all types are at least as well off as in the purely-random assignment. If* $0 < y < 1$ *and* $q_Y(y) > 1/n$, *then all types are strictly better off than in the purely-random assignment.*

The proof of the "at-least-as-well" part follows from Remark 1. The proof of the "strictly" part of Proposition 2 is as follows. Consider an equilibrium y and consider any agent with a given type. Suppose that this type of the agent deviates from the equilibrium by volunteering with probability y and not volunteering with probability $1-y$. Because the agent mimics the average behavior of any other agent, she will be selected with probability $1/n$; in this event, her payoff is the same as in a purely-random assignment. In the complementing event that the agent is not selected, her payoff is the same as her ex-ante expected payoff when she follows the equilibrium strategy, conditioning on the same event. This payoff equals the equilibrium expected ability of the selected agent, which is higher than the expected ability in a random assignment if $q_Y > q_N$. The formal proof can be found in the Appendix.

Proposition 2 does not answer the question whether or not a strict improvement over the purely-random assignment is possible at all. Proposition 3 gives an affirmative answer for all group sizes $n > 2$.

A threshold rule is called *non-extreme* if the assignment probability to a volunteer is below pure randomness if there is a single volunteer (i.e., $p_1 < 1/n$), and the assignment probability to a non-volunteer is below pure randomness if there is a single non-volunteer (i.e., $p_{n-1} > 1-1/n$). This condition is satisfied for all threshold rules with $2 \leq i^* \leq n-2$. A non-extreme threshold rule exists if and only if $n > 2$.

Proposition 3 shows that any non-extreme threshold rule has an equilibrium that satisfies the strict-improvement conditions stated in Proposition 2 and, thus, is not equivalent to a purely-random assignment.[9] This conclusion

[9] Among the extreme threshold rules are the any-volunteer rule (where $p_1 = 1$) and the all-volunteer rule (where $p_{n-1} = 0$). These rules lead to fundamentally different incentives from the non-extreme threshold rules. We discuss the any-volunteer rule at the end of this section. In the all-volunteer rule, a threshold equilibrium with a positive level of volunteering does not exist because $q_Y(y) < 1/n$ for all $y < 1$.

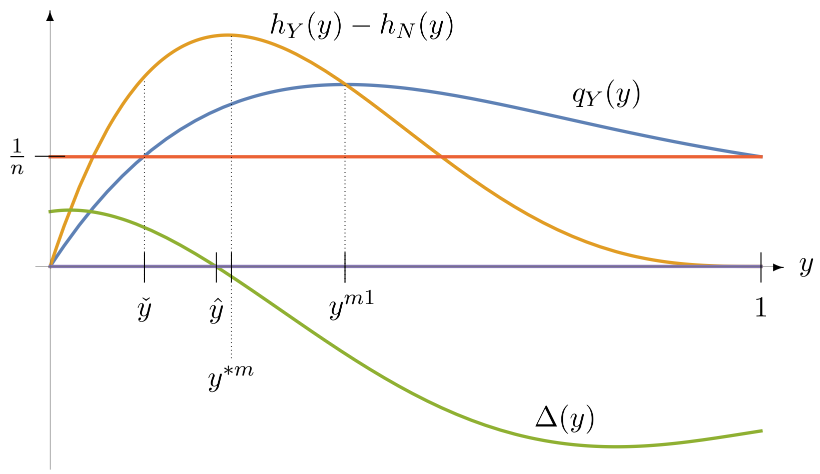

Figure 1: An example of a task assignment problem. There are $n = 5$ agents. The ability of each agent is uniformly distributed on $[0, 1]$. The performance cost is $c = 1$. The diagram shows several functions of the volunteering rate y, for the case of the threshold rule that requires two volunteers (i.e., $i^* = 2$ and $p_{i^*} = 1$). The function $h_Y(y) - h_N(y)$ captures the impact of an agent's switch from non-volunteering to volunteering on the probability (computed from the switching agent's point of view) that the task gets assigned to a volunteer. The function $q_Y(y)$ captures the probability an agent assigns to the event of being selected if she volunteers. The function $\Delta(y)$ captures an agent's payoff gain from volunteering. The volunteering rate $\hat{y}$ is an equilibrium that satisfies the strict-improvement conditions stated in Proposition 2.

holds no matter how large the cost c is.

Proposition 3. *For any non-extreme threshold rule, the set* $\{y \in [0, 1] \mid \Delta(y) = 0\}$ *is non-empty and its maximal element,* $\hat{y}$*, is an equilibrium with* $0 < \hat{y} < 1$ *and* $q_Y(\hat{y}) > 1/n$*.*

The main step towards proving Proposition 3 is Lemma 5. This result concerns the impact of an agent's switch from non-volunteering to volunteering on the probability $h_Y - h_N - q_Y$ that a volunteer other than herself gets assigned the task. Suppose this impact is strictly positive if the volunteering rate is small, is strictly negative if the volunteering rate is large, and the quasiconcavity condition (28) is satisfied. The Lemma shows that there is only one volunteering rate such that the impact equals 0, that is, the impact function changes its sign only once.

Lemma 5. *Consider any binary mechanism such that (28) holds. Assume that*

$$h_Y(y) - h_N(y) - q_Y(y) > 0 \text{ for all } y > 0 \text{ sufficiently close to 0,} \quad (31)$$
$$\text{and } h_Y(y) - h_N(y) - q_Y(y) < 0 \text{ for all } y < 1 \text{ sufficiently close to 1.} \quad (32)$$

Then there exists a unique $y^{m1} \in (0, 1)$ *such that*

$$h_Y(y^{m1}) - h_N(y^{m1}) - q_Y(y^{m1}) = 0.$$

The proof of Lemma 5 can be found in the appendix. Here is a sketch. We have to show that the functions q_Y and $h_Y - h_N$ intersect only once (cf.

Figure 1). By assumption (32), q_Y lies above $h_Y - h_N$ at $y = 1$. Because q_Y lies below $h_Y - h_N$ at small values of y, and lies above at large values, there is a maximal intersection point y^{m1}. What we have to show is that another, earlier, intersection point cannot exist. We prove this in two steps. First, there exists no interval bounded by intersection points $\underline{y}^1$ and $\overline{y}^1$ such that at the points in the interior of the interval q_Y lies above. Second, q_Y actually lies below $h_Y - h_N$ at all points smaller than y^{m1}. The crucial tool for both steps is (10).

As for the first step, by construction of the supposed interval, $h_Y - h_N$ is at most as steep as q_Y at the left boundary point $\underline{y}^1$, and is at least as steep as q_Y at the right boundary point $\overline{y}^1$. Because both points are intersection points, the function q_Y has a horizontal tangent at these points by (10). Therefore, $h_Y - h_N$ must have a non-positive derivative at $\underline{y}^1$ and a non-negative derivative at $\overline{y}^1$, contradicting the quasilinearity assumption (28). Thus, the supposed interval cannot exist, showing that $h_Y - h_N$ is at least as large as q_Y at all points to the left of y^{m1}.

As for the second step, suppose that there exists an intersection point smaller than y^{m1}. At this point, by the first step, the functions $h_Y - h_N$ and q_Y must have the same slope, which by (10) equals 0. Now using the quasilinearity assumption (28), this intersection point must be the point y^{*m}. Again using (10), the function q_Y is strictly increasing on the interval $[y^{*m}, y^{m1}]$, contradicting the fact that on this same interval the function $h_Y - h_N$ is strictly decreasing by the quasilinearity assumption (28). This completes the proof of Lemma 5.

The proof of Proposition 3 begins by showing that every non-extreme threshold rule satisfies the assumptions of Lemma 5. By Lemma 4, (28) holds. To get the intuition for why (31) is satisfied, consider a threshold rule with $p_{i^*} = 1$. Concerning both functions, the pivotality $h_Y(y) - h_N(y)$ and the individual assignment probability q_Y, the relevant event is that at least $i^* - 1$ other agents choose to volunteer. Conditioning on this event, and considering a small volunteering rate y, it is then extremely likely that the threshold i^* is reached exactly, so that the agent is almost certainly pivotal, but she herself gets assigned the task only with probability $1/i^*$. Thus, for all $i^* \geq 2$ the function $h_Y(y) - h_N(y)$ lies above the function q_Y if y is small; in the case $i^* = 1$ this argument breaks down and the non-extremeness assumption becomes relevant.

To see why (32) is satisfied, consider any agent who believes that everybody else volunteers ($y = 1$). Then switching her action from non-volunteering to volunteering changes the probability that a volunteer is selected from p_{n-1} to 1. Thus, $h_Y(1) - h_N(1) = 1 - p_{n-1}$, whereas the individual assignment probability is $q_Y(1) = 1/n$, independently of the underlying rule. Hence, (32) is immediate from the non-extremeness assumption.

Given that the conditions of Lemma 5 are satisfied, the next observation is that $q_Y(y^{m1}) > 1/n$; this follows because $q_Y(1) = 1/n$ and q_Y is strictly decreasing on the interval $[y^{m1}, 1]$ by (10). Lowering y further below y^{m1}, we reach a point $\check{y} < y^{m1}$ where $q_Y(\check{y}) = 1/n$ (because $q_Y(0) < 1/n$ by non-extremeness). Note that q_Y lies strictly above $1/n$ on the open interval $(\check{y}, 1)$. Using Remark 3, $q_N(\check{y}) = 1/n$. That is, at the point $\check{y}$, the agent's action has no impact on the probability that she gets assigned the task. On the other hand, Lemma 5 implies that $h_Y - h_N$ lies above q_Y at the point $\check{y}$. That is, switching from non-volunteering to volunteering increases the probability that the task gets assigned to a volunteer other than herself. Thus, the payoff gain from switching is strictly positive at the point $\check{y}$. Finally, the payoff gain is clearly strictly negative at the point where everybody else volunteers ($y = 1$). Thus, the maximal point $\hat{y}$ where the payoff gain equals 0, lies strictly between $\check{y}$ and 1, implying that the strict-improvement conditions stated in Proposition 2 are satisfied at this equilibrium. The formal proof of Proposition 3 is relegated to the Appendix.

The following result is immediate from Proposition 2 and Proposition 3.

Corollary 2. *Suppose that there are $n \geq 3$ agents. Then the purely-random assignment does not solve the binary second-best problem.*

In the case $n = 2$, it is straightforward to verify that $q_Y(y) - q_N(y) = p_1 - 1/2$ for all $y \in [0, 1]$ and any mechanism $p_1 \in [0, 1]$, and (14) simplifies to $U_Y - U_N = (p_1 - 1/2)(\theta - c - E[\theta])$. Thus, a threshold equilibrium y with $0 < y < 1$ and $q_Y(y) - q_N(y) > 0$ (or, equivalently, $q_Y(y) > 1/2$) exists if and only if $p_1 > 1/2$ and $c < \overline{\theta} - E[\theta]$. Thus, in the case $n = 2$ the purely-random assignment is optimal if and only if $c \geq \overline{\theta} - E[\theta]$.

We end this section with a discussion of the any-volunteer rule. We show that there always exists an equilibrium in threshold form. If the cost is low, then a strict improvement over the purely-random assignment is achieved in equilibrium; if the cost is high, then the threshold will be such that nobody volunteers and the purely-random assignment obtains. Thus, the incentives in the any-volunteer rule differ fundamentally from the incentives in a non-extreme threshold rule: a non-extreme threshold rule always allows for an improvement over the purely-random assignment, while the any-volunteer rule does not.

Proposition 4. *If $c < \overline{\theta} - E[\theta]$, then the any-volunteer rule has a threshold equilibrium y such that $0 < y < 1$ and $q_Y(y) > q_N(y)$.*

If $c \geq \overline{\theta} - E[\theta]$, then the unique equilibrium of the any-volunteer rule is $y = 0$, so that the purely-random allocation obtains. If $c < \overline{\theta} - E[\theta]$, then any equilibrium y satisfies $0 < y < 1$.

The reason the any-volunteer rule can lead to the breakdown of volunteering can be understood if we consider an agent of highest ability who believes that nobody else will volunteer. Switching her action from non-volunteering to volunteering raises the probability that she herself gets assigned the task by $1 - 1/n$. At the same time, the switch reduces, by the same amount, the probability that a non-volunteer other than herself is selected. Thus, the agent faces an equal-probability tradeoff between the payoff from volunteering herself, $\overline{\theta} - c$, and the payoff from letting somebody else do the job, $E[\theta]$. Thus, she will not volunteer if the cost is high. This argument shows that in case $c \geq \overline{\theta} - E[\theta]$ there exists no equilibrium $y \neq 0$ in the vicinity of 0. To provide a complete proof of Proposition 4, we must also exclude equilibria y arbitrarily far away from 0. All the remaining steps can be found in the Appendix.

3.2 Optimality of a threshold rule

In this section, we show that the solution to the planner's problem always involves a threshold rule (Proposition 5). Towards proving this, it is useful to know that the equilibrium condition can be relaxed so that it becomes an inequality.

Lemma 6. *Any solution to the binary second-best problem also solves the relaxed problem in which the constraint* $\Delta(y) = 0$ *is replaced by the inequality* $\Delta(y) \geq 0$.

Proof. Let $(p_1, \ldots, p_{n-1}, y)$ be a solution to the relaxed problem. Suppose first that $q_Y(y) = q_N(y)$. Then $q_Y(y) = q_N(y) = 1/n$ by (9), implying $\mathcal{E} = E[\theta]$ by the law of iterated expectations. Thus, the value at the optimum of the relaxed problem equals the value at the optimum of the planner's problem. This implies the desired conclusion.

Now consider cases in which $q_Y(y) > q_N(y)$. Suppose that $y = 1$. Applying the equation (45) at $\theta = \underline{\theta} = F^{-1}(1-y)$,

$$\Delta(y) = U_Y(y, F^{-1}(1-y)) - U_N(y, F^{-1}(1-y)) = (\frac{1}{n} - 1 + p_{n-1}) \underbrace{(\underline{\theta} - c - E[\theta])}_{<0}.$$

The right-hand side is < 0 because the constraint $q_Y > q_N$ implies $\frac{1}{n} - 1 + p_{n-1} > 0$, yielding a contradiction to the relaxed inequality constraint.

We conclude that $y < 1$. Suppose that $\Delta(y) > 0$. Applying Lemma 3, $d\mathcal{E}/dy > 0$. This is a contradiction to optimality because none of the constraints on y is binding. □

Proposition 5. *Any solution to the planner's problem involves a threshold rule.*

Proof of Proposition 5. Consider any solution $(p_1 \ldots, p_{n-1}, y)$.

If $n = 2$, then we have nothing to prove because any binary mechanism is a threshold rule.

Assume that $n \geq 3$. Corollary 2 implies that $0 < y < 1$ because otherwise $(p_1 \ldots, p_{n-1}, y)$ would be equivalent to a purely-random assignment. Similarly, $q_Y(y) > q_N(y)$ because otherwise $q_Y(y) = 1/n = q_N(y)$ from (9), implying purely-random-assigment payoffs by Remark 1.

By Lemma 6, $(p_1 \ldots, p_{n-1}, y)$ solves the relaxed problem.

Fixing y, the remaining relaxed maximization problem over $(p_1, \ldots, p_{n-1})$ is a linear problem. Hence the Lagrange conditions are necessary and sufficient, without any qualification. Let $\lambda \geq 0$ denote the Lagrange multiplier for the constraint $U_Y(y, F^{-1}(1-y)) - U_N(y, F^{-1}(1-y)) \geq 0$. Due to $q_Y(y) > q_N(y)$, the Lagrange multiplier for the constraint $q_Y(y) - q_N(y) \geq 0$ equals 0.

Let $\hat{\theta} = F^{-1}(1-y)$. For all $j = 1, \ldots, n-1$, consider

$$\begin{aligned}\hat{s}_j &= \binom{n}{j} y^j (1-y)^{n-j} (E_Y - E_N) \\ &+ \binom{n-1}{j-1} y^{j-1}(1-y)^{n-j} \left(\lambda\left(\frac{j-1}{j}E_Y + \frac{1}{j}(\hat{\theta}-c) - E_N\right)\right) \\ &- \binom{n-1}{j} y^j (1-y)^{n-1-j} \left(\lambda\left(E_Y - \frac{n-j-1}{n-j}E_N - \frac{1}{n-j}(\hat{\theta}-c)\right)\right).\end{aligned}$$

The Lagrange conditions require:

$$\begin{array}{l}\text{if } \hat{s}_j > 0, \text{ then } p_j = 1, \\ \text{if } \hat{s}_j < 0, \text{ then } p_j = 0.\end{array} \tag{33}$$

Moreover,

$$\text{if } U_Y(y, \hat{\theta}) > U_N(y, \hat{\theta}), \text{ then } \lambda = 0. \tag{34}$$

The sign of $\hat{s}_j$ is preserved if instead of $\hat{s}_j$ we consider the variable

$$s_j = \frac{\hat{s}_j}{\binom{n}{j} y^{j-1}(1-y)^{n-j-1}} \quad \text{for all } j.$$

Thus,

$$\begin{aligned}s_j &= y(1-y)(E_Y - E_N) + \lambda\frac{j}{n}(1-y)\left(\frac{j-1}{j}E_Y + \frac{1}{j}(\hat{\theta}-c) - E_N\right) \\ &\quad -\lambda\frac{n-j}{n}y\left(E_Y - \frac{n-j-1}{n-j}E_N - \frac{1}{n-j}(\hat{\theta}-c)\right) \\ &= y(1-y)(E_Y - E_N) + \lambda\frac{1}{n}(1-y)\Big((j-1)E_Y + (\hat{\theta}-c) - jE_N\Big) \\ &\quad -\lambda\frac{1}{n}y\Big((n-j)E_Y - (n-j-1)E_N - (\hat{\theta}-c)\Big).\end{aligned} \tag{35}$$

Consider the case that $\lambda > 0$.

$$s_j = \underbrace{\lambda\frac{1}{n}(E_Y - E_N)}_{>0} j + [\text{terms independent of } j].$$

If $s_j < 0$ for all j, then (33) implies that $(p_1 \ldots, p_{n-1}) = (0, \ldots, 0)$, a threshold rule.

Otherwise let i^* be the smallest integer such that $s_j \geq 0$. Then (33) implies that $(p_1 \ldots, p_{n-1})$ is an i^*-threshold rule.

It remains to consider the case $\lambda = 0$. Then (35) implies

$$s_j = y(1-y)(E_Y - E_N).$$

That is, s_j is independent of j and $s_j > 0$, implying $p_j = 1$ for all j, that is, the solution entails the any-volunteer rule. □

The remaining question is which threshold mechanism and threshold equilibrium solves the planner's problem. We have already seen (Corollary 1) that the any-volunteer rule is uniquely optimal at $c = 0$. We will show below that threshold rules with arbitrarily large i^* can be optimal as the group size n becomes large.

3.3 Volunteering in a large population

In this section, we characterize equilibrium volunteering levels of threshold rules when the population is large (Proposition 6) and demonstrate how the first best can be approximated in a large population (Proposition 7). To simplify the notation, we only consider "pure" threshold rules, that is, we assume that $p_{i^*} = 1$, where $i^* \geq 1$ is the threshold.

Proposition 6 considers sequences of equilibria that are indexed by the population size. We show that any pure i^*-threshold rule with a sufficiently large threshold i^* has a sequence of equilibria along which the expected number of volunteers remains bounded away from 0 as the population becomes arbitrarily large; we derive a formula for the large-population limit of the expected number of volunteers. Furthermore, we obtain a formula for the limit probability that the task is assigned to a volunteer, which in turn yields a formula for the limit expected-ability of the selected agent.

Proposition 6 considers thresholds i^* so high that the inequality $c/i^* < \overline{\theta} - E[\theta]$ is satisfied.[10] This inequality is crucial towards proving the existence of equilibria with volunteering rates that stay bounded away from 0 as the population grows large. To understand why, assume a large population and consider an agent who believes that the marginally volunteering type among the other agents is close to $\overline{\theta}$; the expected ability among the other volunteers is then close to $\overline{\theta}$ as well. Conditional on the event that the required number of other volunteers $i^* - 1$ is not reached, the agent is essentially indifferent between volunteering or not because the population is large and most likely she will not be selected. Now consider the event that the required number of other volunteers $i^* - 1$ is reached. Then, for a type close to $\overline{\theta}$, the benefit from volunteering is approximately equal to the quality change from the job being done at average quality ($E[\theta]$) to the job being done at top quality ($\overline{\theta}$). On the other hand, if the number of other volunteers is exactly equal to $i^* - 1$, then the cost of volunteering is c/i^* because the agent because will be selected with probability $1/i^*$.

A central role is played by the Poisson distribution. For any $z > 0$, let $\text{Pois}(z)(i) = z^i e^{-z}/i!$ denote the probability of the realization $i = 0, 1, \ldots$ according to the Poisson distribution with expectation z. The corresponding hazard-rate function,[11]

$$h^{\text{Pois}(z)}(i) = \frac{\text{Pois}(z)(i)}{\sum_{j=i}^{\infty} \text{Pois}(z)(j)} = \frac{1}{i! \sum_{j=i}^{\infty} \frac{z^{j-i}}{j!}}, \tag{36}$$

will be used in the characterization of equilibrium volunteering.

[10] Bergstrom and Leo (2015) obtain formulas similar to those in Proposition 6 in the case $i^* = 1$, in a setting without private information. They define the coordinated volunteer's dilemma as the game in which, similar to the any-volunteer mechanism, the task is performed by a randomly selected volunteer if and only if at least one volunteer comes forward; if nobody volunteers, then the task is not performed at all. The task has a commonly known benefit b to each agent; thus, equilibria are in mixed strategies. Denoting by r^* the large-population-limit probability that at least one individual volunteers in equilibrium, formulas analogous to those in Proposition 6 hold with $i^* = 1$ and $\overline{\theta} - E[\theta]$ replaced by b.

[11] The denominator in this definition can also be written by using the upper incomplete gamma function, which is given in terms of an integral instead of an infinite sum. The infinite sum is the more useful representation for our analysis.

Proposition 6. *Consider the threshold rule with parameter* $i^* \geq 1$ *and* $p_{i^*} = 1$. *Assume that* $c/i^* < \overline{\theta} - E[\theta]$. *Given any sequence of threshold equilibria* $(\hat{y}_n)$ *defined for all population sizes* $n \geq i^*$, *let* $z_n = n\hat{y}_n$ *denote the corresponding expected number of volunteers.*

There exists a sequence of equilibria such that $\liminf_n z_n > 0$.

For any such sequence,

$$z_n \to z^*, \quad \text{where } h^{Pois(z^*)}(i^*) = \frac{c/i^*}{\overline{\theta} - E[\theta]}.$$

Let (r_n) *denote the sequence of equilibrium probabilities that the task is assigned to a volunteer, that is,* $r_n = n\hat{y}_n q_Y(\hat{y}_n)$ *for all* $n \geq i^*$. *Then*

$$r_n \to r^* \in (0,1), \quad \text{where } r^* = \sum_{j=i^*}^{\infty} Pois(z^*)(j).$$

The sequence of equilibrium levels of the expected ability of the selected agent converges to $r^*\overline{\theta} + (1 - r^*)E[\theta]$.

The intuition behind Proposition 6 is as follows (for proof details see the Appendix). First, recall that the Poisson distribution is the limit of binomial distributions as the number of draws grows large and the expected number of successes stabilizes. For large n, the number of volunteers therefore approximately follows a Poisson distribution. The intuition behind the formula for z^* stated in Proposition 6 is that for the marginally volunteering type ($\approx \bar{\theta}$ in a large population), the benefit of volunteering must be equal to the cost of volunteering. The benefit converges to the limit probability of being pivotal,

$$e^{-z^*} \frac{(z^*)^{i^*-1}}{(i^*-1)!},$$

times the induced change in the quality of the job done, which converges to $\overline{\theta} - E[\theta]$ for the marginal type. The cost of volunteering equals c times the probability of being selected, assuming that other agents use the equilibrium strategy. With a Poisson distributed number of volunteers with mean z^*, the expected cost of volunteering is

$$ce^{-z^*} \sum_{j=i^*-1}^{\infty} \frac{(z^*)^j}{j!(j+1)}.$$

Setting this equal to the above formula for the benefit of volunteering yields the formula for z^* that is stated in the proposition. The formula for r^* is straightforward from the fact that the binomial distribution converges to the Poisson distribution.

We remark that, because the function $z \mapsto h^{\text{Pois}(z)}(i)$ is strictly decreasing, the expected number of volunteers, z^*, is strictly decreasing in the ratio $c/(\overline{\theta} - E[\theta])$. Moreover, because the function $z \mapsto h^{\text{Pois}(z)}(i)$ approaches the value 1 as $z \to 0$, z^* is close to 0 if $c/(\overline{\theta} - E[\theta])$ is close to i^*.

The following result shows that the first-best optimal assignment can be approximated arbitrarily closely via an i^*-threshold rule if i^* is chosen sufficiently large and the population is sufficiently large. This result is important because it shows that binary mechanisms, although being very simple with just two possible actions for each agent, are sufficient to approximate the first best in a large population. The reason a binary mechanism *may be* good enough is that the information to be extracted from the agents is binary as well: each agent is essentially asked whether or not her type is close to the highest feasible type. The nonobvious feature of the i^*-threshold rule with large i^* is that in equilibrium it becomes almost certain that at least i^* volunteers will come forward if the population is sufficiently large.

Proposition 7. *Consider* r^* *as defined in Proposition 6. Then* $\lim_{i^* \to \infty} r^* = 1$.

Because the proof is relatively short and is best understood in algebraic terms, we present it here.

Proof of Proposition 7. From Proposition 6,

$$h^{\mathrm{Pois}(z^*)}(i^*) \quad = \quad \frac{c/i^*}{\overline{\theta} - E[\theta]}.$$

Using the shortcut $\kappa = c/(\overline{\theta} - E[\theta])$, the above equality can also be written as

$$i^* \mathrm{Pois}(z^*)(i^*) \quad = \quad \kappa \sum_{j=i^*}^{\infty} \mathrm{Pois}(z^*)(j)$$

or, using the definitions of $\mathrm{Pois}(z^*)(i^*)$ and of r^*,

$$i^* \frac{e^{-z^*}(z^*)^{i^*}}{(i^*)!} \quad = \quad \kappa r^*. \tag{37}$$

We will use the following (Chernoff) bounds for tail probabilities as applied to a Poisson random variable with mean z:

$$\sum_{j=i}^{\infty} \mathrm{Pois}(z)(j) \quad \leq \quad \frac{e^{-z}(ez)^i}{i^i} \quad \text{for all } i \geq z, \tag{38}$$

$$\sum_{j=0}^{i} \mathrm{Pois}(z)(j) \quad \leq \quad \frac{e^{-z}(ez)^i}{i^i} \quad \text{for all } i \leq z. \tag{39}$$

To prove (38), let X denote a Poisson distributed random variable with mean z. Then

$$E[\left(\frac{i}{z}\right)^X] = \sum_{k=0}^{\infty} \left(\frac{i}{z}\right)^k \frac{z^k e^{-z}}{k!} = \sum_{k=0}^{\infty} \frac{i^k}{k!} e^{-z} = e^{i-z}.$$

Thus, (38) follows from the Markov inequality:

$$E[X \geq i] = \Pr[\left(\frac{i}{z}\right)^X \geq \left(\frac{i}{z}\right)^i] \leq \frac{E[\left(\frac{i}{z}\right)^X]}{\left(\frac{i}{z}\right)^i} = e^{i-z}\left(\frac{z}{i}\right)^i.$$

The proof of (39) is analogous.

We begin by showing that

$$z^* \geq i^* \text{ for all sufficiently large } i^*. \tag{40}$$

Suppose that $z^* < i^*$. Using (38) and the definition of r^*,

$$r^* \leq \frac{e^{-z^*}(ez^*)^{i^*}}{{i^*}^{i^*}}.$$

Using (37), we can substitute r^* and obtain

$$i^* \frac{e^{-z^*}(z^*)^{i^*}}{(i^*)!} \leq \kappa \frac{e^{-z^*}(ez^*)^{i^*}}{{i^*}^{i^*}}.$$

After cancelling terms,

$$\frac{{i^*}^{i^*+1}}{(i^*)! e^{i^*}} \leq \kappa.$$

By Stirling's formula, the left-hand side tends to infinity as $i^* \to \infty$, yielding a contradiction. This shows (40).

In particular, $\sqrt{i^* - 1}/z^* \to 0$ as $i^* \to \infty$. Because the right-hand side of (37) is bounded by κ, it also follows that

$$\frac{e^{-z^*}(z^*)^{i^*-1}\sqrt{i^* - 1}}{(i^* - 1)!} \to 0.$$

By Stirling's formula,

$$\frac{e^{i^*-1-z^*}(z^*)^{i^*-1}}{(i^* - 1)^{(i^*-1)}} \to 0.$$

Thus, using (39) with $i = i^* - 1$,

$$1 - r^* \leq \frac{e^{i^*-1-z^*}(z^*)^{i^*-1}}{(i^* - 1)^{(i^*-1)}} \to 0,$$

implying $\lim_{i^* \to \infty} r^* = 1$. □

Conclusion

If a task is to be assigned among a group of agents with heterogenous abilities, the multiple-volunteers principle turns out to be a powerful tool for improving welfare. Many possible extensions and variations of the model come to mind, including heterogeneous costs, other preferences that may include altruism or behavioral elements like regret aversion, tasks that can be avoided, a task that is pleasant instead of being costly, or more complex mechanisms (e.g., sequential mechanisms). Moreover, volunteering for a lottery may be a way to signal to the public that one's own ability type is high. These extensions are left for future research.

4 Appendix

Proof of Lemma 1. Using the definition (4),

$$
\begin{aligned}
q_Y(y) &= \sum_{j=0}^{n-1} \frac{(n-1)!y^j(1-y)^{n-(j+1)}}{(j+1)!(n-(j+1))!} p_{j+1} \\
&= \frac{1}{ny} \sum_{j=0}^{n-1} B_y^n(j+1) p_{j+1}. \qquad (41)
\end{aligned}
$$

Applying the quotient rule,

$$
q_Y'(y) = \frac{\sum_{j=0}^{n-1} \left(\frac{d}{dy} B_y^n(j+1)y - B_y^n(j+1) \right) p_{j+1}}{ny^2}.
$$

Thus, using the following standard identity about the derivative of a Bernstein polynomial,

$$
\frac{d}{dy} B_y^n(j) = n \left(B_y^{n-1}(j-1) - B_y^{n-1}(j) \right), \qquad (42)
$$

and using again (41), we obtain

$$
q_Y'(y) = \frac{n \sum_{j=0}^{n-1} \left(B_y^{n-1}(j) - B_y^{n-1}(j+1) \right) p_{j+1} y - nyq_Y}{ny^2}.
$$

Now (10) follows from the definitions (2) and (3). The proof of (11) is analogous. □

Proof of Remark 1. Suppose first that $U_Y(\sigma, \theta) - U_N(\sigma, \theta) > 0$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$. Then the equilibrium condition implies that all agents always play Y, yielding a purely-random assignment, as was to be shown. The same conclusion obtains if $U_Y(\sigma, \theta) - U_N(\sigma, \theta) < 0$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$.

The remaining possibility is that $U_Y(\sigma, \theta) - U_N(\sigma, \theta)$ changes its sign, that is, there exists $\hat{\theta} \in [\underline{\theta}, \overline{\theta}]$ such that $U_Y(\sigma, \hat{\theta}) - U_N(\sigma, \hat{\theta}) = 0$. Now using (14) with $\theta = \hat{\theta}$ and recalling that $q_Y = q_N = 1/n$, it follows that

$$
0 = (h_Y - h_N - \frac{1}{n})(E_Y - E_N).
$$

In the case $E_Y = E_N$, the law of iterated expectations implies that $E_Y = E_N = E[\theta]$. Thus, equivalence holds because the right-hand sides of (12) and (13) are equal to the right-hand side of (15).

In the case $E_Y \neq E_N$, we conclude that $h_Y - h_N - \frac{1}{n} = 0$. On the other hand, (7) implies that $y = yh_Y + (1-y)h_N$. Solving the system of these two equations leads to the formulas in (6). Plugging these into (12) yields that

$$
\begin{aligned}
U_Y(\sigma, \theta) &= y \frac{n-1}{n} E_Y + \frac{1}{n} \cdot (\theta - c) + (1-y) \frac{n-1}{n} E_N \\
&= (1 - \frac{1}{n}) E[\theta] + \frac{1}{n} (\theta - c),
\end{aligned}
$$

by the law of iterated expectations. Thus, the payoff from playing Y is the same as in the random-assignment rule. The analogous statement for the action N follows from a similar computation. □

Proof of Lemma 2. Consider any mechanism-equilibrium combination

$$(p'_1, \ldots, p'_{n-1}, \sigma').$$

Analogously to (1), let $y' = \int \sigma'(\theta)\mathrm{d}F(\theta)$ denote the probability that an agent plays Y.

Any equilibrium with $y' = 0$ or $y' = 1$ yields the random-assignment allocation. Thus, an equivalent mechanism-equilibrium combination in threshold form is given by the purely-random-assignment rule together with any threshold strategy y, that is, we can set $(p_1, \ldots, p_{n-1}, y) = (1/n, \ldots, 1 - 1/n, y)$. The desired conclusion holds due to (6) and (15).

Suppose that $0 < y' < 1$. Then there exists a type $\hat{\theta} \in (\underline{\theta}, \overline{\theta})$ such that $U_Y(\sigma', \hat{\theta}) = U_N(\sigma', \hat{\theta})$. Thus, using (14),

$$U_Y(\sigma', \theta) - U_N(\sigma', \theta) \quad = \quad (q_Y(y') - q_N(y'))(\theta - \hat{\theta}). \tag{43}$$

There are three cases, (i) $q_Y(y') > q_N(y')$, (ii) "$<$", and (iii) "$=$".

In case (i), (43) implies that the strategy σ' is the threshold strategy y'. Thus, we can set $(p_1, \ldots, p_{n-1}, y) = (p'_1, \ldots, p'_{n-1}, y')$.

In case (ii), (43) implies that

$$\sigma'(\theta) = 1 \text{ for all } \theta < \hat{\theta},$$
$$\sigma'(\theta) = 0 \text{ for all } \theta > \hat{\theta}.$$

We obtain an equivalent mechanism-equilibrium combination $(p_1, \ldots, p_{n-1}, \sigma)$ by setting $p_j = 1 - p'_{n-j}$ and $\sigma(\theta) = 1 - \sigma'(\theta)$. Thus σ has the threshold form, showing that the desired conclusion holds with $y = 1 - y'$.

In case (iii), (9) implies that $q_Y(y') = 1/n = q_N(y')$. Thus, the conclusion follows from Remark 1. □

Proof of Remark 2. Given any mechanism $(p_1, \ldots, p_{n-1})$, the threshold strategy 1 leads to the following values of the conditional-expectation functions and probability-selection functions:

$$E_Y(1) = E[\theta], \;\; E_N(1) = \underline{\theta}, \;\; h_Y(1) = 1, \;\; h_N(1) = p_{n-1}, \;\; q_Y(1) = \frac{1}{n}, \;\; q_N(1) = 1 - p_{n-1}. \tag{44}$$

Plugging all of that into (14), we obtain

$$U_Y(1, \theta) - U_N(1, \theta) \quad = \quad (\frac{1}{n} - 1 + p_{n-1}) \; \underbrace{(\theta - c - E[\theta])}_{<0 \quad \text{for all } \theta \approx \underline{\theta}.} \tag{45}$$

Suppose that $y = 1$. By equilibrium, $U_Y(1, \theta) - U_N(1, \theta) \geq 0$ for all θ. But the assumption $q_Y(1) > q_N(1)$ implies that $\frac{1}{n} - 1 + p_{n-1} > 0$. This yields a contradiction with (45). □

Proof of Remark 4. There exists a sequence (y_n) such that $y_n \to 0$ and $(1 - y_n)^n \to 0$. For example, taking $y_n = 1/\sqrt{n}$, it holds that $(1 - y_n)^{\sqrt{n}} \to 1/e$ by definition of the Euler number e, and $(1/e)^{\sqrt{n}} \to 0$.

Using (22), we have $n y_n q_Y(y_n) \to 1$.

This together with $E_Y \to \overline{\theta}$ implies that $\mathcal{E} \to \overline{\theta}$. Because $\mathcal{E}^{b*} \geq \mathcal{E}$ by optimality, we conclude that $\mathcal{E}^{b*} \to \overline{\theta}$.

This implies that $y^{b*} \to 0$ and $ny^{b*}q_Y(y^{b*}) \to 1$ because otherwise $\liminf_n E_Y < \overline{\theta}$ or $\liminf_n ny^{b*}q_Y(y^{b*}) < 1$, implying $\liminf_n \mathcal{E}^{b*} < \overline{\theta}$. □

Proof of Lemma 4. Assume first that $n = 2$. Then $i^* = 1$, and (27) implies that $h_Y - h_N = y(1 - p^{i^*}) + (1 - y)p^{i^*}$. Thus, $(h_Y - h_N)'(y) = 1 - 2p^{i^*}$. If $p^{i^*} = 1/2$, then the difference $h^Y - h^N$ is constant and equal to 1/2. If $p^{i^*} < 1/2$, then $(h_Y - h_N)'(y) > 0$ for all $y \in [0, 1]$, so that we can set $y^{*m} = 1$. If $p^{i^*} > 1/2$, then $(h_Y - h_N)'(y) < 0$ for all $y \in [0, 1]$, so that we can set $y^{*m} = 0$.

Now assume that $n \geq 3$. Suppose that $i^* = 1$. Using (27), it is straightforward to verify that

$$(h_Y - h_N)'(y) \quad = \quad (1 - y)^{n-3}(n - 1)l(y),$$

where we use the auxiliary function

$$l(y) \quad = \quad 1 - 2p^{i^*} - y(n(1 - p^{i^*}) - 1),$$

which is linear in y. If $p^{i^*} \geq 1 - 1/n$, then $l(0) = 1 - 2p^{i^*} < 0$ and $l(1) = (n-2)(p^{i^*} - 1) \leq 0$, implying that $l(y) < 0$ for all $y \in [0, 1)$. Thus, (28) holds with $y^{*m} = 0$. If $p^{i^*} < 1 - 1/n$, then $l(y)$ is strictly decreasing in y, implying (28).

The case $i^* = n - 1$ is treated analogously to the case $i^* = 1$.

Suppose that $1 < i^* < n - 1$. Using (27), it is straightforward to verify that

$$(h_Y - h_N)'(y) \quad = \quad \frac{(n-1)!y^{i^*-2}(1-y)^{n-2-i^*}}{(i^*)!(n-i^*)!}l(y),$$

where we use the auxiliary function

$$\begin{aligned} l(y) \quad = \quad & y(1 - y)(1 - 2p^{i^*})i^*(n - i^*) \\ & - y^2(1 - p^{i^*})(n - i^*)(n - i^* - 1) \; + \; (1 - y)^2 p^{i^*} i^*(i^* - 1). \end{aligned}$$

Note that $l(0) = (i^* - 1)i^*p^{i^*} > 0$ if $p^{i^*} > 0$, and $l'(0) = i^*(n - i^*) > 0$ if $p^{i^*} = 0$, implying that

$$l(y) > 0 \quad \text{for all } y > 0 \text{ that are sufficiently close to } 0.$$

Similarly, $l(1) = -(n - i^*)(n - i^* - 1)(1 - p^{i^*}) < 0$ if $p^{i^*} < 1$, and $l'(1) = i^*(n - i^*) > 0$ if $p^{i^*} = 1$, implying that

$$l(y) < 0 \quad \text{for all } y < 1 \text{ that are sufficiently close to } 1.$$

Thus, by the mean-value theorem, $l(y^{*m}) = 0$ for some $y^{*m} \in (0, 1)$. Moreover, y^{*m} is unique because l is quadratic in y. Hence, for all $y \in (0, 1)$,

$$l(y) > 0 \text{ if } y < y^{*m}, \text{ and } l(y) < 0 \text{ if } y > y^{*m},$$

showing (28). □

Proof of Proposition 2. If $y = 0$ or $y = 1$, then the random assignment obviously obtains. In the case $q_Y(y) = 1/n$, (9) implies that $q_N(y) = 1/n$, so that the conclusion follows from Remark 1.

It remains to consider cases in which $0 < y < 1$ and $q_Y(y) > q_N(y)$. Using (9), $q_Y > \frac{1}{n}$ and $q_N < \frac{1}{n}$. Thus, $nyq_Y > y$ and $n(1-y)q_N < (1-y)$. Hence, (19) together with (16) implies

$$\mathcal{E} \quad > \quad yE_Y + (1-y)E_N = E[\theta],$$

where the last equality relies on the law of iterated expectations.

Thus, using (12), (13), (7), and (8), and summarizing terms, for any type θ,

$$\begin{aligned} yU_Y(y,\theta) + (1-y)U_N(y,\theta) &= (1-\frac{1}{n})\mathcal{E} + \frac{1}{n}(\theta - c) \\ &> (1-\frac{1}{n})E[\theta] + \frac{1}{n}(\theta - c), \end{aligned}$$

which is the agent's payoff from random assignment. Therefore, also the equilibrium payoff $\max\{U_Y(y,\theta), U_N(y,\theta)\}$ is strictly larger than the payoff from random assignment.

□

Proof of Lemma 5. Let

$$S \quad = \quad \{y \in (0,1) \mid q_Y(y) = (h_Y - h_N)(y)\}.$$

The set S is finite because q_Y and $h_Y - h_N$ are non-identical polynomials. From (31), (32), and the intermediate-value theorem, the set S is non-empty. Define

$$y^{m1} \quad = \quad \max S.$$

In particular, using (32), $(h_Y - h_N)(y) < q_Y(y)$ for all $y > y^{m1}$.

Next we prove that

$$\forall\ 0 < y < y^{m1}:\ (h_Y - h_N)(y) \geq q_Y(y). \tag{46}$$

Suppose (46) fails, that is, $(h_Y - h_N)(y^1) < q_Y(y^1)$ for some $0 < y^1 < y^{m1}$. From (31), the set $S \cap [0, y^1] \neq \emptyset$. Also $S \cap [y^1, 1] \neq \emptyset$ because this set contains y^{m1}. Thus, we can define

$$\underline{y}^1 = \max\left(S \cap [0, y^1]\right), \quad \overline{y}^1 = \left(\min S \cap [y^1, 1]\right).$$

By construction,

$$\forall y \in (\underline{y}^1, \overline{y}^1):\ (h_Y - h_N)(y) < q_Y(y).$$

Moreover, using (10),

$$q_Y'(\underline{y}^1) = 0 \quad \text{and} \quad q_Y'(\overline{y}^1) = 0.$$

Using this together with the fact that $(h_Y - h_N)(\underline{y}^1) = q_Y(\underline{y}^1)$ from $\underline{y}^1 \in S$, we obtain an upper bound for the derivative of $h_Y - h_N$ at $\underline{y}^1$, by taking the limit from the right:

$$\begin{aligned}(h_Y - h_N)'(\underline{y}^1) &= \lim_{y \searrow \underline{y}^1} \frac{(h_Y - h_N)(y) - (h_Y - h_N)(\underline{y}^1)}{y - \underline{y}^1} \\ &\leq \lim_{y \searrow \underline{y}^1} \frac{q_Y(y) - q_Y(\underline{y}^1)}{y - \underline{y}^1} \;=\; q_Y'(\underline{y}^1) \;=\; 0.\end{aligned}$$

Similarly, we obtain a lower bound for the derivative at $\overline{y}^1$ by taking the limit from the left and using that $(h_Y - h_N)(\overline{y}^1) = q_Y(\overline{y}^1)$:

$$\begin{aligned}((h_Y - h_N))'(\overline{y}^1) &= \lim_{y \nearrow \overline{y}^1} \frac{(h_Y - h_N)(y) - (h_Y - h_N)(\overline{y}^1)}{y - \overline{y}^1} \\ &\geq \lim_{y \nearrow \overline{y}^1} \frac{q_Y(y) - q_Y(\overline{y}^1)}{y - \overline{y}^1} \;=\; q_Y'(\overline{y}^1) \;=\; 0,\end{aligned}$$

where the inequality was reversed because $y - \underline{y}^1 < 0$.

Using (28), the above upper bound implies that $\underline{y}^1 \geq y^{*m}$, whereas the lower bound implies that $\overline{y}^1 \leq y^{*m}$, contradicting the fact that $\underline{y}^1 < \overline{y}^1$.

Now we can improve upon (46) by showing that

$$\forall\; 0 < y < y^{m1} : \; (h_Y - h_N)(y) > q_Y(y). \tag{47}$$

Suppose (46) fails, that is, $(h_Y - h_N)(y^1) = q_Y(y^1)$ for some $0 < y^1 < y^{m1}$. In particular, $q_Y'(y^1) = 0$ from (10).

Taking the limit from the right and using (46), we find that

$$\begin{aligned}((h_Y - h_N))'(y^1) &= \lim_{y \searrow y^1} \frac{(h_Y - h_N)(y) - (h_Y - h_N)(y^1)}{y - y^1} \\ &\geq \lim_{y \searrow y^1} \frac{q_Y(y) - q_Y(y^1)}{y - y^1} \;=\; q_Y'(y^1) \;=\; 0.\end{aligned}$$

Similarly, taking the limit from the left,

$$\begin{aligned}((h_Y - h_N))'(y^1) &= \lim_{y \nearrow y^1} \frac{(h_Y - h_N)(y) - (h_Y - h_N)(y^1)}{y - y^1} \\ &\leq \lim_{y \nearrow y^1} \frac{q_Y(y) - q_Y(y^1)}{y - y^1} \;=\; q_Y'(y^1) \;=\; 0.\end{aligned}$$

Thus, $((h_Y - h_N))'(y^1) = 0$, showing that $y^1 = y^{*m}$ by (28).

However, from (46) and (10), the function q_Y is strictly increasing on $[0, y^{m1}]$, implying

$$(h_Y - h_N)(y^{m1}) = q_Y(y^{m1}) > q_Y(y^{*m}) = (h_Y - h_N)(y^{*m}),$$

contradicting the fact that $(h_Y - h_N)$ is strictly decreasing on $[y^{*m}, y^{m1}]$ by (28). Thus, (47) is true. This completes the proof. □

Proof of Proposition 3. We begin by verifying the assumptions of Lemma 5. We have $n \geq 3$ because otherwise no non-extreme threshold rule exists. By Lemma 4, (28) holds.

Let i^* be minimal such that $p_{i^*} > 0$. Using (27) and (29),

$$h_Y(y) - h_N(y) = \binom{n-1}{i^*-1} y^{i^*-1} p_{i^*} + \mathcal{O}(y^{i^*}), \quad q_Y(y) = \binom{n-1}{i^*-1} y^{i^*-1} \frac{p_{i^*}}{i^*} + \mathcal{O}(y^{i^*}),$$

Thus, (31) holds if $i^* \geq 2$. If $i^* = 1$, then

$$\begin{aligned} h_Y(y) - h_N(y) &= p_1 + (n-1)y(1-p_1) - (n-1)yp_1 + \mathcal{O}(y^2), \\ q_Y(y) &= p_1 + (n-1)y\frac{1}{2} - (n-1)yp_1 + \mathcal{O}(y^2). \end{aligned}$$

Thus, (31) follows from $1 - p_1 > 1 - 1/n > 1/2$.

Using (44) and the assumption $p_{n-1} > 1 - \frac{1}{n}$,

$$h_Y(1) - h_N(1) - q_Y(1) = 1 - p_{n-1} - \frac{1}{n} < 0,$$

implying (32).

Thus, there exists y^{m1} as stated in Lemma 5. Because $q_Y(0) = p_1 \leq 1/n$, we can define

$$\check{y} = \max\{y \in (0,1] | q_Y(y) \leq 1/n\}.$$

Using (44), $q_Y(1) = 1/n$. From (10) and Lemma 5, the function q_Y is strictly increasing on $[0, y^{m1}]$ and is strictly decreasing on $[y^{m1}, 1]$. Thus,

$$q_Y(y) > 1/n \quad \text{for all} \quad y \in (\check{y}, 1), \tag{48}$$

implying $\check{y} < y^{m1}$, and Lemma 5 implies that

$$h_Y(\check{y}) - h_N(\check{y}) - q_Y(\check{y}) > 0. \tag{49}$$

Define $\check{\theta} = F^{-1}(1 - \check{y})$. From Remark 3, $q_N(\check{y}) = 1/n$. Using (14) with $y = \check{y}$ and $\theta = \check{\theta}$,

$$\Delta(\check{y}) = \underbrace{(h_Y(\check{y}) - h_N(\check{y}) - q_Y(\check{y}))}_{>0 \text{ by } (49)} \underbrace{(E_Y - E_N)}_{>0 \text{ by } (16)}.$$

On the other hand, a straightforward computation using (44) shows that

$$\Delta(1) = (p_{n-1} - 1 + \frac{1}{n})(\underline{\theta} - c - E[\theta]) < 0.$$

Thus, $\hat{y}$ as defined in the statement of the proposition satisfies $\check{y} < \hat{y} < 1$. Combining this with (48), all claims are proved. □

Proof of Proposition 4. Consider the any-volunteer rule $(p_1, \ldots, p_{n-1}) = (1, \ldots, 1)$.

For any volunteering rate $y > 0$, using (22) and (23),

$$q_Y(y) - q_N(y) = \frac{1}{ny}\left(1 - (1-y)^n - y(1-y)^{n-1}\right) = \frac{1}{ny}\left(1 - (1-y)^{n-1}\right) > 0.$$

Moreover $q_Y(0) = 1$ and $q_N(0) = \frac{1}{n}$. We conclude that $q_Y(y) > q_N(y)$ for all $0 \le y \le 1$. Thus, the right-hand side in (14) is strictly increasing in θ, showing that any equilibrium has the threshold form.

Suppose that $c < \overline{\theta} - E[\theta]$. If we had $\Delta(1) \ge 0$, then $y = 1$ would be an equilibrium, contradicting Remark 2. Thus, $\Delta(1) < 0$.

Now consider the volunteering rate $y = 0$. Note that $E_N(0) = E[\theta]$. Using (21), $h_Y(0) - h_N(0) = 1$. Recall $q_Y(0) = 1$ and $q_N(0) = \frac{1}{n}$. Thus, (14) implies

$$\Delta(0) \;=\; U_Y(0, \overline{\theta}) - U_N(0, \overline{\theta}) \;=\; (1 - \frac{1}{n})(\overline{\theta} - c - E[\theta]) \;>\; 0.$$

Thus, by the intermediate-value theorem, there exists $0 < y < 1$ such that $\Delta(y) = 0$.

Finally, suppose that $c \ge \overline{\theta} - E[\theta]$ and that some $0 < y < 1$ is an equilibrium. Let $\hat{\theta} < \overline{\theta}$ be such that $1 - F(\hat{\theta}) = y$. We will derive a contradiction to the equilibrium condition (17).

Using (14),

$$\begin{aligned} \Delta(y) &= U_Y(y, \hat{\theta}) - U_N(y, \hat{\theta}) \\ &= (h_Y - h_N)(E_Y - E_N) \\ &\quad + q_Y(-E_Y + \hat{\theta}) - q_N(-E_N + \hat{\theta}) - c(\underbrace{q_Y - q_N}_{>0}). \end{aligned} \tag{50}$$

Thus, using the assumption $c \ge \overline{\theta} - E[\theta]$,

$$\begin{aligned} \Delta(y) &\le (h_Y - h_N)(E_Y - E_N) \\ &\quad + q_Y(E[\theta] - E_Y + \hat{\theta} - \overline{\theta}) - q_N(E[\theta] - E_N + \hat{\theta} - \overline{\theta}) \\ &< (h_Y - h_N)(E_Y - E_N) \;+\; q_Y(E[\theta] - E_Y) - q_N(E[\theta] - E_N). \end{aligned}$$

Using the law of iterated expectations ($E[\theta] = yE_Y + (1 - y)E_N$), the inequality derived above can also be written as

$$\begin{aligned} \Delta(y) &< (h_Y - h_N)(E_Y - E_N) \;+\; q_Y \cdot (1 - y)(E_N - E_Y) - q_N \cdot y(E_Y - E_N) \\ &= \frac{1}{ny} g(y)(E_Y - E_N), \end{aligned}$$

where we use the auxiliary function

$$g(y) \;=\; (h_Y - h_N)yn - q_Y \cdot (1 - y)yn - q_N \cdot y^2 n.$$

Due to (16), the desired contradiction $\Delta(y) < 0$ to the equilibrium condition (17) is obtained once we show that $g(y) \le 0$ for all $y > 0$.

Using (21), (22), and (23),

$$g(y) \;=\; (1 - y)^{n-1} yn - (1 - (1 - y)^n)\,(1 - y) - (1 - y)^{n-1} y^2.$$

It is straightforward to verify that $g(y) = 0$ for all y if $n = 2$. Consider the case $n \ge 3$. Then one can verify that $g(0) = 0$, $g(1) = 0$, $g'(0) = 0$, $g'(1) = 1$, and $g''(y) = (n - 2)(n - 1)(1 - y)^{n-3}(ny - 1)$. Thus, $g''(y) < 0$ if $y < 1/n$ and $g''(y) > 0$ if $y > 1/n$. In particular, the function g' is strictly decreasing on the interval $[0, 1/n]$ and is strictly increasing on the interval $[0, 1/n]$. This together with $g'(0) = 0$ and $g'(1) = 1$ shows that there exists a threshold $\breve{y}$ such that $g'(y) < 0$ if $y < \breve{y}$ and $g'(y) > 0$ if $y > \breve{y}$. In other words, g is strictly decreasing on the interval $[0, \breve{y}]$ and is strictly increasing on the interval $[\breve{y}, 1]$. This together with the equations $g(0) = 0$ and $g(1) = 0$ show that $g(y) < 0$ for all $0 < y < 1$. □

Proof of Proposition 6. Before presenting the proof, we provide a roadmap. *Step 0* recalls the Poisson approximation of the binomial distribution.

Using the assumption $c/i^* < \overline{\theta} - E[\theta]$, *Step 1* shows that there exists a sequence of strategies along which the expected number of volunteers does not vanish and the marginally volunteering type strictly prefers volunteering over non-volunteering. Lowering the marginal type until she becomes indifferent then yields a sequence of equilibria with non-vanishing expected numbers of volunteers z_n (this is *Step 2*).

In *Step 3* we show that z_n is bounded. Suppose otherwise, that is, $z_n \to \infty$ on some subsequence. That is, the expected number of volunteers tends to infinity. Given that only i^* volunteers are needed, the probability that the task is assigned to a non-volunteer falls to zero so fast along the subsequence that the probability tends to 0 even if it is first telescoped by z_n. On the other hand, an agent expects that she is selected with a probability approximately equal to $1/z_n$ if she volunteers. Therefore, after telescoping payoffs by z_n and defining the marginal type $\hat{\theta}_n = F^{-1}(1 - \hat{y}_n)$, type $\hat{\theta}_n$'s limit payoff difference between volunteering and non-volunteering is $-E_y + \hat{\theta}_n - c \leq -c$, contradicting equilibrium.

Step 4 considers any large-population limit point z^* of the sequence z_n (existence of a limit point follows from *Step 3*). We show that z^* satisfies the equation that is stated in the proposition. The left-hand-side is strictly decreasing by (36), showing that the limit point is unique. Elementary properties of the Poisson distribution are useful: the probability $\text{Pois}(z^*)(i^* - 1)$ differs by a factor of z^*/i^* from the probability that there are exactly i^* volunteers; this probability differs by a factor of z^*/i^* from the probability that there are at least i^* volunteers.

Step 5 establishes the formula for r^*.

Step 0. Consider any sequence of numbers (y_n), $y_n \in [0, 1]$ and a number $z > 0$ such that $z_n \to z$, where we use the shortcuts $z_n = ny_n$. Then

$$\lim_n B_{y_n}^{n-1}(j) \quad = \quad \frac{z^j}{j!} e^{-z} \quad \text{for } j = 0, 1, \ldots, n, \tag{51}$$

and

$$\lim_n \sum_{j=i^*-1}^{n-1} B_{y_n}^{n-1}(j) \frac{1}{j+1} \quad = \quad e^{-z} \sum_{j=i^*-1}^{\infty} \frac{z^j}{(j+1)!}. \tag{52}$$

Formula (51) is the well-known Poisson limit theorem. To see (52), define

$B^{n-1}_{y_n}(j) = 0$ for all $j > n-1$ and note that

$$\sum_{j=i^*-1}^{\infty} \frac{z^j}{(j+1)!} e^{-z} \overset{(51)}{=} \sum_{j=i^*-1}^{\infty} \lim_n B^{n-1}_{y_n}(j) \frac{1}{j+1}$$

$$= \lim_n \sum_{j=i^*-1}^{\infty} B^{n-1}_{y_n}(j) \frac{1}{j+1} = \lim_n \sum_{j=i^*-1}^{n-1} B^{n-1}_{y_n}(j) \frac{1}{j+1}.$$

This completes *Step 0.*

Now fix any number $\underline{z}$ such that

$$0 < \underline{z} < \ln\left(\frac{(\overline{\theta} - E[\theta])i^*}{c}\right). \quad (53)$$

We will use the function Δ defined in (17), as applied to the threshold rule with parameter i^* and $p_{i^*} = 1$.

Step 1. For all n, define $\underline{y}_n = \underline{z}/n$. Then $\Delta(\underline{y}_n) > 0$ for all sufficiently large n.

We take the limit $n \to \infty$ in (17) with $y = \underline{y}_n$.

Because $\underline{y}_n \to 0$, only the type with the highest possible ability volunteers in the large-population limit. Thus, $E_Y \to \overline{\theta}$ and $E_N \to E[\theta]$.

Applying (51) with $j = i^* - 1$ to (27) with $p_{i^*} = 1$,

$$\lim_n h_Y(\underline{y}_n) - h_N(\underline{y}_n) = \frac{(\underline{z})^{i^*-1}}{(i^*-1)!} e^{-\underline{z}}.$$

Applying (52) to (29) with $p_{i^*} = 1$,

$$\lim_n q_Y(\underline{y}_n) = e^{-z} \sum_{j=i^*-1}^{\infty} \frac{z^j}{(j+1)!}.$$

Using (30),

$$q_N(\underline{y}_n) \le \frac{1}{n-i^*} \to 0 \text{ as } n \to \infty.$$

Thus, using (17) and cancelling terms,

$$\lim_n \Delta(\underline{y}_n) = \frac{(\underline{z})^{i^*-1}}{(i^*-1)!} e^{-\underline{z}} (\overline{\theta} - E[\theta]) + e^{-\underline{z}} \sum_{j=i^*-1}^{\infty} \frac{(\underline{z})^j}{(j+1)!}(-c). \quad (54)$$

Note that

$$\sum_{j=i^*-1}^{\infty} \frac{(\underline{z})^j}{(j+1)!} \le \sum_{j=i^*-1}^{\infty} \frac{(\underline{z})^{i^*-1}(\underline{z})^{j-i^*+1}}{(i^*)!(j-i^*+1)!} = \frac{(\underline{z})^{i^*-1}}{(i^*)!} e^{\underline{z}}.$$

Thus, (54) implies

$$\begin{aligned}
\lim_n \Delta(\underline{y}_n) &\ge \frac{(\underline{z})^{i^*-1}}{(i^*-1)!} e^{-\underline{z}} (\overline{\theta} - E[\theta]) + \frac{(\underline{z})^{i^*-1}}{(i^*)!}(-c) \\
&= \frac{(\underline{z})^{i^*-1}}{(i^*-1)!} \left(e^{-\underline{z}} (\overline{\theta} - E[\theta]) - \frac{c}{i^*}\right) \\
&> 0,
\end{aligned}$$

where the last inequality follows from (53). This completes *Step 1*.

Step 2. For all sufficiently large n, there exists a threshold equilibrium y_n such that $z_n > \underline{z}$, where we define $z_n = ny_n$.

For all $i^* \geq 2$, because $\Delta(1) < 0$, the desired claim follows from *Step 1* and Proposition 3, using the Intermediate Value Theorem. Consider $i^* = 1$. It is straightforward to verify that $\Delta(1) < 0$ and $q_Y(y) > q_N(y)$ for all $y \in (0,1)$. Thus, the desired claim is immediate from *Step 1*, using the Intermediate Value Theorem.

Now consider any sequence of equilibria (y_n) such that $\liminf_n z_n > 0$.

Using (27), (29), and (30),

$$\begin{aligned}\Delta(y) \quad = \quad & \left(B_y^{n-1}(i^*)(1-p_{i^*}) + B_y^{n-1}(i^*-1)p_{i^*}\right)(E_Y - E_N) \\ & + \left(\sum_{j\geq i^*} B_y^{n-1}(j)\frac{1}{j+1} + B_y^{n-1}(i^*-1)p_{i^*}\frac{1}{i^*}\right)(-E_Y + F^{-1}(1-y) - c) \\ & - \left(\sum_{j\leq i^*-1} B_y^{n-1}(j)\frac{1}{n-j} + B_y^{n-1}(i^*)(1-p_{i^*})\frac{1}{n-i^*}\right)(-E_N + F^{-1}(1-y) - c).\end{aligned}$$

Setting $p_{i^*} = 1$, and multiplying the equilibrium condition $\Delta(y_n) = 0$ with z_n, we obtain

$$\begin{aligned}0 \quad = \quad & z_n B_{y_n}^{n-1}(i^*-1)(E_Y - E_N) \\ & + z_n \sum_{j\geq i^*-1} B_{y_n}^{n-1}(j)\frac{1}{j+1}(-E_Y + F^{-1}(1-y_n) - c) \\ & - z_n \sum_{j\leq i^*-1} B_{y_n}^{n-1}(j)\frac{1}{n-j}(-E_N + F^{-1}(1-y_n) - c). \qquad (55)\end{aligned}$$

Step 3. The sequence (z_n) is bounded.

Suppose otherwise, that is, along some subsequence, $z_n \to \infty$. Then, along this subsequence,

$$B_{y_n}^n(j) \to 0 \quad \text{for all } j = 0, 1, \ldots. \qquad (56)$$

To see this, note that, due to elementary properties of the binomial distribution,

$$B_{y_n}^n(j) \leq n^j(y_n)^j(1-y_n)^{n-j} = (ny_n)^j(1-y_n)^{n-j},$$

implying

$$\ln\left(B_{y_n}^n(j)\right) \leq j\ln(ny_n) + (n-j)\ln(1-y_n).$$

Hence, using the elementary inequalities $\ln(1-y_n) \leq -y_n$ and $y_n \leq 1$,

$$\ln\left(B_{y_n}^n(j)\right) \leq j\ln(ny_n) - (n-j)y_n \leq j\ln(z_n) - z_n + j \to -\infty.$$

This implies (56).

From (56) it follows that

$$z_n B_{y_n}^{n-1}(j) \to 0 \quad \text{for all } j = 0, 1, \ldots. \qquad (57)$$

To see this, note that, by elementary properties of the binomial distribution,

$$ny_n B_{y_n}^{n-1}(j) \quad = \quad (j+1)B_{y_n}^n(j+1). \tag{58}$$

From (57) it follows that

$$z_n \sum_{j=0}^{i^*-1} B_{y_n}^{n-1}(j) \to 0. \tag{59}$$

Now we consider the limit $n \to \infty$ in (55). From (57) and (59), the first and third terms vanish, and in the second term the range of the sum can be replaced by $\sum_{j \geq 0}$. Using (58),

$$\lim_n z_n \sum_{j=0}^{n-1} B_{y_n}^{n-1}(j) \frac{1}{j+1} = \lim_n \sum_{j=0}^{n-1} B_{y_n}^n(j+1) = 1 - \lim_n B_{y_n}^n(0) \overset{(56)}{=} 1.$$

Plugging this into (55) yields $\lim_n -E_Y + F^{-1}(1-y_n) - c = 0$, contradicting the fact that the average volunteer's type must be larger than the marginal volunteer's type, $E_Y > F^{-1}(1-y_n)$. This completes *Step 3*.

Step 4. Consider any limit point z^* of (z_n). Then $h^{Pois(z^*)}(i^*) = \frac{c/i^*}{\overline{\theta}-E[\theta]}$.

To see this, consider a subsequence $z_{n_k} \to z^*$ as $k \to \infty$. A computation analogous to that leading to (54) implies

$$\lim_k \Delta(y_{n_k}) \quad = \quad \frac{(z^*)^{i^*-1}}{(i^*-1)!} e^{-z^*}(\overline{\theta} - E[\theta]) + e^{-z^*} \sum_{j=i^*-1}^{\infty} \frac{(z^*)^j}{(j+1)!}(-c).$$

Applying the equilibrium condition $\Delta(y_{n_k}) = 0$,

$$0 \quad = \quad \frac{(z^*)^{i^*-1}}{(i^*-1)!} e^{-z^*}(\overline{\theta} - E[\theta]) + \sum_{j=i^*-1}^{\infty} \frac{(z^*)^j}{(j+1)!} e^{-z^*}(-c).$$

After multiplying by z^*/i^* and switching to the variable $j' = j+1$ in the sum,

$$0 \quad = \quad \frac{(z^*)^{i^*}}{i^*!} e^{-z^*}(\overline{\theta} - E[\theta]) + \sum_{j'=i^*}^{\infty} \frac{(z^*)^{j'}}{j'!} e^{-z^*}(-c/i^*).$$

Thus,

$$0 \quad = \quad \text{Pois}(z^*)(i^*)(\overline{\theta} - E[\theta]) + \sum_{j'=i^*}^{\infty} \text{Pois}(z^*)(j')(-c/i^*).$$

This implies the claimed formula, completing the proof of *Step 4*.

From (36) one sees that, for any i, the function $z \mapsto h^{\text{Pois}(z)}(i)$ is strictly decreasing. Thus, the limit point z^* established in *Step 3* is unique, showing that the sequence (z_n) converges to z^*.

Step 5. The formula for r^*.

The probability that the task is assigned to a volunteer in equilibrium y_n is

$$r_n = \sum_{j=i^*}^{n} B_{y_n}^{n}(j) \overset{(58)}{=} \sum_{j=i^*}^{n} \frac{z_n}{j} B_{y_n}^{n-1}(j-1)$$

Thus, using *Step 0*,

$$\lim_n r_n = z^* \sum_{j=i^*}^{\infty} \frac{(z^*)^{j-1}}{j!} e^{-z^*} = \sum_{j=i^*}^{\infty} \frac{(z^*)^{j}}{j!} e^{-z^*}$$

This completes the proof of the proposition. □

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