Discussion Paper Series – CRC TR 224

Discussion Paper No. 313
Project B 03

English Versus Vickrey Auctions With Loss-Averse Bidders

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July 2021

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Funding by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through CRC TR 224 is gratefully acknowledged.

English versus Vickrey Auctions with Loss-Averse Bidders*

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July 14, 2021

Abstract

Evidence suggests that people evaluate outcomes relative to expectations. I analyze this expectations-based loss aversion à la Kőszegi and Rabin in the context of dynamic and static auctions, where the reference point is given by the (endogenous) equilibrium outcome. If agents update their reference point during the auction, the arrival of information crucially affects equilibrium behavior. Consequently, I show that—even with independent private values—the Vickrey auction yields strictly higher revenue than the (ascending clock) English auction, violating the well-known revenue equivalence.

Keywords: Vickrey auction, English auction, Japanese auction, expectations-based loss aversion, revenue equivalence, dynamic loss aversion, personal equilibrium

JEL classification: D03, D44

*I thank Yves Breitmoser, Françoise Forges, Matthias Hammer, Paul Heidhues, Radosveta Ivanova-Stenzel, Johannes Johnen, Thomas Mariotti, Pietro Ortoleva, Antonio Rosato, Thomas Schacherer, Ran Spiegler, Roland Strausz, Georg Weizsäcker, and three anonymous referees for helpful comments, as well as participants at the 11th World Congress of the Econometric Society (Montreal), the 2015 EEA conference (Mannheim), the Applied Theory Workshop at Toulouse School of Economics, the 2016 EARIE conference (Lisbon), the CRC conference in Berlin, the ZEW Research Seminar, and the 2017 Annual Conference of the Verein für Socialpolitik (Vienna). Funding by the German Research Foundation (DFG) through CRC TR 190 and 224 (Project B03) is gratefully acknowledged.

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1 Introduction

Auctions are a universal tool to organize sales in markets. At the core of auction theory stand the well-known revenue equivalence results. Vickrey (1961) notes the strategic equivalence between the dynamic English and the static Vickrey auction: if values are private, there is no effect of sequential information and, independent of risk attitudes, it is a weakly dominant strategy to bid (up to) one's private valuation in both formats.[1] These powerful theoretical predictions, however, stand in contrast to the experimental literature, which mostly finds lower revenues for the English auction.[2] I identify endogenous preferences in the form of expectations-based loss aversion as a possible explanation for this phenomenon.

In my model, I study the effect of dynamic information on bidding behavior when bidders are expectations-based loss averse: bidders evaluate the auction outcome relative to their reference point, formed by rational expectations. A strong belief in winning increases the emotional attachment to the object, and hence the willingness to pay. This is the well-established attachment effect.[3] I model the English auction as ascending-clock button (i.e., "Japanese") auction. The price ascends in increments. Bidders eventually drop out until only one bidder remains and the auction ends. At each increment a bidder updates her beliefs based on whether the auction has ended.[4] As this arrival of new information changes expectations during the auction, the strategic equivalence between the English auction and the Vickrey auction breaks down. In the English auction the belief in winning eventually declines for each losing bidder as the clock price approaches her bidding limit. Consequently, any time-consistent bidding plan for such a bidder features lower bids in the English auction than in the Vickrey auction, where bids are chosen optimally with respect to ex-ante beliefs. Since prices in both auctions are determined by the second-highest—losing—bidder, this discouragement effect leads to lower prices in the English auction.

[1] Myerson (1981) extends the results to show that all main auction formats give rise to the same expected revenue.

[2] See, e.g., Kagel and Levin (1993), Harstad (2000), and Li (2017).

[3] Banerji and Gupta (2014) and Rosato and Tymula (2019) provide experimental evidence that expectations indeed affect bidding in auction-like environments in the predicted way. Delgado et al. (2008) show that in auction environments the blood oxygen level in the brain responds more strongly to losses than gains and correlates to overbidding.

[4] In Section 4.4 and Appendix 6.3 I discuss an extension where dropouts of individual opponents are observable.

The model of reference-dependent preferences follows Kőszegi and Rabin (2006, 2009). Each bidder has a reference point formed by rational expectations over final transfers. Whenever new information arises, a bidder updates her reference point. At any update the bidder instantaneously experiences psychological utility of gains and losses from changes in the winning probability and the expectation on how much to pay. I assume that bidders bracket narrowly, meaning that they assign gains and losses separately to the money dimension and the goods dimension.

In the Vickrey auction, where there is no information update during the auction, bidders compare the outcome to ex-ante beliefs. If they win, they feel a gain in the goods dimension and a loss in the money dimension, both proportional to how unexpected it was for them to win. Conversely, if they lose, they feel a gain in money and a loss in good, proportional to their expectation of winning. The idea that losses are weighted more strongly than gains gives rise to an *attachment effect*: the stronger a bidder believes in winning the less willing she is to give up the good and experience a loss. Hence, similar to Lange and Ratan (2010), high types with high expectations of winning bid more than their intrinsic valuation. Low types, who feel more attached to keeping their money, underbid.

In the English auction, a bidder learns at each increment whether she might win the auction. This information permanently updates her expectations, and gives rise to reference-dependent utility in each increment.[5] Each bidder forms a time-consistent bidding plan, taking into account the effects of information revelation on expected reference-dependent utility. In my model, I take the continuous-time limit by letting the increment size go to zero, and identify two main effects of dynamic information revelation.

First, any bidder necessarily experiences a discouragement effect during the auction: as long as the auction is ongoing a bidder's chance to win declines gradually.[6] As the price approaches a bidder's bidding limit, her belief in winning approaches zero. Bidders feel no attachment effect for the object any more, since they eventually perceive themselves as a low type with respect to the remaining

[5] Kőszegi and Rabin interpret an agent's reference point as her lagged beliefs. As discussed in the literature section, recent experimental findings, however, suggest that the reference point adjusts quickly to new information. For this paper, I consider the natural and important benchmark of instantaneous updating. Whether instantaneous reference-point updating is a realistic approximation may depend on the exact auction environment, e.g., the speed at which the price augments.

[6] This effect is reminiscent of the discouragement effect in sequential auctions with reference-depend preferences in Rosato (2020).

bidders. As bidders anticipate this discouragement effect when forming their bidding plan, any time-consistent plan features lower bids in the English auction than in the Vickrey auction, where bids are optimal with respect to ex-ante beliefs.

Second, since losses are weighted more strongly than gains, expected gain-loss utility is always negative. Consequently, bidders dislike fluctuations in beliefs. They would prefer not to observe the auction process and would rather use proxies to bid on their behalf. This logic is related to Benartzi et al. (1995) and Pagel (2016), who explain the equity premium puzzle using loss aversion: since stock prices fluctuate, an investor who regularly checks her portfolio will experience negative expected gain-loss utility. This disutility makes stocks less attractive relative to bonds.

Since bidders are forward-looking, they account for these costs when they choose their equilibrium bidding strategy among the set of time-consistent plans. Intuitively, bidders would like to tend to the extremes in their bids, as such behavior exhibits an *insurance effect* on total belief fluctuations. Similar as in Mermer (2021), the ex-ante optimal plan would amount to underbidding for low types and excessive overbidding for high types. However, while low bids for low types are time consistent, high types anticipate that, due to the discouragement effect, any plan featuring high bids is time inconsistent. Consequently, in equilibrium all types bid lower in the English auction than in the Vickrey auction. While bids of low types in the English auction are determined by the insurance effect, bids of high types are bound by the discouragement effect.

As reviewed in the literature section, the theoretical predictions of my model are in line with most of the experimental literature. It is worth emphasizing, however, that all results in this paper rely on the assumption that bidders practice mental accounting (Kahneman and Tversky (1984), Thaler (1985, 1999)) with respect to gains and losses in money and goods. For most laboratory experiments the auctioned object is a money voucher. In that case mental accounting would imply that bidders regard the two sources of money as non-fungible, and fail to take compound lotteries.[7] In practice, there is various evidence of such behavior.[8]

[7]This point was first raised by Lange and Ratan (2010) who argue that if bidders bracket widely in induced value auctions, then inference from lab experiments does not carry over to commodity auctions in the field.

[8]For instance, in Rabin and Weizsäcker (2009) subjects fail to take compound lotteries over money, whereas Milkman and Beshears (2009) and Abeler and Marklein (2017) provide evidence that subjects regard different sources of money as non-fungible.

Schindler (2003) provides the only laboratory-controlled experiment that I am aware of, which tests the revenue equivalence for commodities.[9] She reports 14 percent lower revenues in English auctions, therefore confirming the findings of the induced-value literature as well as my theoretical predictions.

The contribution of my paper is twofold. First, it provides a novel rationale to explain the observed revenue gap between the two auction formats. Second, it contributes to the small body of literature on strategic interaction between loss-averse agents in a dynamic framework.

The remainder of the paper is structured as follows: Section 2 discusses the related literature, Section 3 introduces the model. Section 4 analyzes and compares equilibrium behavior in both auction formats, while Section 5 concludes. All proofs and various extensions are relegated to the appendix.

2 Related Literature

Kagel et al. (1987) first report a failure in the strategic equivalence between the Vickrey and English auction, and in particular significant overbidding in the Vickrey auction for affiliated values. Kagel and Levin (1993) and Harstad (2000) replicate these results for independent private values. More recent replications are found in Li (2017) and in Breitmoser and Schweighofer-Kodritsch (2019) for observable and non-observable dropouts.

As the violation of the strategic equivalence between the English and Vickrey auction cannot be explained with standard risk preference it is often labeled as a cognitive mistake (see Kagel et al. (1987), Harrison (1989), and Li (2017)). Overbidding in the Vickrey auction has also been attributed to non-standard preferences such as "spite" (Morgan et al. (2003)) or "joy of winning" (Cox et al. (1983)).[10] However, these preferences cannot explain a violation of the strategic equivalence between the two formats. There is a small body of literature that suggests other non-standard preferences as the source of the violation. Chew (1989) shows that the strategic equivalence breaks down if the bidders' values are

[9] The only field experiment I am aware of was conducted by Lucking-Reiley (1999), who trades collectable cards on an internet auction platform. He finds no significant difference in revenues, though he admits himself that he cannot entirely control for a potential selection bias and endogenous entry.

[10] Notably, for private values bounded rationality in the sense of "cursed equilibrium" (Eyster and Rabin (2005)) cannot explain overbidding in the Vickrey auction. K-level reasoning (Crawford and Iriberri (2007)) can only explain it when the type distribution does not follow the uniform distribution used in most experiments.

uncertain and they are implicit weighted utility maximizers. Relatedly, Karni and Safra (1989) show that equivalence holds if and only if bidders are expected utility maximizers. In a recent contribution, Auster and Kellner (2020) show that the equivalence between Dutch and First-Price auction fails for the case of ambiguity-averse bidders.

Kőszegi and Rabin (2006) suggest recent rational expectations as a reference point. The hypothesis that expectations play a role in an individual's preferences have been supported in experiments (Ericson and Fuster (2011) and Abeler et al. (2011)), as well as challenged (Heffetz and List (2014)).[11] In the context of auctions, Banerji and Gupta (2014) and Rosato and Tymula (2019) provide evidence that expectations-based loss aversion affects bidding behavior. Banerji and Gupta (2014) manipulate rational expectations by changing the support of the opponents' draw in a BDM auction. Rosato and Tymula (2019) vary the number of bidders in a Vickrey auction between treatments. Both experiments find that bids significantly increase in the induced expectations to win, as predicted by the model of Kőszegi and Rabin (2006).

The idea that the reference point is determined by recent beliefs leads to the natural question of the speed of reference-point adjustment. Strahilevitz and Loewenstein (1998) provide early evidence that the time span for which individuals hold beliefs has an impact on the reference point. Gill and Prowse (2012) use a real-effort task to measure loss aversion and find that in their framework "the adjustment process is essentially instantaneous." Smith (2019) induces different probabilities of winning an item across groups of individuals. After the uncertainty is resolved, he measures the willingness to pay for the item among bidders who have not won. In contrast to Ericson and Fuster (2011), who elicit valuations *before* the uncertainty resolves, Smith finds no significant difference between different groups. This suggests that the reference point is not so much determined by lagged beliefs, but rather adjusts quickly to the new information.[12]

For static environments Kőszegi and Rabin (2006) has arguably become the standard model of reference-dependent preferences. It has been successfully applied to various fields, like mechanism design (Eisenhuth (2019)), contract theory (see Kőszegi (2014) for a review, and Herweg et al. (2010) in particular), industrial organization (for instance Heidhues and Kőszegi (2008, 2014), Herweg and Mierendorff (2013), Karle and Peitz (2014), Rosato (2016)), and labor markets

[11] For a literature review on related evidence, see Ericson and Fuster (2014).

[12] Smith's confidence intervals are, however, rather wide.

(Eliaz and Spiegler (2014)).

There is a small, but growing, body of literature concerning strategic interaction between multiple loss-averse players. Dato et al. (2017) extend the equilibrium concepts of Kőszegi and Rabin (2006) to strategic interaction. Mermer (2021) analyzes contests with loss-averse agents. Similar to my results in the Vickrey auction, she finds that the willingness to invest is increasing in the winning probability.

In the context of static auctions Lange and Ratan (2010) were the first to point out that the attachment effect affects bidding behavior. They show that the bidding behavior depends on whether bidders bracket widely or narrowly. For the Vickrey auction they find similar results to mine for a slightly different equilibrium concept.[13] Eisenhuth and Grunewald (2020) show that an all-pay auction yields higher payoffs than a first-price auction for narrow-bracketing bidders, since loss-averse bidders dislike payment uncertainty. Balzer and Rosato (2021) derive equilibria in an environment with interdependent values and independent signals.

For dynamic environments, Kőszegi and Rabin (2009) propose a model of dynamic loss aversion, where updates of expectations carry reference-dependent utility. This model has so far only been applied sparsely. Ehrhart and Ott (2014) introduce a model of the Dutch and English auction. They use a simplified version of Kőszegi and Rabin (2009) in the sense that sequential information updates the reference point, but—in contrast to Kőszegi and Rabin (2009)—does not induce gain-loss utility. As a result, in equilibrium there is never any feeling of loss in the English auction, since by the time a bidder drops out she expects to lose. Rosato (2020) uses a two-period dynamic model to show that revenues are decreasing in sequential auctions with loss-averse bidders, due to a similar discouragement effect, as I identify in the English auction. Macera (2018) shows for a two-period moral hazard model with loss-averse agents that for the optimal contract wages are fixed and incentives are deferred to the future. To my best knowledge, Pagel is the first to rigorously apply Kőszegi and Rabin (2009) to dynamic problems with a longer time horizon. Pagel (2016) shows that dynamic reference-dependent preferences can explain the historical levels of equity premiums and premium volatility in asset prices. Related to the logic in the English auction, loss-averse agents dislike price fluctuations, which makes assets relatively unattractive. Pagel (2017) shows that dynamic reference-dependent preferences can explain empirical

[13] For more details on the equilibrium concept see Appendix 6.1.

observations about saving schemes for life-cycle consumption.

3 The Model

3.1 Auction Rules

There are $n \geq 2$ loss-averse bidders participating in an auction for a non-divisible good. Bidder i's intrinsic valuation θ_i is privately observed and independently drawn from a common distribution G with a strictly positive, differentiable density g on positive support $[\theta^{\min}, \theta^{\max}]$.

For the Vickrey (second-price) auction, every bidder submits a sealed bid after learning her private valuation. Then the auction is resolved: the bidder with the highest bid receives the object and has to pay the amount of the second-highest bid.

For the English auction, I am considering an Ascending Clock ("Japanese") Auction. A clock starts at a price of zero and is raised incrementally. Bidders observe the ascending clock and signal to the auctioneer when they wish to drop out. Once a bidder drops out she cannot bid again. Once there is only one active bidder left the auction ends and the remaining bidder has to pay the price at which the last of her opponents dropped out.

3.2 Strategies and Preferences

Definition 1. A **bidding plan** b specifies an available action for every point in time and every possible history of information revelation. A **bidding strategy** $b(\theta)$ assigns to each possible type θ a bidding plan.

Hence, as bidders cannot observe individual opponent dropouts, a bidding plan in both the English and Vickrey auction is described by a (maximum) bid $b \in \mathbb{R}_+$. Note that for standard preferences any profile of bidding plans induces the same utilities in both auction formats.

The model of dynamic reference-dependent utility follows Kőszegi and Rabin (2009). The bidders' reference point is determined by rational beliefs about final payoffs in both dimensions. For the reference point formation, I take the interim approach: first, each bidder learns her valuation θ and forms a bidding plan $b(\theta)$. Then, rational beliefs H_0 about the opponents' bidding plans define the bidder's initial reference point. Then, the auction takes place.

Any change in beliefs during the auction updates the reference point and instantaneously induces a psychological utility of gains or losses (henceforth gain-loss utility). Let $F_t^k \equiv F_t^k(b, \theta, H_t)$ be a bidder's beliefs over final payoffs in $k \in \{money, good\}$ at time t. Denote the gain-loss utility of an update from F_{t-1}^k to F_t^k with $N(F_t^k | F_{t-1}^k)$.

For the evaluation of an update $N(F_t^k | F_{t-1}^k)$ bidders assign gains and losses to changes in the respective quantiles of the distribution function. Intuitively, they rank possible outcomes from worst to best and then evaluate changes to the worst, the second worst ,..., until the best outcome. I define the quantile function $c_{F_t^k}(p)$ with quantile $p \in [0, 1]$ as usual as the left-continuous inverse of F_t^k. Then

$$N(F_t^k | F_{t-1}^k) = \int_0^1 \mu_k(c_{F_t^k}(p) - c_{F_{t-1}^k}(p))dp,$$

where the function μ_k measures feelings of gain and loss for respective belief changes. As a key feature, loss-averse bidders weight losses with respect to their reference distribution stronger than gains. Following Section IV in Kőszegi and Rabin (2006) and most of the literature, I assume μ_k to be piecewise linear,

$$\mu_k(y) = \begin{cases} \eta_k y & y \geq 0, \\ \lambda_k \eta_k y & y < 0, \end{cases}$$

where $\eta_k > 0$, $\lambda_k > 1$. Moreover, I assume $\Lambda_k := \lambda_k \eta_k - \eta_k < 1$ for $k \in \{m, g\}$.[14] As it allows for a significantly simpler exposition, I first focus on the case in which bidders are loss averse in the goods dimension only, i.e., $\eta_m = 0$.[15] In Section 4.4 and Appendix 6.2 I show that similar effects on equilibrium bidding can be derived from loss aversion in the money dimension as well. Hence, the main results strengthen for loss aversion in both dimensions.

[14] The condition $\Lambda < 1$ is referred to as "no dominance of gain-loss utility" by Herweg et al. (2010). Whereas a large Λ is innocuous in the Vickrey auction, $\Lambda > 1+\eta$ in the English auction would lead to a nonexistence of symmetric equilibria, as the lowest types would never choose to participate, cf. the discussion in 6.3.

[15] Horowitz and McConnell (2002) conclude in their summary that the endowment effect is "lowest for experiments involving forms of money." Kőszegi and Rabin (2009) rationalize this fact with the idea that a loss in money is foregone future consumption, whose reference-dependent utility may be discounted. In this sense it may be plausible that loss aversion mainly applies to the goods dimension.

Utility in the Vickrey auction

In the static Vickrey auction there is no reference point updating until the auction is resolved. Hence, the timing is as follows: First, the bidder learns her type θ and forms a plan to bid b^*, which—together with rational beliefs on opponents' behavior—defines her reference point. Then, she submits a bid b. Finally, the auction is resolved, transfers are made, and the bidder enjoys reference-dependent utility by comparing the outcome to her reference point. To keep notational similarity with the English auction, denote for a given maximal opponent bid x with $F_T^k(b, \theta, H_T)$ the distribution of final payoffs from a bid b. Then, if the bidder plans to bid b^* but deviates to bid b instead, her utility is

$$u_0(b, \theta | b^*) = \underbrace{\sum_{k \in \{m,g\}} N(F_T^k(b, \theta, H_T) | F_0^k(b^*, \theta, H_0))}_{\text{gain-loss utility}} + \underbrace{\mathbb{1}_{b>x}(\theta - x)}_{\text{classical utility}}. \tag{1}$$

Utility in the English auction

In the English auction, a bidder at each increment updates the reference point with respect to new information or a deviation to another bidding plan.

- At each period t, before the clock increases, a bidder may deviate from plan b^* to another plan b. The associated belief change instantaneously induces gain-loss utility $N(F_t^k(b, \theta, H_t) | F_t^k(b^*, \theta, H_t))$ in both dimensions.
- At each increment a bidder observes whether she won at the respective price. The corresponding belief update instantaneously induces gain-loss utility $N(F_{t+1}^k(b, \theta, H_{t+1}) | F_t^k(b, \theta, H_t))$ in both dimensions.

After the auction is terminated, transfers are made according to the auction rules.[16] The timing is summarized in Figure 1. The assumption that a bidder immediately updates her reference point with respect to a deviation is a slight departure from Kőszegi and Rabin (2009), who suggest that the agent should update only at the end of the period, i.e. after the information in the period has resolved. This modeling choice reflects the idea of instantaneous updating in the English auction. Technically, it eliminates an asymmetry between information

[16]For mathematical convenience, I abstract from tie-breaking rules in the main model and assume that the good is not sold, if the remaining bidders drop out simultaneously. With the assumption of continuous density of types, as the increment size goes to zero, this becomes equivalent to a tie-breaking rule by coin-flip.

updates in the current and all future periods. This allows me to isolate the effect of dynamic information and gives scope for the insurance effect. Importantly, the revenue ranking between the two formats, which results from a discouragement effect, does not rely on that assumption.

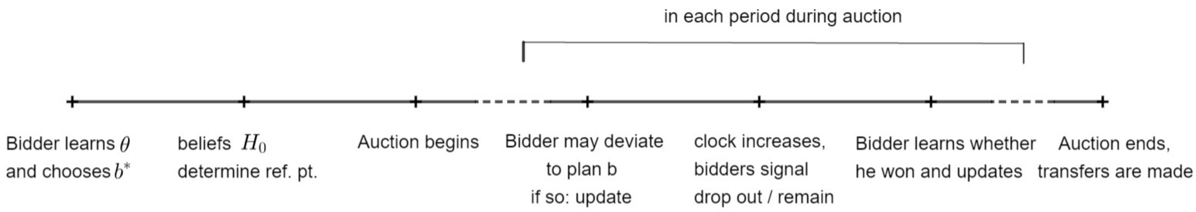

Figure 1: Timeline in the English auction

Suppose the English auction runs at most for T increments. If a bidder deviates at time t from plan b^* to plan b, then total utility $u_t(b,\theta|b^*)$ from time t onward is

$$u_t(b,\theta|b^*) = u_c + \sum_{k\in\{m,g\}} \left(N\left(F_t^k(b,\theta,H_t)|F_t^k(b^*,\theta,H_t)\right) + \sum_{s=t+1}^{T} N\left(F_s^k(b,\theta,H_s)|F_{s-1}^k(b,\theta,H_{s-1})\right)\right). \tag{2}$$

The first term $u_c \equiv \mathbb{1}_{b>x}(\theta - x)$ denotes classical utility for a maximal opponent bid x, the second term is gain-loss utility from the deviation, and the third term is total remaining gain-loss utility from news.[17] Note that the distributions F_T^k are degenerate with both mass one on zero if $b \leq x$, and with mass one on θ for F_T^g (respectively, mass one on $-x$ for F_T^m) if $b > x$.

Before I define the appropriate equilibrium concept and derive optimal bidding strategies, the following example illustrates how gain-loss utility is formed during the auction. It shows why loss-averse bidders prefer a Vickrey auction to an English auction for given strategies.

Example 1. Consider a bidder, who is loss averse in the goods dimension, in an English auction with one opponent. Suppose the bidder plans to bid up to $b = 8$ and expects an opponent drop-out price uniformly distributed on $[0, 10]$. Hence, for any clock price $t < 8$ where the opponent is still active, the updated losing

[17]The upper bound of T in the sum is without loss of generality; if the auction terminates early, all subsequent periods can be regarded as uninformative, and carry no further reference-dependent utility.

probability is $\frac{2}{10-t}$, giving rise to a quantile function of

$$c_{F_t^g}(p) = \begin{cases} 0 & p \leq \frac{2}{10-t}, \\ \theta & p > \frac{2}{10-t}. \end{cases}$$

Suppose the opponent drops out at a price of 6. Figure 2 shows an example of the quantile functions before the auction begins (dotted), at a clock price of 4 (dashed), an arbitrary small increment before 6 (solid), and after the dropout at 6 (solid constant function at θ). As long as the price increases and the opponent

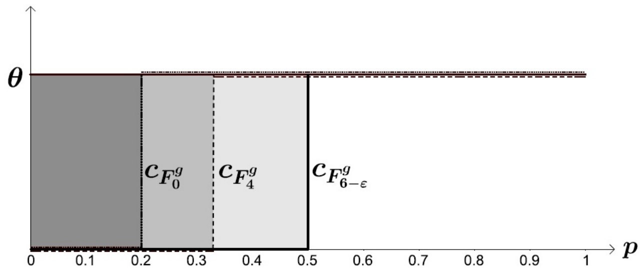

Figure 2: Updating in the English Auction

does not drop out, the losing probability increases. The discontinuity point in the quantile function gradually shifts to the right, inducing a loss in each increment. At a price of 4 the accumulated loss is $\lambda\eta$ times the medium gray shaded area. Just before the opponent drops out at 6 the accumulated loss is $\lambda\eta$ times the medium and light shaded area. When the opponent drops out at a price of 6, and the quantile function jumps to the constant function $c_{F_6^g} = \theta$, the update induces a feeling of gain of η times the three combined shaded areas. Thus, the net gain-loss utility in the goods dimension for the English auction is $(0.2\eta+0.3(\eta-\lambda\eta))\theta = (0.2\eta - 0.3\Lambda)\theta$.

In contrast, in a Vickrey auction with the same bidding strategies—and hence same ex-ante beliefs—the only update takes place after the auction has been resolved, and the belief jumps to the constant quantile function $c_{F_6^g}$. Thus, total gain-loss utility is given by $0.2\eta\theta$, proportional to the dark gray area only.

Intuitively, since losses are weighted stronger than gains, belief fluctuations in the English auction generate a net loss of $-0.3\Lambda\theta$ compared to the Vickrey auction. If the bidder could use a bidding proxy that enabled her to ignore new information during the English auction, she would forgo unpleasant variation in beliefs, and receive the same utility as in a Vickrey auction. This logic is due to

Kőszegi and Rabin (2009), who formally show in their Proposition 1 that, ceteris paribus, any collapse of information signals weakly increases agents' utility. The result, that bidders in the English auction would prefer proxies to bid on their behalf, is a testable prediction.

3.3 Equilibrium Concept

In the following I denote at any time t with

$$U_t(b,\theta|b^*) \equiv \mathbb{E}_{H_t} u_t(b,\theta|b^*) \tag{3}$$

expected utility for the remaining auction, and

$$L_t(b) \equiv L_t(b,\theta|b) \equiv \mathbb{E}_{H_t}\left(\sum_{k\in\{m,g\}}\sum_{s=t+1}^{T} N(F_s^k(b,\theta,H_s)|F_{s-1}^k(b,\theta,H_{s-1}))\right) \tag{4}$$

the expected gain-loss utility at time t from all future information updates when the bidding plan is b.

I apply the equilibrium concept of Kőszegi and Rabin (2009). For full details and a psychological justification of the specific dynamic modeling choices, I refer to their paper.

Definition 2. Define a credible plan in the following backward recursive way: A bidding plan b^* is **credible** at time t, if — given rational expectations derived from the plan— at all times $s \geq t$ and for all possible information revelations

$$U_s(b^*,\theta|b^*) \geq U_s(b,\theta|b^*) \tag{5}$$

holds with respect to all deviation plans b that are credible at time s. A **personal equilibrium** (PE) is a plan that is credible at time zero. A PE is a **preferred personal equilibrium** (PPE) if it maximizes utility at time zero among all PE.

Hence, a bidding plan is a personal equilibrium if, given the reference point resulting from the plan, it maximizes expected utility at any point in time among all credible deviations. In practice, the set of credible plans for a given belief must be determined by thinking backwards. Crucially, the credibility assumption implies that bidders do not have commitment power to their future selves in the sense that they cannot plan profitable, but non-credible strategies. Committing to ex-post unfavorable actions could be profitable ex ante, because it would alter beliefs and therefore change gain-loss utility received during the auction.

In the Vickrey auction the only decision is made at $t = 0$. Hence, applying Kőszegi's and Rabin's definition of a PE to this setting simply calls for a bid b^* such that

$$U_0(b^*, \theta|b^*) \geq U_0(b, \theta|b^*)$$

for all $b \in \mathbb{R}_+$. This definition of a personal equilibrium for the special case of a single individual decision under uncertainty exactly coincides with the definition of an *unacclimating personal equilibrium (UPE)* in Kőszegi and Rabin (2007), as I formally show in Appendix 6.1.

Kőszegi's and Rabin's notion of a UPE contrasts their concept of a *choice-acclimating personal equilibrium (CPE)*, which requires $U_0(b^*, \theta|b^*) \geq U_0(b, \theta|b)$. for all $b \in \mathbb{R}_+$.

Both concepts are frequently used in the literature. I focus on the UPE as a special case of the dynamic PE, in order to isolate the effect of dynamic information revelation in my comparison between the Vickrey auction and the English auction. In Appendix 6.1, I discuss the equilibrium concepts more broadly and show that the revenue ranking does not rely on my choice of equilibrium concept.

In general, the set of personal equilibria depends on the belief about other players' actions. To analyze the strategic interaction between multiple bidders, I focus on symmetric personal equilibria.

Definition 3. A bidding strategy $b(\theta)$ is a **(preferred) symmetric equilibrium** if for each type θ and the belief that all opponents bid according to strategy $b(\theta)$, the bidding plan $b(\theta)$ is a (preferred) personal equilibrium.

4 Analysis

4.1 The Vickrey Auction

Fixing a bidder of type θ, denote with $H(b)$ the distribution over the maximal opponent bid. Since the bidder receives a payoff of θ if and only if her bid exceeds the highest opponent bid x, gain-loss utility associated with a bid b reads

$$N(F_T^g(b,\theta)|F_0^g(b^*,\theta,H)) = \begin{cases} H(b^*)\mu(-\theta) & b \le x, \\ \big(1-H(b^*)\big)\mu(\theta) & b > x. \end{cases}$$

The first line describes the feeling of loss if the agent loses the auction, and the second line describes the feeling of gain if she wins. By Equation (3),

$$\begin{aligned} U_0(b,\theta|b^*) &= \mathbb{E}_H\Big(H(b^*)\mu(-\theta)\mathbb{1}_{x\ge b} + \big(1-H(b^*)\big)\mu(\theta)\mathbb{1}_{x<b} + \mathbb{1}_{x<b}(\theta - x)\Big) \\ &= \underbrace{\big(1-H(b)\big)}_{\text{Prob to lose}}\underbrace{H(b^*)\mu(-\theta)}_{\text{feeling of loss}} + \underbrace{H(b)}_{\text{Prob to win}}\underbrace{\big(1-H(b^*)\big)\mu(\theta)}_{\text{feeling of gain}} + \underbrace{\int_0^b(\theta - s)dH(s)}_{\text{classical utility}}. \end{aligned}$$

Recall that b^* is a personal equilibrium for belief H if it maximizes $U_0(b,\theta|b^*)$ for all $b \in \mathbb{R}_+$. In a symmetric equilibrium the belief H is determined by the symmetric equilibrium bidding function, hence $H(b(\theta)) = G^{n-1}(\theta)$.

Proposition 1. *1. For any continuous belief $H(b)$ about the maximal opponent bid a bidder's PE satisfies*

$$b(\theta) = \Big(1 + \eta\big(1 - H(b(\theta))\big) + \lambda\eta H(b(\theta))\Big)\theta.$$

2. Consider n bidders who are loss averse only in the goods dimension. The unique symmetric increasing continuously differentiable PE in the Vickrey auction is given by

$$b(\theta) = \Big(1 + \eta\big(1 - G^{n-1}(\theta)\big) + \lambda\eta G^{n-1}(\theta)\Big)\theta.$$

All types overbid with respect to their intrinsic valuation θ, but increasingly in their type. The reason is the well-established attachment effect. As with standard preferences, the optimal bid reflects the opportunity value of receiving the good. With expectations-based loss aversion this opportunity value is belief-dependent. As high types strongly believe in winning, losing would induce a feeling of loss, which they try to prevent with an aggressive bid. The fact that a bidder is indifferent between winning at a price of her bid and the loss perceived from losing implies that a bidder may pay such a high price that the utility from winning is negative.[18] The finding that the attachment effect in UPE environments can

[18]Overbidding for all types stems from the fact that loss aversion is assigned only to goods,

lead to purchase decisions which are unpreferable from an ex-ante perspective is a central motif in various papers in the behavioral IO literature.[19]

4.2 The English Auction

In contrast to the Vickrey auction, in the English auction a bidder experiences gain-loss utility in each period when she learns whether she is winning at the current price. Since the information in each period is binary the following lemma is central for calculating expected gains and losses.

Lemma 1. *Suppose that a loss-averse agent's payoff is distributed according to F_1 with probability Δ, and according to F_2 with probability $1 - \Delta$. Let $[a, b]$ contain the support of F_1 and F_2. Denote with $F = \Delta F_1 + (1 - \Delta)F_2$ the ex-ante distribution of payoffs. Then the ex-ante expected reference-dependent utility from learning, whether the distribution is F_1 or F_2, is given by*

$$\mathbb{E}(N(F_i|F)) = -\Delta\Lambda \int_a^b |F(x) - F_1(x)|dx.$$

Since losses loom larger than gains, the expected news utility is always negative. With this result I can derive a closed-form solution for the expected utility when increments become small. From now the subscript t denotes the current price rather than the increment, whereas F is the ex-ante distribution of the maximal opponent bid.

Proposition 2. *Suppose the distribution F has a density f. Suppose the bidder plans to drop out at b, and the auction has not yet ended at price $t < b$. Then, for increment size ε going to zero, in the limit the ex-ante expected marginal gain-loss utility at price t is given by*

$$\ell_t(b, \theta, F) = \frac{-f(t)}{(1 - F(t))^2}(1 - F(b))\Lambda\theta.$$

which makes the good relatively more important, compared to money. In Section 4.4 and Appendix 6.2, I discuss loss aversion in both dimensions. I show that for equal-sized loss aversion in both dimensions low types underbid and high types overbid. The intuition that bidders may obtain negative utility from winning remains, however, unchanged. Depending on the type distribution, even the expected utility can be negative.

[19]In this literature firms exploit this behavior, and strategically induce a high attachment effect via random sales prices (Heidhues and Kőszegi (2014)), informational advertisement (Karle and Schumacher (2017)), or limited supply of bargains (Rosato (2016)).

Expected utility for the remainder of the auction at time t is in the limit given by

$$U_t(b,\theta,F|b^*) = \underbrace{\frac{\int_t^b (\theta - s)dF(s)}{1-F(t)}}_{\text{classical utility}} + \underbrace{\frac{\mu(F(b)-F(b^*))}{1-F(t)}\theta}_{\substack{\text{one-time gain/loss} \\ \text{from deviating}}} + \underbrace{L_t(b),}_{\substack{\text{expected gain-loss utility} \\ \text{of remaining auction}}} \tag{6}$$

where

$$L_t(b) \equiv L_t(b,\theta,F|b) = \ln\left(\frac{1-F(b)}{1-F(t)}\right)\frac{1-F(b)}{1-F(t)}\Lambda\theta.$$

In the following, I refer to the limit result when the increment size goes to zero as the *continuous English auction.*[20]

The deviation utility in Equation 6 is easily determined, as $\frac{F(b)-F(b^*)}{1-F(t)}$ is the change in winning probability. More interestingly, note that the amount of marginal disutility $\ell_t(b,\theta,F)$ is decreasing in b: an aggressive strategy induces less belief fluctuation at each information update, and hence less marginal disutility in the continuous English auction. Thus, an aggressive bid exerts an *insurance effect* against high gain-loss disutility. There is, however, a countervailing effect on total gain-loss disutility: the higher the bidder's drop-out price, the longer she may stay in the auction and be exposed to gain-loss disutility. Figure 3 shows total expected gain-loss disutility at the beginning of the auction for different bids. Losses are the strongest for intermediate bids. Bidding 0 or 1 induces no uncertainty and therefore no gain-loss utility.

An optimal strategy must account for this incentive to insure against belief fluctuations. I start by looking at ex-ante optimal bidding plans by ignoring the restriction that plans must be credible.

Lemma 2. *If n loss-averse bidders could commit ex ante to a (possibly non-credible) bidding strategy in the continuous English auction, the lowest symmetric increasing differentiable equilibrium would satisfy*

$$b(\theta) = \Big(1 + \eta - \Lambda\big(1 + \ln(1 - G^{n-1}(\theta))\big)\Big)\theta.$$

Figure 4 shows the ex-ante optimal strategy as a solid function. The *insurance*

[20]This notion does not intend to refer to the concept of *continuous games* by Simon and Stinchcombe (1989). One should still regard the game as one with discrete increments on the clock which are, however, arbitrarily small.

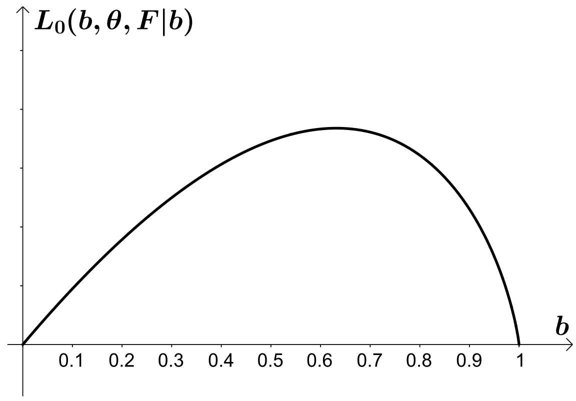

Figure 3: Total Expected Loss for $F \sim U[0, 1]$

effect leads to low bids for low types, while high types wish to strongly overbid. Intuitively, bidders want to reduce expected gain-loss utility, and therefore try to reduce uncertainty. Note, however, that since $L_t(b, \theta, F, b)$ is decreasing in t, the incentive to insure against belief fluctuations declines as the auction unfolds, creating a conflict of interest between an ex-ante self and a future self that has to execute the bidding strategy. Next, I look at the constraints on bidding behavior induced by the time-consistency assumptions that bids must be credible.

Lemma 3. *Let the opponent's drop-out price be distributed according to distribution F with non-zero density f on some positive support $[a, c]$. Then, for the continuous English auction any credible bidding strategy $b \in (a, c)$ satisfies*

$$b \leq (1 + \eta)\theta.$$

Lemma 3 illustrates the essential *discouragement effect* as the driving force for low bidding in the English auction. The belief in winning eventually declines, and by the time a bidder has to execute her maximum bid it is virtually zero, leaving no place for any attachment effect. At that time the bidder perceives the remaining auction similarly as to a Vickrey auction, where she has the lowest possible type. Hence, at that point in time, her optimal bidding strategy resembles that of the lowest type in the Vickrey auction, i.e., she bids no more than $b = (1 + \eta)\theta$.

Figure 4 illustrates the boundary of time-consistent strategies as a dashed line, showing that this constraint is binding for high types.

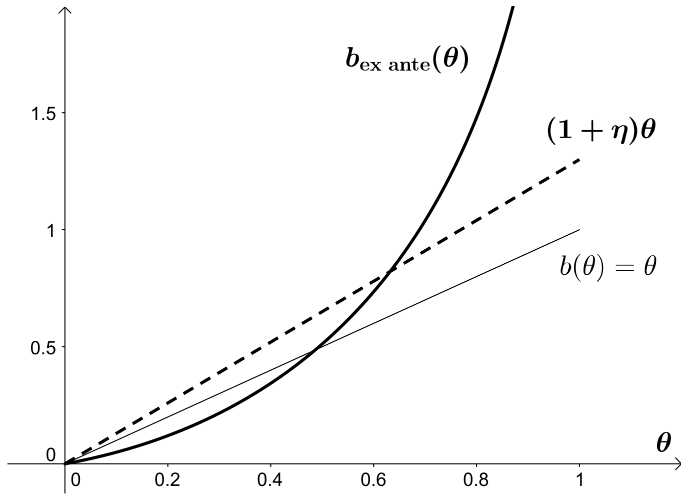

Figure 4: $G^{n-1}(\theta) \sim U[0,1]$, $\eta = 0.3$, $\lambda = 4$

I have so far only considered constraints on equilibrium behavior at time 0 and at time b. It turns out that these are the binding constraints:

Lemma 4. *If a bidder's belief F of the maximal opponent bid attains a non-zero density f with some positive support $[a, c]$, then a strategy $b^* \in (a, c)$ is a PE if and only if*

1. $b^* \leq (1+\eta)\theta$;
2. $U_0(b^*, \theta, F|b^*) \geq U_0(b, \theta, F|b^*)$ *for any* $b \in [b^*, (1+\eta)\theta]$.

Hence, for low types, individual optimal behavior is determined by the insurance effect, whereas high types' bids are bound by the discouragement effect. Notably, Lemma 4 does not rely on the strategic interaction or loss aversion of other bidders. Effectively, a continuous belief is sufficient for the discouragement effect which binds optimal time-consistent bids below $(1+\eta)\theta$. For strategic interaction I obtain:

Proposition 3. *An increasing, almost everywhere differentiable function $b(\theta)$ is a symmetric equilibrium in the continuous English auction with n loss-averse bidders if and only if for all θ*

1. $b(\theta) \leq (1+\eta)\theta$;

2. $b(\theta) \geq \min\Big\{(1+\eta)\theta \ ; \ \Big(1+\eta-\Lambda\big(1+\ln(1-G^{n-1}(\theta))\big)\Big)\theta \ \Big\}$.

Thus, any increasing smooth function in the gray shaded area of Figure 5 constitutes a symmetric equilibrium.

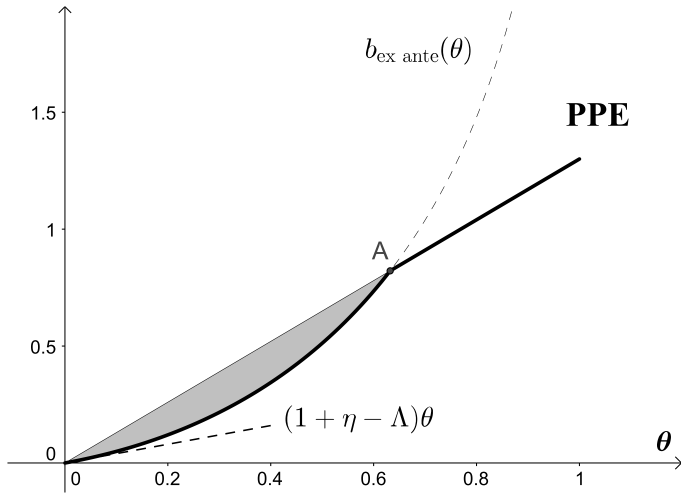

Figure 5: $G^{n-1}(\theta) \sim U[0,1]$, $\eta = 0.3$, $\lambda = 4$

The thick line indicates the preferred symmetric equilibrium (PPE). Point A, where the PPE hits the boundary of time-consistent strategies can be easily determined:

$$\Big(1+\eta-\Lambda\big(1+\ln(1-G^{n-1}(\theta))\big)\Big)\theta = (1+\eta)\theta$$

if and only if $G(\theta) = (1-1/e)^{\frac{1}{n-1}}$.

Corollary 1. *The symmetric PPE in the continuous English auction with n loss-averse bidders is given by*

$$b_{PPE}(\theta) = \begin{cases} \Big(1+\eta-\Lambda\big(1+\ln(1-G^{n-1}(\theta))\big)\Big)\theta & G(\theta) \leq (1-1/e)^{\frac{1}{n-1}}, \\ (1+\eta)\theta & G(\theta) > (1-1/e)^{\frac{1}{n-1}}. \end{cases}$$

Low types bid below their intrinsic value θ in the PPE if and only if $\lambda > 2$.

Note that the PPE is tangent to $(1+\eta-\Lambda)\theta$ at the lowest type. Hence, there is underbidding for low types if and only if $\eta - \Lambda < 0$, thus if and only if $\lambda > 2$.

4.3 Revenue Comparison

Figure 6 summarizes the results on symmetric equilibrium bidding behavior as established in Proposition 1 for the Vickrey auction and in Proposition 3 and Corollary 1 for the English auction. The function $b_E(\theta)$ indicates the PPE in the English auction. The shaded area indicates the other potential symmetric equilibria in the English auction, which are bounded by the line $(1+\eta)\theta$. Both are below b_V, the unique PE in the Vickrey auction, indicating strictly higher revenues in the Vickrey auction.

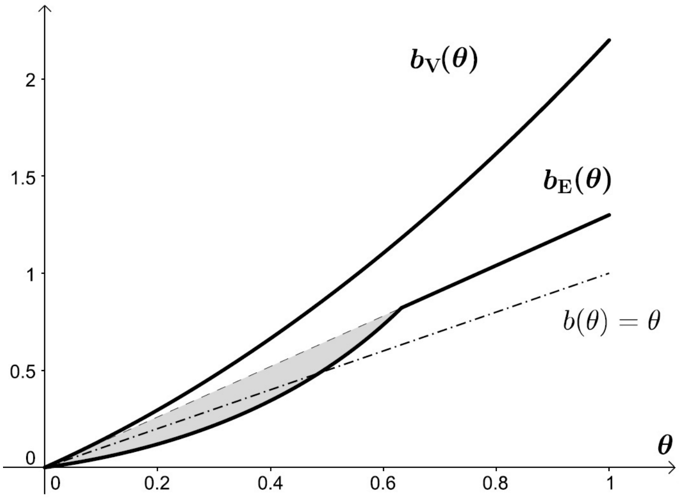

Figure 6: $G^{n-1}(\theta) \sim U[0,1]$, $\eta = 0.3$, $\lambda = 4$

Note that $b_V(\theta)$ at the lowest type is tangent to $(1+\eta)\theta$—the upper bound of equilibria in the English auction. The intuition is that for low types the decision problem in both auction formats becomes increasingly similar: bidders in the English auction only learn whether there are opponents with a lower valuation than their own. Hence, for low types the information difference between the two auction formats at the time the bidder places her (maximal) bid is small.

It is worth emphasizing that nothing about this intuition relies on the fact that bidders hold correct beliefs or that all bidders share the same degree of loss aversion. Indeed, as shown in Lemma 4, the discouragement effect binds bidding in the English auction below $(1+\eta)\theta$ for *any* continuous belief on opponent dropouts. Conversely, in the Vickrey auction (1) of Proposition 1 shows that in the Vickrey auction for *any* (potentially different) continuous belief a PE requires bidding weakly above $(1+\eta)\theta$, and strictly so if bidders expect a positive probability to win.[21] Hence, Proposition 1, Lemma 4, and Proposition 3 immediately imply the following Corollary on the revenue ranking.

Corollary 2. *If a loss-averse bidder updates her reference point instantaneously during the auction process, any continuous belief about opponents' strategies lead to higher personal equilibrium bids in the Vickrey auction than in the English auction.*

For symmetric equilibria only bids for the lowest type may coincide for both auction formats. All other types bid strictly higher in the Vickrey auction, and the Vickrey auction attains strictly higher revenue.

4.4 Robustness

In this section, I discuss various robustness checks with respect to the modeling assumptions. The formal derivation of the results can be found in the respective appendices.

Modeling Concept of Dynamic Reference-Dependent Utility

I have employed the news utility model of Kőszegi and Rabin (2009) to analyze the impact of dynamic information in the English auction. The idea that any news induces reference-dependent utility about prospective gains and losses gave scope for the insurance effect: bidders take this expected news utility into account when they form their bidding strategy. Notably, the revenue ranking is independent of news utility, and only relies on the fact that the belief of winning decreases with dynamic information during the English auction (discouragement effect). In Appendix 6.1 I show formally that this effect remains unchanged in a model without news utility, where bidders experience reference-dependent utility only in the last period where the auction is resolved.

[21] The fact that high types in the English auction bid $(1+\eta)\theta$ immediately implies that with heterogeneity in η there can be no symmetric equilibrium that is monotone in type θ. Optimal bidding remains, however, monotone in the opportunity value of winning. For a bidder with sufficiently high θ this opportunity value is $(1+\eta)\theta$ in the English auction. As shown in (1) of Proposition 1, the same holds for the Vickrey auction, where the (belief-dependent) opportunity value of winning $\big(1+\eta(1-H(b))+\lambda\eta H(b)\big)\theta$ is increasing in λ and η.

Loss Aversion in Money

In Appendix 6.2 I extend all results to a setting where bidders are also loss averse with respect to money. While the technical derivations are much more challenging, all intuitions and results on the comparison of the English auction and Vickrey auction remain qualitatively exactly the same.

Moreover, a model of equal-sized loss aversion in both dimensions provides a meaningful comparison to the risk-neutral benchmark. In the Vickrey auction, low types have a low belief in winning and perceive paying money as an unexpected loss. Hence, they underbid with respect to their intrinsic valuation. Conversely, high types, who expect to win with high probability, fear losing the object and overbid with respect to their intrinsic valuation.

Again, due to the discouragement effect, any credible bidding plan in the English auction is determined by the incentives of the bidder when she is on the verge of dropping out. At that time such a bidder has virtually zero belief in winning. Hence, winning at a price of b would create a feeling of loss of $\lambda\eta b$ in the money domain, and a feeling of gain of $\eta\theta$ in the object domain. Incorporating classical utility, bidding up to b can only be time consistent if $(1+\lambda\eta)b \leq (1+\eta)\theta$, hence only if $b \leq \frac{1+\eta}{1+\lambda\eta}$. Again, this behavior mirrors that of the lowest type in the Vickrey auction, who also has a belief of zero in winning.

Observable Opponent Dropouts

To mirror the Vickrey auction as closely as possible, I assumed that in the English auction bidders do not observe individual opponent dropouts, but only whether there are any active bidders at a price. With this assumption both auction formats are strategically equivalent in the sense that they share the same strategy space and payoff matrix for standard preferences. Hence, it allowed me to isolate the effect of dynamic information from other strategic considerations. In Appendix 6.3 I consider an extension where bidders can observe individual dropouts. First, I show that in such a model equilibrium strategies necessarily depend on the history of dropouts. Second, I show that with more than three bidders symmetric and increasing equilibria fail to exist. Intuitively, with instantaneous updating belief fluctuation creates such strong disutility that the participation constraint of low bidders is violated. Third, I show that the discouragement effect still holds in the following sense: whenever a bidder holds a continuous belief about her opponents' drop-out prices, her belief eventually approaches zero. Hence, again any credible strategy involves only bids that satisfy the same upper bound as in the baseline model.

5 Conclusion

I studied the effects of dynamic information on expectations-based preferences in the dynamic English auction compared to the static Vickrey auction. If bidders update their reference point instantaneously with respect to new information, dynamic information in the English influences the bidders' endogenous preferences, and thus their bidding strategies. The classical strategic equivalence between the two auction formats breaks down and the English auction attains strictly lower revenue than the Vickrey auction.

This finding highlights the importance of understanding the evolution of the reference point in dynamic environments. In particular, research about the speed of reference point adaptation with respect to new information is still in its infancy and deserves further study.

The non-equivalence of the two auction formats stands in sharp contrast to the revenue equivalence principles by Vickrey (1961) and Myerson (1981). Indeed, the powerful approach of mechanism design and the revelation principle relies on the assumption that agents' valuations are exogenously given and do not depend on the choice of mechanism. This assumption is violated if bidders have endogenous preferences that depend on expectations induced by the mechanism itself. In particular, if agents update their reference point with respect to new information in a multi-stage mechanism, such a mechanism cannot be replaced by a simple direct mechanism without changing agents' incentives. The failure of the revelation principle naturally leads to the question of optimal mechanism design in dynamic environments with expectations-based loss-averse agents. The study of optimal expectation management in these environments is an interesting question for future research.

6 Appendix

6.1 Equilibrium Concepts

In this section I formally derive how the concept of a personal equilibrium (PE) in Kőszegi and Rabin (2009) for the special case of a single decision under uncertainty coincides with the concept of an unacclimated personal equilibrium (UPE) for static decision problems in Kőszegi and Rabin (2007). I then compare this equilibrium concept with the dynamic extension of the CPE concept as developed in Rosato (2020). I show that the revenue ranking would remain under this modeling choice.

Since utility is additively separable across different commodity dimensions, it suffices to consider one dimension. For the framework of Kőszegi and Rabin (2009) suppose that a person in period 0 chooses an action from some choice set D. The action is characterized by its distribution G of payoffs in period 1. The distribution G determines the reference point for the payoffs. Utility from a realized payoff x in period 1 is then given by

$$u_1(x) = x + \int_0^1 \mu(x - c_G(p))dp.$$

Choosing some action with payoff distribution F when the reference point is G therefore induces expected utility of

$$U(F|G) = \int_{-\infty}^{\infty} \left(x + \int_0^1 \mu(x - c_G(p))dp \right) dF(x). \tag{7}$$

By the definition in Kőszegi and Rabin (2009) an action with distribution G is a personal equilibrium if it maximizes expected utility, given its induced beliefs, i.e. if $U(G|G) \geq U(F|G)$ for all $F \in D$.

Similarly, by Equation (2) in Kőszegi and Rabin (2007) expected utility of a payoff distribution F when the reference point is G is given by

$$U(F|G) = \int_{-\infty}^{\infty} \left(x + \int_{-\infty}^{\infty} \mu(x - s)dG(s) \right) dF(x), \tag{8}$$

and G is a UPE if $U(G|G) \geq U(F|G)$ for all $F \in D$. It therefore remains to show that

$$\int_0^1 \mu(x - c_G(p))dp = \int_{-\infty}^{\infty} \mu(x - s)dG(s),$$

such that the definitions of $U(F|G)$ in equation (7) and (8) coincide. For continuously increasing distributions G this is an immediate consequence of integration by substitution. For general distributions it follows from the fact that integration $\int \cdot \, dG(x)$ is the pushforward measure of the Lebesgue measure under $c_G : (0, 1) \to \mathbb{R}$

(c.f. Theorem 1.104 in Klenke (2013)), which concludes the proof.

The competing equilibrium concept to UPE for static decision problems is CPE (Kőszegi and Rabin (2007)) where a bidder's reference point changes when contemplating deviations, i.e. an equilibrium b^* satisfies $U(b^*, \theta|b^*) \geq U(b, \theta|b)$ for any deviation b. Rosato (2020) provides an extension of this static CPE concept to dynamic decision problems: gains and losses materialize only when transfers are made. Anticipating future decisions and using backward induction a bidder at each point in time then faces a (static) CPE decision problem, where she evaluates expected utility $U_t(b, \theta|b)$ as the sum of expected classical utility and (final period) expected gain-loss utility using current beliefs about future plans and uncertainty.

As an important difference to the PE concept used in my paper, Rosato's concept of sequential CPE (SCPE) does not provide a model of news utility since bidders don't perceive gains and losses from belief changes during the auction. Hence, it is unable to draw a comparison to the Vickrey auction on the effect of belief fluctuations and does not give scope for strategic incentives to mitigate surprises (insurance effect).

I now show that the discouragement effect and the implied revenue ranking between the two auction formats obtains under SCPE. This demonstration serves two purposes at once. First, it shows that my results are not driven by the modeling choice in Kőszegi and Rabin (2009), i.e. that the reference point is fixed when contemplating deviations. Second, it shows that the discouragement effect as a driver for the revenue ranking depends only on the dynamically decreasing belief to win, but not on the model of news utility for belief changes during the auction.

For the static Vickrey auction SCPE reduces to standard CPE. Proposition 3 in Lange and Ratan (2010) provides the respective symmetric equilibrium bidding function

$$b(\theta) = \Big(1 - \Lambda\big(1 - 2G^{n-1}(\theta)\big)\Big)\theta.$$

Again, due to the attachment effect, high types overbid, increasingly so in their belief to win.

For the English auction call $\Delta_t = \frac{F(b)-F(t)}{1-F(t)}$ the winning probability of plan b at price t. An SCPE bidding plan b necessarily implies that a bidder does not want to deviate and drop out at any price $t < b$. Since a dropout yields a utility of zero, by Lemma 1 this condition reads

$$-\Delta_t(1-\Delta_t)\Lambda + \Delta_t(\theta - t) \geq 0,$$

which solves to

$$t \leq (1-(1-\Delta_t)\Lambda)\theta.$$

For a continuous belief F and $t \to b$ one has $\Delta_t \to 0$, and hence

$$b(\theta) \leq (1-\Lambda)\theta.$$

Hence, for all types $\theta > \theta^{\min}$ bids are strictly higher in the Vickrey auction, and the Vickrey auction yields higher revenue.

6.2 Loss Aversion in the Money Dimensions

When bidders are loss averse with respect to both commodity dimensions, all intuitions about the insurance effect, the discouragement effect, and the revenue ranking remain unchanged.

The Vickrey Auction

Denote in short with $F_b \equiv F_0^m(b, \theta, H)$ the distribution of payments for a submitted bid b given the continuous distribution of the highest opponent bid H. Since with probability $1 - H(b)$ a bidder loses and pays nothing, the distribution F_b is given by

$$F_b(s) = \begin{cases} 1 - H(b) + H(s) & s \leq b, \\ 1 & s > x, \end{cases}$$

For a reference point induced by bid b^* a realized payment of x induces gain-loss utility

$$\begin{aligned}\int_0^1 \mu_m(c_{F_{b^*}}(p) - x)dp &= \int_0^\infty \mu_m(s-x)dF_{b^*}(s) \\ &= \big(1 - H(b^*)\big)\mu_m(-x) + \int_0^{b^*} \mu_m(s-x)dH(s),\end{aligned}$$

where the first equality follows from the fact that integration by dF_{b^*} is the pushforward of Lebesgue integration under $c_{F_{b^*}} : (0,1) \to \mathbb{R}^+$ (see, for instance, Theorem 1.104 in Klenke (2013)).

Hence, expected gain-loss utility in the money dimension from a bid b when the reference point is given by bid b^* is

$$\begin{aligned} L_0^m(b, \theta, H|b^*) &= \int_0^\infty \left((1 - H(b^*))\mu_m(-x) + \int_0^{b^*} \mu_m(s - x)dH(s) \right) dF_b(x) \\ &= \int_0^b \left((1 - H(b^*))\mu_m(-x) + \int_0^{b^*} \mu_m(s - x)dH(s) \right) dH(x) \\ &\quad + (1 - H(b)) \int_0^{b^*} \mu_m(s)dH(s), \end{aligned}$$

where I used that x is zero with probability $1 - H(b)$. Intuitively, the first summand is the loss from winning and paying unexpectedly, the second summand is gain-loss utility from winning at a price different than expected, and the third summand is the gain from losing unexpectedly and not paying. For total expected utility I put together the derived gain-loss utility in the money dimension with classical utility and gain-loss utility in the good dimension as derived in Section 4.1.

$$\begin{aligned} U_0(b, \theta|b^*) = &\int_0^b (\theta - x)dH(x) + \big(1 - H(b)\big)H(b^*)\mu_g(-\theta) + H(b)\big(1 - H(b^*)\big)\mu_g(\theta) \\ &+ (1 - H(b^*) \int_0^b \mu_m(-x)dH(x) + \int_0^b \int_0^{b^*} \mu_m(s - x)dH(s)dH(x) \\ &+ (1 - H(b)) \int_0^{b^*} \mu_m(s)dH(s). \end{aligned}$$

In equilibrium the order statistic H is again endogenously determined by the opponents' equilibrium bids. Using the opponents' response functions, it is straightforward to calculate the symmetric equilibrium bidding function:

Proposition 4. *Consider n loss-averse bidders in the Vickrey auction. Then the unique symmetric increasing continuously differentiable UPE for is given by*

$$\begin{aligned} b(\theta) = &\frac{1 + \eta_g + \Lambda_g G^{n-1}(\theta)}{1 + \lambda_m \eta_m} \theta \\ &+ \int_{\theta^{min}}^{\theta} \frac{\Lambda_m (1 + \eta_g + \Lambda_g G^{n-1}(x))}{(1 + \lambda_m \eta_m)^2} x \exp\left(\frac{\Lambda_m}{1 + \lambda_m \eta_m} \big(G^{n-1}(\theta) - G^{n-1}(x)\big) \right) dG^{n-1}(x). \end{aligned}$$

Note that $b(\theta^{\min}) = \frac{1+\eta_g}{1+\lambda_m\eta_m}\theta^{\min}$, while any $\theta > \theta^{\min}$ yields $b(\theta) > \frac{1+\eta_g+\Lambda_g G^{n-1}(\theta)}{1+\lambda_m\eta_m}\theta >$

$\frac{1+\eta_g}{1+\lambda_m \eta_m}\theta$. In particular, for equally weighted loss aversion in both dimensions, there is now a meaningful comparison to the model with risk neutrality. Low types underbid compared to the risk-neutral benchmark, while

$$b(\theta^{\max}) > \frac{1+\eta+\Lambda G^{n-1}(\theta^{\max})}{1+\lambda\eta}\theta^{\max} = \frac{1+\eta+\Lambda}{1+\lambda\eta}\theta^{\max} = \theta^{\max}$$

shows that high types overbid. The reason is again the attachment effect.

The English Auction

I concisely develop the equilibrium characterization along the lines of the baseline model. Similar to Proposition 2 the following Lemma derives reference-dependent utility from deviations and expected news utility from the auction process.

Lemma 5. *At time t total expected deviation utility with loss aversion in both dimensions is*

$$U_t(b,\theta,F|b^*) = \frac{\int_t^b (\theta - x)dF(x)}{1-F(t)} + \frac{\mu_g(F(b)-F(b^*))\theta}{1-F(t)} + \frac{\mu_m\left(\int_{b^*}^b -xdF(x)\right)}{1-F(t)} + L_t^g(b,\theta,F|b) + L_t^m(b,\theta,F|b),$$

where $L_t^g(b,\theta,F|b)$ is expected news utility in good as in Proposition 2, and

$$L_t^m(b,\theta,F|b) = -\Lambda_m \int_t^b \frac{f(s)}{1-F(t)}\left(\frac{1-F(b)}{1-F(s)}s + \int_s^b \frac{F(b)-F(x)}{1-F(s)}dx\right)ds.$$

Figure 7 shows expected gain-loss utility in money for a uniform maximal opponent bid. Similar to Figure 3 losses are strongest for intermediate bids. Again, expected loss around the highest bid is decreasing with unbounded slope. Intuitively, a strong bid shifts mass in the payment distribution from zero to high payments, which reduces payment uncertainty in the late auction for bidders who are likely to pay a high price.

Similar to Lemma 2 and Lemma 3, I derive the ex-ante optimum and the time consistency constraint.

Lemma 6. *If bidders could commit to a strategy ex ante, the lowest symmetric PE would satisfy*

$$b_{ex\ ante}(\theta) = \left(1+\lambda_m\eta_m+\Lambda_m-\Lambda_m \ln(1-G^{n-1}(\theta))\right)\int_0^\theta \left(\frac{1+\eta_g-\Lambda_g-\Lambda_g\ln(1-G^{n-1}(s))+\Lambda_g\frac{(n-1)g(s)G^{n-2}s}{1-G^{n-1}(s)}}{(1+\lambda_m\eta_m+\Lambda_m-\Lambda_m\ln(1-G^{n-1}(s)))^2}\right)ds.$$

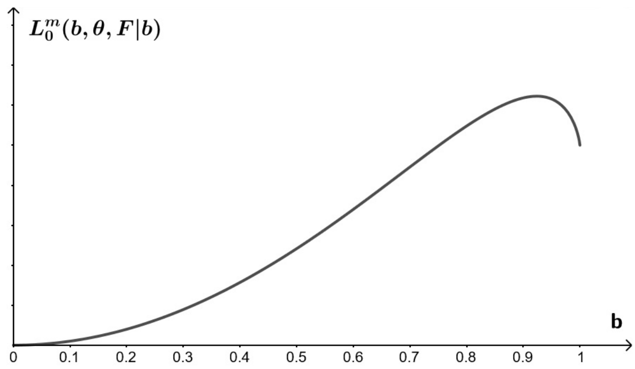

Figure 7: Total Expected Loss in money for $F \sim U[0,1]$

Any time-consistent bidding plan necessarily satisfies

$$b(\theta) \leq \frac{1+\eta_g}{1+\lambda_m \eta_m}\theta.$$

Again, bidders have an incentive to insure against belief fluctuations ex ante. In particular, high types would like to excessively overbid. However, again bidders eventually become discouraged, and time consistency requires bidding at most $\frac{1+\eta_g}{1+\lambda_m \eta_m}\theta$, identical to the lowest type in the Vickrey auction.

I define

$$\underline{b}(\theta) = \min\left\{ b_{\text{ex ante}}(\theta), \frac{1+\eta_g}{1+\lambda_m \eta_m}\theta \right\}.$$

Proposition 5. *Any symmetric PE in the English auction is bounded from above by $\frac{1+\eta_g}{1+\lambda_m \eta_m}\theta$ and bounded from below by $\underline{b}(\theta)$. If $\eta_g \geq \eta_m$ and $\lambda_g \geq \lambda_m$ then the lower bound is attained, and $\underline{b}(\theta)$ is the lowest symmetric PE.*[22]

Figure 8 summarizes the above results. The gray shaded area together with its upper bound line depict the region of potential PE. For equally pronounced loss aversion in both dimensions the upper bound is below the risk-neutral bidding function, and all types underbid in the English auction. Since in the Vickrey auction $b_V(\theta) \geq \frac{1+\eta_g}{1+\lambda_m \eta_m}\theta$, with equality only for $\theta^{\min}$, and in the English auction by time consistency $b_E(\theta) \leq \frac{1+\eta_g}{1+\lambda_m \eta_m}\theta$, it is immediate that the Vickrey auction remains to generate higher revenue.

[22]Intuitively, more pronounced losses in the good dimension ensure that the insurance effect is strongest at time zero, such that it suffices to look at ex-ante incentives for upward deviations.

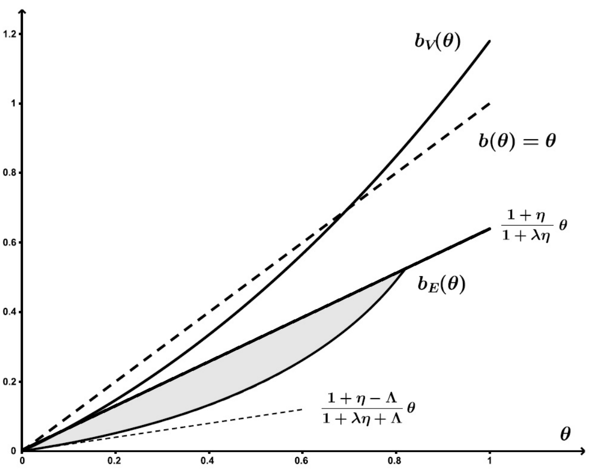

Figure 8: $G^{n-1}(\theta) \sim U[0,1]$, $\eta_g = \eta_m = 0.4$, $\lambda_g = \lambda_m = 3$

6.3 Observable Dropouts

In this section, I analyze a model where opponents' dropout prices are observable. I show that any PE strategy is necessarily history dependent. I establish the discouragement effect and the revenue ranking for any PE with continuous beliefs. Then I derive the unique symmetric equilibrium for the three-bidder case and show non-existence of symmetric equilibria for more than three bidders.

In order to deal with strategies contingent on histories, I introduce the following notation:

Definition 4. For any n-bidder auction, define for all $k \in \{0, ..., n-2\}$

$$H_k = \{(t_1, ..., t_k) | 0 \leq t_1 \leq, ..., \leq t_k\}$$

as the set of histories / future contingencies with k dropouts at the respective prices $t_1, ..., t_k$, with the convention $H_0 = \{\emptyset\}$.

With this notation, a complete bidding plan prescribes for each history and future contingency the price at which a bidder of type θ plans to drop out:

Definition 5. A pure strategy bidding plan prescribes a bidding strategy

$$b : \bigcup_{0 \leq k \leq n-2} H_k \times [\theta^{\min}, \theta^{\max}] \to \mathbb{R}_+,$$

with the restriction that for any $(t_1, ..., t_k, \theta)$

$$b(t_1, ..., t_k, \theta) > t_k.$$

The latter condition on the bidding function ensures that bidders cannot condition their dropout on events that happen after the dropout. For notational convenience, I usually omit the type as an argument of the strategy $b(h_k)$.

Denote for any history $h_k \in H_k$ with $F_{h_k}(s)$ the belief about the $k+1$-th drop-out price s as held at price t_k. I call a bidder's belief continuous if for all possible histories h_k the distributions $F_{h_k}(s)$ are continuous in s.

I now derive an iterative formula for expected news utility of a bidder of type θ at the time of an opponent dropout. For $k = n-2$ the two-bidder auction after some dropout history h_k yields by Proposition 2 an expected news utility of

$$L_{h_k}(b(h_k)) = \Lambda \ln\Big(1 - F_{h_k}(b(h_k))\Big)\Big(1 - F_{h_k}(b(h_k))\Big)\theta. \tag{9}$$

I call for $h_k = (t_1, ..., t_k)$

$$P_{h_k}(s) = \frac{F_{h_k}(b(h_k)) - F_{h_k}(s)}{1 - F_{h_k}(s))} \tag{10}$$

the belief to win this two-bidder auction as held at price $s > t_k$ for history h_k.

For $k \in \{0, ..., n-1\}$ I define (backward) inductively

$$P_{h_k}(s) = \int_s^{b(h_k)} P_{(h_k, t_{k+1})}(t_{k+1}) dF_{h_k}(t_{k+1}|s), \tag{11}$$

where $F_{h_k}(t|s) = \frac{F_{h_k}(t) - F_{h_k}(s)}{1 - F_{h_k}(s)}$ is the belief about the next dropout at t as held at s with dropout history h_k.

Notice that for any history $h_k = (t_1, ..., t_k)$ and increments of ε a dropout at $t_{k+1} > t_k$ updates the winning probability from $P_{h_k}(t_{k+1} - \varepsilon)$ to $P_{(h_k, t_{k+1})}(t_{k+1})$.[23] For continuous strategies in the continuous English auction Lemma 1 then implies the following formula for total news utility:

$$L_{h_k}(b(h_k)) = \int_{t_k}^{b(h_k)} -\Lambda\theta \big|P_{h_k}(t_{k+1}) - P_{(h_k, t_{k+1})}(t_{k+1})\big| + L_{(h_k, t_{k+1})}(b(h_k, t_{k+1})) dF_{h_k}(t_{k+1}) \tag{12}$$

[23] This formulation does not account for the possibility of multiple dropouts at the same time. With the natural assumption that any opponent dropout increases the winning probability, there is no loss in assuming multiple distinct updates in this case. Moreover, for any continuous belief in a continuous auction the event of multiple dropouts at the same price is an event of measure zero, and does not enter considerations about utility maximization.

Proposition 6. *For any belief the set of PE and PPE is nonempty. For any continuous belief as the increment size goes to zero any PE approaches a function bounded from above by* $(1+\eta)\theta$*.*

Proposition 6 captures the central intuition that was already present without observable dropouts. Any equilibrium bidding limit must be a time-consistent plan. Since at the time the price has reached the limit, the belief to win has decreased to zero for any continuous belief. Thus, the attachment effect has vanished, and bidding above $(1+\eta)\theta$ is time inconsistent.

Symmetric Equilibria

For symmetric increasing bidding functions it follows for types s, t that $P_{h_k}(b(h_k, s)) = \left(\frac{G(\theta)-G(s)}{1-G(s)}\right)^{n-k-1}$ and $F_{h_k}(b(h_k, s)|b(h_k, t)) = 1 - \left(\frac{1-G(s)}{1-G(t)}\right)^{n-k-1}$. It is then convenient to index the expected loss function with the number of bidders, rather than histories, and take types as argument, rather than bids. With this slight change in notation I define $G_n(s|t) \equiv 1 - \left(\frac{1-G(s)}{1-G(t)}\right)^n$, and obtain from Equation (12) the following formula for total expected gain-loss utility for the remaining auction with n opponents at a price where type t would have dropped out:

$$L_n(t,\theta) = \int_t^\theta g_n(s|t)\left[-\Lambda\theta\left(\left(\frac{G(\theta)-G(s)}{1-G(s)}\right)^{n-1} - \left(\frac{G(\theta)-G(s)}{1-G(s)}\right)^n\right) + L_{n-1}(s,\theta)\right] ds \tag{13}$$

Note that for $t = \theta^{\min}$ I obtain a formula for total expected loss, which is independent of the bidding function. Indeed, belief fluctuations in a symmetric increasing equilibrium are fully determined by the realization of types. Since only a dropout that terminates the auction affects the price, it follows that any symmetric increasing equilibrium can only be determined up to monotone transformations of the bidding functions $b(h_k, \theta)$ for any $k < n-2$.

The symmetric equilibrium for 3 bidders

As any symmetric continuously increasing equilibrium $\big(b(\theta), b(s,\theta)\big)$ does not essentially change under monotone transformations of $b(\theta)$, any such equilibrium for the three-bidder auction is "essentially uniquely" described by $b(s,\theta)$, the maximal bid of a type-θ bidder, when the first dropout is of type s.

Denote with $\hat{\theta}(s)$ a deviation from equilibrium that prescribes a dropout price $b(s, \hat{\theta}(s))$ for all s. Using (12) and (13), total expected news utility with three active bidders from price $b(t)$ onwards is

$$L_2(t,\hat{\theta}(s)) = \int_t^{\theta} \frac{2g(s)(1-G(s))}{(1-G(t))^2} \left[-\Lambda\theta \left(\frac{G(\hat{\theta}(s))-G(s)}{1-G(s)} - \int_s^{\theta} \frac{2g(x)}{(1-G(s))^2} \big(G(\hat{\theta}(x))-G(x)\big)dx \right) + \frac{\theta L_1(s,\hat{\theta}(s))}{\hat{\theta}(s)} \right] ds.$$

Proposition 7. *The essentially unique symmetric, continuously increasing equilibrium of the continuous English auction with three bidders and observable dropouts is given by*

$$b(s,\theta) = \min\left\{ (1+\eta)\theta, \left(1+\eta-\Lambda\left(1+\ln\left(\frac{1-G(\theta)}{1-G(s)}\right)\right)\right)\theta \right\}.$$

Figure 9 shows equilibrium bidding functions for various lowest realizations s. The dashed function indicates the benchmark with unobservable dropouts. Any symmetric equilibrium plan $b(s,\theta)$ is constrained by the conditions of equilibria for the two-bidder auction following the dropout of type s, as outlined in Proposition 3.

Since beliefs fluctuate more compared to the model with unobservable dropouts, expected losses from news are strictly higher. In particular the additional update from the first dropout creates an incentive to bid low in the two-bidder subgame to mitigate this update. As a result, the lowest PE for the subsequent two bidder auction remains as the only time-consistent strategy. Since the two-bidder auction itself is subject to the insurance effect as outlined in Lemma 2, and increasingly so in the duration of that auction, the bid $b(s,\theta)$ depends on s, the time where the first dropout occurs two-bidder part of the auction starts. Hence, equilibrium bids weakly decrease in the type of the first dropout, and are therefore history dependent. It is worth emphasizing that the *set* of PE at any time t looking forward is history independent. However, the ex-ante time-consistent choice *within* that set depends on the dropout history before time t.

No symmetric equilibria for more than 3 bidders

With more than three bidders, observable dropouts, and instantaneous updating, belief fluctuations lead to substantial losses in news utility, which exceed expected gains from trade for low types. Hence, participation becomes a central issue. Strictly increasing equilibria fail to exist, since the lowest types would always prefer to stay out of the auction. Therefore, I focus on weakly increasing symmetric equilibria, in order to allow for bunching and collective non-participation

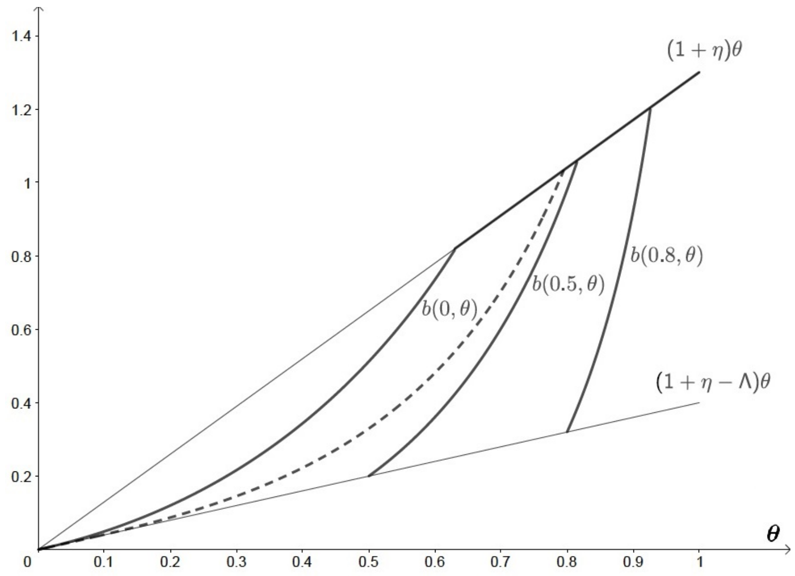

Figure 9: $G(\theta) \sim U[0,1]$, $\eta = 0.3$, $\lambda = 4$

of types below some threshold. But even then the following result holds:

Proposition 8. *With more than three bidders, observable dropouts, and instantaneous updating there exist no weakly increasing symmetric equilibria.*

The idea of the proof is to show that for any potential equilibrium there must be some interval of types for which the bidding function in some contingency does not have atoms. But then as the winning probability P of a θ-type bidder from that interval approaches zero, expected losses from news utility are of order $P\Lambda^n\theta$, with n the number of active opponents. These losses exceed the potential gains from trade, and the bidder prefers to drop out immediately rather than follow the equilibrium plan.

The non-existence result is driven by the stark assumption of instantaneous updating. While this assumption is innocuous in the baseline model, since updating will unambiguously bring bad news until the bidder wins, it may be inappropriate in settings with strong belief fluctuations.

As shown in Appendix 6.1, the modeling choices of news utility are not the force behind the discouragement effect, which drives low bids in the English auction. Hence, the logic of Proposition 6 remains unchanged in a model without news utility or with less frequent updating.

6.4 Proofs

Proof of Proposition 1. Suppose that all opponents bid according to some increasing, continuously differentiable bidding function $b(\theta)$. Since $G(\theta)$ is a distribution with strictly positive, continuous density g, it follows that the distribution of the maximal opponent bid, $H(x) = G^{n-1}(b^{-1}(x))$, is a differentiable distribution with positive, continuous density $h(x)$ on $[b(\theta^{\min}), b(\theta^{\max})]$ as well.

The bidding function $b(\theta)$ constitutes a PE if and only if the utility function $U_0(x, \theta|b(\theta))$ attains its maximum at $x = b(\theta)$ for all θ. Differentiation with respect to x yields

$$\frac{\partial U_0(x, \theta|b(\theta))}{\partial x} = h(x)\big(1 - H(b(\theta))\big)\mu(\theta) - h(x)H(b(\theta))\mu(-\theta) + (\theta - x)h(x).$$

Dividing by $h(x)$ and evaluating at $x = b(\theta)$ yields the first-order condition

$$0 = \big(1 - H(b(\theta))\big)\eta\theta + H(b(\theta))\lambda\eta\theta + (\theta - b(\theta)).$$

Rearranging yields

$$b(\theta) = \Big(1 + \eta\big(1 - H(b(\theta))\big) + \lambda\eta H(b(\theta))\Big)\theta. \tag{14}$$

Since $H(b(\theta)) = G^{n-1}(\theta)$ one obtains

$$b(\theta) = \Big(1 + \eta\big(1 - G^{n-1}(\theta)\big) + \lambda\eta G^{n-1}(\theta)\Big)\theta$$

as the unique equilibrium candidate. For sufficiency note first that

$$h(b(\theta)) = \frac{(G^{n-1})'(\theta)}{b'(\theta)} = \frac{(n-1)G^{n-2}(\theta)g(\theta)}{1 + \eta\big(1 - G^{n-1}(\theta)\big) + \lambda\eta G^{n-1}(\theta) + \Lambda(n-1)G^{n-2}(\theta)g(\theta)\theta}$$

is differentiable since $g(\theta)$ is differentiable. Now it is immediate that

$$\frac{\partial^2 U_0(x, \theta|b(\theta))}{(\partial x)^2}\Big|_{x=b(\theta)} = -h\big(b(\theta)\big) + h'\big(b(\theta)\big)\underbrace{\Big[\theta - b(\theta) + \big(1 - H(b(\theta))\big)\mu(\theta) - H(b(\theta))\mu(-\theta)\Big]}_{=0}$$
$$< 0.$$

□

Proof of Lemma 1. For calculating the ex-ante expected gain-loss utility, it is more convenient to work with distribution functions rather than with quantile functions. This is possible, since they are generalized inverses of each other, and the integral between functions equals the integral between their inverses up to the sign:

Lemma 7. *Let F_1 and F_2 be distributions on some interval $[a, b]$ and let c_{F_1}, c_{F_2} be the respective quantile functions. Then*

$$\int_a^b \mu(F_1(x) - F_2(x))dx = \int_0^1 \mu(c_{F_2}(p) - c_{F_1}(p))dp.$$

Proof of Lemma 7. Suppose first that F_1 and F_2 are invertible on $[a, b]$. By the theorem of the integral over inverse functions (e.g. Theorem 1 in Key (1994)), any invertible distribution F on $[a, b]$ satisfies

$$\int_a^b F(x)dx = bF(b) - aF(a) - \int_0^1 c_F(p)dp = b - \int_0^1 c_F(p)dp,$$

which implies

$$\int_a^b (F_1(x) - F_2(x))dx = (b - b) - \int_0^1 c_{F_1}(p)dp + \int_0^1 c_{F_2}(p)dp = \int_0^1 (c_{F_2}(p) - c_{F_1}(p))dp.$$

Define now

$$F_1^+(x) = \begin{cases} F_1(x) & F_1(x) > F_2(x), \\ F_2(x) & F_1(x) \leq F_2(x), \end{cases}$$

and similarly

$$F_1^-(x) = \begin{cases} F_1(x) & F_1(x) \leq F_2(x), \\ F_2(x) & F_1(x) > F_2(x). \end{cases}$$

By construction, F_1^+ and F_1^- are invertible and satisfy $F_1^+(x) \geq F_2(x) \geq F_1^-(x)$ for all $x \in [a, b]$, and moreover

$$c_{F_1^+}(p) = \begin{cases} c_{F_1}(p) & F_1(c_{F_1}(p)) > F_2(c_{F_1}(p)), \\ c_{F_2}(p) & F_1(c_{F_1}(p)) \leq F_2(c_{F_1}(p)), \end{cases}$$

and

$$c_{F_1^-}(p) = \begin{cases} c_{F_1}(p) & F_1(c_{F_1}(p)) \leq F_2(c_{F_1}(p)), \\ c_{F_2}(p) & F_1(c_{F_1}(p)) > F_2(c_{F_1}(p)), \end{cases}$$

for all $p \in [0,1]$. With these constructions I obtain

$$\begin{aligned}
&\int_a^b \mu(F_1(x) - F_2(x))dx \\
&= \int_a^b \eta(F_1^+(x) - F_2(x))dx + \int_a^b \lambda\eta(F_1^-(x) - F_2(x))dx \\
&= \int_0^1 \eta(c_{F_2}(p) - c_{F_1^+}(p))dp + \int_0^1 \lambda\eta(c_{F_2}(p) - c_{F_1^-}(p))dp \\
&= \int_0^1 \mu(c_{F_2}(p) - c_{F_1}(p))\mathbb{1}_{F_1(c_{F_1}(p))>F_2(c_{F_1}(p))}dp + \int_0^1 \mu(c_{F_2}(p) - c_{F_1}(p))\mathbb{1}_{F_1(c_{F_1}(p))\leq F_2(c_{F_1}(p))}dp \\
&= \int_0^1 \mu(c_{F_2}(p) - c_{F_1}(p))d(p),
\end{aligned}$$

which proves the lemma for invertible distributions. I now show the lemma for general distribution functions, when the quantile functions c_{F_i} are defined as usual by

$$c_{F_i}(p) = \inf\{x \in \mathbb{R} | p \leq F(x)\}.$$

Take two arbitrary sequences of continuously increasing distribution functions $(F_{1,n})$, $(F_{2,n})$ which converge pointwise $F_{i,n} \to F_i$ everywhere outside the null-set of discontinuity points of F_i for $i = 1, 2$.[24] By Theorem 1.1.1 in De Haan and Ferreira (2007), $\lim_{n\to\infty} F_{i,n}(x) = F_i(x)$ for all continuity points of F_i implies $\lim_{n\to\infty} c_{F_{i,n}}(p) = c_{F_i}(p)$ for all continuity points of $c_{F_i}(p)$. Using Lebesgue's dominated convergence theorem and that the set of discontinuity points is a null-set,

$$\begin{aligned}
\int_a^b \mu(F_1(x) - F_2(x))dx &= \int_a^b \lim_{n\to\infty} \mu(F_{1,n}(x) - F_{2,n}(x))dx \\
&= \lim_{n\to\infty} \int_a^b \mu(F_{1,n}(x) - F_{2,n}(x))dx \\
&= \lim_{n\to\infty} \int_0^1 \mu(c_{F_{2,n}}(p) - c_{F_{1,n}}(p))d(p) \\
&= \int_0^1 \lim_{n\to\infty} \mu(c_{F_{2,n}}(p) - c_{F_{1,n}}(p))d(p) \\
&= \int_0^1 \mu(c_{F_2}(p) - c_{F_1}(p))d(p),
\end{aligned}$$

[24]To see existence of such a sequence, take a positive sequence $\varepsilon_n \to 0$ and define $F_{i,n} = (1 - \varepsilon_n)G_{i,n} + \varepsilon_n \frac{x-a}{b-a}$, where $G_{i,n}$ is the continuous function which is linear on ε_n-balls around any discontinuity point of F_i and coincides with F_i elsewhere.

which concludes the proof for arbitrary distributions. □

By applying Lemma 7, and using the fact that μ is piecewise linear, one can write

$$\begin{aligned}
\mathbb{E}(N(F_i|F)) &= \Delta N(F_1|F) + (1-\Delta)N(F_2|F) \\
&= \Delta \int_0^1 \mu(c_{F_1}(p) - c_F(p))dp + (1-\Delta)\int_0^1 \mu(c_{F_2}(p) - c_F(p))dp \\
&= \Delta \int_a^b \mu(F(x) - F_1(x))dx + (1-\Delta)\int_a^b \mu(F(x) - F_2(x))dx \\
&= \Delta \int_a^b \mu(F(x) - F_1(x))dx + \int_a^b \mu\big((1-\Delta)F(x) - (1-\Delta)F_2(x)\big)dx \\
&= \Delta \int_a^b \mu(F(x) - F_1(x))dx + \int_a^b \mu\big((1-\Delta)F(x) - (F(x) - \Delta F_1(x))\big)dx \\
&= \Delta \int_a^b \mu(F(x) - F_1(x))dx + \int_a^b \mu\big(-\Delta F(x) + \Delta F_1(x)\big)dx \\
&= \Delta \int_a^b \mu(F(x) - F_1(x))dx + \Delta\int_a^b \mu(-F(x) + F_1(x))dx \\
&= \Delta(-\lambda\eta + \eta)\int_a^b |F(x) - F_1(x)|dx \\
&= -\Delta\Lambda \int_a^b |F(x) - F_1(x)|dx.
\end{aligned}$$

□

Proof of Proposition 2. Suppose the current clock price is t and the opponent hasn't dropped out yet. I start with calculating the deviation utility $N(F_t(b,\theta,F)|F_t(b^*,\theta,F))$.

At time t the winning probability is the probability that the opponent drops out between t and b^*, given she didn't drop out before t, thus $\frac{F(b^*)-F(t)}{1-F(t)}$. The update changes the probability of getting a utility of θ by

$$\frac{F(b)-F(t)}{1-F(t)} - \frac{F(b^*)-F(t)}{1-F(t)} = \frac{F(b)-F(b^*)}{1-F(t)}.$$

Hence,

$$N(F_t^b|F_t^{b^*}) = \mu\left(\frac{F(b)-F(b^*)}{1-F(t)}\theta\right) = \frac{\mu(F(b)-F(b^*))}{1-F(t)}\theta.$$

Now I turn to the expected gain-loss utility from a bid b. If the clock increases in increments of ε, then the conditional probability that the opponent drops out at the next increment is given by

$$\Delta_t := \frac{F(t+\varepsilon) - F(t)}{1 - F(t)}.$$

Given her strategy b and that the opponent hasn't dropped out until t, the bidder faces the conditional probability of $\frac{1-F(b)}{1-F(t)}$ to lose the auction. Denote with F_t^b the belief about payoffs in the good dimension at time t given strategy b, i.e.

$$F_t^b(z) = \begin{cases} \frac{1-F(b)}{1-F(t)} & z < \theta, \\ 1 & z \geq \theta. \end{cases}$$

If the bidder wins in the next increment, the belief will update to

$$F_{t+\varepsilon}^b(z) = \begin{cases} 0 & z < \theta, \\ 1 & z \geq \theta. \end{cases}$$

According to Lemma 1, expected gain-loss utility of the increment from t to $t+\varepsilon$ is then given by

$$\mathbb{E}(N(F_{t+\varepsilon}^b | F_t^b)) = -\Delta_t \Lambda \int |F_t^b(z) - F_{t+\varepsilon}^b(z)| dz = -\Delta_t \Lambda \frac{1 - F(b)}{1 - F(t)} \theta.$$

Now, the marginal loss at time t if ε goes to zero reads

$$\ell_t(b, \theta, F) = \lim_{\varepsilon \to 0} \frac{-\Delta_t \Lambda \frac{1-F(b)}{1-F(t)} \theta}{\varepsilon} = \frac{-f(t)}{(1 - F(t))^2} (1 - F(b)) \Lambda \theta.$$

To calculate total expected gain-loss utility starting at time t, note that any information update at time $s > t$ is only informative and carries gain-loss utility if the opponent hasn't already dropped out between t and s, which holds true with the conditional probability $\frac{1-F(s)}{1-F(t)}$. Thus

$$\begin{aligned}
L_t(b,\theta,F|b) &= \lim_{\varepsilon\to 0}\sum_{i=0}^{\lfloor\frac{b-t}{\varepsilon}\rfloor-1} N(F^b_{t+(i+1)\varepsilon}|F^b_{t+i\varepsilon}) \\
&= \lim_{\varepsilon\to 0}\sum_{i=0}^{\lfloor\frac{b-t}{\varepsilon}\rfloor-1} -\frac{1-F(t+i\varepsilon)}{1-F(t)}\Delta_{t+i\varepsilon}\Lambda\frac{1-F(b)}{1-F(t+i\varepsilon)}\theta \\
&= \int_t^b \frac{-f(s)}{1-F(s)}\frac{1-F(b)}{1-F(t)}\Lambda\theta ds \\
&= \Big(\ln(1-F(b))-\ln(1-F(t))\Big)\frac{1-F(b)}{1-F(t)}\Lambda\theta \\
&= \ln\left(\frac{1-F(b)}{1-F(t)}\right)\frac{1-F(b)}{1-F(t)}\Lambda\theta.
\end{aligned}$$

□

Proof of Lemma 2. Given the distribution of the maximal opponent bid F and bidder's type θ, a bid $b(\theta)$ is a personal equilibrium in the auction with commitment if and only if

$$U_0\big(y,\theta,F|b(\theta)\big)\leq U_0\big(b(\theta),\theta,F|b(\theta)\big)$$

for all y. In particular, it is necessary that

$$\lim_{y\searrow b(\theta)}\frac{\partial U_0\big(y,\theta,F|b(\theta)\big)}{\partial y}\leq 0.$$

By Equation (6) the utility for $y>b(\theta)$ at time zero reads

$$U_0\big(y,\theta,F|b(\theta)\big)=\int_0^y(\theta-s)dF(s)+\eta\big(F(y)-F(b(\theta))\big)\theta+\ln\big(1-F(y)\big)\big(1-F(y)\big)\Lambda\theta.$$

Hence, the necessary condition is equivalent to

$$f(b(\theta))\Big(\theta-b(\theta)+\eta\theta-\Lambda\big(1+\ln(1-F(b(\theta)))\big)\theta\Big)\leq 0.$$

In any symmetric equilibrium, the opponents bid according to $b(\theta)$ as well, and therefore $F(b(\theta))=G^{n-1}(\theta)$. From $(G^{n-1})'(\theta)=f(b(\theta))b'(\theta)$ and the restriction that b is increasing it follows that $f(b(\theta)) > 0$. Hence,

$$b(\theta) \geq \Big(1 + \eta - \Lambda\big(1 + \ln(1 - G^{n-1}(\theta))\big)\Big)\theta$$

for any equilibrium candidate. Note that for any such candidate the participation constraint is satisfied for every bidder, since, by the assumption that $\Lambda < 1 + \eta$, we have

$$b(\theta) \geq \big(1 + \eta - \Lambda\big)\theta > 0.$$

It remains to verify that

$$b(\theta) = \Big(1 + \eta - \Lambda\big(1 + \ln(1 - G^{n-1}(\theta))\big)\Big)\theta \tag{15}$$

is a personal equilibrium, given opponent's response $b(\theta)$. For this it is sufficient to show that

$$\frac{\partial U_0\big(y, \theta, F|b(\theta)\big)}{\partial y} \leq 0$$

for all $y > b(\theta)$, and

$$\frac{\partial U_0\big(y, \theta, F|b(\theta)\big)}{\partial y} \geq 0$$

for all $y < b(\theta)$. Note that I can without loss of generality restrict to $y \in [b(\theta^{\min}), b(\theta^{\max})]$.

For any such y there exists some $\tilde{\theta}$ with $y = b(\tilde{\theta})$, since the bidding function is continuous.

Consider first $y > b(\theta)$, thus $\tilde{\theta} > \theta$. Then

$$\begin{aligned}
\frac{\partial U_0\big(y, \theta, F|b(\theta)\big)}{\partial y}\Big|_{y=b(\tilde{\theta})} &= f(b(\tilde{\theta}))\Big(\theta - b(\tilde{\theta}) + \eta\theta - \Lambda\big(1 + \ln(1 - F(b(\tilde{\theta})))\big)\theta\Big) \\
&< f(b(\tilde{\theta}))\Big(\tilde{\theta} - b(\tilde{\theta}) + \eta\tilde{\theta} - \Lambda\big(1 + \ln(1 - F(b(\tilde{\theta})))\big)\tilde{\theta}\Big) \\
&= \lim_{y \searrow b(\tilde{\theta})} \frac{\partial U_0\big(y, \tilde{\theta}, F|b(\tilde{\theta})\big)}{\partial y} \\
&= 0,
\end{aligned}$$

where the last equality is due to equality in (15). Similarly, for $y < b(\theta)$, thus

$\tilde{\theta} < \theta$,

$$\frac{\partial U_0\big(y,\theta,F|b(\theta)\big)}{\partial y}|_{y=b(\tilde{\theta})} = f(b(\tilde{\theta}))\Big(\theta - b(\tilde{\theta}) + \lambda\eta\theta - \Lambda\big(1+\ln(1-F(b(\tilde{\theta})))\big)\theta\Big)$$
$$> f(b(\tilde{\theta}))\Big(\tilde{\theta} - b(\tilde{\theta}) + \eta\tilde{\theta} - \Lambda\big(1+\ln(1-F(b(\tilde{\theta})))\big)\tilde{\theta}\Big)$$
$$= \lim_{y\searrow b(\tilde{\theta})} \frac{\partial U_0\big(y,\tilde{\theta},F|b(\tilde{\theta})\big)}{\partial y}$$
$$= 0.$$

□

Proof of Lemma 3. The bidder does not want do deviate to a lower strategy y at any time t, given plan b if and only if

$$U_t(y,\theta,F|b) \le U_t(b,\theta,F|b)$$

for all $t \le y \le b$. In particular it is necessary that for all $t < b$ the derivative from the left satisfies

$$0 \le \lim_{y\nearrow b} \frac{\partial U_t(y,\theta,F|b)}{\partial y}$$
$$= \frac{f(b)}{1-F(t)}\left(\theta - b + \lambda\eta\theta - \Lambda\left(1+\ln\left(\frac{1-F(b)}{1-F(t)}\right)\right)\theta\right).$$

This expression is well defined, since $F(t) < F(b) < 1$. Now, as t approaches b one obtains

$$0 \le \lim_{t\to b} \frac{f(b)}{1-F(t)}\left(\theta - b + \lambda\eta\theta - \Lambda\left(1+\ln\left(\frac{1-F(b)}{1-F(t)}\right)\right)\theta\right)$$
$$= \frac{f(b)}{1-F(b)}\left(\theta - b + \lambda\eta\theta - \Lambda\theta\right).$$

Since, by assumption, $f(b) > 0$, this means that necessarily

$$b \le (1+\lambda\eta-\Lambda)\theta = (1+\eta)\theta.$$

□

Proof of Lemma 4. Consider a bidding strategy b^*.

Claim 1: If and only if $b^* \le (1+\eta)\theta$, it is at no time $t < b^*$ profitable to deviate to a lower strategy $b \in [t, b^*)$.

Proof: Necessity has been proved in Lemma 3. For sufficiency suppose that $b^* \leq (1+\eta)\theta$. Consider a deviation at some time $t < b^*$ from b^* to $b \in [t, b^*)$. I first look at the change in expected gain-loss disutility: term A can be interpreted as the change due to different expectations at each time between t and b, while term B is forgone gain-loss disutility, since the auction necessarily ends at b:

$$
\begin{aligned}
& L_t(b, \theta, F|b) - L_t(b^*, \theta, F|b^*) \\
=& \Lambda\theta \left(\ln\left(\frac{1-F(b)}{1-F(t)}\right) \frac{1-F(b)}{1-F(t)} - \ln\left(\frac{1-F(b^*)}{1-F(t)}\right) \frac{1-F(b^*)}{1-F(t)} \right) \\
=& \Lambda\theta \left(\int_t^b \frac{-f(s)}{1-F(s)} ds \frac{1-F(b)}{1-F(t)} - \int_t^{b^*} \frac{-f(s)}{1-F(s)} ds \frac{1-F(b^*)}{1-F(t)} \right) \\
=& \Lambda\theta \left(\int_t^b \frac{-f(s)}{1-F(s)} ds \frac{1-F(b)}{1-F(t)} - \int_t^{b} \frac{-f(s)}{1-F(s)} ds \frac{1-F(b^*)}{1-F(t)} - \int_b^{b^*} \frac{-f(s)}{1-F(s)} ds \frac{1-F(b^*)}{1-F(t)} \right) \\
=& \Lambda\theta \left(\underbrace{\int_t^b \frac{-f(s)}{1-F(s)} ds \frac{F(b^*)-F(b)}{1-F(t)}}_{A} - \underbrace{\int_b^{b^*} \frac{-f(s)}{1-F(s)} ds \frac{1-F(b^*)}{1-F(t)}}_{B} \right) \\
\leq& \Lambda\theta \int_b^{b^*} \frac{f(s)}{1-F(s)} ds \frac{1-F(b^*)}{1-F(t)} \\
<& \Lambda\theta \int_b^{b^*} f(s) ds \frac{1-F(b^*)}{(1-F(b^*))(1-F(t))} \\
=& \Lambda\theta \frac{F(b^*)-F(b)}{1-F(t)}.
\end{aligned}
$$

Now, I have

$$
\begin{aligned}
& U_t(b,\theta,F|b^*) - U_t(b^*,\theta,F|b^*) \\
<& \frac{1}{1-F(t)}\left(-\int_b^{b^*}(\theta - s)dF(s) + \mu(F(b)-F(b^*))\theta + \Lambda\theta(F(b^*)-F(b))\right) \\
<& \frac{F(b^*)-F(b)}{1-F(t)}\left(-\theta + b^* - \lambda\eta\theta + \Lambda\theta\right) \\
=& \frac{F(b^*)-F(b)}{1-F(t)}\left(-(1+\eta)\theta + b^*\right) \\
\leq & 0.
\end{aligned}
$$

Thus, there is no profitable deviation to $b < b^*$ at any time, which concludes the proof of Claim 1.

Claim 1 directly shows the necessity of 1. in Lemma 4 for any PE. Certainly, 2. is necessary as well.

To show sufficiency I start with showing the following claim.

Claim 2: If it is not profitable to deviate to a strategy $b > b^*$ at time $t = 0$, then it is not profitable at any time $t \leq b^*$.

Proof: It is not profitable to deviate to a strategy $b > b^*$ at time t if and only if

$$0 \geq U_t(b,\theta,F|b^*) - U_t(b^*,\theta,F|b^*)$$

Now,

$$
\begin{aligned}
&U_t(b,\theta,F|b^*) - U_t(b^*,\theta,F|b^*)\\
=&\frac{1}{1-F(t)}\left(\int_{b^*}^{b}(\theta - s)dF(s) + \mu(F(b)-F(b^*))\theta\right)\\
&+\Lambda\theta\left(\frac{1-F(b)}{1-F(t)}\ln\left(\frac{1-F(b)}{1-F(t)}\right) - \frac{1-F(b^*)}{1-F(t)}\ln\left(\frac{1-F(b^*)}{1-F(t)}\right)\right)\\
=&\frac{1}{1-F(t)}\Bigg(\int_{b^*}^{b}(\theta - s)dF(s) + \mu(F(b)-F(b^*))\theta \;...\\
&...+\Lambda\theta\Big((1-F(b))\ln(1-F(b)) - (1-F(b^*))\ln(1-F(b^*)) + (F(b)-F(b^*))\ln(1-F(t))\Big)\Bigg).
\end{aligned}
$$

Note that the expression in the big brackets is decreasing in t. Thus, if it is negative for $t = 0$, then it is as well negative for all $t > 0$. Hence, if

$$0 \geq U_0(b,\theta,F|b^*) - U_0(b^*,\theta,F|b^*)$$

then

$$0 \geq U_t(b,\theta,F|b^*) - U_t(b^*,\theta,F|b^*)$$

for all $t > 0$, which concludes the proof of Claim 2.

I am ready to show sufficiency: suppose 1. and 2. hold. Then by Claim 1 it can't be profitable to deviate to a lower strategy at any time. To show that there is no profitable deviation to a higher strategy, take any time-consistent strategy $b \geq b^*$. By Claim 1 this necessarily means $b \in [b^*, (1+\eta)\theta]$. From 2. it follows that $U_0(b^*,\theta,F|b^*) \geq U_0(b,\theta,F|b^*)$. Then, by Claim 2, the bidder does not want to deviate to a higher strategy at any time, and b^* is indeed a PE. □

Proof of Proposition 3. Take some increasing equilibrium function. By Lemma 4, it satisfies $b(\theta) \leq (1+\eta)\theta$ for all $\theta \in (\theta^{\min}, \theta^{\max})$. If $b(\theta) < (1+\eta)\theta$ for some θ, then—again by Lemma 4—any $y \in [b(\theta), (1+\eta)\theta]$ satisfies $U_0(b(\theta),\theta,F|b(\theta)) \geq U_0(y,\theta,F|b(\theta))$. This means that for the right-derrivative

$$\lim_{y \searrow b(\theta)} \frac{\partial U_0(y,\theta,F|b(\theta))}{\partial y} \leq 0,$$

which—as derived in the proof of Lemma 2—straightforwardly solves to

$$b(\theta) \geq \Big(1+\eta-\Lambda\big(1+\ln(1-G^{n-1}(\theta))\big)\Big)\theta.$$

This shows that any increasing equilibrium satisfies 1. and 2. for all $\theta \in (\theta^{\min}, \theta^{\max})$. By continuity it also holds for all $\theta \in [\theta^{\min}, \theta^{\max}]$. Conversely, assume that $b(\theta)$ satisfies 1. and 2. By Lemma 4 it only remains to show that for any

$$y \in [b(\theta), (1+\eta)\theta]$$

it holds that

$$U_0\big(b(\theta), \theta, F|b(\theta)\big) \geq U_0\big(y, \theta, F|b(\theta)\big).$$

This condition is trivially satisfied for any θ with $b(\theta) = (1+\eta)\theta$. Consider therefore θ with $b(\theta) < (1+\eta)\theta$. It suffices to show that

$$\frac{\partial U_0(y, \theta, F|b(\theta))}{\partial y} \leq 0$$

for all $y \in (b(\theta), (1+\eta)\theta)$. Let $\tilde{y}$ be any of such y. Since

$$b(\theta^{\max}) = (1+\eta)\theta^{\max} > (1+\eta)\theta > \tilde{y} > b(\theta),$$

and b is continuous, there exists some $\tilde{\theta} > \theta$ with $b(\tilde{\theta}) = \tilde{y}$. Since $(1+\eta)\tilde{\theta} > (1+\eta)\theta > \tilde{y} = b(\tilde{\theta})$, it follows from 2. that

$$b(\tilde{\theta}) \geq \Big(1+\eta-\Lambda\big(1+\ln(1-G^{n-1}(\tilde{\theta}))\big)\Big)\tilde{\theta}.$$

Now,

$$\begin{aligned}
\frac{\partial U_0(y, \theta, F|b(\theta))}{\partial y}\Big|_{y=\tilde{y}} &= [(1+\eta)\theta - \tilde{y} - \Lambda\theta(1+\ln(1-F(\tilde{y})))]f(\tilde{y}) \\
&= \Big[\underbrace{\Big(1+\eta-\Lambda\big(1+\ln(1-F(b(\tilde{\theta})))\big)\Big)}_{>0}\theta - b(\tilde{\theta})\Big]f(b(\tilde{\theta})) \\
&\leq \Big[\underbrace{\Big(1+\eta-\Lambda\big(1+\ln(1-F(b(\tilde{\theta})))\big)\Big)\tilde{\theta}}_{\leq b(\tilde{\theta})} - b(\tilde{\theta})\Big]f(b(\tilde{\theta})) \\
&\leq 0.
\end{aligned}$$

□

Proof of Corollary 1. The condition

$$\Big(1+\eta-\Lambda\big(1+\ln(1-G^{n-1}(\theta))\big)\Big)\theta \leq (1+\eta)\theta$$

holds if and only if $-\big(1+\ln(1-G^{n-1}(\theta))\big) \leq 0$, which is equivalent to $G^{n-1}(\theta) \leq 1-1/e$. Therefore, by Proposition 3, a function $b(\theta)$ is a symmetric equilibrium if and only if

- $b(\theta) \in \Big[\Big(1+\eta-\Lambda\big(1+\ln(1-G^{n-1}(\theta))\big)\Big)\theta, \big(1+\eta\big)\theta\Big]$ for $G(\theta) \leq (1-1/e)^{\frac{1}{n-1}}$,
- $b(\theta) = (1+\eta)\theta$ for $G(\theta) > (1-1/e)^{\frac{1}{n-1}}$.

Next, I need to show that for any belief F the lowest PE is the one that maximizes utility among all PE. Indeed, let $b > b^*$ be two PE for a given belief F and type θ. Then, by Equation 6 and the definition of PE,

$$\begin{aligned}
U_0(b^*, \theta, F|b^*) \geq U_0(b, \theta, F|b^*) &= \int_0^b (\theta - s)dF(s) + \mu(F(b) - F(b^*))\theta + L_t(b) \\
&> \int_0^b (\theta - s)dF(s) + L_t(b) \\
&= U_0(b, \theta, b).
\end{aligned}$$

Finally, since for the PPE

$$b(\theta^{\min}) = \Big(1+\eta-\Lambda\big(1+\ln(1-G(\theta^{\min}))\big)\Big)\theta^{\min} = (1+\eta-\Lambda)\theta^{\min},$$

there is underbidding for low types in the PPE if and only if

$$0 > \eta - \Lambda = 2\eta - \lambda\eta,$$

hence if and only if $\lambda > 2$. □

Proof of Proposition 4. The structure of the proof is similar to the one of Proposition 3 in Lange and Ratan (2010). Suppose that all opponents bid according to some increasing, continuously differentiable bidding function $b(\theta)$. Since $G(\theta)$ is a distribution with strictly positive, continuous density g, the distribution of the maximal opponent bid $H(x) = G^{n-1}(b^{-1}(x))$ is a differentiable distribution with positive, continuous density $h(x)$ on $[b(\theta^{\min}), b(\theta^{\max})]$ as well. The bidding function $b(\theta)$ constitutes a PE if and only if $U_0(y, \theta|b(\theta))$ attains a maximum at

$y = b(\theta)$ for all θ. Differentiation of the utility function with respect to y yields

$$\begin{aligned}\frac{\partial U_0(y,\theta|b(\theta))}{\partial y} =&(\theta - y)h(y) + h(y)H(b(\theta))\lambda_g\eta_g\theta + h(y)\big(1 - H(b(\theta))\big)\eta_g\theta \\ &+ \big(1 - H(b(\theta))\big)\mu_m(-y)h(y) + \int_0^{b(\theta)} \mu_m(s-y)h(y)dH(s) \\ &- h(y)\int_0^{b(\theta)} \mu_m(s)dH(s).\end{aligned}$$

By dividing by $h(y)$ and evaluating at $y = b(\theta)$, I obtain the first-order condition

$$\begin{aligned}0 =&(\theta - b(\theta)) + \big(1 - H(b(\theta))\big)\eta_g\theta + H(b(\theta))\lambda_g\eta_g\theta \\ &+ \big(1 - H(b(\theta))\big)\mu_m(-b(\theta)) + \int_0^{b(\theta)} \mu_m(s - b(\theta))dH(s) - \int_0^{b(\theta)} \mu_m(s)dH(s) \\ =&(\theta - b(\theta)) + \big(1 - H(b(\theta))\big)\eta_g\theta + H(b(\theta))\lambda_g\eta_g\theta \\ &+ \big(1 - H(b(\theta))\big)(-\lambda_m\eta_m b(\theta)) - \lambda_m\eta_m\int_0^{b(\theta)} (b(\theta) - s)dH(s) - \eta_m\int_0^{b(\theta)} sdH(s),\end{aligned}$$

which simplifies to

$$0 = (1 + \eta_g)\theta - (1 + \lambda_m\eta_m)b(\theta) + \Lambda_m\int_0^{b(\theta)} sdH(s) + \Lambda_g H(b(\theta))\theta. \tag{16}$$

Using that $H(b(\theta)) = G^{n-1}(\theta)$ one can rewrite this equation to

$$0 = (1 + \eta_g)\theta - (1 + \lambda_m\eta_m)b(\theta) + \Lambda_m\int_0^{\theta} b(s)dG^{n-1}(s) + \Lambda_g G^{n-1}(\theta)\theta.$$

Differentiation with respect to θ yields

$$0 = (1 + \eta_g) - (1 + \lambda_m\eta_m)b'(\theta) + \Lambda_m b(\theta)(G^{n-1})'(\theta) + \Lambda_g(G^{n-1}(\theta)\theta)'.$$

The rearranged equation

$$b'(\theta) = \frac{\Lambda_m(G^{n-1})'(\theta)}{1 + \lambda_m\eta_m}b(\theta) + \frac{1 + \eta_g + \Lambda_g(\theta G^{n-1}(\theta))'}{1 + \lambda_m\eta_m}$$

is a first-order linear differential equation, which solves to

$$b(\theta) = \exp\left(\frac{\Lambda_m}{1 + \lambda_m\eta_m}G^{n-1}(\theta)\right)\left(\int_0^{\theta} \frac{1 + \eta_g + \Lambda_g(xG^{n-1}(x))'}{1 + \lambda_m\eta_m}\exp\left(-\frac{\Lambda_m}{1 + \lambda_m\eta_m}G^{n-1}(x)\right)dx + C\right),$$

where C is the constant of integration. Since $G(x) = 0$ for $x \leq \theta^{min}$,

$$b(\theta^{\min}) = \exp(0)\left(\int_0^{\theta^{\min}} \frac{1+\eta_g}{1+\lambda_m\eta_m}\exp(0)dx + C\right) = \frac{1+\eta_g}{1+\lambda_m\eta_m}\theta^{\min} + C.$$

To determine C, I insert $\theta^{\min}$ into equation (16) and obtain that

$$0 = (\theta^{\min} - b(\theta^{\min})) + (-\lambda_m\eta_m b(\theta^{\min})) + \eta_g\theta^{\min},$$

or equivalently

$$b(\theta^{\min}) = \frac{1+\eta_g}{1+\lambda_m\eta_m}\theta^{\min},$$

which shows that $C = 0$. Now I can use integration by parts in order to rewrite the solution into

$$b(\theta) = \frac{1+\eta_g+\Lambda_g G^{n-1}(\theta)}{1+\lambda_m\eta_m}\theta + \int_0^\theta \frac{\Lambda_m(1+\eta_g+\Lambda_g G^{n-1}(x))}{(1+\lambda_m\eta_m)^2} x\exp\left(\frac{\Lambda_m}{1+\lambda_m\eta_m}(G^{n-1}(\theta)-G^{n-1}(x))\right)dG^{n-1}(x).$$

Since $G(x) = 0$ for all $x \leq \theta^{\min}$, I finally obtain

$$b(\theta) = \frac{1+\eta_g+\Lambda_g G^{n-1}(\theta)}{1+\lambda_m\eta_m}\theta + \int_{\theta^{\min}}^\theta \frac{\Lambda_m(1+\eta_g+\Lambda_g G^{n-1}(x))}{(1+\lambda_m\eta_m)^2} x\exp\left(\frac{\Lambda_m}{1+\lambda_m\eta_m}(G^{n-1}(\theta)-G^{n-1}(x))\right)dG^{n-1}(x).$$

For sufficiency note first that $b'(\theta)$ is differentiable, since $g(\theta)$ is differentiable. It follows that

$$h(b(\theta)) = \frac{(G^{n-1})'(\theta)}{b'(\theta)}$$

is differentiable as well. Now it is immediate that

$$\begin{aligned}
&\left.\frac{\partial^2 U_0(x,\theta|b(\theta))}{(\partial x)^2}\right|_{x=b(\theta)} \\
=&\frac{\partial}{\partial x}\left(h(x)\frac{\partial U_0(x,\theta|b(\theta)/\partial x)}{h(x)}\right)\Bigg|_{x=b(\theta)} \\
=&h'(b(\theta))\underbrace{\left(\frac{\partial U_0(x,\theta|b(\theta)/\partial x)}{h(x)}\right)\Bigg|_{x=b(\theta)}}_{=0} \\
&+ h(b(\theta))\underbrace{\left[-1+\int_0^{b(\theta)} -\lambda_m\eta_m dH(s) - \lambda_m\eta_m\big(1-H(b(\theta))\big)\right]}_{<0} \\
&< 0.
\end{aligned}$$

□

Proof of Lemma 5. Let F be the distribution of the highest opponent bid. I derive $F^m_{t,b}$, the payment distribution for a plan b at price $t < b$. Losing and paying nothing has a probability of $\frac{1-F(b)}{1-F(t)}$, winning at some price $x \in [t,b]$ has marginal probability $\frac{f(x)}{1-F(t)}$. Hence, the payment distribution for payments $x \in \mathbb{R}_+$ is

$$F^m_{t,b}(x) = \begin{cases} \frac{1-F(b)}{1-F(t)} & x < t \\ \frac{1-F(b)+F(x)-F(t)}{1-F(t)} & x \in [t,b] \\ 1 & x > b. \end{cases}$$

Note that for notational convenience payments in $F^m_{t,b}$ are defined on positive support. Since a payment induces a loss, a deviation from b^* to $b > b^*$ yields instantaneous reference-dependent utility of

$$\begin{aligned}\mathbb{E}N(F^m_{t,b}|F^m_{t,b^*}) &= \int_0^{b^*} \mu_m \left(\frac{1-F(b)-(1-F(b^*))}{1-F(t)}\right) dx + \int_{b^*}^{b} \mu_m \left(\frac{1-F(b)+F(x)-F(t)}{1-F(t)} - 1\right) dx \\ &= \frac{\lambda_m \eta_m}{1-F(t)} \left(\int_0^{b^*} (F(b^*)-F(b))\, dx + \int_{b^*}^{b} (F(x)-F(b))\, dx\right) \\ &= -\lambda_m \eta_m \int_{b^*}^{b} \frac{xf(x)}{1-F(t)} dx,\end{aligned}$$

where the last step is by integration by parts. Analogously, a deviation from $b^* > t$ to $b \in [t, b^*)$ yields

$$\mathbb{E}N(F^m_{t,b}|F^m_{t,b^*}) = \eta_m \int_b^{b^*} \frac{xf(x)}{1-F(t)} dx,$$

and hence for both cases in one formula

$$\mathbb{E}N(F^m_{t,b}|F^m_{t,b^*}) = \frac{\mu_m \left(\int_{b^*}^{b} -x dF(x)\right)}{1-F(t)}.$$

Next, I calculate expected news disutility from money. Consider the expected news utility from an information update from an ε-increment from price $t-\varepsilon$ to price t. Call Δ_t the probability that the auction ends in that increment, such that the bidder has to pay $t-\varepsilon$ (i.e. payoff distribution $\mathbb{1}_{x \geq t-\varepsilon}$). By Lemma 1 the expected news utility from that update is

$$
\begin{aligned}
&\mathbb{E}N(F^m_{t,b}|F^m_{t-\varepsilon,b})\\
=&-\Delta_t\Lambda_m\int_0^b \left|\mathbb{1}_{[t-\varepsilon,b]}-F^m_{t-\varepsilon,b}(x)\right|dx\\
=&-\Delta_t\Lambda_m\left(\int_0^{t-\varepsilon}\frac{1-F(b)}{1-F(t)}dx+\int_{t-\varepsilon}^b\left|1-\frac{1-F(b)+F(x)-F(t)}{1-F(t)}\right|dx\right)\\
=&-\Delta_t\Lambda_m\left((t-\varepsilon)\frac{1-F(b)}{1-F(t)}+\int_{t-\varepsilon}^b\frac{F(b)-F(x)}{1-F(t)}dx\right).
\end{aligned}
$$

For aggregated news utility from time t note that at t the probability that an increment at $s>t$ is reached before the auction is over is $\frac{1-F(s)}{1-F(t)}$. For increment size to zero, this event has marginal probability $\lim_{\varepsilon\to 0}\frac{\Delta_s}{\varepsilon}\frac{1-F(s)}{1-F(t)}=\frac{f(s)}{1-F(t)}$, hence expected news utility from t onward is

$$
L^m_t(b,\theta|b)=-\int_t^b\frac{f(s)}{1-F(t)}\Lambda_m\left(s\frac{1-F(b)}{1-F(s)}+\int_s^b\frac{F(b)-F(x)}{1-F(s)}dx\right)ds.
$$

Plugging this together with classical utility and the reference-dependent deviation utility in the good dimension as calculated in Proposition 2 yields the result.

□

Proof of Lemma 6. As a necessary condition for b to be an ex-ante PE given type θ and belief F the right-derivative at $b'=b$ must satisfy $\frac{\partial_+ U_0(b',\theta|b)}{\partial b'}\leq 0$, hence

$$
(\theta-b)f(b)+\eta_g f(b)\theta-\lambda_m\eta_m bf(b)-\Lambda_g\Big(1+\ln(1-F(b))\Big)f(b)\theta-\Lambda_m\left(f(b)b+\int_0^b\frac{f(s)}{1-F(s)}(b-2s)f(b)ds\right)\leq 0.
$$

In any continuous symmetric increasing equilibrium, it must be that $b=b(\theta)$, $F(b(\theta))=G^{n-1}(\theta)$, and $f(b(\theta))b'(\theta)=(G^{n-1})'(\theta)$, and moreover $f(b(\theta))>0$ for all θ. Since after dividing by $f(b)$ the necessary condition is violated for $b=0$ it follows by continuity that for the minimal b where it is satisfied (if any) it holds with equality. Rearranging yields for this b

$$
\Big(1+\eta_g-\Lambda_g\big(1+\ln(1-F(b(\theta)))\big)\Big)\theta-(1+\lambda_m\eta_m+\Lambda_m)b(\theta)-\Lambda_m\int_0^{b(\theta)}\frac{f(s)}{1-F(s)}(b(\theta)-2s)ds=0.
$$

The fact that for $\theta=\theta^{\min}$ the probability $F(b(\theta))$ as well as the integral in this condition equal zero immediately implies

$$b(\theta^{\min}) = \frac{1+\eta_g-\Lambda_g}{1+\lambda_m\eta_m+\Lambda_m}\theta^{\min}. \tag{17}$$

To obtain the lower bound for all other θ, I differentiate the binding necessary condition with respect to θ, and obtain the differential equation

$$1+\eta_g - \Lambda_g\big(1+\ln(1-G^{n-1}(\theta))\big) + \Lambda_g\frac{(G^{n-1})'(\theta)}{1-G^{n-1}(\theta)}\theta - (1+\lambda_m\eta_m+\Lambda_m)b'(\theta)...$$
$$... + \Lambda_m\frac{(G^{n-1})'(\theta)}{1-G^{n-1}(\theta)}b(\theta) + \Lambda_m\ln(1-G^{n-1}(\theta))b'(\theta) = 0,$$

or equivalently

$$b'(\theta) + \frac{-\Lambda_m\frac{(G^{n-1})'(\theta)}{1-G^{n-1}(\theta)}}{1+\lambda_m\eta_m+\Lambda_m-\Lambda_m\ln(1-G^{n-1}(\theta))}b(\theta) = \frac{1+\eta_g-\Lambda_g-\Lambda_g\ln(1-G^{n-1}(\theta))+\Lambda_g\frac{(G^{n-1})'(\theta)\theta}{1-G^{n-1}(\theta)}}{1+\lambda_m\eta_m+\Lambda_m-\Lambda_m\ln(1-G^{n-1}(\theta))}. \tag{18}$$

Together with the initial value for $\theta^{\min}$ from (17) this first-order linear differential equation solves to

$$b(\theta) = \Big(1+\lambda_m\eta_m+\Lambda_m-\Lambda_m\ln(1-G^{n-1}(\theta))\Big)\int_0^\theta\left(\frac{1+\eta_g-\Lambda_g-\Lambda_g\ln(1-G^{n-1}(s))+\Lambda_g\frac{(G^{n-1})'(s)}{1-G^{n-1}(s)}}{(1+\lambda_m\eta_m+\Lambda_m-\Lambda_m\ln(1-G^{n-1}(s)))^2}\right)ds.$$

For sufficiency I show that a θ-type bidder does not profit from a (global) deviation to $b = b(\tilde{\theta})$ for any $\tilde{\theta} \neq \theta$. It is easy to check that for two upward deviations with different b^* (and similarly for downward deviations) $\frac{\partial U_0(b,\theta|b^*)}{\partial b}$ does not depend on b^*. Hence, for any $\tilde{\theta} > \theta$ and $b = b(\tilde{\theta})$

$$\frac{\partial U_0(b,\theta|b(\theta))}{\partial b} = \frac{\partial_+ U_0(b,\theta|b(\tilde{\theta}))}{\partial b} < \frac{\partial_+ U_0(b,\tilde{\theta}|b(\tilde{\theta}))}{\partial b} = 0,$$

where the inequality follows from

$$\frac{\partial\Big(\partial_+ U_0(b,\theta|b^*)\Big)}{\partial b\partial\theta} = f(b)\big(1+\eta_g-\Lambda_g-\Lambda_g\ln(1-F(b))\big) \geq f(b)(1+\eta_g-\Lambda_g) > 0.$$

Similarly, for downward deviations to $b = b(\tilde{\theta})$ with $\tilde{\theta} < \theta$

$$\frac{\partial U_0(b,\theta|b(\theta))}{\partial b} = \frac{\partial_- U_0(b,\theta|b(\tilde{\theta}))}{\partial b} > \frac{\partial_- U_0(b,\tilde{\theta}|b(\tilde{\theta}))}{\partial b} > 0,$$

where the first inequality follows from

$$\frac{\partial\Big(\partial_+ U_0(b,\theta|b(\theta))\Big)}{\partial b\partial\theta} = f(b)\big(1+\lambda_g\eta_g-\Lambda_g-\Lambda_g\ln(1-F(b))\big) \geq f(b)(1+\lambda_g\eta_g-\Lambda_g) > 0.$$

For the time consistency constraint, I look at the constraint that—given a plan b—a bidder does not want to deviate and drop out at t, the increment before b. Call again $\Delta_t = \frac{F(t+\varepsilon)-F(t)}{1-F(t)}$ the winning probability of bidding for another increment until $t \geq b$. With the belief of Δ_t of winning and paying $t-\varepsilon$, dropping out at t rather than $t+\varepsilon$ is not profitable if and only if

$$-\Delta_t\lambda_g\eta_g\theta + \Delta_t\eta_m(t-\varepsilon) \leq \Delta_t(\theta-(t-\varepsilon)) - \Delta(1-\Delta)(\Lambda_g\theta+\Lambda_m(t-\varepsilon),$$

where the right-hand side is due to Lemma 1. For increment size ε going to zero and a continuous F one has $\Delta_t \to 0$ and $t-\varepsilon \to b$, and obtains

$$-\Delta_t\lambda_g\eta_g\theta + \Delta_t\eta_m b \leq \Delta_t(\theta-b) - \Delta_t(\Lambda_g\theta+\Lambda_m g),$$

which, after dividing by Δ_t, can be rearranged to $b \leq \frac{1+\eta_g}{1+\lambda_m\eta_m}\theta$. □

Proof of Proposition 5. **Claim:** If for continuous belief F and $b^* \leq \frac{1+\eta_g}{1+\lambda_m\eta_m}\theta$ a bidder profits at some $t < b^*$ from a downward deviation to $b \in [t, b^*)$, then she does so for $t = 0$.

Proof of claim: Define for given b and b^*

$$\delta(t) = \delta(t,b,b^*) = (1-F(t))(U_t(b,\theta,F|b^*) - U_t(b^*,\theta,F|b^*)).$$

Clearly, a deviation from b^* to $b < b^*$ is profitable at t if and only if for these b^* and b the condition $\delta(t) > 0$ holds. Further,

$$\begin{aligned}
\delta(b) &= \int_b^{b^*} (-(1+\lambda_g\eta_g)\theta + (1+\eta_m)x)\,dF(x) - (1-F(b))(L_b^g(b^*,\theta,F|b^*) + L_b^m(b^*,\theta,F|b^*)) \\
&< (F(b^*)-F(b))\Big(-(1+\lambda_g\eta_g-\Lambda_g)\theta + (1+\eta_m)b^*\Big) \\
&\quad + \Lambda_m\int_b^{b^*}\frac{f(s)}{1-F(s)}\left((1-F(b^*))b^* + \underbrace{\int_0^{b^*}(F(b^*)-F(s))dx}_{<b^*(F(b^*)-F(s))}\right)ds \\
&< (F(b^*)-F(b))\Big(-(1+\lambda_g\eta_g-\Lambda_g)\theta + (1+\eta_m+\Lambda_m)b^*\Big) \\
&= (F(b^*)-F(b))\Big(-(1+\eta_g)\theta + (1+\lambda_m\eta_m)b^*\Big) < 0.
\end{aligned}$$

Moreover,

$$\delta'(t) = \frac{f(t)}{1-F(t)}\left(\big(-F(b)+F(b^*)\big)(\Lambda_g\theta+\Lambda_m t)+\Lambda_m\int_t^b(F(b)-F(x))dx-\Lambda_m\int_t^{b^*}\big(F(b^*)-F(x)\big)dx\right).$$

Since the term in the big brackets is decreasing in t it follows that $\delta'(t) < 0$ for some $t > 0$ only if $\delta'(s) < 0$ for all $s \in [0,t]$. Now, suppose there is some $t < b < b^*$ for which $\delta(t) > 0$. Then $\delta(t) > \delta(b)$, hence there is some $s \in [t,b]$ with $\delta'(s) < 0$. Then $\delta'(t) < 0$ for all $t < b$, and hence also $\delta(0) > 0$, which proves the claim.

I show the bounds for the symmetric PEs. Clearly, by Lemma 6 any symmetric PE satisfies $b(\theta) \leq \frac{1+\eta_g}{1+\lambda_m\eta_m}\theta$. It remains to show that any $b < \underline{b}(\theta)$ cannot be part of a symmetric PE. By Lemma 6 an upward deviation to $\underline{b}(\theta)$ at time $t = 0$ would be profitable for such a bid. Hence, $b < \underline{b}(\theta)$ can only be a PE if it is time consistent, but $\underline{b}(\theta)$ is not. I bring this case to a contradiction. If $\underline{b}(\theta)$ were time inconsistent with respect to downward deviations then by the above claim it would by suboptimal at $t = 0$, contradicting Lemma 6. Hence, suppose that at some t for some $\tilde{b} > \underline{b}(\theta)$ one has $\delta(t,\tilde{b},\underline{b}(\theta)) > 0$, but $\delta(t,\tilde{b},b) \leq 0$. This is a contradiction since for all $b < \frac{1+\eta_g}{1+\lambda_m\eta_m}\theta$ one has

$$\frac{\partial\delta(t,\tilde{b},b)}{\partial b} = f(b)\Big(-(1+\eta_g)\theta+(1+\lambda_m\eta_m)b\Big) < 0.$$

Hence, any symmetric PE must satisfy $b(\theta) \geq \underline{b}(\theta)$.

It remains to show that $\underline{b}(\theta)$ is a symmetric PE if $\eta_g \geq \eta_m$ and $\lambda_g \geq \lambda_m$. By Lemma 6 no bidder can profit from a time-consistent deviation at $t = 0$. By Claim 1 downward deviations are then not profitable at any time. For upward deviations at $t = 0$, again by Lemma 6, $\frac{\partial\delta(t,b,\underline{b}(\theta))}{\partial b} < 0$ for any $b > \underline{b}(\theta)$. To obtain this result for all t, note that

$$\frac{\partial^2\delta(t,b,\underline{b}(\theta))}{\partial t\partial b} = \frac{f(b)f(t)}{1-F(t)}\big(\Lambda_m(-2t+b)-\Lambda_g\theta\big) < \frac{f(b)f(t)}{1-F(t)}\left(\Lambda_m\frac{1+\eta_g}{1+\lambda_m\eta_m}\theta-\Lambda_g\theta\right) < 0,$$

since, by assumption, $\Lambda_g > \Lambda_m$. □

Proof of Proposition 6. For completeness, I prove the proposition for the case of loss aversion in both commodity dimension, the result then follows as special case with $\eta_g = \eta$ and $\lambda_m = 0$.

Recall that by definition the set of PEs is determined by thinking all discrete increments backwards. For any history h_k at any sufficiently high price (e.g. for prices $s > (1 + \lambda_g \eta_g)\theta$) where remaining would win with positive probability in the next increment, the unique PE action is to drop out.[25] Hence, for any continuation game with prices above $(1 + \lambda_g \eta_g)\theta$ there exists a finite set of time consistent plans, and in each the bidder loses with probability one.

Thinking iteratively backwards from there, suppose a bidder at some decision node has determined a non-empty set of time-consistent continuation plans for any future contingency following her sequential action. A plan b at that node consists of an action (remain/drop) and a time-consistent continuation plan. I must show that there is a plan b for which $U_t(b, \theta|b) \geq U_t(b', \theta|b)$ with respect to any other plan b'.

Call —with some abuse of notation—P_b the winning probability and T_b the expected payment as induced by plan b. Then the instantaneous update from deviating from b to b' induces instantaneous utility bounded from above by $\eta_g(P_{b'} - P_b)\theta - \eta_m(T_{b'} - T_b)$. Hence, for any plan $argmax_b\{U_t(b, \theta|b) + \eta P_b \theta - \eta_m T_b\}$ one obtains

$$U_t(b, \theta|b) \geq U_t(b', \theta|b') + \eta_g(P_{b'} - P_b)\theta - \eta_m(T_{b'} - T_b) \geq U_t(b', \theta|b),$$

which shows existence of a PE.

A PPE exists as a utility maximizing element in the finite set of PEs.

I now show that equilibrium bids are bounded by $\frac{1+\eta_g}{1+\lambda_m \eta_m}\theta$. Consider a continuous belief and suppose for the sake of contradiction that there is some history $h_k = (t_1, ..., t_k)$ for which $b(h_k, \theta) > \frac{1+\eta_g}{1+\eta_m \lambda_m}\theta$ is part of a PE. By the definition of continuous beliefs for any $\Delta > 0$ there is some price $s \in [t_k, b(h_k, \theta)]$ such that the bidder's probability to win at s with history h_k is below Δ. Using Lemma 1 and that by Proposition 1 in Kőszegi and Rabin (2009) a bidder weakly increases her utility when she obtains all information revelation at once, I obtain

$$\begin{aligned} &U_s(b(h_k), \theta|b(h_k)) - U_s(s, \theta|b(h_k)) \\ \leq &\Delta\Big(\theta - (1 - \Delta)\Lambda_g\theta - s - (1 - \Delta)\Lambda_m s\Big) - \Delta\Big(- \lambda_g\eta_g\theta + \eta_m s\Big) \\ = &\Delta\Big((1 + \eta_g + \Delta\Lambda_g)\theta - (1 + \lambda_m\eta_m - \Delta\Lambda_m)s\Big). \end{aligned}$$

With sufficiently small increments and $s \to b(h_k)$ it follows that $\Delta \to 0$, and

[25] If the probability to win in that increment is zero the decision whether to drop out is inconsequential, and both actions are consistent with equilibrium behavior. Yet, the bidder will drop out before she would otherwise win with positive probability.

hence the term in brackets on the right hand side approaches
$(1+\eta_g)\theta - (1+\lambda_m\eta_m)b(h_k) < 0$, a contradiction. □

Proof of Proposition 7. By integration by parts

$$\int_t^\theta \frac{2g(s)}{1-G(s)} \int_s^\theta 2g(x)\big(G(\hat\theta(x)) - G(x)\big)dxds = -\int_t^\theta 2g(s)\ln\left(\frac{(1-G(s))^2}{(1-G(t))^2}\right)\big(G(\hat\theta(s)) - G(s)\big)ds.$$

Hence, I obtain

$$\begin{aligned} L_2(t,\hat\theta(s)) = &-\Lambda\theta\int_t^\theta \frac{2g(s)}{(1-G(t))^2}\left(1+\ln\left(\frac{(1-G(s))^2}{(1-G(t))^2}\right)\right)(G(\hat\theta(s)) - G(s))ds \\ &+\Lambda\theta\int_t^\theta \frac{2g(s)(1-G(s))}{(1-G(t))^2}\left(\frac{1-G(\hat\theta(s))}{1-G(s)}\right)\ln\left(\frac{1-G(\hat\theta(s))}{1-G(s)}\right)ds, \end{aligned}$$

where the first line is expected loss from the three-bidder part of the auction, and the second line is expected loss from the subsequent two-bidder auction.

Consider first downward deviations in the sense that $\hat\theta(s)$ induces a smaller winning probability than the truthful strategy. Let $U_t(\hat\theta(s)|\theta)$ be the expected utility of type θ at price $b(t)$ in the auction with three active bidders when deviating to strategy $\big(b(\theta), b(s,\hat\theta(s))\big)$. Incorporating the instantaneous update and the change in classical utility, the change in utility from the deviation reads

$$\begin{aligned} &\Delta_t(\hat\theta(s)|\theta) := U_t(\hat\theta(s)|\theta) - U_t(\theta|\theta) \\ &= \int_t^\theta \frac{2g(s)}{(1-G(t))^2}\left(\left(1+\ln\left(\frac{(1-G(s))^2}{(1-G(t))^2}\right)\right)(-\Lambda\theta)\Big(G(\hat\theta(s)) - G(s)\Big) + \int_\theta^{\hat\theta(s)}\Big((1+\lambda\eta)\theta - b(s,x)\Big)g(x)dx\right)ds \\ &+ \int_t^\theta \frac{2g(s)(1-G(s))}{(1-G(t))^2}\left[\Lambda\theta\left(\frac{1-G(\hat\theta(s))}{1-G(s)}\right)\ln\left(\frac{1-G(\hat\theta(s))}{1-G(s)}\right) - L_1(s,\theta)\right]ds. \end{aligned} \tag{19}$$

Notice that the choice of $\hat\theta(s)$ enters the integrand pointwise for every s, hence optimization with respect to the optimal $\hat\theta(s)$ can be done pointwise. Since for every s the integrand of $(1-G(t))^2\Delta_t(\hat\theta(s))$ satisfies $\frac{\partial^2}{\partial\hat\theta(s)\partial t}(\ldots) = -\frac{4\theta\Lambda g(t)g(s)g(\hat\theta(s))}{1-G(t)} < 0$, a downward deviation of $\hat\theta(s)$ for some s is profitable at any price if and only if it is so at price $b(s)$.[26] Next, notice that it is sufficient to consider local deviations in $\hat\theta(s)$. Indeed, the first summand of the integrand in the first line of (19) is linear in $G(\hat\theta(s))$ whereas the other summands describe the deviation utility for the two-bidder auction following the first dropout, for which I have verified sufficiency of local deviations in the main model. It follows that a bidding strategy is optimal with respect to downward deviations in $\hat\theta(s)$ for any $s \in [\underline{\theta}, \theta]$ if

[26] This is precisely the insurance effect: A lower belief magnifies the update at a first dropout, and hence induces more expected during the auction, hence stronger incentives for higher bids early in the auction.

and only if at $\hat{\theta}(s) = \theta$ the necessary condition for the integrand at $t = s$

$$0 \leq \frac{2g(s)}{(1-G(s))^2}\left[(-\Lambda\theta)\left(1+\ln\left(\frac{(1-G(s))^2}{(1-G(s))^2}\right)\right) + (1+\lambda\eta)\theta - b\big(s,\hat{\theta}(s)\big) + \Lambda\theta\left(-1-\ln\left(\frac{1-G(\hat{\theta}(s))}{1-G(s)}\right)\right)\right] g(\hat{\theta}(s))$$

is satisfied. This condition simplifies to

$$b(s,\theta) \leq \left(1+\eta-\Lambda\left(1+\ln\left(\frac{1-G(\theta)}{1-G(s)}\right)\right)\right)\theta. \tag{20}$$

Together with the constraints on equilibrium behavior for the two-bidder auction following a drop-out at $b(s)$ as outlined in Proposition 3,

$$b(s,\theta) = \min\left\{(1+\eta)\theta, \left(1+\eta-\Lambda\left(1+\ln\left(\frac{1-G(\theta)}{1-G(s)}\right)\right)\right)\theta\right\} \tag{21}$$

remains as the only equilibrium candidate that is optimal with respect to downwards deviations during the three-bidder auction and time consistent with respect to the two-bidder auction, following a first dropout.

For sufficiency notice that upward deviations $\hat{\theta}(s)$ from the equilibrium candidate in (21) are no concern since any such deviation, by (20), would be time inconsistent.

It remains to show that a bidder cannot profit from a deviation to an earlier or later dropout in the three-bidder part of the auction. Since $(1-x)\ln(1-x) > -x$ for $0 < x < 1$ it follows that $L_1(s,\theta) > -\Lambda\theta\frac{G(\theta)-G(s)}{1-G(s)}$. From (13) it then follows that

$$L_2(t,\theta) > \int_t^\theta \frac{2g(s)(1-G(s))}{(1-G(t))^2}\left[-\Lambda\theta\frac{G(\theta)-G(s)}{1-G(s)} - \Lambda\theta\frac{G(\theta)-G(s)}{1-G(s)}\right] ds = -2\Lambda\theta\left(\frac{1-G(\theta)}{1-G(t)}\right)^2,$$

where $\left(\frac{1-G(\theta)}{1-G(t)}\right)^2$ is the winning probability at time $b(t)$. Since the price in case of winning is bounded below $(1+\eta-\Lambda)\theta$, utility on equilibrium path satisfies

$$U_t(\theta|\theta) > \left(\frac{1-G(\theta)}{1-G(t)}\right)^2\Big(\theta - 2\Lambda\theta - (1+\eta-\Lambda)\theta\Big) = -\lambda\eta\theta\left(\frac{1-G(\theta)}{1-G(t)}\right)^2,$$

where the right-hand side is the utility of an earlier dropout.

An upward deviation is inconsequential as it does not change the winning probability. Indeed, if the first dropout is at a price above $b(\theta)$ then the remaining opponent is necessarily of higher type and wins the auction, since (20) is increasing in θ. □

Proof of Proposition 8. Suppose, by way of contradiction, there is a symmetric, weakly monotone equilibrium $b(\cdot,\theta)$ for some $n \geq 4$.

I first show that for any history h_k the equilibrium function $b(h_k,\theta)$ must be strictly increasing a neighborhood of $\theta^{\max}$.

Suppose otherwise that for some history the function $b(h_k,\theta)$ is constant on some interval $[\hat{\theta},\theta^{\max}]$. Let P be the probability that (for a given tie breaking rule) a bidder is assigned the good in case of the simultaneous dropout of the remaining $n-k$ bidders at price $b(h_k,\theta^{\max})$. The fact that a bidder of type $\theta \in [\theta_{h_k},\theta^{\max}]$ does not benefit from dropping out earlier implies

$$-\lambda\eta P\theta \leq P\Big(\big(1+\eta(1-P)\big)\theta - b(h_k,\theta^{\max})\Big) - \lambda\eta(1-P)P\theta$$

$$\Leftrightarrow \qquad b(h_k,\theta^{\max}) \leq \Big(1+P\lambda\eta+(1-P)\eta\Big)\theta.$$

The fact that she does not prefer to proceed and win with certainty implies

$$P\big((1+\eta(1-P))\theta - b(h_k,\theta^{\max})\big) - \lambda\eta(1-P)P\theta \geq \big(1+(1-P)\eta\big)\theta - b(h_k,\theta^{\max})$$

$$\Leftrightarrow \qquad b(h_k,\theta^{\max}) \geq \Big(1+P\lambda\eta+(1-P)\eta\Big)\theta.$$

Hence, $b(h_k,\theta^{\max}) = \big(1+P\lambda\eta+(1-P)\eta\big)\theta$ for different values of θ, a contradiction. Since $b(h_k,\theta)$ is strictly increasing around $\theta^{\max}$ for every of the finite possible histories there exists an interval $[\hat{\theta},\theta^{\max}]$ on which $b(h_k,\theta)$ is increasing for each history. Suppose the smallest of all n type realization satisfies $\theta > \hat{\theta}$. Then, since for any price above $b(\theta_0)$ with $\theta_0 > \hat{\theta}$ the bidding function is strictly increasing, expected loss from news utility at price $b(\theta_0)$ is given by $L_{n-1}(\theta_0,\theta)$ in (13). Next, I show by induction that $\lim_{\theta_0\to\theta} \frac{L_n(\theta_0,\theta)}{-n\Lambda\theta\left(\frac{G(\theta)-G(\theta_0)}{1-G(\theta_0)}\right)^n} = 1$. For $n=1$ by L'Hospital's rule

$$\lim_{\theta_0\to\theta} \frac{L_1(\theta_0,\theta)}{-\Lambda\theta\frac{G(\theta)-G(\theta_0)}{1-G(\theta_0)}} = \lim_{\theta_0\to\theta} \frac{\ln\left(\frac{1-G(\theta)}{1-G(\theta_0)}\right)\frac{1-G(\theta)}{1-G(\theta_0)}}{\frac{G(\theta_0)-G(\theta)}{1-G(\theta_0)}} = \lim_{\theta_0\to\theta} \frac{\frac{g(\theta_0)(1-G(\theta))}{(1-G(\theta_0))^2}\left(1+\ln\left(\frac{1-G(\theta)}{1-G(\theta_0)}\right)\right)}{\frac{g(\theta_0)(1-G(\theta))}{(1-G(\theta_0))^2}} = 1.$$

Suppose the claim holds for $n-1$. Then, by (13),

$$\begin{aligned}
&\lim_{\theta_0\to\theta} \frac{L_n(\theta_0,\theta)}{-n\Lambda\theta\left(\frac{G(\theta)-G(\theta_0)}{1-G(\theta_0)}\right)^n} \\
&= \lim_{\theta_0\to\theta} \frac{\int_{\theta_0}^{\theta} g_n(s|\theta_0)\left[-\Lambda\theta\left(\left(\frac{G(\theta)-G(s)}{1-G(s)}\right)^{n-1} - \left(\frac{G(\theta)-G(s)}{1-G(s)}\right)^n\right) + L_{n-1}(s)\right] ds}{-n\Lambda\theta\left(\frac{G(\theta)-G(\theta_0)}{1-G(\theta_0)}\right)^n} \\
&= \lim_{\theta_0\to\theta} \frac{\int_{\theta_0}^{\theta} \frac{ng(s)(1-G(s))^{n-1}}{(1-G(\theta))^n}\left[-\Lambda\left(\frac{G(\theta)-G(s)}{1-G(s)}\right)^{n-1} - (n-1)\Lambda\left(\frac{G(\theta)-G(s)}{1-G(s)}\right)^{n-1}\right] ds}{-n\Lambda\left(\frac{G(\theta)-G(\theta_0)}{1-G(\theta_0)}\right)^n} \\
&= \lim_{\theta_0\to\theta} \frac{-\int_{\theta_0}^{\theta} \frac{ng(s)n\Lambda\left(G(\theta)-G(s)\right)^{n-1}}{(1-G(\theta))^n} ds}{-n\Lambda\left(\frac{G(\theta)-G(\theta_0)}{1-G(\theta_0)}\right)^n} \;=\; 1.
\end{aligned}$$

Since for $\theta > \theta_0$ I have the case of strictly increasing bidding functions, time-consistency and Proposition 7 imply that the winning price approaches $(1-\eta+\Lambda)\theta_0$ for $\theta_0 \to \theta$. Hence, for $P_{\theta_0} := \left(\frac{G(\theta)-G(\theta_0)}{1-G(\theta_0)}\right)^{n-1}$, i.e. the winning probability at a clock price of $b(\theta_0)$ with $n-1$ opponents, expected utility from the equilibrium plan as θ_0 approaches θ goes to

$$P_{\theta_0}\big(\theta - (1-\eta+\Lambda)\theta\big) - (n-1)\Lambda\theta P_{\theta_0} = P_{\theta_0}\Big(-\lambda\eta - (n-3)\Lambda\Big)\theta < -\lambda\eta P_{\theta_0}\theta.$$

As the expression on the right-hand side is the utility of an immediate dropout at $b(\theta_0)$, this is a contradiction. □

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