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Full Surplus Extraction From Colluding Bidders

Daniil Larionov [1]

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[1] Department of Economics, University of Mannheim, Email: daniil.larionov@gess.uni-mannheim.de

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Full Surplus Extraction from Colluding Bidders*

Daniil Larionov†

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Abstract. I consider a repeated auction setting with colluding buyers and a seller who adjusts reserve prices over time without long-term commitment. To model the seller's concern for collusion, I introduce a new equilibrium concept: *collusive public perfect equilibrium (cPPE)*. For every strategy of the seller I define the corresponding "*buyer-game*" in which the seller is replaced by Nature who chooses the reserve prices for the buyers in accordance with the seller's strategy. A public perfect equilibrium is collusive if the buyers cannot achieve a higher symmetric public perfect equilibrium payoff in the corresponding buyer-game. In a setting with symmetric buyers with private binary *iid* valuations and publicly revealed bids, I find a *collusive public perfect equilibrium* that allows the seller to extract the entire surplus from the buyers in the limit as the discount factor goes to 1. I therefore show that a patient, non-committed seller can effectively fight collusion even when she can only set reserve prices and has to satisfy stringent public disclosure requirements.

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†Department of Economics, University of Mannheim. daniil.larionov@gess.uni-mannheim.de

1 Introduction

Auctions rarely involve a one-shot interaction, often buyers and sellers face each other repeatedly. Procurement decisions for road construction and maintenance, to take one example, have to be made regularly and public authorities often have to deal with the same pool of potential suppliers. Auctions for electromagnetic spectrum, although less regular, often involve the same pool of potential buyers.

I model a seller who is concerned about colluding buyers and her own lack of commitment power. I assume that the seller offers an infinite sequence of first-price auctions with adjustable reserve prices and has to satisfy stringent public disclosure requirements: both the reserve prices and the buyers' bids are publicly disclosed after each round of trading. The seller can commit to her chosen reserve prices within every period, but does not have enough commitment power to fix the whole dynamic sequence of reserve prices. With respect to collusion, the seller takes a rather pessimistic stance: she expects the buyers to take her chosen strategy as given and try to collectively maximize their own payoff. To model the seller's concern for collusion, I introduce a subclass of public perfect equilibria, which I call *collusive public perfect equilibria*. For every public strategy of the seller I define the corresponding dynamic game among the buyers ("buyer-game") in which the reserve prices are chosen by Nature in accordance with the seller's strategy; I select only those public perfect equilibria of the repeated first-price auction game, in which the buyers' payoff is no smaller than the payoff they could achieve in the maximal strongly symmetric public perfect equilibrium of the corresponding buyer-game. I call the selected public perfect equilibria *collusive*. My main goal is to determine the highest payoff that the seller can obtain in a *collusive public perfect equilibrium* of the repeated auction game.

I consider buyers whose valuations are binary, independent and identically distributed across them and over time. The buyers in my model employ strongly symmetric strategies in any public perfect equilibrium of any buyer-game. In essence, the buyers are prohibited from using more complex asymmetric collusive schemes which might involve communication and/or bidding strategies dependent on each buyer's identity. While it is possible that the seller has less power against a more sophisticated cartel, it should be noted that asymmetric strategies (due to their complexity) might require explicit coordination among the buyers, and explicit coordination could be more easily detected and prevented via the traditional instruments of anti-trust policy. This paper finds a seller's strategy that is robust to collusive schemes that are simpler and more tacit, and thus harder to detect and prove to a court.

I study equilibrium outcomes as the discount factor goes to 1 and show that collusion in repeated auctions can be dealt with rather effectively: I establish that there is a *collusive public perfect equilibrium* that achieves full surplus extraction in the limit as the discount factor goes to 1, even though the seller can only set reserve prices, and stringent public disclosure requirements force her to publicly reveal bids in the end of each period. This *full-surplus-extracting collusive public perfect equilibrium* is stationary along the equilibrium path, features higher reserve prices than the static outcome and forces the buyers to bid even if their valuation is below the offered reserve price in the current period. Note that, since I am studying a restricted class of public perfect equilibria, my full surplus extraction results do not rely on any of the existing folk theorems. Since these theorems refer to the full set of public equilibrium payoffs, even the mere possibility of full surplus extraction by any *collusive public perfect equilibrium* (let alone by a cPPE of any particular structure) is not implied by the existing folk theorems.

In the full-surplus-extracting equilibrium the seller forces the buyer types to separate and punishes any off-equilibrium path deviations she can detect. In the corresponding buyer-game the buyers take the seller's threat as given and might try to deviate to a lower bidding profile. The key to the construction of the optimal equilibrium is in identifying the optimal symmetric joint deviation for the buyers and making sure that the original construction renders this joint deviation unprofitable. Since the full-surplus-extracting cPPE forces any low-type buyer to bid even when his valuation is below the reserve price, the optimal joint deviation will involve the low-type buyers abstaining from participating and receiving the punishment of zero continuation payoffs, and the high-type buyers bidding at the reserve price. There are three cases corresponding to different parameter values. In all three cases the seller extracts full surplus from the buyers. In Cases 1 and 2, the buyers' payoff in the full-surplus-extracting cPPE is exactly equal to the payoff of the optimal joint deviation. In Case 3 the proportion of the low-type buyers is so high that the optimal joint deviation would provide the buyers with a strictly lower payoff than the one they obtain along the equilibrium path.

Beyond addressing purely theoretical concerns, my results shed light on how collusion can be dealt with in practice. Note that dealing with collusion in repeated first-price auctions is especially challenging because of a fundamental conflict between revenue maximization and fighting collusion. A seller, who wants to maximize her revenue, must force the different valuation types of the buyers to separate, making the higher types bid relatively high. But separation of the different valuation types creates scope for collusion since, absent punishments, the buyers would try to coordinate on a lower bidding profile. Higher patience will only make this coordination process easier for them. What my results suggest, however, is that higher patience also allows the seller to come up with very effective punishments for colluding buyers. To effectively fight collusion, a revenue-maximizing seller should force the buyers to pay "upfront" for the continuation of favorable terms of trade, which is achieved by making the relatively low-valuation types participate even when they have to bid above their current valuations. Penalization of non-participation makes sure that the buyers cannot improve their payoff by making the lower types abstain from the auction altogether and making the higher types take their place in bidding low. Since the higher valuation types also want to avoid (inefficiently) pooling with the lower valuation types, they can do nothing but bid high.

1.1 Related literature

The dynamic nature of the interaction presents formidable challenges for an auction designer. Some of those challenges (e.g. intertemporal dependence of agents' private information) have been addressed by the dynamic mechanism design literature (see e.g. Pavan et al. (2014), and Bergemann and Välimäki (2019) for a review). Other important issues however remain. It is well-known that dynamic games often exhibit a multiplicity of equilibria, which makes the classical mechanism design assumption of favorable equilibrium selection harder to justify. For example, in repeated auction settings, collusive outcomes with lower revenue can be supported in equilibrium (see e.g. Skrzypacz and Hopenhayn (2004), who analyze equilibria of repeated first-price auctions and conclude that a bid rotation scheme, which leaves the seller with less revenue than optimal, can be supported even under limited observability of bids and auction outcomes). Moreover, collusive equilibria seem to be practically relevant as collusive bidding patterns are observed in many different repeated auction settings around the world (see e.g Chassang et al. (2021)).

Repeated auctions are special cases of general repeated games. Equilibria of repeated games were studied by Abreu et al. (1990), who provide a recursive characterization of equilibrium payoffs for repeated games with imperfect monitoring, and Fudenberg et al. (1994) who prove a folk theorem for these games. Athey et al. (2004) introduce (*iid*) private information into a repeated Bertrand game with imperfect monitoring and apply the recursive characterization of Abreu et al. (1990) to their game. They show that patient players can sustain high rigid prices in the optimal equilibrium, thus extracting a lot of surplus from the consumers. Their model can be translated to an auction setting with a passive seller who chooses a reserve price once and for all in the beginning of the game. In the buyer-optimal equilibrium with patient buyers such a seller would be forced to sell the good at her chosen reserve price in every period.

Even though the literature on collusion in repeated auctions and oligopolies with private information is very extensive (see Correia-da Silva (2017) for a review), very few papers are concerned with the study of how the seller's or auction designer's behavior might affect the buyers' collusion. Abdulkadiroglu and Chung (2004) consider a stage game design problem in which a committed seller proposes a mechanism that will become the stage game played repeatedly by a set of tacitly colluding buyers. The seller in their model is concerned with buyers coordinating on the buyer-optimal sequential equilibrium and designs the stage game accordingly. Similarly to my paper, Abdulkadiroglu and Chung (2004) find that there is a mechanism which extracts the entire surplus from the buyers. In the optimal mechanism all the buyers pay the same participation fee to the seller and then the partnership dissolution mechanism of Cramton et al. (1987) is run. Abdulkadiroglu and Chung (2004) however note that a non-committed seller will fall far short of full surplus extraction: in the buyer-optimal sequential equilibrium of the repeated game in which the seller moves first and proposes a mechanism, the seller's revenue will be zero. In this paper I propose a less pessimistic (from the seller's point of view) model of equilibrium coordination. While the seller in my model lacks long-term commitment, she is able to control her own strategy and does not have to coordinate on the worst equilibrium for herself. She cannot however guarantee that the buyers will coordinate on her preferred equilibrium either. The buyers could take her strategy as given and tacitly coordinate on a lower bidding profile using their continuation values to enforce collusive behavior, hence her equilibrium strategy must make such coordination unprofitable for the buyers. Although the seller has a more active role in equilibrium coordination in my model, she is more constrained in terms of feasible mechanisms: she must offer a first-price auction in every period and can only adjust reserve prices over time. The first-price auctions are widely used in practice, but give rise to severe challenges when it comes to collusive behavior under private information. A seller who wants to obtain a higher revenue should try to force the buyer types to separate, but that very separation creates a scope for collusion. I show that this conflict is resolved in favor of the seller when she is patient enough.

A few other papers study similar settings, but none of them (to the best of my knowledge) simultaneously deals with the lack of seller's commitment and equilibrium coordination in a satisfactory way. Thomas (2005) notices that a seller could make collusion harder for the buyers by raising reserve prices, but assumes that the seller moves only once, in the beginning of time, and chooses one reserve price for the entirety of the repeated game between the buyers. Zhang (2021) studies a class of collusive agreements between bidders in a model of repeated first-price auctions, and, as a side note to his main results, shows how a revenue-maximizing seller should respond to collusion. His seller, much like the seller in Thomas (2005), moves only once and commits to a single reserve price. As the discount factor goes to 1, the seller is forced to tolerate "full collusion", in which all bids are suppressed down to the reserve price, and thus essentially makes an optimal take-it-or-leave-it offer to the colluding bidders. In contrast to the results in my paper, the revenue of a patient seller, who is restricted to choose only one reserve price once and for all, is lower than the revenue achieved under the infinite repetition of the competitive static outcome, and is therefore of course far below full surplus.

Ortner et al. (2020) are concerned with mitigating the effects of collusion in repeated procurement auctions. They propose a model with a regulator who observes the whole (infinite) bidding history and can punish colluding bidders. They construct tests for detecting collusive patterns of behavior which only allow for false negatives – therefore competitive bidders pass them with probability one. The regulator can then use the outcomes of the tests to punish the colluding bidders. My seller only has access to finite histories of bids and can only use reserve prices to punish colluding bidders.

Bergemann and Hörner (2018) also study a binary type model of first-price auctions similar to mine. The seller in their model is however passive and does not set a reserve price at all, and the buyers' valuations are perfectly persistent. They are concerned with disclosure regimes regarding the bid and winning history. In contrast to the findings in my paper, they show that the maximal disclosure regime leads to inefficient equilibria with low revenues. I show that an active seller who can adjust reserve prices over time can extract full surplus even when the full history of bids and identities of the winning buyers is publicly disclosed.

My paper is also related to the literature on collusion in static auctions. This literature was started by McAfee and McMillan (1992), who study outcomes of explicit before-auction communication in a first-price auction setting. They solve for optimal collusive schemes with ("strong collusion") and without transfers ("weak collusion"). In the optimal weak collusion scheme, the bidders bid at the reserve price as long as their valuation exceeds it and abstain otherwise. In the optimal strong collusion scheme, the colluding buyers can obtain a higher expected payoff by running a "knock-out" auction among themselves. The winner of the knock-out auction bids at the reserve price (as long as it exceeds his valuation) in the legitimate auction, and the losers are compensated for abstaining from the legitimate auction. It is however known now, that in the static setting the seller can avoid the dramatic losses from collusion via more sophisticated auction design. Che and Kim (2009) show that the second-best auction can be made collusion-proof, even when the bidders can use transfers to collude.

Finally, this paper speaks to the large literature on robustness in mechanism design (see Carroll (2019) for a comprehensive review). In my paper the seller aims to be robust to collusive behavior of the buyers.

1.2 Roadmap

The rest of the paper is organized as follows: Section 2 introduces the model of a repeated first-price auction game. In Section 3, I introduce the definitions of a *buyer-game* and a *collusive public perfect equilibrium*. In Section 4, I show how supporting collusive public perfect equilibria can be constructed to punish the seller and the buyers for deviations from the equilibrium path of *full-surplus-extracting cPPE* constructed in Sections 5 and 6. Section 7 briefly discusses the optimal reserve prices of the seller. Finally, Section 8 concludes.

2 Model

2.1 Setup

There is a seller (player 0) and $n \geq 2$ buyers (players $1, \ldots, n$) who interact over infinitely many periods. The seller sells one unit of a private good in every period via a first price auction with a reserve price. Each buyer is privately informed about his valuation type, which is drawn from a binary set $\Theta = \{\underline{\theta}, \overline{\theta}\}$, with $0 \leq \underline{\theta} < \overline{\theta}$, *iid* across periods and buyers. The probability of the low type $\underline{\theta}$ is $q \in (0, 1)$. The players share a common discount factor $\delta \in [0, 1)$.

The players play a repeated extensive form game with imperfect public monitoring. The timing of each period is as follows:

1. Seller announces a reserve price r.
2. Buyers privately learn their valuations for the good in the current stage.
3. Buyers bid or abstain ($\emptyset$) in the first price auction with the reserve price r.
4. The winner (if any) is determined, the buyers' choices are publicly disclosed.

The action set of the seller is $A_0 = \mathbb{R}_+$, the action set of each buyer is $A = \{\emptyset\} \cup \mathbb{R}_+$.

Buyer i's payoff is equal to his valuation θ_i net of his bid b_i if he wins the auction and zero otherwise. Ties are broken by a fair coin toss. Formally,

$$u_i(r, b, \theta_i) = \begin{cases} \frac{1}{\#(\text{win})}(\theta_i - b_i), & \text{if } b_i \geq r \ \& \ \big(b_i = \max\{b_1, ..., b_n\} \text{ or } b_{-i} = \emptyset\big) \\ 0, & \text{otherwise} \end{cases},$$

where #(win) stands for the number of winners in the auction, i.e. the number of buyers who placed the highest bid.

The seller's revenue is equal to the highest bid if there is a buyer who bids above his reserve price, and zero otherwise:

$$\mathcal{R}(r,b) = \begin{cases} b_i, & \text{if } b_i \geq r \ \& \ \big(b_i = \max\{b_1, ..., b_n\} \text{ or } b_{-i} = \emptyset\big) \\ 0, & \text{otherwise} \end{cases}.$$

2.2 One-shot auctions

Before we turn our attention to the repeated auction problem, we have to consider subgame perfect equilibria of the stage game. The intuition here is rather straightforward. If there are relatively few low types in the population (the probability q of having a low valuation is small), then the seller will prefer to trade with high types only, and will therefore set the reserve price equal to the high valuation $\overline{\theta}$. The low-type buyers will abstain while the high-type buyers will bid their valuation $\overline{\theta}$. If there are relatively many low types in the population, then the seller will prefer to trade with both types, and will therefore set the reserve price to the low valuation $\underline{\theta}$. The low-type buyers will bid their valuation while the high-type buyers will play a mixed strategy whose support lies above $\underline{\theta}$. The following proposition applies:

Proposition 1. One-shot auction equilibria

- *If the parameters of the model fall into the* ***High-reserve-price region*** $\big(q < \frac{n(\overline{\theta}-\underline{\theta})}{\underline{\theta}+n(\overline{\theta}-\underline{\theta})}\big)$, *then the seller sets* $r^*_{os} = \overline{\theta}$ *and generates revenue* $\mathcal{R}^*_{os} = (1-q^n)\overline{\theta}$; *the buyers get the ex ante payoff* $v^*_{os} = 0$.

- *If the parameters of the model fall into the* ***Low-reserve-price region*** $\big(q \geq \frac{n(\overline{\theta}-\underline{\theta})}{\underline{\theta}+n(\overline{\theta}-\underline{\theta})}\big)$, *then the seller sets* $r^*_{os} = \underline{\theta}$ *and generates revenue* $\mathcal{R}^*_{os} = (1-q^n)\overline{\theta}+q^n\underline{\theta}-n(1-q)q^{n-1}(\overline{\theta}-\underline{\theta})$; *the buyers get the ex ante payoff* $v^*_{os} = (1-q)q^{n-1}(\overline{\theta}-\underline{\theta})$.

Its proof along with other details of equilibrium characterization is provided in Appendix A.

3 Collusive Public Perfect Equilibrium

3.1 Motivation

Let us consider the Low-reserve-price region and the infinite repetition of the associated one-shot equilibrium. Clearly, it is an equilibrium of the infinitely repeated auction game, but there is no reason to believe that the players will actually coordinate on it. In fact, buyers' collusion is a good reason to believe otherwise. Suppose that the seller sets the reserve price equal to the low valuation $\underline{\theta}$, but the buyers, instead of coordinating on their one-shot equilibrium strategies, use a different bidding profile, in which high-type buyers bid $\overline{b} = \underline{\theta}$ and the low-type buyers abstain $\underline{b} = \emptyset$ in every period. This bidding profile gives a lower revenue of $(1-q^n)\underline{\theta}$ to the seller and a higher payoff of $\frac{1}{n}(1-q^n)(\overline{\theta}-\underline{\theta})$ to the buyers. The buyers can support their new bidding profile using a "grim-trigger" strategy, which punishes deviations by moving back to the one-shot equilibrium strategies of the Low-reserve-price region; the buyers only have to make sure that the high types do not want to deviate to $\underline{\theta}+\epsilon$, i.e. whenever

$$\underbrace{(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\underline{\theta})+\delta\frac{1}{n}(1-q^n)(\overline{\theta}-\underline{\theta})}_{\text{Payoff from } \overline{b}=\underline{\theta},\ \underline{b}=\emptyset} \geq \underbrace{(1-\delta)(\overline{\theta}-\underline{\theta})}_{\text{Today's deviation payoff}} + \underbrace{\delta(1-q)q^{n-1}(\overline{\theta}-\underline{\theta})}_{\text{Grim punishment payoff}},$$

which can be satisfied for high enough values of δ.

As we can see, the infinite repetition of the one-shot equilibrium in the Low-reserve-price region is not *"collusive"* because the buyers do not exploit their ability to collude to the fullest extent possible. A seller who has concerns about buyers' collusion should not hope to end up in such an equilibrium and needs to consider more sophisticated strategies. The seller's equilibrium strategy should however always guarantee that the buyers cannot improve their payoff similarly to how they did it in the above example. I formalize this requirement by introducing the concept of *collusive public perfect equilibrium*.

3.2 Definition

A *collusive public perfect equilibrium* is a strongly symmetric public perfect equilibrium that satisfies two novel requirements:

1. **Collusiveness on path.** The buyers must collude given the seller's on-path play of her equilibrium strategy. Central to this requirement is the notion of a *buyer-game* I introduce below. Buyer-game is a stochastic first price auction game between the buyers, in which the reserve prices are determined according to the seller's strategy. Collusiveness requires that the buyers be unable to improve their payoff by moving to a different strongly symmetric public perfect equilibrium in the buyer-game induced by the seller's equilibrium strategy. In the above example of the infinite repetition of the one-shot equilibrium of the Low-reserve-price region *collusiveness* was violated since the buyers could improve their payoff by moving to a different equilibrium between themselves.
2. **Collusiveness off path.** The continuation play must be collusive on path in the above sense in the above sense regardless of seller's actions as long as the buyers stick to their equilibrium strategies. This requirement formalizes the idea that buyers' collusive agreements cannot be broken by seller's actions. It does however allow non-collusive equilibria to be played following buyers' deviations and thus imposes no restriction on the buyers' ability to collude.

Strongly symmetric public perfect equilibrium is a public perfect equilibrium, in which buyers take symmetric actions on and off the equilibrium path. Public perfect equilibrium is an equilibrium in *public strategies*, i.e. strategies which map *public histories* into players' actions. A *public history* in the beginning of period $t+1$ is a sequence that includes all the actions taken by each player up to that period: $\big(\varnothing,(r_0,b_{10},\dots,b_{n0}),\dots,(r_{t-1},b_{1t},\dots,b_{nt})\big)$, where $\varnothing$ denotes the initial history. The set of those histories is given by $\mathcal{H}_0 \equiv \cup_{t=0}^{\infty}\big(A_0\times A^n\big)^t$, with a typical period-$t$ history denoted h_0^t. Since buyers additionally observe the action taken by the seller in every period, the set of public histories at which they get to make a move is given by $\mathcal{H}\equiv \cup_{t=0}^{\infty}\big[\big(A_0\times A^n\big)^t\big]\times A_0$ with a typical period-t history denoted h^t. A pure public strategy for the seller is a mapping $\sigma_0:\mathcal{H}_0\to A_0$, for the buyers it is $\sigma_i:\mathcal{H}\times\Theta\to A$.

The expected payoff of the seller in the repeated auction game is given by:

$$U_0(\sigma)=(1-\delta)\,\mathbb{E}\sum_{t=0}^{\infty}\delta^t\mathcal{R}\big(\sigma_0(h_0^t),\sigma_i(h^t,\theta_{it}),\sigma_{-i}(h^t,\theta_{-it})\big).$$

The expected payoff of the buyers $i = 1, 2, ..., n$ in the repeated auction game is given by:

$$U_i(\sigma) = (1-\delta)\,\mathbb{E}\sum_{t=0}^{\infty}\delta^t u_i\big(\sigma_0(h_0^t), \sigma_i(h^t, \theta_{it}), \sigma_{-i}(h^t, \theta_{-it}), \theta_{it}\big).$$

The above definitions extend naturally to behavioral strategies. We can now state the following definition:

Definition 1. Strongly symmetric public perfect equilibrium

A strategy profile $(\sigma_0^*, \sigma_1^*, ..., \sigma_n^*)$ is a strongly symmetric public perfect equilibrium if

1. *it induces a Nash equilibrium after every public history $h_0 \in \mathcal{H}_0$ and $h \in \mathcal{H}$;*
2. *$\sigma_i^*(h, \theta) = \sigma_j^*(h, \theta)$ after any public history $h \in \mathcal{H}$ for any two buyers i, j and any θ.*

The first condition of Definition 1 rules out non-credible threats at every public history much like subgame perfect equilibrium rules out non-credible threats in every subgame. The second condition makes sure that the buyers use symmetric bidding actions on and off the equilibrium path. Note that strongly symmetric public perfect equilibria have recursive structure: the continuation play after any public history is itself a strongly symmetric public perfect equilibrium.

All strongly symmetric public perfect equilibria I construct below, except the infinite repetition of the one-shot equilibrium in the Low-reserve-price region, satisfy the following additional assumption:

Assumption 1(a). Pure bidding actions along the equilibrium path

Buyers use pure bidding actions along the equilibrium path, i.e. after any public history $h \in \mathcal{H}$ consistent with the on-path play of $(\sigma_0^*, \sigma^*, \ldots, \sigma^*)$, the action $\sigma^*(h, \theta)$ is pure for both types $\theta \in \{\underline{\theta}, \overline{\theta}\}$.

Assumption 1(a) itself is not restrictive since we can find a full-surplus-extracting strongly symmetric public perfect equilibrium that belongs to the class of equilibria allowed by Assumption 1(a). However, I make a similar assumption in the next subsection (Assumption 1(b)), which forces the buyers to play the same class of equilibria in any buyer-game, restricting the set of collusive schemes they could use. It remains an open question whether Assumptions 1(a) and 1(b) could be dispensed with.

3.2.1 Collusiveness on path

To define *collusiveness on path* formally, we have to introduce the notion of a *buyer-game* induced by a seller's strategy. To define the states in the buyer-game, we need to define the *path automaton* of a seller's strategy[1]. In order to do that, fix a particular pure public strategy[2] of the seller σ_0. Let $\tilde{\mathcal{H}}_0(\sigma_0)$ be the set of histories consistent with the seller's play of σ_0 and any profile of buyers' strategies[3]. Two histories h_0 and h_0' from $\tilde{\mathcal{H}}_0(\sigma_0)$ are called σ_0-equivalent if they prescribe the same continuation play for the seller according to σ_0, i.e. $\sigma_0|_{h_0} = \sigma_0|_{h_0'}$. Let Ω be the resulting set of equivalence classes with ω_0 being the equivalence class of the initial history $\emptyset$. The path automaton representation of σ_0 is defined as follows:

Definition 2. Path automaton of a seller's strategy

The path automaton of σ_0 *is the tuple* $\left(\Omega, \omega^0, r, \tau\right)$, *where*

- $r : \Omega \to A_0$ *is the decision rule satisfying* $r(\omega) = \sigma_0(h_0)$ *for any* $h_0 \in \omega$.
- $\tau : \Omega \times A^n \to \Omega$ *is the transition function satisfying* $\tau(\omega, b) = w'$ *iff for any history* $h_0 \in w$ *the concatenated history* $(h_0, r(\omega), b) \in w'$.

We can now introduce the definition of the buyer-game induced by σ_0:

Definition 3. Buyer-game

Let $\left(\Omega, \omega^0, r, \tau\right)$ *be the path automaton of* σ_0. *The buyer-game induced by* σ_0 *is a stochastic game between the buyers where:*

- *The set of states is* Ω, *with the initial state* ω_0. *State transitions occur according to* τ.
- *The set of actions for each buyer is* A, *i.e. is as defined in the repeated auction game.*

[1] Unlike an automaton representation, the path automaton of a seller's strategy assumes that the seller never deviates from σ_0, and therefore represents only part of her repeated game strategy. See also Kandori and Obara (2006) who employ a similar definition of a path automaton in the context of repeated games with private monitoring.

[2] It is without loss of generality to restrict attention to pure strategies of the seller, since our goal is to construct a full-surplus-extracting collusive public perfect equilibrium, which can be achieved under this restriction.

[3] A typical element of $\tilde{\mathcal{H}}_0(\sigma_0)$ can be written as $h_0^t = \left(\emptyset,\ (\sigma_0(\emptyset),\ b_0),\ (\sigma_0(h_0^1),\ b_1),\ \ldots,\ (\sigma_0(h_0^{t-1}),\ b_{t-1})\right)$; where $h_0^1 = \left(\sigma_0(\emptyset),\ b_0\right)$, $h_0^2 = \left((\sigma_0(\emptyset),\ b_0),\ (\sigma_0(h_0^0),\ b_1)\right)$, etc.

- *The set of valuations for each buyer is Θ, i.e. is as defined in the repeated auction game.*
- *The utility of buyer i with type θ_i bidding b_i in state ω is*

$$\tilde{u}_i(\omega, b, \theta_i) = \begin{cases} \frac{1}{\#(\text{win})}(\theta_i - b_i), & \text{if } b_i \geq r(\omega) \ \& \ \big(b_i = \max\{b_1, ..., b_n\} \text{ or } b_{-i} = \emptyset\big) \\ 0, & \text{otherwise} \end{cases},$$

where $\#(\text{win})$ stands for the number of winners in the auction.

Let us look at the strongly symmetric public perfect equilibria of the buyer-game induced by σ_0. A public history at period $t+1$ in the buyer-game includes all states and bids up to period $t+1$: $\big(\omega_0, (b_{10}, \ldots, b_{n0}), \ldots, \omega_t, (b_{1t}, \ldots, b_{nt}), \omega_{t+1}\big)$. Let $\mathbf{H}(\sigma_0)$ be the set of these public histories. A public strategy in the buyer game is a function $\rho_i : \mathbf{H}(\sigma_0) \times \Theta \to A$. This definition of public strategy extends naturally to behavior strategies. A strongly symmetric public perfect equilibrium in the buyer-game induced by a seller's strategy σ_0 is defined as follows:

Definition 4. Strongly symmetric public perfect equilibrium in the buyer-game

A strategy profile $(\rho_1^*, \ldots, \rho_n^*)$ is a strongly symmetric public perfect equilibrium equilibrium of the buyer-game induced by σ_0 if

1. *It induces a Nash equilibrium after any public history $\mathbf{h} \in \mathbf{H}(\sigma_0)$.*
2. *$\rho_i^*(\mathbf{h}, \theta) = \rho_j^*(\mathbf{h}, \theta)$ after any public history $\mathbf{h} \in \mathbf{H}(\sigma_0)$ for any two buyers i, j and any θ.*

Recall that by Assumption 1(a) the buyers use pure bidding actions along the equilibrium path of any strongly symmetric public perfect equilibrium of the repeated auction game. The following Assumption 1(b) restrict the buyers to play equilibria from the same class in the buyer game.

Assumption 1(b). Pure bidding actions along the equilibrium path of a buyer-game

Buyers use pure bidding actions along the equilibrium path in the buyer-game induced by σ_0, i.e. after any public history $\mathbf{h} \in \mathbf{H}(\sigma_0)$ consistent with the on-path play of $(\rho^*, \ldots, \rho^*)$, the action $\rho^*(\mathbf{h}, \theta)$ is pure for both types $\theta \in \{\underline{\theta}, \overline{\theta}\}$.

Assumption 1(b) does not allow the buyers to collude by moving to a strongly symmetric public perfect equilibrium of the buyer game that exhibits mixed actions along the equilibrium path. It is in principle possible that the buyers could collectively benefit from using mixed actions along the equilibrium of the buyer-game induced by the full-surplus-extracting collusive equilibrium constructed below. It can be shown that the simplest collusive schemes with mixed actions do not help the buyers to improve their payoff[4]. The larger question of whether Assumption 1(b) could be dispensed with remains open.

We can now use the above definitions to formally introduce the notion of *collusiveness on path*.

Definition 5. Collusiveness on path

A strongly symmetric public perfect equilibrium $(\sigma_0^*, \sigma^*, ..., \sigma^*)$ *of the repeated auction game is collusive on path if there is no strongly symmetric public perfect equilibrium with pure actions along the equilibrium path (i.e. satisfying Assumption 1(b)) in the buyer-game induced by* σ_0^**, whose equilibrium payoff exceeds the buyer payoff from* $(\sigma_0^*, \sigma^*, ..., \sigma^*)$ *in the repeated auction game.*

3.2.2 Collusiveness off path

Recall that the requirement of *collusiveness off path* formalizes the idea that buyers' collusive agreements cannot be broken by seller's actions. More specifically, if the buyers have played their equilibrium actions up to the current period, then they must collude on path from the next period on no matter what the seller has played. The formal definition is as follows:

Definition 6. Collusiveness off path

Suppose $(\sigma_0^*, \sigma^*, \ldots, \sigma^*)$ *is a strongly symmetric public perfect equilibrium of the repeated auction game. Consider an alternative seller's strategy* σ_0' *and let* $h_0^t \in \mathcal{H}_0$ *be a period-t history consistent with the on-path play of* $(\sigma_0', \sigma^*, \ldots, \sigma^*)$*. If the continuation equilibrium* $(\sigma_0^*|_{h_0^t}, \sigma^*|_{h_0^t}, \ldots, \sigma^*|_{h_0^t})$ *is collusive on path for any such* h_0^t *and* σ_0'*, then* $(\sigma_0^*, \sigma^*, \ldots, \sigma^*)$ *is collusive off path.*

We can now state the main definition:

[4]For example, some stationary schemes, in which the high types mix over two bidding actions on path, do not improve the buyers' payoff because of their efficiency loss vis-à-vis fully separating behavior

Definition 7. Collusive public perfect equilibrium

A strongly symmetric public perfect equilibrium of the repeated auction game is a collusive public perfect equilibrium if it is collusive on and off path.

Remark 1. *Observe that the infinite repetition of the one-shot equilibrium in the High-reserve-price region is a collusive public perfect equilibrium in the sense of Definition 7. First of all it is clearly a strongly symmetric public perfect equilibrium. To show collusiveness on path, observe that the buyers get zero payoff along the equilibrium path, and it is not possible for them to improve their payoff once the seller's on path play is fixed: bidding below $\overline{\theta}$ leads to a zero payoff as well, bidding above $\overline{\theta}$ can only lead to losses. Since after a deviation by any player, the players return to the same equilibrium in the next period, it is also collusive off path.*

4 Supporting collusive equilibria

A seller who intends to actively fight collusion has to come up with punishment strategies for the buyers who are suspected of coordinating their bidding behavior. Since our ultimate goal is to construct a *collusive public perfect equilibrium*, in which the seller extracts the entire surplus from the buyers in the limit as the discount factor goes to 1, the punishment has to be as severe as possible. The most severe punishment that the seller could construct in principle involves leaving zero payoff to the buyers. Our goal in this section is to establish that a threat of such an severe punishment can be made credible if the seller is patient enough.

4.1 Repetition of static equilibrium in the High-reserve-price region

It is easy to see that the threat of severe punishment is immediately available to the seller if the parameters belong to High-reserve-price region. Since the one-shot equilibrium payoff of the buyers is already equal to zero, the seller can always reduce the equilibrium payoff of the buyers to zero, no matter what the value of δ is by switching to the infinite repetition of the one-shot equilibrium. Moreover since the equilibrium reserve price is extremely high, there is no room for collusion in this equilibrium:

Lemma 1. *Suppose that the parameters of the model belong to then the infinite repetition of the equilibrium of Proposition 1 (with $r^* = \overline{\theta}$ in every period) is a collusive public perfect equilibrium in the sense of Definition 7.*

Proof. The buyers get zero payoff along the equilibrium path. It is not possible for them to improve their payoff once the seller's strategy is fixed: bidding below $\overline{\theta}$ is impossible, bidding above $\overline{\theta}$ can only lead to losses □

4.2 Low-revenue collusive equilibria in the Low-reserve-price region

Suppose now that the parameters of the model belong to the Low-reserve-price region. Unlike in the High-reserve-price region, it might be harder for the seller to reduce the buyers' payoff to zero when she prefers trading with both types in the one-shot auction game. It nevertheless turns out to be possible when the seller is patient enough. To provide the appropriate punishments to the seller, I first construct *collusive public perfect equilibria* which leave the seller with little revenue. I will then use these equilibria to support a high reserve price equilibrium, in which the seller sets $r = \overline{\theta}$ along the equilibrium path and the buyers get zero equilibrium payoffs. This high reserve price equilibrium equilibrium will then be used to support the full-surplus-extracting equilibrium in Section 6.

4.2.1 Low-revenue separating equilibrium

I will now construct a separating equilibrium with low (but non-zero) revenue that can be supported for high enough discount factors. Since our aim is to find a low revenue equilibrium, it is reasonable to try to force the seller to set $r = 0$ along the equilibrium path and have the low type of each buyer bid zero in every period. I denote the high type's bid by $\overline{b}$.

First, we have to make sure that the on-schedule incentive compatibility conditions are satisfied, i.e. that the low type does not want to emulate the behavior of the high type and vice versa. A low type $\underline{\theta}$ obtains in every period: $\frac{q^{n-1}}{n}\underline{\theta}$ and a high type's payoff in each period is: $\frac{1-q^n}{n(1-q)}(\overline{\theta} - \overline{b})$[5] If a low type buyer attempts to mimic a high type buyer's behavior, his payoff is

[5]Interested readers will find the calculation of separating equilibrium payoffs in Appendix B.

going to be: $\frac{1-q^n}{n(1-q)}(\underline{\theta}-\overline{b}))$, thus the low type incentive compatibility is given by:

$$\frac{q^{n-1}}{n}\underline{\theta} \geq \frac{1-q^n}{n(1-q)}(\underline{\theta}-\overline{b}),$$

which is equivalent to $\overline{b} \geq \frac{1-q^{n-1}}{1-q^n}\underline{\theta}$. Since we are attempting to minimize the seller's revenue, it is reasonable to select the minimal possible bid for a high type buyer:

$$\overline{b}^* = \frac{1-q^{n-1}}{1-q^n}\underline{\theta}.$$

The *ex ante* equilibrium payoff of each buyer:

$$v^*_{\text{lrs}} = \frac{1}{n}\left[(1-q^n)\left(\overline{\theta} - \frac{1-q^{n-1}}{1-q^n}\underline{\theta}\right) + q^n\underline{\theta}\right] = \frac{1}{n}\big[(1-q^n)(\overline{\theta}-\underline{\theta}) + q^{n-1}\underline{\theta}\big] \tag{1}$$

The resulting revenue of the seller:

$$\mathcal{R}^*_{\text{lrs}} = (1-q^n)\frac{1-q^{n-1}}{1-q^n}\underline{\theta} + q^n 0 = (1-q^{n-1})\underline{\theta}.$$

Recall that in the one-shot equilibrium of Proposition **??**, the *ex ante* equilibrium payoff for each bidder is given by $v^*_{\text{os}} = (1-q)q^{n-1}(\overline{\theta}-\underline{\theta})$. Comparing the static equilibrium payoff in Proposition 1 and the payoff in (1), we obtain:

$$\begin{aligned}
v^*_{\text{lrs}} - v^*_{\text{os}} &= \frac{1}{n}\big[(1-q^n)(\overline{\theta}-\underline{\theta}) + q^{n-1}\underline{\theta}\big] - (1-q)q^{n-1}(\overline{\theta}-\underline{\theta}) \\
&= \frac{1}{n}\big[(1-q^n)(\overline{\theta}-\underline{\theta}) + q^{n-1}\underline{\theta} - n(1-q)q^{n-1}(\overline{\theta}-\underline{\theta})\big] \\
&= \frac{1}{n}\big[(1-q^n - n(1-q)q^{n-1})(\overline{\theta}-\underline{\theta}) + q^{n-1}\underline{\theta}\big] \\
&= \frac{1}{n}\left[\left((1-q)\sum_{k=0}^{n-1} q^k - n(1-q)q^{n-1}\right)(\overline{\theta}-\underline{\theta}) + q^{n-1}\underline{\theta}\right] \\
&= \frac{1}{n}\left[(1-q)\left(\sum_{k=0}^{n-1} q^k - nq^{n-1}\right)(\overline{\theta}-\underline{\theta}) + q^{n-1}\underline{\theta}\right] \\
&> \frac{1}{n}\big[(1-q)\big(nq^{n-1} - nq^{n-1}\big)(\overline{\theta}-\underline{\theta}) + q^{n-1}\underline{\theta}\big] = \frac{1}{n}q^{n-1}\underline{\theta} > 0,
\end{aligned}$$

which suggests that the chosen on-path behavior of the buyers can be supported by the threat of switching to the infinite repetition of the one-shot equilibrium. We can now formulate the full definition of the strategy profile:

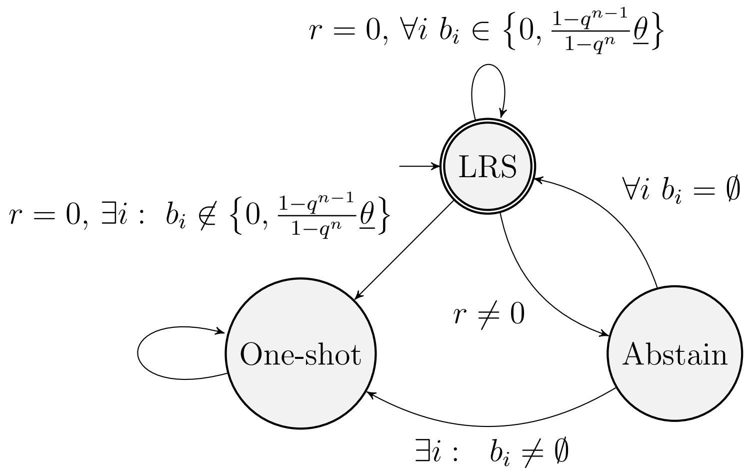


Figure 1: Low-revenue separating (LRS) strategy profile

Definition 8. Low-revenue separating strategy profile

(i) Along the equilibrium path:

(a) Seller sets $r^* = 0$*,*

(b) Any low-type buyer bids $\underline{b}^* = 0$*,*

(c) Any high-type bids $\overline{b}^* = \frac{1-q^{n-1}}{1-q^n}\underline{\theta}$*,*

(ii) If at any history following $r = 0$ *in every period a bid outside of* $\{\underline{b}^*, \overline{b}^*\}$ *is made, then the game switches to the infinite repetition of the one-shot equilibrium of Low-reserve-price region forever.*

(iii) Both buyer types abstain whenever $r > 0$*.*

(iv) After any history along which a positive bid has been observed following $r > 0$*, the game switches the infinite repetition of the one-shot equilibrium of the Low-reserve-price region forever.*

The low-revenue separating strategy profile is illustrated by Figure 1. The following proposition shows that the low-revenue separating strategy profile is a strongly symmetric public perfect equilibrium of the repeated auction game for high values of the discount factor.

Proposition 2. *Suppose that the parameters of the model belong to the Low-reserve-price region. There exists δ^* such that for all $\delta \in [\delta^*, 1)$ the low-revenue separating strategy profile is a strongly symmetric public perfect equilibrium of the repeated auction game. Along the equilibrium path the buyers will obtain the payoff of $v^*_{lrs} = \frac{1}{n}\big[(1-q^n)(\overline{\theta} - \underline{\theta}) + q^{n-1}\underline{\theta}\big]$, and the seller will get $\mathcal{R}^*_{lrs} = (1-q^{n-1})\underline{\theta}$.*

Proof. Consider first the incentives of the seller. It is clear that the seller does not want to deviate: if she attempts a one-shot deviation to $r > 0$, her revenue will become $(1-\delta)0 + \delta(1 - q^{n-1})\underline{\theta} = \delta(1-q^{n-1})\underline{\theta}$ (because all the buyers will abstain following $r > 0$), which can never exceed his equilibrium revenue of $(1-q^{n-1})\underline{\theta}$.

Now turn to the buyers. Consider first the public histories along which neither of the players has deviated. Incentive compatibility will require for a high-type buyer:

$$(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta} - \overline{b}^*) + \delta\frac{1}{n}\big[(1-q^n)(\overline{\theta} - \underline{\theta}) + q^{n-1}\underline{\theta}\big] \\ \geq (1-\delta)\max\{q^{n-1}\overline{\theta}, \overline{\theta} - \overline{b}^*\} + \delta(1-q)q^{n-1}(\overline{\theta} - \underline{\theta}),$$

and for a low type buyer:

$$(1-\delta)\frac{q^{n-1}}{n}\underline{\theta} + \delta\frac{1}{n}\big[(1-q^n)(\overline{\theta} - \underline{\theta}) + q^{n-1}\underline{\theta}\big] \\ \geq (1-\delta)\max\{q^{n-1}\underline{\theta}, \underline{\theta} - \overline{b}^*\} + \delta(1-q)q^{n-1}(\overline{\theta} - \underline{\theta}).$$

Before dealing with these constraints, consider a public history along which the seller has deviated to $r > 0$ in the current period. The equilibrium strategy of the buyers prescribes abstaining from participation if the reserve price is set above zero. The associated incentive compatibility condition of a high-type buyer is given by:

$$\delta\frac{1}{n}\big[(1-q^n)(\overline{\theta} - \underline{\theta}) + q^{n-1}\underline{\theta}\big] > (1-\delta)(\overline{\theta} - r) + \delta(1-q)q^{n-1}(\overline{\theta} - \underline{\theta}).$$

For a low-type buyer it is given by:

$$\delta\frac{1}{n}\big[(1-q^n)(\overline{\theta} - \underline{\theta}) + q^{n-1}\underline{\theta}\big] \geq (1-\delta)(\underline{\theta} - r) + \delta(1-q)q^{n-1}(\overline{\theta} - \underline{\theta}).$$

The best deviation for $r > 0$ is the one for the high type and when $r \approx 0$. This deviation is unprofitable whenever:

$$\delta \frac{1}{n}\left[(1-q^n)(\overline{\theta}-\underline{\theta}) + q^{n-1}\underline{\theta}\right] \geq (1-\delta)\overline{\theta} + \delta(1-q)q^{n-1}(\overline{\theta}-\underline{\theta}). \tag{2}$$

Notice that the incentive compatibility condition in (2) implies all of the above incentive compatibility conditions since the on-path payoff in each of them can only be higher and the deviation payoff can only be lower than in (2). The incentive compatibility condition in (2) is satisfied for all δ such that:

$$\delta > \frac{n\overline{\theta}}{n\overline{\theta} + \left(1-q^n-n(1-q)q^{n-1}\right)(\overline{\theta}-\underline{\theta}) + q^{n-1}\underline{\theta}} \equiv \delta^*. \tag{3}$$

Since $1-q^n-n(1-q)q^{n-1} > 0$, we can conclude that $\delta^* \in [0,1)$. □

4.2.2 Zero-revenue pooling equilibrium

It is natural to ask the question whether the seller can be forced to give away the good in every period for free (clearly the worst possible outcome for the seller in this setup). That would require the seller to set the reserve price $r = 0$ along the equilibrium path and the buyers to bid $b^* = 0$ along the equilibrium path. The buyers' payoff would be equal to:

$$v^*_{\text{zrp}} = (1-q)\frac{1}{n}(\overline{\theta}-r) + q\frac{1}{n}(\underline{\theta}-r) = \frac{\mathbb{E}(\theta)}{n}. \tag{4}$$

Comparing the buyer's payoff in 4 to the *ex ante* payoff of the buyers in the low-revenue separating equilibrium, we get:

$$\begin{aligned}
v^*_{\text{lrs}} - v^*_{\text{zrp}} &= \frac{1}{n}\left[(1-q^n)(\overline{\theta}-\underline{\theta}) + q^{n-1}\underline{\theta}\right] - (1-q)\frac{1}{n}\overline{\theta} - q\frac{1}{n}\underline{\theta} \\
&= \frac{1}{n}\left[(1-q^n)(\overline{\theta}-\underline{\theta}) + q^{n-1}\underline{\theta} - (1-q)\overline{\theta} - q\underline{\theta}\right] \\
&= \frac{1}{n}\left[(q-q^n)\overline{\theta} + (-1+q^n+q^{n-1}-q)\underline{\theta}\right] \\
&= \frac{1}{n}\left[q(1-q^{n-1})\overline{\theta} - (1+q)(1-q^{n-1})\underline{\theta}\right] \\
&= \frac{1-q^{n-1}}{n}\left[q\overline{\theta} - (1+q)\underline{\theta}\right],
\end{aligned}$$

which means that $v^*_{\text{lrs}} < v^*_{\text{zrp}}$ whenever $q\overline{\theta} - (1+q)\underline{\theta} < 0$ or

$$q < \frac{\underline{\theta}}{\overline{\theta} - \underline{\theta}}. \tag{5}$$

We can now formulate the full definition of the zero-revenue pooling strategy profile:

Definition 9. Zero-revenue pooling strategy profile

(i) Along the equilibrium path

(a) Seller sets $r^* = 0$,

(b) Both buyer types bid 0,

(ii) If at any history following $r = 0$ in every period a bid $b \neq 0$ is placed, then the game switches to the infinite repetition of the one-shot equilibrium of the Low-reserve-price region forever.

(iii) Both buyer types abstain whenever $r > 0$.

(iv) After any history along which a bid has been observed following $r > 0$, the play of the game switches to the infinite repetition of the one-shot equilibrium of the Low-reserve-price region forever.

The zero-revenue pooling strategy profile is illustrated by Figure 2. The following proposition shows that the zero-revenue pooling strategy profile is a strongly symmetric public perfect equilibrium of the repeated auction game condition in (5) is satisfied.

Proposition 3. *Suppose that the parameters of the model belong to the Low-reserve-price region, and suppose further that the condition in (5) is satisfied, then there exists $\delta^* \in [0, 1)$ such that for all $\delta > \delta^*$ the zero-revenue pooling strategy profile is a strongly symmetric public perfect equilibrium of the repeated auction game.*

Proof. Consider first the seller's incentives. The seller does not have any incentive to deviate because she would end up with zero revenue regardless of the reserve price, which makes setting $r = 0$ one of the optimal choices.

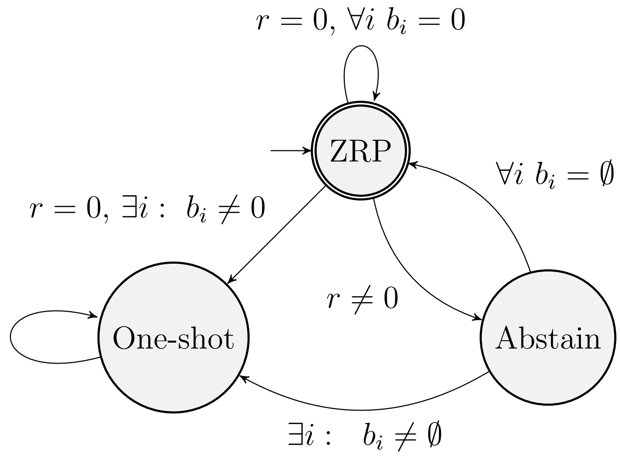


Figure 2: Zero-revenue pooling (ZRP) strategy profile

Consider now one of the buyers who is contemplating a deviation. Consider first a public history along which neither player has deviated, the best available deviation after such a history is for the high type to bid $0 + \epsilon$ for some small ϵ. This deviation will be detected by both the seller and the competing buyer. The competing buyer would then have to punish the deviator by switching to the infinite repetition of the one-shot equilibrium with $r = \underline{\theta}$ and competitive bidding, enforcing the continuation value of $(1-q)q^{n-1}(\overline{\theta} - \underline{\theta})$. The associated incentive compatibility condition for the high type is then given by:

$$(1-\delta)\frac{1}{n}\overline{\theta} + \delta\frac{\mathbb{E}(\theta)}{n} \geq (1-\delta)\overline{\theta} + \delta(1-q)q^{n-1}(\overline{\theta} - \underline{\theta}). \tag{6}$$

Consider now a public history along which the seller has deviated to $r > 0$ in the current period. The equilibrium strategy prescribes abstaining from participation for both buyers in the current period. The payoff from following the equilibrium strategy is thus $\delta\,\mathbb{E}(\theta)/n$. The best deviation available to the buyers is for the high type to bid r and get the good with the payoff of $\overline{\theta} - r$. Since this deviation is automatically detected by the seller and the competing buyers, the game then switches to the infinite repetition of the one-shot equilibrium, thus resulting in the incentive compatibility condition given by:

$$\delta\frac{\mathbb{E}(\theta)}{n} \geq (1-\delta)(\overline{\theta} - r) + \delta(1-q)q^{n-1}(\overline{\theta} - \underline{\theta}).$$

Clearly this deviation is most profitable when $r \approx 0$, therefore we could rule out all such deviations if we made sure that the following condition holds:

$$\delta \frac{\mathbb{E}(\theta)}{n} \geq (1-\delta)\overline{\theta} + \delta(1-q)q^{n-1}(\overline{\theta} - \underline{\theta}). \tag{7}$$

Recall now the no-deviation condition in (6). Clearly its left-hand side is strictly above the left-hand side of (7). As the respective right-hand sides are identical, it is obvious then that (7) implies (6). The condition in 7 is satisfied whenever

$$\delta \geq \frac{n\overline{\theta}}{n\overline{\theta} + q\underline{\theta} + (1-q)\overline{\theta} - n(1-q)q^{n-1}(\overline{\theta} - \underline{\theta})} \equiv \delta^*. \tag{8}$$

Note that the critical value of the discount factor δ^* defined in (8) is in $[0, 1)$ as long as $q\underline{\theta} + (1-q)\overline{\theta} - n(1-q)q^{n-1}(\overline{\theta} - \underline{\theta}) = n(v^*_{\text{zrp}} - v^*_{\text{os}})$ is strictly positive. Recall that the payoff from the low revenue separating equilibrium v^*_{lrs} always exceeds the one-shot equilibrium payoff v^*_{os}. Under the assumption that $q < \frac{\underline{\theta}}{\overline{\theta} - \underline{\theta}}$ in (5) we have $v^*_{\text{zrp}} > v^*_{\text{lrs}} > v^*_{\text{os}}$, which establishes the claim. □

Observe that both the low-revenue separating equilibrium and the zero-revenue pooling equilibrium lead to the same buyer-game. This buyer game is a repeated first-price auction game in which the reserve price is set to zero. In an optimal strongly symmetric public perfect equilibrium of this game the buyers either pool or separate along the equilibrium path. If they pool, then their optimal equilibrium payoff is equal to the buyers' payoff in the zero-revenue pooling equilibrium. If they separate, then their optimal equilibrium payoff is equal to the buyer's payoff in the low-revenue pooling equilibrium. Thus, depending on the parameter values, either the zero-revenue pooling equilibrium is collusive, or the low-revenue separating equilibrium is collusive. The following proposition, whose proof is relegated to Appendix C, establishes this claim formally.

Proposition 4. *If $q \geq \frac{\underline{\theta}}{\overline{\theta} - \underline{\theta}}$, then the low-revenue separating equilibrium of Proposition 2 is collusive in the sense of Definition 7, otherwise the zero-revenue pooling equilibrium of Proposition 3 is collusive in the sense of Definition 7.*

Proof. See Appendix C. □

4.3 High-reserve-price equilibrium in the Low-reserve-price region

Having constructed equilibria with low revenue in the previous sections, we can now proceed to characterize some of the high(er) revenue equilibria in which the seller actively fights collusion among the buyers. Suppose that the seller sets $r = \overline{\theta}$ along the equilibrium path. Clearly the optimal response of the buyers is to bid $\overline{\theta}$ for the high type and to abstain for the low type. This equilibrium therefore leaves zero rents to the buyers, but is inefficient and therefore does not allow the seller to extract full surplus. It does, however, allow the seller to credibly threaten the buyers with zero continuation value (as does the repetition of the one-shot equilibrium in the High-reserve-price region). In the full-surplus-extracting equilibria of Section 6 the buyers can therefore be incentivized to give up almost the entire surplus along the equilibrium path.

The on-path behavior in this equilibrium can be supported either by the threat of switching to the low-revenue separating equilibrium or by the threat of switching to the zero-revenue separating equilibrium. The full definition of the strategy profile is as follows:

Definition 10. High-reserve-price strategy profile

(i) At any history in which the seller has always set $r^* = \overline{\theta}$

  (a) The seller sets $r^* = \overline{\theta}$,

  (b) Any low-type buyer abstains,

  (c) Any high-type buyer bids $\overline{\theta}$.

(ii) If $q \geq \frac{\underline{\theta}}{\overline{\theta}-\underline{\theta}}$ (low-revenue separating equilibrium is collusive), then

- *Following any observation of $r < \overline{\theta}$ in period t, the buyers abstain in period t and the low-revenue separating equilibrium is played from period $t+1$ on.*
- *Following any observation of $r < \overline{\theta}$ in period t, if any of the buyers fails to abstain in period t, the one-shot equilibrium of the Low-reserve-price region is infinitely repeated from period $t+1$ on.*

(iii) If $q < \frac{\underline{\theta}}{\overline{\theta}-\underline{\theta}}$ (zero-revenue pooling equilibrium is collusive), then

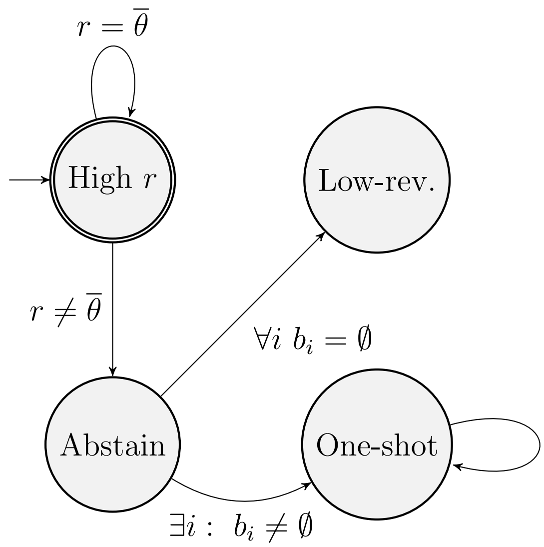


Figure 3: High-reserve-price (High r) strategy profile

- *Following any observation of $r < \overline{\theta}$ in period t, the buyers abstain in period t and the zero-revenue pooling equilibrium is played from period $t+1$ on.*
- *Following any observation of $r < \overline{\theta}$ in period t, if any of the buyers fails to abstain in period t and places a positive bid above r, the one-shot equilibrium of the Low-reserve-price region is infinitely repeated from period $t+1$ on.*

The high-reserve-price strategy profile is illustrated by Figure 3. The following proposition shows that it is a strongly symmetric public perfect equilibrium of the repeated auction game for high values of the discount factor.

Proposition 5. *Suppose that the parameters of the model belong to the , then there exists $\delta^* \in [0, 1)$ such that for all $\delta > \delta^*$ the high-reserve-price strategy profile is a strongly symmetric public perfect equilibrium of the repeated auction game. The buyers get the payoff $v^*_{\text{hrp}} = 0$, the seller gets the revenue of $\mathcal{R}^*_{\text{hrp}} = (1 - q^n)\overline{\theta}$.*

Proof. (ii) *Low-revenue separating equilibrium is collusive*

It is easy to see that the seller does not want to deviate in any period. Along the equilibrium path, her revenue is equal to $\mathcal{R}^*_{\text{hrp}} = (1 - q^n)\overline{\theta}$. If she deviates to any $r < \overline{\theta}$, then her revenue is $(1-\delta)0+\delta(1-q^{n-1})\underline{\theta}=\delta(1-q^{n-1})\underline{\theta}<(1-q^n)\overline{\theta}$.

Buyers get zero payoffs along the equilibrium path. Following $r=\overline{\theta}$ neither type wants to deviate: bidding leads to a negative payoff for the low type in the current period, and abstaining does not improve the payoff of the high type in the current period. It remains to make sure that buyers do not want to deviate from the proposed strategy following an observation of a lower reserve price $r<\overline{\theta}$. It is required that both types prefer abstaining in the current period and playing the low-revenue separating equilibrium to bidding r (the lowest possible bid) and playing the one-shot equilibrium in the continuation game, i.e. for type $\theta_i \in \{\underline{\theta},\overline{\theta}\}$

$$\delta\frac{1}{n}\left[(1-q^n)(\overline{\theta}-\underline{\theta})+q^{n-1}\underline{\theta}\right]>(1-\delta)(\theta_i-r)+\delta(1-q)q^{n-1}(\overline{\theta}-\underline{\theta}).$$

The best deviation obtains for the high type at $r=0$:

$$\delta\frac{1}{n}\left[(1-q^n)(\overline{\theta}-\underline{\theta})+q^{n-1}\underline{\theta}\right]>(1-\delta)\overline{\theta}+\delta(1-q)q^{n-1}(\overline{\theta}-\underline{\theta}),$$

which is the same condition as in (2) satisfied for all δ defined in (3).

(iii) *Zero-revenue pooling equilibrium is collusive*

Just as in the previous case, the seller's revenue is equal to $\mathcal{R}^*_{\text{hrp}}=(1-q^n)\overline{\theta}$. She does not want to deviate since deviation leads to zero revenue forever.

As before the best deviation is for a high type buyer whenever the seller deviates to a reserve price $r>0$ near zero. The condition is:

$$\delta\frac{\mathbb{E}(\theta)}{n}\geq(1-\delta)\overline{\theta}+\delta(1-q)q^{n-1}(\overline{\theta}-\underline{\theta}),$$

which is identical to the no deviation condition in 7, and therefore leads to the same threshold for the discount factors as in (8).

□

The following corollary is immediate:

Corollary 1. *The high-reserve-price equilibrium of Proposition 5 is collusive in the sense of Definition 7.*

Proof. Holding the seller's equilibrium strategy fixed, it is impossible for the two buyers to improve their payoff even if they perfectly coordinate: bidding higher leads to negative payoffs, bidding lower is impossible. □

Since this equilibrium leaves the buyers with zero payoffs, we can now use it to construct full-surplus-extracting equilibria by threatening the buyers who deviate off-schedule with zero continuation values.

5 High-revenue collusive equilibria

In this section I will introduce a class of collusive public perfect equilibria that allow the seller to extract full surplus in the limit as δ goes to 1. These equilibria are stationary and separating along the equilibrium path, i.e. in each of them any low-type buyer bids $\underline{b}$, and a high-type buyer bids $\bar{b}$, while the seller sets the reserve price to $r = \underline{b}$ in every period along the equilibrium path. The full description of the class of strategy profiles I am considering is given by the following definition.

Definition 11. High-revenue strategy profile

Fix a pair of bids $(\bar{b}, \underline{b})$. *The corresponding* ***high-revenue strategy profile*** *is described as follows.*

(i) Along the equilibrium path

- *Seller sets a reserve price equal to the equilibrium bid of a low type buyer* $r = \underline{b}$.
- *Any low-type buyer bids* $\underline{b}$
- *Any high-type buyer bids* $\bar{b}$

(ii) If the parameters of the model belong to the High-reserve-price-region, then

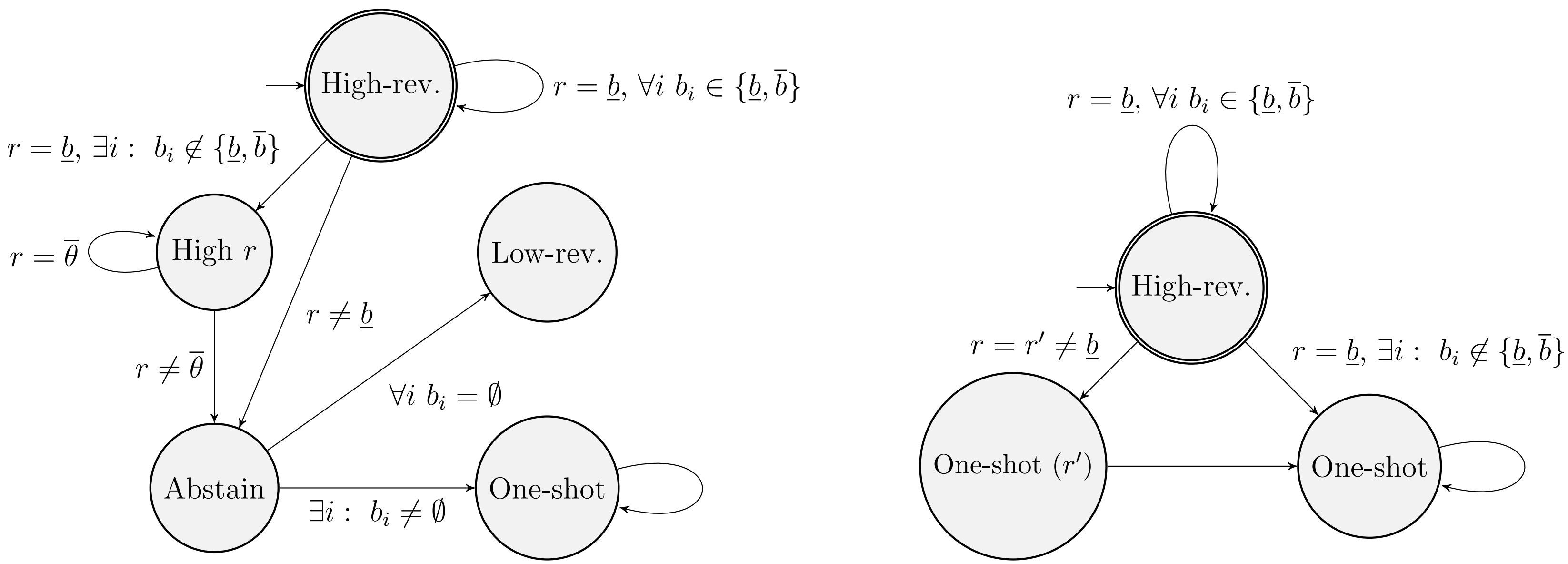
- *If at any history following* r *in every period a bid outside of* $\{\underline{b}, \bar{b}\}$ *is placed, the play of the game switches to the infinite repetition of the one-shot equilibrium of High-reserve-price region forever.*

- *If in period t the seller sets $r' \neq r$, then the buyers play the one-shot equilibrium with reserve price r' in period t, and the play of the game switches to the infinite repetition of the one-shot equilibrium of High-reserve-price region forever.*

(iii) If the parameters of the model belong to the Low-reserve-price region, then

- *If at any history following r in every period a bid outside of $\{\underline{b}, \overline{b}\}$ is placed, then the play of the game switches to the high-reserve-price equilibrium of Proposition 5 forever.*
- *both types abstain in period t if $r' \neq r$ is observed in period t, and from $t+1$ on the play of the game game switches to the low-revenue separating equilibrium when it is collusive, (i.e. when $q \geq \frac{\underline{\theta}}{\overline{\theta}-\underline{\theta}}$) or to the zero-revenue pooling equilibrium when it is collusive (i.e. when $q < \frac{\underline{\theta}}{\overline{\theta}-\underline{\theta}}$).*
- *After any history along which a bid has been observed following $r' \neq r$, the game switches to the infinite repetition of the one-shot equilibrium of he Low-reserve-price region forever.*

The high-revenue strategy profile is illustrated by Figure 4.



(a) Low-reserve-price region

(b) High-reserve-price region

Figure 4: High-revenue strategy profile

Having discussed the structure of the high-revenue strategy profiles, I can set up the following *revenue maximization problem*:

$$\mathcal{RM}: \ \mathcal{R}^*_{\text{fse}} \equiv \max_{\bar{b},\underline{b},v} (1-q^n)\bar{b} + q^n \underline{b}, \quad \text{s.t.}$$

$$\text{(Eq-payoff)} \qquad v = \frac{1}{n}\big[(1-q^n)(\bar{\theta}-\bar{b}) + q^n(\underline{\theta}-\underline{b})\big];$$

Incentive constraints:

$$\text{(LowIC)} \qquad (1-\delta)\frac{q^{n-1}}{n}(\underline{\theta}-\underline{b}) + \delta v \geq 0,$$

$$\text{(HighIC-up)} \qquad (1-\delta)\frac{1-q^n}{n(1-q)}(\bar{\theta}-\bar{b}) + \delta v \geq (1-\delta)(\bar{\theta}-\bar{b}),$$

$$\text{(HighIC-down)} \qquad (1-\delta)\frac{1-q^n}{n(1-q)}(\bar{\theta}-\bar{b}) + \delta v \geq (1-\delta)q^{n-1}(\bar{\theta}-\underline{b}),$$

$$\text{(HighIC-on-sch)} \qquad \frac{1-q^n}{n(1-q)}(\bar{\theta}-\bar{b}) \geq \frac{q^{n-1}}{n}(\bar{\theta}-\underline{b});$$

No-collusion constraints:

$$\text{(No-col-sep-1)} \qquad v \geq \frac{(1-\delta)(1-q^n)(\bar{\theta}-\underline{b})}{n(1-\delta(1-q)^n)},$$

$$\text{(No-col-sep-2)} \qquad v \geq \frac{(1-\delta)\big[(1-q^n)(\bar{\theta}-\underline{b}) + q^n(\underline{\theta}-\underline{b})\big]}{n(1-\delta q^n)},$$

$$\text{(No-col-pool)} \qquad v \geq \frac{1}{n}\big[(1-q)(\bar{\theta}-\underline{b}) + q(\underline{\theta}-\underline{b})\big];$$

A solution to the revenue maximization problem in $\mathcal{RM}$ is a pair of bids $(\bar{b}^*, \underline{b}^*)$ together with a buyer payoff v^*_{fse}. In the next lemma, I will show that the high-revenue strategy profile corresponding to $(\bar{b}^*, \underline{b}^*)$ is a *collusive public perfect equilibrium* for high enough values of δ as long as the solution to $\mathcal{RM}$ induces a well-defined separating equilibrium (i.e. $\bar{b}^* > \underline{b}^*$), the low-type buyers bid strictly above their valuation (i.e. $\underline{b}^* > \underline{\theta}$), and the seller achieves a higher revenue than in the high reserve price equilibrium (i.e. $\mathcal{R}^*_{\text{fse}} \geq (1-q^n)\bar{\theta}$). In Section 6, I will solve $\mathcal{RM}$, verify that its solution satisfies the aforementioned conditions for sufficiently high values of δ, and show that the maximal revenue goes to full surplus as δ goes to 1.

Lemma 2. *Suppose* $\big(\bar{b}^*, \underline{b}^*, v^*_{\text{fse}}\big)$ *solve the revenue maximization problem* $\mathcal{RM}$*. Suppose further that* $\underline{\theta} < \underline{b}^* < \bar{b}^*$ *and* $\mathcal{R}^*_{\text{fse}} \geq (1-q^n)\bar{\theta}$*, then the high-revenue strategy profile corresponding to*

$\left(\overline{b}^*, \underline{b}^*\right)$ *(as defined by 11) is a collusive public perfect equilibrium of the repeated auction game if*

(i) the parameters of the model belong to the High-reserve-price region; or

(ii) the parameters of the model belong to the Low-reserve-price region and δ satisfies

- *condition (3) if $q \geq \frac{\underline{\theta}}{\overline{\theta}-\underline{\theta}}$ (low-revenue separating equilibrium is collusive),*
- *condition (8) if $q < \frac{\underline{\theta}}{\overline{\theta}-\underline{\theta}}$ (zero-revenue pooling equilibrium is collusive).*

Proof. Let us show first that the high-revenue strategy profile is a strongly symmetric public perfect equilibrium of the repeated auction game. Strong symmetry follows from Definition 11, thus we only need to check the players' incentives. I start with the buyers.

Incentive compatibility of the buyers. Consider histories in which every player has stayed on the equilibrium path up to period t. Suppose first that the parameters of the model fall into i.e. $r = \overline{\theta}$ is optimal in the one-shot game). If the seller deviates in period t, the play from $t+1$ is a public perfect equilibrium by construction. Since the buyers receive zero continuation values from $t+1$ on, they will play the one-shot equilibrium in period t for a given reserve price as if the game ends tomorrow, hence the buyers do not want to deviate in period t. Suppose now that the parameters of the model fall into the Low-reserve-price region (i.e. $r = \underline{\theta}$ is optimal in the one-shot game). If the seller deviates in period t, then the equilibrium strategy dictates that the buyers abstain in period t. Since a buyer's deviation triggers the switch to the infinite repetition of the one-shot equilibrium of Low-reserve-price region, it is not profitable for the buyers as long as δ satisfies conditions (3) or (8) by the argument employed in the construction of the low-revenue separating or zero-revenue pooling equilbria respectively.

Suppose now that the seller does not deviate in period t, and consider the buyers' incentives. Let us start with on-schedule deviations, i.e. attempts to mimic the behavior of the other type. The on-schedule deviation is unprofitable of a low-type buyer as long as:

$$\underbrace{\frac{q^{n-1}}{n}(\underline{\theta} - \underline{b}^*)}_{\text{Equilibrium reward}} \geq \underbrace{\frac{1-q^n}{n(1-q)}(\underline{\theta} - \overline{b}^*)}_{\text{Mimic the high type}}.$$

This incentive compatibility condition is satisfied since $\underline{\theta} < \underline{b}^* < \overline{b}^*$ by assumption: if a low-type buyer deviates to $\overline{b}^*$, then he receives a lower payoff with a higher probability, which cannot be profitable. The on-schedule deviation is unprofitable for a high-type buyer as long as:

$$\underbrace{\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*)}_{\text{Equilibrium reward}} \geq \underbrace{\frac{q^{n-1}}{n}(\overline{\theta}-\underline{b}^*)}_{\text{Mimic the low type}},$$

which is the incentive constraint (HighIC-on-sch) of the revenue maximization problem $\mathcal{RM}$ evaluated at $(\overline{b}^*, \underline{b}^*)$, and is therefore satisfied.

Consider now off-schedule deviations. First of all, we must make sure that a low-type buyer is actually willing to participate in the auction as opposed to abstaining and getting the forever punishment of high reserve price, i.e. that the following condition is satisfied:

$$\underbrace{(1-\delta)\frac{q^{n-1}}{n}(\underline{\theta}-\underline{b}^*)+\delta v^*_{\text{fse}}}_{\text{Equilibrium payoff of a low-type buyer}} \geq (1-\delta) \underbrace{0}_{\text{Abstain today}} + \delta \underbrace{0}_{\text{Switch to } r=\overline{\theta} \text{ forever}} = 0,$$

which is the incentive constraint (LowIC) of the revenue maximization problem $\mathcal{RM}$ evaluated at $(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}})$. If a low-type buyer deviates to a higher off-schedule bid, then he receives a negative expected reward in the period of the attempted deviation (since $\underline{\theta} < \underline{b}^*$) and zero continuation value, which cannot be profitable for someone who receives a positive payoff along the equilibrium path. We can therefore conclude that the remaining off-schedule incentive constraints of a low-type buyer are satisfied at $\left(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}}\right)$.

Consider now high-type buyers who contemplate off-schedule deviations. A high-type buyer could deviate upwards which would guarantee him winning the auction with probability 1. The best upward deviation is to $\overline{b}^* + \epsilon$ which gives the deviating high-type buyer a payoff almost equal to $\overline{\theta} - \overline{b}^*$. For this deviation to be unprofitable, we must have:

$$\underbrace{(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*)+\delta v^*_{\text{fse}}}_{\text{Equilibrium payoff of a high-type buyer}} \geq (1-\delta) \underbrace{(\overline{\theta}-\overline{b}^*)}_{\text{Deviate to } \overline{b}^*+\epsilon} + \delta \underbrace{0}_{\text{Switch to } r=\overline{\theta} \text{ forever}} = (1-\delta)(\overline{\theta}-\overline{b}^*),$$

which is the incentive constraint (HighIC-up) of the revenue maximization problem $\mathcal{RM}$ evaluated at $(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}})$.

A high-type buyer could also deviate downwards and win the auction only in the case when all his competitors are low-type buyers, that is with probability q^{n-1}. In this case the best deviation is to $\underline{b}^* + \epsilon$ with a payoff almost equal to $\overline{\theta} - \underline{b}^*$. For this deviation to be unprofitable, we must have:

$$\underbrace{(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*) + \delta v^*_{\text{fse}}}_{\text{Equilibrium payoff of a high type buyer}} \geq (1-\delta)\underbrace{q^{n-1}(\overline{\theta}-\underline{b}^*)}_{\text{Deviate to } \underline{b}^*+\epsilon} + \delta \underbrace{0}_{\text{Switch to } r=\overline{\theta} \text{ forever}} = (1-\delta)q^{n-1}(\overline{\theta}-\overline{b}^*),$$

which is the incentive constraint (HighIC-down) of the revenue maximization problem $\mathcal{RM}$ evaluated at $(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}})$.

Incentive compatibility of the seller. Consider now the seller's incentives. Recall that we have $\mathcal{R}^*_{\text{fse}} \geq (1-q^n)\overline{\theta}$ by assumption. If the parameters of the model belong to the High-reserve-price region, a deviating seller would receive the payoff of $(1-\delta)\,\mathcal{R}^*_r + \delta(1-q^n)\overline{\theta}$ where $\mathcal{R}^*_r$ is the revenue achieved by the seller in the one-shot auction game with the reserve price equal to r. In the High-reserve-price region the optimal reserve price for the seller is $r = \overline{\theta}$ with the associated revenue of $(1-q^n)\overline{\theta}$. Thus a deviating seller would not be able to get more than $(1-\delta)(1-q^n)\overline{\theta} + \delta(1-q^n)\overline{\theta} = (1-q^n)\overline{\theta}$ which cannot exceed $\mathcal{R}^*_{\text{fse}}$. If the parameters of the model belong to Low-reserve-price region, a deviating seller would receive either 0 (if the zero-revenue pooling equilibrium is collusive), or $\delta(1-q^{n-1})\underline{\theta}$ (if the low revenue separating equilibrium is collusive), neither of which can exceed $\mathcal{R}^*_{\text{fse}}$.

Other histories. Neither the seller nor the buyers want to deviate after any of the other histories by construction of continuation equilibria, hence the high-revenue strategy profile corresponding to the bids $(\overline{b}^*, \underline{b}^*)$ is a strongly symmetric public perfect equilibrium.

Buyer-game. We must make sure that the public perfect equilibrium we have constructed is indeed collusive in the sense of Definition 7. To do that, we shall consider the buyer-game induced by the seller's equilibrium strategy. This buyer game is a stochastic game with two states. The game starts in the low reserve price state ω^l, in which the reserve price is equal to $r(\omega^l) = \underline{b}^*$, and remains in that state unless a bid outside of $\{\underline{b}^*, \overline{b}^*\}$ is placed by at least one buyer, in which the game transitions to the high reserve price state ω^h, in which the reserve price is $r(\omega^h) = \overline{\theta}$. The high reserve price state is absorbing, i.e. once the high reserve price state is achieved, the game remains in that state forever. The full definition of this high-revenue buyer-game is as follows:

Definition 12. ***High-revenue buyer-game***

- *The set of states is $\Omega = \{\omega^l, \omega^h\}$, the initial state is $\omega^0 = \omega^l$.*
- *The set of actions for each buyer is A, i.e. as defined in the repeated auction game*
- *The transitions between states occur according to τ:*

$$\tau(\omega^l, b) = \begin{cases} \omega^l, & \text{if } b \in \{\underline{b}^*, \overline{b}^*\}^n \\ \omega^h, & \text{otherwise} \end{cases},$$

$$\tau(\omega^h, b) = \omega^h, \ \forall b.$$

- *The set of valuations for each buyer is Θ, i.e. is as defined in the repeated auction game.*
- *The utility of buyer i with type θ_i bidding b_i in state ω is*

$$\tilde{u}_i(\omega, b, \theta_i) = \begin{cases} \frac{1}{\#(\text{win})}(\theta_i - b_i), & \text{if } b_i \geq r(\omega) \ \& \ \big(b_i = \max\{b_1, ..., b_n\} \text{ or } b_{-i} = \emptyset\big) \\ 0, & \text{otherwise} \end{cases},$$

where $\#(\text{win})$ *stands for the number of winners in the auction.*

The definition of collusive public perfect equilibria (Definition 7) requires that the buyers be unable to play a strongly symmetric public perfect equilibrium of the high-revenue buyer-game in Definition 12 that improves their payoff. I first show that the buyers' strategy in any strongly symmetric public perfect equilibrium of the buyer-game must be monotonic:

Lemma 3. ***Monotonicity lemma***

Consider the high-revenue buyer-game in Definition 12. Any strongly symmetric public perfect equilibrium of this buyer-game satisfies monotonicity: pick any history of play that leads to state ω_l, if $\overline{b}$ is the equilibrium bidding action of a high-type buyer and $\underline{b}$ is the equilibrium bidding action of a low-type buyer after that history, then $\overline{b} \geq \underline{b}$.

Proof. See Appendix D. □

The Monotonicity lemma shows any high-type buyer must always place a higher bid than any low type buyer in any symmetric public perfect equilibrium of the buyer-game whenever the current state is ω^l. Recall that when the current state is ω^h, the reserve price is equal to $\overline{\theta}$, and thus the buyers cannot get more than zero in any continuation equilibrium is that state. Since they cannot get a negative payoff in any continuation equilibrium either, they must be getting zero once the game is stuck in state ω^h. As I restrict attention to pure strategies along the equilibrium path, the resulting *ex ante* payoff from bidding $(\overline{b}, \underline{b})$ in state ω^l is given by:

$$\hat{u}_{\omega^l}(\overline{b}, \underline{b}) \equiv \begin{cases} \frac{1}{n}\left[(1-q^n)(\overline{\theta}-\overline{b}) + q^n(\underline{\theta}-\underline{b})\right] & \text{if } \overline{b} > \underline{b} \\ \frac{1}{n}\left[(1-q)(\overline{\theta}-\underline{b}) + q(\underline{\theta}-\underline{b})\right] & \text{if } \overline{b} = \underline{b} \end{cases}$$

where whenever $b < r(\omega^l)$, the convention is to set $\theta - b = 0$ for the respective type $\theta \in \{\overline{\theta}, \underline{\theta}\}$

Consider now the optimal collusion problem in the high-revenue buyer-game and ignore all the aspects of incentive compatibility except monotonicity. Since all the remaining incentive compatibility constraints are ignored, the following maximization problem provides an upper bound on symmetric equilibrium payoffs in the buyer-game:

$$\max_{\{\overline{b}_t, \underline{b}_t\}_{t=0}^{+\infty}} (1-\delta) \sum_{t=0}^{+\infty} \delta^t \hat{u}_{\omega}(\overline{b}_t, \underline{b}_t) \quad \text{s.t.} \tag{9}$$

$$(i)\ \overline{b}_t \geq \underline{b}_t,$$

$$(ii)\ \text{Transition function } \tau.$$

where $\hat{u}_{\omega^l}(\overline{b}, \underline{b})$ is defined above, and $\hat{u}_{\omega^h}(\overline{b}, \underline{b})$ is assumed to be equal to zero without loss of generality. The optimization problem in (9) is a Markov decision problem. It follows from

Blackwell (1965) that, if this problem has a solution, it must also have a stationary solution. I therefore consider two kinds of stationary monotonic bidding profiles: separating and pooling.

Separating profiles. Suppose first that the buyers coordinate on a separating bidding profile in the high-revenue buyer-game under consideration. If both types bid on schedule, then clearly there is only one option: $\underline{b} = \underline{b}^*$ and $\overline{b} = \overline{b}^*$ with the payoff equal to v^*_{fse}. *If all buyers of type $\overline{\theta}$ bid on schedule and all buyers of type $\underline{\theta}$ bid off schedule*, then the off-schedule action of any low-type buyer will be immediately detected by the seller and punished with zero continuation values. Since the punishment will not occur if and only if all buyers have high types (i.e. with probability $(1-q)^n$), the resulting payoff will be:

$$v = (1-\delta)\frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}) + q^n(\underline{\theta}-\underline{b})\big] + \delta(1-q)^n v.$$

Recall that we assume $\underline{b}^* > \underline{\theta}$, hence by incentive compatibility we must have $\overline{b}^* < \overline{\theta}$. Then the optimal solution here is to coordinate on the bidding profile in which any high-type buyer bids the low equilibrium bid $\underline{b}$ and any low-type buyer abstains, i.e. choose $\underline{b}^* = \emptyset$ and $\overline{b} = \underline{b}^*$, which results in the payoff:

$$v(\underline{b}^*, \emptyset) = \frac{(1-\delta)\big[(1-q^n)(\overline{\theta}-\underline{b}^*)\big]}{n(1-\delta(1-q)^n)}.$$

The no-collusion constraint (No-col-sep-1) of the revenue maximization problem $\mathcal{RM}$ evaluated at $(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}})$ gives us $v^*_{\text{fse}} \geq v(\underline{b}^*, \emptyset)$.

If all buyers of type $\overline{\theta}$ bid off schedule and all buyers of type $\underline{\theta}$ bid on schedule, then the off-schedule action of any high-type buyer will be immediately detected by the seller and punished with zero continuation values. Since the punishment will not occur if and only if all buyers have low types (i.e. with probability q^n), the resulting payoff will be:

$$v = (1-\delta)\frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}) + q^n(\underline{\theta}-\underline{b})\big] + \delta q^n v,$$

which can be solved for v' to get:

$$v = \frac{(1-\delta)\big[(1-q^n)(\overline{\theta}-\overline{b}) + q^n(\underline{\theta}-\underline{b})\big]}{n(1-\delta q^n)}.$$

The optimal solution here is for the low types to choose $\underline{b} = \underline{b}^*$ and for the high types to choose $\overline{b} = \underline{b}^* + \epsilon$, with the resulting payoff of:

$$v(\underline{b}^* + \epsilon, \underline{b}^*) = \frac{(1-\delta)\big[(1-q^n)(\overline{\theta} - \underline{b}^*) + q^n(\underline{\theta} - \underline{b}^*)\big]}{n(1-\delta q^n)}.$$

The no-collusion constraint (No-col-sep-2) of the revenue maximization problem $\mathcal{RM}$ evaluated at $(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}})$ gives us $v^*_{\text{fse}} \geq v(\underline{b}^* + \epsilon, \underline{b}^*)$.

If buyers of both types bid off schedule, then the seller will punish them in the first period with probability 1, and the resulting payoff will be:

$$v = (1-\delta)\frac{1}{n}\big[(1-q^n)(\overline{\theta} - \overline{b}) + q^n(\underline{\theta} - \underline{b})\big] + \delta 0.$$

Since it must be that $\overline{b}^* < \overline{\theta}$, the best bidding profile here is for the high types to choose $\overline{b} = \underline{b}^* + \epsilon$ and for the low types to choose $\underline{b} = \emptyset$ with the payoff of:

$$v(\underline{b}^* + \epsilon, \emptyset) = (1-\delta)\frac{1}{n}(1-q^n)(\overline{\theta} - \underline{b}^*),$$

which is clearly below $v(\underline{b}^*, \emptyset)$ and therefore below v^*_{fse}.

Pooling profiles. The buyers might find it optimal to pool instead of separating. *If the buyers pool on schedule*, then their collusive scheme is never detected by the seller. Clearly the optimal pooling on schedule is achieved at $\underline{b}^*$ with the resulting payoff of:

$$v(\underline{b}^*, \underline{b}^*) = \frac{1}{n}\big[(1-q)(\overline{\theta} - \underline{b}^*) + q(\underline{\theta} - \underline{b}^*)\big] \tag{10}$$

The no-collusion constraint (No-col-pool) of the revenue maximization problem $\mathcal{RM}$ evaluated at $(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}})$ gives us $v^*_{\text{fse}} \geq v(\underline{b}^*, \underline{b}^*)$.

Note that the payoff from *pooling off-schedule* cannot exceed v^*_{fse}. If the buyers coordinate on any off-schedule bid above $\underline{b}^*$ they will get a fraction of the payoff in 10 since they will be punished by the seller with probability 1. Abstaining from the auction altogether cannot be optimal as long as $v^*_{\text{fse}} \geq 0$, which it is by incentive compatibility.

We therefore conclude that no strongly symmetric public perfect equilibrium payoff in the high-revenue buyer game corresponding to $(\underline{b}^*, \overline{b}^*)$ can exceed v^*_{fse}, and therefore the high-revenue strategy profile corresponding to $(\underline{b}^*, \overline{b}^*)$ is a *collusive public perfect equilibrium* of the repeated auction game in the sense of Definition 7. $\square$

6 Full surplus extraction

Let us now solve the revenue maximization problem $\mathcal{RM}$. There are three cases depending on which constraints are binding; the parameter values corresponding to each case are illustrated by Figure 5. In Case 1, (No-col-sep-1) and (LowIC) constraints are binding with both types being indifferent between their payoff in the full-surplus-extracting cPPE and the payoff they could have obtained by coordinating on the bidding profile $(\underline{b}^*, \emptyset)$. Case 1 does not always apply because its solution candidate does not always satisfy the (HighIC-up) incentive compatibility constraint: if n is high enough, the winning probability of a high-type buyer is so low that such a buyer would prefer to win with probability 1 by placing a slightly higher bid and suffer the punishment of zero continuation values. We therefore have to consider Case 2, in which (HighIC-up) and (No-col-sep-1) are binding and the remaining constraints are slack. Case 2 equilibrium candidate in turn does not apply for high values of q: in this case the (HighIC-down) incentive compatibility constraint will be violated. Intuitively, if the mass of low types is sufficiently large, then a high type buyer will have a fairly high chance of winning by bidding just above the low type equilibrium bid even though placing such a bid is severely punished. In Case 3, only (HighIC-up) and (HighIC-down) are binding, and the remaining constraints are slack, which implies that the buyers do not have a strict incentive to collude.

The remaining constraints in the revenue maximization problem are never binding. Consider first the on-schedule incentive compatibility constraint of a high-type buyer (HighIC-on-sch). This constraint essentially puts an upper bound on the high-type equilibrium bid (if a high-type buyer is asked to bid a lot more than a low-type buyer, he might find it profitable to deviate to the low-type bid and get a much higher reward with a smaller winning probability), but we have already included a constraint that does the same, the no-collusion constraint (No-col-sep-1). Indeed, if a high-type buyer is asked to place a very high bid in every period, then the buyers might find it profitable to collude on a lower bidding profile, and such a collusion scheme

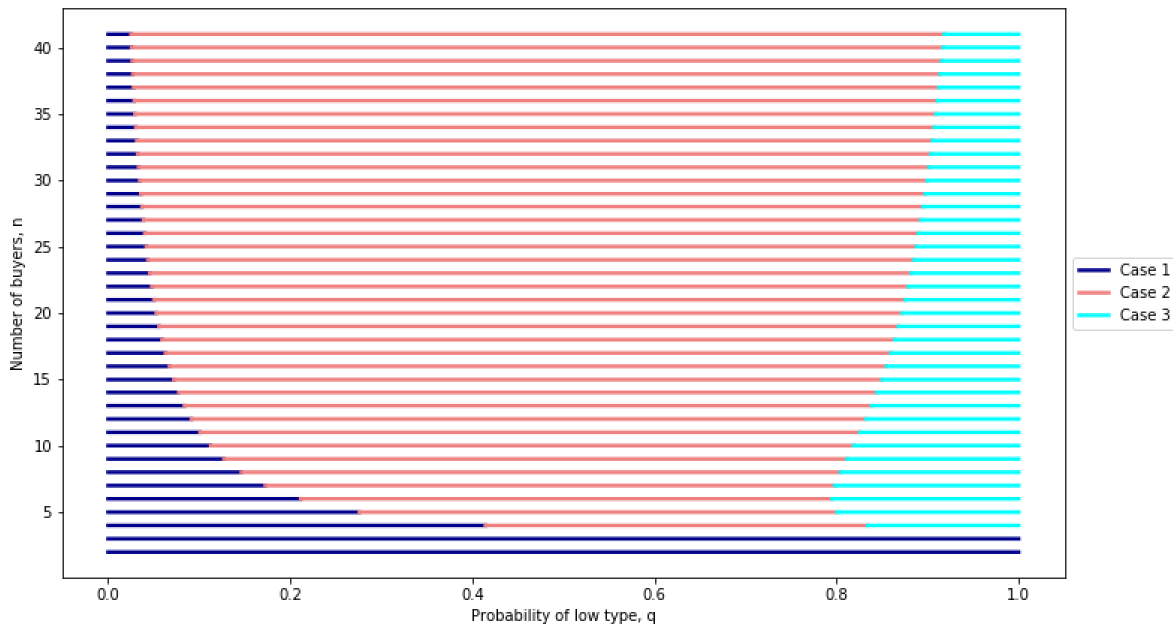


Figure 5: *Parameters corresponding to Cases 1, 2, and 3. For each number of buyers n, the respective line shows which values of q belong to Cases 1, 2, and 3.*

is prevented by (No-col-sep-1). The restriction on equilibrium bids imposed by (No-col-sep-1) is more severe than the one imposed by the on-schedule incentive compatibility of a high type buyer. Clearly, if the more severe restriction were the one imposed by incentive compatibility, we would be unlikely to consider collusion an important problem in an auction setting with adverse selection.

The two remaining no-collusion constraints, (No-col-sep-2) and (No-col-pool), are also non-binding in all three cases, which means that the optimal optimal collusion scheme for the buyers always involves bidding $\underline{b}^*$ for the high types and abstaining for the low types. Collusion by pooling on schedule turns out to be particularly inefficient as it leads to negative payoffs for the buyers for δ close to 1, while the buyers' payoff in the full-surplus-extracting cPPE is non-negative by construction. Collusion by leaving the low types on schedule and moving the high types off schedule does not outperform the optimal collusion scheme because it leads to punishments for the high types, who, as opposed to the low types, get a positive payoff in every period. The gain from bidding lower made by the high types in this collusion scheme is completely offset by the severity of the seller's punishment.

In the following subsections I will construct the solutions to the revenue maximization problem $\mathcal{RM}$ in each of the three cases. I will show that the revenue-maximizing bidding profiles can indeed be supported in the collusive public perfect equilibrium with the corresponding high-revenue strategy profiles (as defined by 11), and derive the conditions on the parameters of the model for each of the three cases. In all three cases the seller will be able to extract full surplus from the buyers in the limit as the discount factor δ goes to 1.

Case 1: High expected valuation/Small number of buyers

Recall that in Case 1, the no-collusion constraint (No-col-sep-1) and the low-type incentive compatibility constraint (LowIC) bind at the optimum of the revenue maximization problem $\mathcal{RM}$.

Full surplus extraction cPPE, Case 1.

- *Equilibrium conditions:*

$$\text{(No-col-sep-1)} \qquad v^*_{\text{fse}} = \frac{(1-\delta)(1-q^n)(\overline{\theta} - \underline{b}^*)}{n(1-\delta(1-q)^n)},$$

$$\text{(LowIC)} \qquad (1-\delta)\frac{q^{n-1}}{n}(\underline{\theta} - \underline{b}^*) + \delta v^*_{\text{fse}} = 0,$$

$$\text{(Eq-payoff)} \qquad v^*_{\text{fse}} = \frac{1}{n}\big[(1-q^n)(\overline{\theta} - \overline{b}^*) + q^n(\underline{\theta} - \underline{b}^*)\big].$$

- *Parameter restriction:*

$$q < \frac{1-q^n}{n(1-q)}.$$

The solution to this system of equilibrium conditions is provided in Appendix E.1. I will derive the condition on the parameters in the course of proving Proposition 6 below. The resulting equilibrium payoff for a low-type buyer conditional upon winning with $\underline{b}^*$ is:

$$\underline{\theta} - \underline{b}^* = \frac{-\delta q\big(1-q^n\big)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)}; \tag{11}$$

for a high-type buyer winning with $\overline{b}^*$ we have:

$$\overline{\theta} - \overline{b}^* = \frac{q^n(1-\delta(1-q))(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)}; \tag{12}$$

and for a high-type buyer winning with $\underline{b}^*$

$$\overline{\theta} - \underline{b}^* = \frac{q^n\big(1-\delta(1-q)^n\big)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)}. \tag{13}$$

The *ex ante* equilibrium payoff is:

$$v^*_{\text{fse}} = \frac{1}{n}\frac{(1-\delta)q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)}. \tag{14}$$

The equilibrium bids can be immediately computed from the payoffs in 11 and 12:

$$\underline{b}^* = \underline{\theta} + \frac{\delta q\big(1-q^n\big)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)}, \tag{15}$$

$$\overline{b}^* = \overline{\theta} - \frac{q^n(1-\delta(1-q))(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)}. \tag{16}$$

I first show that the equilibrium bids in (15) and (16) satisfy the condition of Lemma 2.

Lemma 4. *$\underline{\theta} < \underline{b}^* < \overline{b}^*$.*

Proof. (i) $\underline{\theta} < \underline{b}^*$ is equivalent to $\underline{\theta} - \underline{b}^* < 0$, which is true since $-\delta q(1-q^n)(\overline{\theta}-\underline{\theta}) < 0$.

(ii) $\underline{b}^* < \overline{b}^*$ is equivalent to $\overline{\theta} - \underline{b}^* > \overline{\theta} - \overline{b}^*$, which is true since $1-\delta(1-q)^n > 1-\delta(1-q)$ because $(1-q)^n < (1-q)$ for any $q \in (0,1)$ and $n \geq 2$.

□

I now show that the bids in (15) and (16) can in fact be supported in a *collusive public perfect equilibrium* for a high values of δ:

Proposition 6. *Suppose that* $q < \frac{1-q^n}{n(1-q)}$. *Suppose further that* $\underline{b}^*$ *and* $\overline{b}^*$ *are as defined in (15) and (16) respectively, then there exists a critical discount factor* δ^*, *such that for all* $\delta \in [\delta^*, 1)$ *the high-revenue strategy profile corresponding to* $(\overline{b}^*, \underline{b}^*)$ *(as defined by 11) is a collusive public perfect equilibrium of the repeated auction game in the sense of Definition 7. Moreover, the seller achieves full surplus extraction in the limit as* δ *goes to* 1.

Proof sketch. The complete proof is provided in Appendix G.1. Here I briefly sketch the main arguments. Recall that by Lemma 2 and Lemma 4, it is enough to check that $\mathcal{R}^*_{\text{fse}} \geq (1-q^n)\overline{\theta}$ and that the remaining constraints in the revenue maximization problem $\mathcal{RM}$ are satisfied at $(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}})$ for high enough δ. I start with the seller's revenue.

Seller's revenue. The seller's revenue is equal to the full surplus net of the equilibrium payoff of the buyers:

$$\mathcal{R}^*_{\text{fse}} = (1-q^n)\overline{\theta} + q^n\underline{\theta} - nv^*_{\text{fse}}.$$

Recall that nv^*_{fse} is given by:

$$nv^*_{\text{fse}} = \frac{(1-\delta)q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big) + q^n\big(1-\delta(1-q)^n\big)} \xrightarrow[\delta \to 1]{} 0.$$

and therefore the seller extracts full surplus in the limit as δ goes to 1 and $\mathcal{R}^*_{\text{fse}} \approx (1-q^n)\overline{\theta} + q^n\underline{\theta}$ for δ close enough to 1, which clearly exceeds $(1-q^n)\overline{\theta}$.

Incentive constraints. All of the remaining incentive constraints in in the revenue maximization problem $\mathcal{RM}$ are non-binding at $(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}})$ for all δ high enough and all values of q and n, except the incentive constraint (HighIC-up). There is a region of q and n, where this constraint cannot be satisfied even for δ close to 1. To see why, observe that (HighIC-up) can be rewritten as:

$$\delta v^*_{\text{fse}} \geq (1-\delta)\left(1 - \frac{1-q^n}{n(1-q)}\right)(\overline{\theta} - \overline{b}^*)$$

Plugging the respective payoffs from (12) and (14) in, we obtain:

$$\frac{\delta}{n}\frac{(1-\delta)q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)} \geq (1-\delta)\left(1-\frac{1-q^n}{n(1-q)}\right)\frac{q^n(1-\delta(1-q))(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)},$$

which simplifies to:

$$\delta \geq \frac{1}{1-q}-\frac{1-q^n}{n(1-q)^2}. \tag{17}$$

The condition on δ identified in (17) can only be satisfied if the right-hand side of this inequality is strictly below 1, which is only true whenever:

$$q < \frac{1-q^n}{n(1-q)},$$

which gives is satisfied in Case 1 by assumption.

No-collusion constraints. We check that the no-collusion constraints (No-col-sep-2) and (No-col-pool) are satisfied, or, in other words, that in the corresponding buyer-game pooling at $\underline{b}^*$ and bidding $(\underline{b}^*+\epsilon, \underline{b}^*)$ does not improve the buyers' payoff. If the buyers decide to bid $(\underline{b}^*+\epsilon, \underline{b}^*)$ in the buyer-game, their payoff will be:

$$\begin{aligned} v(\underline{b}^*+\epsilon, \underline{b}^*) &= \frac{(1-\delta)\big[(1-q^n)(\overline{\theta}-\underline{b}^*)+q^n(\underline{\theta}-\underline{b}^*)\big]}{n(1-\delta q^n)} \\ &= \frac{(1-\delta)q^n(1-q^n)\big(1-\delta(1-q)^n-\delta q\big)(\overline{\theta}-\underline{\theta})}{n(1-\delta q^n)\big(\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)\big)}. \end{aligned}$$

We must make sure that that $v^*_{\text{fse}} \geq v(\underline{b}^*+\epsilon, \underline{b}^*)$, which is equivalent to:

$$1 \geq \frac{1-\delta(1-q)^n-\delta q}{1-\delta q^n} \Leftrightarrow (1-q)^n \geq -q+q^n,$$

which is true since the right-hand side of $(1-q)^n \geq -q+q^n$ is strictly negative, and the left-hand side is strictly positive.

If the buyers coordinate on pooling at $\underline{b}^*$ in the buyer-game, they will obtain:

$$\begin{aligned} v(\underline{b}^*, \underline{b}^*) =& \frac{1}{n}\big[(1-q)(\overline{\theta}-\underline{b}^*)+q(\underline{\theta}-\underline{b}^*)\big] = \\ &= \frac{\big((1-q)q^n\big(1-\delta(1-q)^n\big)-\delta q^2(1-q^n)\big)(\overline{\theta}-\underline{\theta})}{n\big(\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)\big)}. \end{aligned} \tag{18}$$

Consider the numerator of (18) in the limit as δ goes to 1:

$$
\begin{aligned}
&(1-q)q^n\big(1-(1-q)^n\big) - q^2(1-q^n) \\
&= (1-q)\left[q^n\big(1-(1-q)^n\big) - q^2\sum_{k=0}^{n-1} q^k\right] \\
&= (1-q)\left[-q^n(1-q)^n - q^2\sum_{k=0}^{n-3} q^k - q^{n+1}\right] < 0
\end{aligned}
$$

Hence the payoff from pooling at $\underline{b}^*$ in (18) is strictly negative for all δ sufficiently close to 1, and therefore cannot exceed v^*_{fse} for δ around 1. □

Case 1: the restriction on the parameters

The full surplus extraction equilibrium of Proposition 6 can only be sustained if $q < \frac{1-q^n}{n(1-q)}$. It is easy to check that this condition can be satisfied for any q as long as $n = 2$ or $n = 3$, but only for some q if $n \geq 4$. Indeed consider $n = 2$ first:

(I) $n = 2$. In this case the condition becomes:

$$
2q < \frac{1-q^2}{1-q} \Leftrightarrow 2q < 1+q \Leftrightarrow q < 1,
$$

which is obviously true.

(II) $n = 3$. In this case the condition becomes:

$$
3q < \frac{1-q^3}{1-q} \Leftrightarrow 3q < 1+q+q^2 \Leftrightarrow 0 < 1-2q+q^2 \Leftrightarrow 0 < (1-q)^2,
$$

which is also obviously true for any $q \in (0,1)$.

(III) $n = 4$ In this case the condition becomes:

$$
\begin{aligned}
4q < \frac{1-q^4}{1-q} &\Leftrightarrow 4q < 1+q+q^2+q^3 \Leftrightarrow 0 < 1-3q+q^2+q^3 \\
&\Leftrightarrow 0 < (1-q)(-q^2-2q+1) \Leftrightarrow 0 < -q^2-2q+1,
\end{aligned}
$$

which is only true for $q \in (0, -1+\sqrt{2})$.

It it however possible to establish that for any number of players n there will be some values of q falling into Case 1:

Proposition 7. *The equation $1 - q^n = nq(1 - q)$ has a unique solution q^* on $(0, 1)$ for any $n \geq 4$. Moreover for all $q < q^*$ it is true that $q < \frac{1-q^n}{n(1-q)}$ and vice versa.*

Proof. See Appendix H.1. □

The above proposition essentially shows that for every $n \geq 4$ the restriction divides the interval $(0, 1)$ into two parts. In the left part of the segment one will find the values of q that fall into Case 1, and in the right part of the segment one will find the values of q that fall into Cases 2 and 3. Figure 5 provides an illustration and also suggests that, as n goes to infinity, lower and lower values of q fall into Case 1 until there are none left in the limit. Indeed, it is easy to see that

$$\lim_{n \to \infty} nq(1 - q) - (1 - q^n) = +\infty,$$

implying that, for any fixed value of q, the parameter restriction does not hold for all sufficiently high n.

Case 2: Medium expected valuation

Recall that in Case 2, the no-collusion constraint (No-col-sep-1) and the upward incentive compatibility constraint of a high-type buyer (HighIC-up) bind at the optimum of the revenue maximization problem $\mathcal{RM}$.

Full surplus extraction cPPE, Case 2.

- *Equilibrium conditions:*

$$\text{(No-col-sep-1)} \qquad v^*_{\text{fse}} = \frac{(1-\delta)(1-q^n)(\overline{\theta} - \underline{b}^*)}{n(1-\delta(1-q)^n)},$$

$$\text{(HighIC-up)} \qquad (1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta} - \overline{b}^*) + \delta v^*_{\text{fse}} = (1-\delta)(\overline{\theta} - \overline{b}^*),$$

$$\text{(Eq-payoff)} \qquad v^*_{\text{fse}} = \frac{1}{n}\big[(1-q^n)(\overline{\theta} - \overline{b}^*) + q^n(\underline{\theta} - \underline{b}^*)\big].$$

- *Parameter restrictions:*

$$q \geq \frac{1-q^n}{n(1-q)},$$

$$(1-q^n)(1-q) > q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q) - (1-q^n)\big].$$

The complete solution to the system of equilibrium conditions is provided in Appendix E.2. I will derive the restrictions on the parameters in the course of the proof of Proposition 8. First, define $D(\delta)$ as:

$$D(\delta) = q^n\big(1-\delta(1-q)^n\big)\big[n(1-q) - (1-q^n)\big] + (1-q^n)\big[(1-q^n)(1-\delta q) - n(1-\delta)(1-q)\big].$$

The payoff of a low-type buyer who wins by bidding $\underline{b}^*$ is given by:

$$\underline{\theta} - \underline{b}^* = -\frac{1}{D(\delta)}\big[(1-q^n)(1-\delta q) - n(1-\delta)(1-q)\big](1-q^n)(\overline{\theta} - \underline{\theta}). \tag{19}$$

The payoff of a high type buyer who wins by bidding $\overline{b}^*$ is given by:

$$\overline{\theta} - \overline{b}^* = \frac{1}{D(\delta)}\delta q^n(1-q^n)(1-q)(\overline{\theta} - \underline{\theta}), \tag{20}$$

and the payoff of a high type buyer who wins by bidding $\underline{b}^*$ is given by:

$$\overline{\theta} - \underline{b}^* = \frac{1}{D(\delta)}q^n\big(1-\delta(1-q)^n\big)\big[n(1-q) - (1-q^n)\big](\overline{\theta} - \underline{\theta}). \tag{21}$$

The resulting *ex ante* equilibrium payoff of the buyers is:

$$v^*_{\text{fse}} = \frac{1-\delta}{nD(\delta)}q^n(1-q^n)\big[n(1-q) - (1-q^n)\big](\overline{\theta} - \underline{\theta}). \tag{22}$$

Note that as δ goes to 1, $D(\delta)$ goes to:

$$D(1) = q^n\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big] + (1-q^n)(1-q^n)(1-q) > 0,$$

hence we can conclude that $D(\delta)$ is strictly positive for all δ sufficiently close to 1[6].

The equilibrium bids of each type can be computed from the payoffs in (19) and (20):

$$\underline{b}^* = \underline{\theta} + \frac{1}{D(\delta)}\big[(1-q^n)(1-\delta q) - n(1-\delta)(1-q)\big](1-q^n)(\overline{\theta}-\underline{\theta}), \quad (23)$$

$$\overline{b}^* = \overline{\theta} - \frac{1}{D(\delta)}\delta q^n(1-q^n)(1-q)(\overline{\theta}-\underline{\theta}). \quad (24)$$

I first show that the equilibrium bids satisfy the condition of Lemma 2.

Lemma 5. *Suppose $q \geq \frac{1-q^n}{n(1-q)}$, and δ is sufficiently close to 1, then $\underline{\theta} < \underline{b}^* < \overline{b}^*$.*

Proof. (i) To see that $\underline{\theta} < \underline{b}^*$ for sufficiently high δ, observe that

$$\underline{\theta} - \underline{b}^* \xrightarrow[\delta\to 1]{} -\frac{1}{D(1)}(1-q^n)(1-q)(1-q^n)(\overline{\theta}-\underline{\theta}) < 0.$$

(ii) The proof of $\underline{b}^* < \overline{b}^*$ is provided in Appendix F.

□

I now proceed to establish that the bidding profile $(\overline{b}^*, \underline{b}^*)$ can indeed be played along the equilibrium path of a *collusive public perfect equilibrium* of the repeated auction game:

Proposition 8. *Suppose that $q \geq \frac{1-q^n}{n(1-q)}$ and $(1-q^n)(1-q) > q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q) - (1-q^n)\big]$. Suppose further that $\overline{b}^*$ and $\underline{b}^*$ are defined by (23) and (24) respectively, then there exists a critical discount factor δ^*, such that for all $\delta \in [\delta^*, 1)$ the high-revenue strategy profile corresponding to $(\overline{b}^*, \underline{b}^*)$ (as defined by 11) is a collusive public perfect equilibrium of the repeated auction game in the sense of Definition 7. Moreover, the seller achieves full surplus extraction in the limit as δ goes to 1.*

[6] More precisely, for all δ satisfying

$$\delta > \frac{(1-2q^n)\big(n(1-q)-(1-q^n)\big)}{\big(1-q^n\big)^2\big(n(1-q)-q(1-q^{n-1})\big)}$$

Proof sketch. The complete proof is provided in Appendix G.2. As in the previous case, I only provide a sketch of the main argument in the main text. By Lemma 2 and Lemma 5, it is enough to check that $\mathcal{R}^*_{\text{fse}} \geq (1-q^n)\overline{\theta}$ and that the remaining constraints in the revenue maximization problem $\mathcal{RM}$ are satisfied at $(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}})$ for high enough δ. I start with the seller's revenue.

Seller's revenue. The seller's revenue is equal to the full surplus net of the equilibrium payoff of the buyers::

$$\mathcal{R}^*_{\text{fse}} = (1-q^n)\overline{\theta} + q^n\underline{\theta} - nv^*_{\text{fse}}.$$

nv^*_{fse} is equal to:

$$nv^*_{\text{fse}} = \frac{1-\delta}{D(\delta)} q^n(1-q^n)\big[n(1-q) - (1-q^n)\big](\overline{\theta} - \underline{\theta}),$$

which goes to zero in the limit as δ goes to 1 (recall that $D(\delta)$ converges to a strictly positive number). Thus $\mathcal{R}^*_{\text{fse}} \approx (1-q^n)\overline{\theta} + q^n\underline{\theta}$ for δ close enough to 1, which clearly exceeds $(1-q^n)\overline{\theta}$.

Incentive constraints. Since we have relaxed the low type's incentive compatibility constraint (LowIC), we must now make sure that this constraint is satisfied in the relevant parameter region. Recall that a low-type buyer must be willing to participate in the bidding with the bid $\underline{b}^*$ as opposed to abstaining and getting a zero payoff:

$$(1-\delta)\frac{q^{n-1}}{n}(\underline{\theta} - \underline{b}^*) + \delta v^*_{\text{fse}} \geq 0.$$

Plugging the payoffs defined in (19) and (22) into the above constraint, I obtain:

$$-(1-\delta)\frac{q^{n-1}}{n}\frac{1}{D(\delta)}\big[(1-q^n)(1-\delta q) - n(1-\delta)(1-q)\big](1-q^n)(\overline{\theta} - \underline{\theta}) \\ + \delta\frac{1-\delta}{nD(\delta)} q^n(1-q^n)\big[n(1-q) - (1-q^n)\big](\overline{\theta} - \underline{\theta}) \geq 0,$$

which simplifies to:

$$\delta \leq \frac{1}{1-q} - \frac{1-q^n}{n(1-q)^2},$$

which is true since $\frac{1}{1-q} - \frac{1-q^n}{n(1-q)^2} \geq 1$ by assumption that $q \geq \frac{1-q^n}{n(1-q)}$.

The remaining incentive constraints in $\mathcal{RM}$ are all non-binding at $(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}})$ for high values of δ and all values of q and n, except for the constraint associated with a downward deviation of a high-type buyer (HighIC-down). Recall that a high-type buyer could deviate to $\underline{b}^* + \epsilon$ and win whenever all of his competitors are low types. For this deviation to be unprofitable, his payoff must satisfy:

$$(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta} - \overline{b}^*) + \delta v^*_{\text{fse}} \geq (1-\delta)q^{n-1}(\overline{\theta} - \underline{b}^*).$$

Plugging the payoffs defined in , (20), (21), and (22) into the above inequality, I obtain:

$$\begin{aligned}
&(1-\delta)\frac{1-q^n}{n(1-q)}\frac{1}{D(\delta)}\delta q^n(1-q^n)(1-q)(\overline{\theta} - \underline{\theta}) \\
&\quad + \delta\frac{1-\delta}{nD(\delta)}q^n(1-q^n)\big[n(1-q) - (1-q^n)\big](\overline{\theta} - \underline{\theta}) \\
&\quad \geq (1-\delta)q^{n-1}\frac{1}{D(\delta)}q^n\big(1-\delta(1-q)^n\big)\big[n(1-q) - (1-q^n)\big](\overline{\theta} - \underline{\theta}),
\end{aligned}$$

which simplifies to:

$$\delta(1-q^n)(1-q) \geq q^{n-1}\big(1-\delta(1-q)^n\big)\big[n(1-q) - (1-q^n)\big],$$

which can only be satisfied when for δ high enough:

$$(1-q^n)(1-q) > q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q) - (1-q^n)\big],$$

which is the second parameter restriction of Case 2.

No collusion constraints. I check that the no-collusion constraints (No-col-pool) and (No-col-sep-2) are satisfied, or, equivalently, that pooling at $\underline{b}^*$ or bidding $(\underline{b}^* + \epsilon, \underline{b}^*)$ cannot help the buyers to improve their payoff in the buyer-game induced by the seller's equilibrium strategy. Suppose first that the buyers attempt to bid according to $(\underline{b}^* + \epsilon, \underline{b}^*)$, then their payoff will be equal to:

$$\begin{aligned}
v(\underline{b}^* + \epsilon, \underline{b}^*) &= \frac{(1-\delta)\big[(1-q^n)(\overline{\theta} - \underline{b}^*) + q^n(\underline{\theta} - \underline{b}^*)\big]}{n(1-\delta q^n)} \\
&= \frac{(1-\delta)q^n(1-q^n)(\overline{\theta} - \underline{\theta})}{n(1-\delta q^n)D(\delta)} \times \\
&\quad \times \bigg(\big(1-\delta(1-q)^n\big)\big[n(1-q) - (1-q^n)\big] - \big[(1-q^n)(1-\delta q) - n(1-\delta)(1-q)\big]\bigg),
\end{aligned}$$

which is exceeded by v^*_{fse} for δ sufficiently close to 1 as long as q and n satisfy the following inequality:

$$(1-q^n)(1-q) > \big(q^n - (1-q)^n\big)\big[n(1-q) - (1-q^n)\big],$$

which is implied by the the second parameter restriction of Case 2 (also given below in (25)) since $q^n - (1-q)^n < q^{n-1}\big(1-(1-q)^n\big)$.

If the buyers try to coordinate on pooling at $\underline{b}^*$, their payoff will be:

$$\begin{aligned} v(\underline{b}^*, \underline{b}^*) =& \frac{1}{n}\big[(1-q)(\overline{\theta} - \underline{b}^*) + q(\underline{\theta} - \underline{b}^*)\big] \\ =& \frac{1}{n}\Bigg[(1-q)\frac{1}{D(\delta)}q^n\big(1-\delta(1-q)^n\big)\big[n(1-q)-(1-q^n)\big](\overline{\theta}-\underline{\theta}) \\ &- q\frac{1}{D(\delta)}\big[(1-q^n)(1-\delta q) - n(1-\delta)(1-q)\big](1-q^n)(\overline{\theta}-\underline{\theta})\Bigg], \end{aligned}$$

$v(\underline{b}^*+\epsilon, \underline{b}^*)$ converges to a strictly negative number as δ goes to 1. Indeed, in the limit $v(\underline{b}^*+\epsilon, \underline{b}^*)$ is given by:

$$\frac{(1-q)(\overline{\theta}-\underline{\theta})}{nD(1)}\big[q^n\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big] - q(1-q^n)(1-q^n)\big],$$

which is strictly negative since $(1-q^n)(1-q^n) > q^{n-1}\big[n(1-q)-(1-q^n)\big]$ (see Appendix G.2 for the proof of this claim).

□

Case 2: the restrictions on the parameters

Consider the second parameter restriction of Case 2:

$$(1-q^n)(1-q) > q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]. \tag{25}$$

The pairs of q and n satisfying this restriction (together with the restriction $q \geq \frac{1-q^n}{n(1-q)}$) are illustrated by Figure 5. In the following proposition I establish that the set of q satisfying (25) is non-empty for any $n \geq 4$ and that there are values q that do not satisfy (25) for every $n \geq 4$.

Proposition 9. *The equation*

$$(1-q^n)(1-q) = q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]$$

has a solution on $q \in (0,1)$ *for every* $n \geq 4$.

Proof. See Appendix H.2. □

Observe that the range of q expands as n increases. In the next proposition I establish that any $q \in (0,1)$ will satisfy condition (25) for all sufficiently high values of n:

Proposition 10. *For all* $q \in (0,1)$

$$\lim_{n\to\infty}\Big((1-q^n)(1-q) - q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]\Big) = 1-q > 0.$$

Proof. See Appendix H.3. □

Figure 5 also suggests that the restriction in (25) can be satisfied for all $q \leq \frac{1}{2}$. Indeed, this claim can be shown formally:

Proposition 11. *For all* $q \in (0, \frac{1}{2}]$ *it is true that*

$$(1-q^n)(1-q) > q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big].$$

Proof. See Appendix H.4. □

Case 3: Low expected valuation

Recall that in Case 3, both of the incentive compatibility constraints of a high type buyer, i.e. (HighIC-up) and (HighIC-down), bind at the optimum of the revenue maximization problem $\mathcal{RM}$.

Full surplus extraction cPPE, Case 3.

- *Equilibrium conditions:*

$$\text{(HighIC-up)} \qquad (1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*)+\delta v^*_{\text{fse}} = (1-\delta)(\overline{\theta}-\overline{b}^*),$$

$$\text{(HighIC-down)} \qquad (1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*)+\delta v^*_{\text{fse}} = (1-\delta)q^{n-1}(\overline{\theta}-\underline{b}^*),$$

$$\text{(Eq-payoff)} \qquad v^*_{\text{fse}} = \frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\underline{\theta}-\underline{b}^*)\big].$$

- *Parameter restriction*

$$(1-q^n)(1-q) \le q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big].$$

The full solution to the system of equilibrium conditions is provided in Appendix E.3. Here I present the equilibrium bids and equilibrium payoffs. I will derive the restriction on the parameters in the course of the proof of Proposition 12. Observe that the restriction on the parameters has the following implication:

Lemma 6. *For any $q \in (0,1)$ and $n \ge 2$*

$$(1-q^n)(1-q) \le q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big] \Rightarrow q \ge \frac{1-q^n}{n(1-q)}.$$

Proof. See Appendix H.5 □

To write down the expressions for equilibrium bids and payoffs, define $D(\delta)$ as:

$$D(\delta) = (1-q^n)(1-\delta q)+\delta q(1-q)-n(1-\delta)(1-q).$$

A low-type buyer, who wins the auction with the low equilibrium bid $\underline{b}^*$, gets:

$$\underline{\theta}-\underline{b}^* = -\frac{1}{D(\delta)}\big[(1-q^n)(1-\delta q)-n(1-\delta)(1-q)\big](\overline{\theta}-\underline{\theta}). \tag{26}$$

A high-type buyer, who wins the auction with the high equilibrium bid $\overline{b}^*$, gets:

$$\overline{\theta}-\overline{b}^* = \frac{1}{D(\delta)}\delta q^n(1-q)(\overline{\theta}-\underline{\theta}), \tag{27}$$

and a high-type buyer, who wins the auction with the low equilibrium bid $\underline{b}^*$, gets:

$$\overline{\theta} - \underline{b}^* = \frac{1}{D(\delta)} \delta q(1-q)(\overline{\theta} - \underline{\theta}). \tag{28}$$

The resulting *ex ante* equilibrium payoff of the buyers is given by:

$$v^*_{\text{fse}} = \frac{1}{nD(\delta)}(1-\delta)q^n \big[n(1-q) - (1-q^n)\big](\overline{\theta} - \underline{\theta}). \tag{29}$$

Note that as δ goes to 1, $D(\delta)$ goes to:

$$D(1) = (1-q^n)(1-q) + q(1-q) > 1,$$

hence $D(\delta)$ is strictly positive for δ sufficiently close to 1[7] by continuity of $D(\cdot)$.

The equilibrium bids of each type can be immediately obtained from the respective payoffs in (26) and (27):

$$\underline{b}^* = \underline{\theta} + \frac{1}{D(\delta)}\big[(1-q^n)(1-\delta q) - n(1-\delta)(1-q)\big](\overline{\theta} - \underline{\theta}) \tag{30}$$

$$\overline{b}^* = \overline{\theta} - \frac{1}{D(\delta)} \delta q^n (1-q)(\overline{\theta} - \underline{\theta}) \tag{31}$$

As in the previous two cases, I first establish the following lemma:

Lemma 7. *Suppose δ is sufficiently close to 1, then $\underline{\theta} < \underline{b}^* < \overline{b}^*$.*

Proof. (i) To see that $\underline{\theta} < \underline{b}^*$ for sufficiently high values of δ, observe that:

$$\underline{\theta} - \underline{b}^* \xrightarrow[\delta \to 1]{} -\frac{1}{D(1)}(1-q^n)(1-q)(\overline{\theta} - \underline{\theta}) < 0.$$

(ii) $\underline{b}^* < \overline{b}^*$ is equivalent to $\overline{\theta} - \underline{b}^* > \overline{\theta} - \overline{b}^*$ which is equivalent to:

$$\frac{1}{D(\delta)} \delta q(1-q)(\overline{\theta} - \underline{\theta}) > \frac{1}{D(\delta)} \delta q^n (1-q)(\overline{\theta} - \underline{\theta}),$$

which is clearly true since $D(\delta) > 0$ for δ high enough, and $q > q^n$ for all $n \geq 2$ and $q \in (0, 1)$.

□

[7] For values of δ satisfying

$$\delta > \frac{n(1-q) - (1-q^n)}{n(1-q) - q^2(1-q^{n-1})}.$$

I now show that the bidding profile in (30) and (31) can be supported in a *collusive public perfect equilibrium* of the repeated auction game:

Proposition 12. *Suppose that* $(1-q^n)(1-q) \leq q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]$. *Suppose further that* $\underline{b}^*$ *and* $\overline{b}^*$ *are defined by (30) and (31) respectively, then there exists a critical discount factor* δ^*, *such that for all* $\delta \in [\delta^*, 1)$ *the high-revenue strategy profile corresponding to* $(\overline{b}^*, \underline{b}^*)$ *(as defined by 11) is a collusive public perfect equilibrium of the repeated auction game in the sense of Definition 7. Moreover, the seller achieves full surplus extraction in the limit as* δ *goes to* 1.

Proof sketch. The complete proof is provided in Appendix G.3, I briefly sketch the main arguments here. Just as in the previous two cases, by Lemma 2 and Lemma 7, it is enough to check that $\mathcal{R}^*_{\text{fse}} \geq (1-q^n)\overline{\theta}$ and that the remaining constraints in the revenue maximization problem $\mathcal{RM}$ are satisfied at $(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}})$ for high values of δ. I start with the seller's revenue.

Seller's revenue. The seller's revenue is given by:

$$\mathcal{R}^*_{\text{fse}} = (1-q^n)\overline{\theta} + q^n\underline{\theta} - nv^*_{\text{fse}}.$$

nv^*_{fse} is given by:

$$nv^*_{\text{fse}} = \frac{1}{D(\delta)}(1-\delta)q^n\big[n(1-q)-(1-q^n)\big](\overline{\theta}-\underline{\theta}).$$

Observe that $\lim_{\delta\to 1} nv^*_{\text{fse}} = 0$, which means $\mathcal{R}^*_{\text{fse}} \approx (1-q^n)\overline{\theta} + q^n\underline{\theta}$ for δ close enough to 1, which clearly exceeds $(1-q^n)\overline{\theta}$.

Incentive constraints. As in Cases 1 and 2, the on-schedule incentive compatibiliy constraint (HighIC-on-sch) is satisfied. The two off-schedule incentive compatibility constraints (HighIC-up) and (HighIC-down) are satisfied by construction. Hence it remains to check that the low-type incentive compatibility constraint (LowIC) is satisfied. Recall that (LowIC), evaluated at $(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}})$, is given by:

$$(1-\delta)\frac{q^{n-1}}{n}(\underline{\theta}-\underline{b}^*) + \delta v^*_{\text{fse}} \geq 0.$$

Plugging the payoffs from (26) and (29) in, I get:

$$-(1-\delta)\frac{q^{n-1}}{n}\frac{1}{D(\delta)}\big[(1-q^n)(1-\delta q)-n(1-\delta)(1-q)\big](\overline{\theta}-\underline{\theta})$$
$$+\delta\frac{1}{nD(\delta)}(1-\delta)q^n\big[n(1-q)-(1-q^n)\big](\overline{\theta}-\underline{\theta})\geq 0,$$

which is equivalent to:

$$\delta\leq\frac{1}{1-q}-\frac{1-q^n}{n(1-q)^2},$$

which is true whenever $\frac{1}{1-q}-\frac{1-q^n}{n(1-q)^2}\geq 1$ or $q\geq\frac{1-q^n}{n(1-q)}$, which is in turn true in this case by Lemma 6.

No-collusion constraints. Recall that in Cases 1 and 2 the no-collusion constraint (No-col-sep-1) was binding and thus the joint deviation to bidding $(\underline{b}^*,\emptyset)$ could not benefit the buyers by construction. Since we have relaxed (No-col-sep-1) here in Case 3, we must now make sure that it is satisfied at $(\overline{b}^*,\underline{b}^*,v^*_{\text{fse}})$. Recall that the payoff from bidding $(\underline{b}^*,\emptyset)$ is given by

$$v(\underline{b}^*,\emptyset)=\frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{b}^*)}{n\big(1-\delta(1-q)^n\big)}=\frac{(1-\delta)(1-q^n)\delta q(1-q)(\overline{\theta}-\underline{\theta})}{nD(\delta)\big(1-\delta(1-q)^n\big)}.$$

The equilibrium payoff v^*_{fse} exceeds $v(\underline{b}^*,\emptyset)$ as long as

$$q^{n-1}\big[n(1-q)-(1-q^n)\big]\geq\frac{(1-q^n)\delta(1-q)}{\big(1-\delta(1-q)^n\big)}$$
$$\Leftrightarrow\big(1-\delta(1-q)^n\big)q^{n-1}\big[n(1-q)-(1-q^n)\big]\geq\delta(1-q^n)(1-q),$$

which can be satisfied for any $\delta\in(0,1)$ as long as q and n satisfy

$$\big(1-(1-q)^n\big)q^{n-1}\big[n(1-q)-(1-q^n)\big]\geq(1-q^n)(1-q).$$

which is true by assumption.

Just as in Cases 1 and 2, we must check whether the constraints (No-col-pool) and (No-col-sep-2) are satisfied at $(\overline{b}^*,\underline{b}^*,v^*_{\text{fse}})$, or, equivalently, whether the buyers would lose from pooling at $\underline{b}^*$ or bidding $(\underline{b}^*+\epsilon,\underline{b}^*)$ whenever the state is w^l in the buyer-game. Suppose the buyers coordinate on bidding $(\underline{b}^* + \epsilon, \underline{b}^*)$, then their payoff is:

$$\begin{aligned} v(\underline{b}^* + \epsilon, \underline{b}^*) &= \frac{(1-\delta)\big[(1-q^n)(\overline{\theta} - \underline{b}^*) + q^n(\underline{\theta} - \underline{b}^*)\big]}{n\big(1-\delta q^n\big)} \\ &= \frac{(1-\delta)(\overline{\theta} - \underline{\theta})}{nD(\delta)\big(1-\delta q^n\big)}\Bigg[(1-q^n)\delta q(1-q) - q^n\big[(1-q^n)(1-\delta q) - n(1-\delta)(1-q)\big]\Bigg]. \end{aligned}$$

We must show that $v^*_{\text{fse}} \geq v(\underline{b}^* + \epsilon, \underline{b}^*)$ for sufficiently high values of δ, i.e. that

$$\begin{aligned} & q^n\big[n(1-q) - (1-q^n)\big] \\ & \qquad \geq \frac{1}{\big(1-\delta q^n\big)}\Bigg[(1-q^n)\delta q(1-q) - q^n\big[(1-q^n)(1-\delta q) - n(1-\delta)(1-q)\big]\Bigg], \end{aligned}$$

which holds for δ sufficiently close to 1 whenever it holds as a strict inequality at $\delta = 1$, i.e. whenever

$$\begin{aligned} & q^n\big[n(1-q) - (1-q^n)\big] > \frac{1}{\big(1-q^n\big)}\big[(1-q^n)q(1-q) - q^n(1-q^n)(1-q)\big] \\ \Leftrightarrow\ & q^{n-1}\big[n(1-q) - (1-q^n)\big] > (1-q)(1-q^{n-1}). \end{aligned}$$

Now the last line is true since:

$$(1-q)(1-q^{n-1}) < (1-q)(1-q^n) \leq q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q) - (1-q^n)\big],$$

where the strict is inequality is obviously true, and the weak inequality holds true in Case 3 by assumption.

If the buyers attempt to coordinate on pooling at $\underline{b}^*$ instead, then their payoff will become:

$$\begin{aligned} v(\underline{b}^*, \underline{b}^*) &= \frac{1}{n}\big[(1-q)(\overline{\theta} - \underline{b}^*) + q(\underline{\theta} - \underline{b}^*)\big] \\ &= \frac{\overline{\theta} - \underline{\theta}}{nD(\delta)}\Bigg[(1-q)\delta q(1-q) - q\big[(1-q^n)(1-\delta q) - n(1-\delta)(1-q)\big]\Bigg]. \end{aligned}$$

As in Cases 1 and 2, I show that $v(\underline{b}^*, \underline{b}^*)$ converges to a strictly negative number as δ goes to 1. In the limit the payoff from pooling at $\underline{b}^*$ is:

$$\frac{\overline{\theta} - \underline{\theta}}{nD(1)}\big[(1-q)q(1-q) - q(1-q^n)(1-q)\big] = \frac{q(1-q)(\overline{\theta} - \underline{\theta})}{nD(1)}\big[q^n - q\big] < 0.$$

Since v^*_{fse} is weakly positive, the payoff from pooling at $\underline{b}^*$ cannot exceed the equilibrium payoff in Case 3 for values of δ sufficiently close to 1. □

Case 3: the restriction on the parameters

The range of parameters, where Case 3 applies, equilibrium construction is defined by the following inequality:

$$(1-q^n)(1-q) \leq q^{n-1}\Big(1-(1-q)^n\Big)\left[n(1-q)-(1-q^n)\right]$$

The pairs of q and n satisfying the above inequality are illustrated by Figure 5. Recall that in Lemma 6 we have established that this parameter restriction implies $q \geq \frac{1-q^n}{n(1-q)}$. Recall also that $q \geq \frac{1-q^n}{n(1-q)}$ implies that $n \geq 4$ because it cannot be satisfied for any q as long as $n = 2$ or $n = 3$. Combined with the result of Proposition 9, it implies that Case 3 applies to some values of q for all $n \geq 4$, and does not apply to any values of q for $n = 2$ or $n = 3$ (see Figure 5 for an illustration).

7 Revenue-maximizing reserve prices

The reserve prices along the equilibrium path of the full-surplus-extracting *collusive public perfect equilibria* (in the limit as δ goes to 1) are given by:

$$r^* = \begin{cases} \underline{\theta} + \frac{q(1-q^n)(\overline{\theta}-\underline{\theta})}{q(1-q^n)+q^n\Big(1-(1-q)^n\Big)} & \text{in Case 1} \\ \underline{\theta} + \frac{\Big[(1-q^n)\Big]^2(1-q)(\overline{\theta}-\underline{\theta})}{q^n\Big(1-(1-q)^n\Big)\Big[n(1-q)-(1-q^n)\Big]+\Big[(1-q^n)\Big]^2(1-q)} & \text{in Case 2} \\ \underline{\theta} + \frac{(1-q^n)(\overline{\theta}-\underline{\theta})}{1-q^n+q} & \text{in Case 3} \end{cases}$$

They are illustrated by Figure 6. The reserve prices in the full-surplus-extracting *cPPE* of the repeated auction game are decreasing in q, going to $\overline{\theta}$ as q goes to 0 and going to $\underline{\theta}$ as q goes to 1. Indeed, since q is the probability of the low type, when q is close to zero, the buyers all have high valuations with a very high probability, and when q is close to 1, the buyers all have low valuations with a very high probability. Recall that the optimal reserve prices in the one-shot auction problem are also decreasing in q, but the optimal decision is essentially a cutoff rule (for

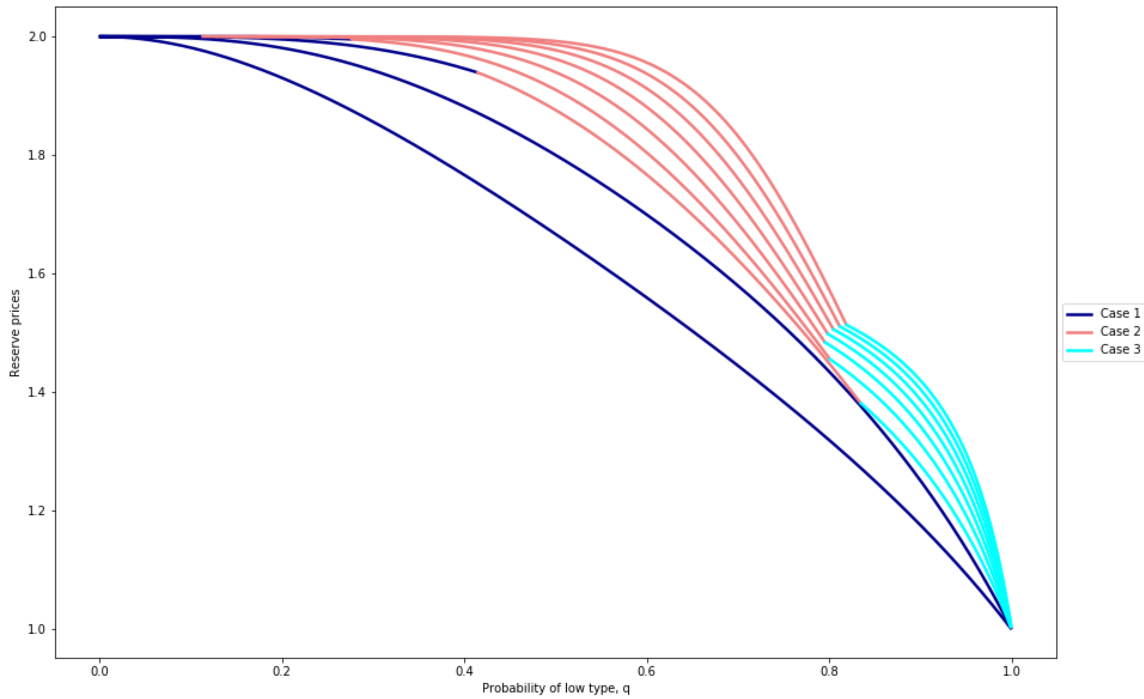


Figure 6: *Reserve prices in the full-surplus-extracting cPPE of the repeated auction game in the limit as* δ *goes to* 1. *Valuations are* $\underline{\theta} = 1$ *and* $\overline{\theta} = 2$. *The curves illustrate the limiting reserve prices for all probabilities of the low type* $q \in (0, 1)$, *and for each* $n \in \{2, ..., 10\}$ *moving from the southwest to the northeast as the number of buyers grows, i.e. the southwesternmost curve illustrates the reserve prices for* $n = 2$, *and the northeasternmost curve illustrates the reserve prices for* $n = 10$. *In the dark-blue, red, and light-blue segments, Cases 1, 2, and 3 apply respectively.*

fixed values of other parameter values): for relatively low values of q the optimal reserve price is $\overline{\theta}$, while for relatively high values of q it is $\underline{\theta}$. Thus, even though the direction of dependence is the same, the functional form of this dependence is much less trivial in the repeated auction setting with collusion. Similarly, the optimal reserve prices in the one-shot auction problem are increasing in the number of buyers, but the dependence takes the form of a cutoff rule (again, when the other parameter values are fixed), where the optimal reserve price is equal to $\underline{\theta}$ when the number of buyers is relatively low, and is equal to $\overline{\theta}$ when it is relatively high. In contrast to the one-shot setting, the reserve prices in the full-surplus-extracting *cPPE*, even though also increasing in n, depend on n in a much less trivial way.

This non-trivial dependence of the reserve prices on q and n can to a certain extent be explained by their very different role in the repeated setting with collusion. In the one-shot auction problem, the role of the reserve prices is to exclude certain valuation types from participation with the purpose of increasing competition among the remaining types. In the repeated setting with colluding buyers, the full-surplus-extracting *cPPE* is efficient and the reserve prices play two crucial roles. First, in the off-path component of the seller's strategy, the reserve prices are chosen to punish the buyers for deviating from the equilibrium path bidding. Second, and more importantly, the on-path component of the reserve prices makes sure that the buyers pay "upfront" for the continuation of favorable terms of trade and at the same time do not have an incentive to collude on a lower bidding profile, resolving the fundamental conflict between revenue-maximization and fighting collusion.

8 Concluding remarks

In this paper, I have considered a repeated first-price auction model with a non-committed seller who dynamically adjusts reserve prices to fight collusion among buyers. To model the interaction between the seller and the colluding buyers, I have proposed the solution concept of *collusive public perfect equilibrium*. A collusive public perfect equilibrium is a public perfect equilibrium that additionally requires that the buyers be unable to improve their equilibrium payoff in the "buyer-game" induced by the seller's equilibrium strategy. Studying the outcomes as the discount factor goes to 1, I find a collusive public perfect equilibrium which allows the seller to extract the entire surplus from the colluding buyers. This result suggests that the problem of collusion in repeated auctions is perhaps less severe than is commonly understood: it turns out that a sufficiently sophisticated seller can come up with rather effective strategies for fighting collusion, even when she has to publicly disclose all the bids in the end of every period.

The buyers in this paper are assumed to have access to symmetric collusive schemes. Such collusive schemes are particularly simple and thus might require no explicit communication among the buyers in practice, which makes them virtually impossible to detect for an antitrust authority. These hard-to-detect collusive schemes must therefore be addressed as part of the repeated auction design problem itself. My results imply that it can be done quite successfully. It is however well-known (see e.g. Mailath and Samuelson (2006)) that more sophisticated asymmetric collusive schemes might allow the buyers to collude more effectively, especially when they can communicate before the start of each auction. Even though asymmetric collusive schemes are often dealt with by conventional means of antitrust policy, it is worth studying if they could also be addressed via more sophisticated auction design.

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A Solution of the one-shot auction problem

Let us first consider the choices made by the buyers who face a reserve price r. Depending on the reserve price chosen by the seller, there are four possible cases to consider:

Case i: $r \leq \underline{\theta}$

In this case both types of each buyer will be willing to participate in the auction. The low types will bid their own valuation $\underline{\theta}$ and receive the payoff of 0. The high types will randomize on $\big(\underline{\theta}, \overline{b}\big]$ where $\overline{b} = (1 - q^{n-1})\overline{\theta} + q^{n-1}\underline{\theta}$ according to

$$G(b) = \frac{q}{1-q}\left[\left(\frac{\overline{\theta} - \underline{\theta}}{\overline{\theta} - b}\right)^{\frac{1}{n-1}} - 1\right],$$

and will get the payoff of $q^{n-1}(\overline{\theta} - \underline{\theta})$. The *ex ante* equilibrium payoff of the buyers is:

$$v^*_{r \leq \underline{\theta}} = (1-q)q^{n-1}(\overline{\theta} - \underline{\theta}).$$

The seller generates revenue:

$$\mathcal{R}^*_{r \leq \underline{\theta}} = (1 - q^n)\overline{\theta} + q^n\underline{\theta} - nv^*_{r \leq \underline{\theta}} = (1 - q^n)\overline{\theta} + q^n\underline{\theta} - n(1-q)q^{n-1}(\overline{\theta} - \underline{\theta}).$$

Case ii: $\underline{\theta} < r < \overline{\theta}$

In this case only the high types are willing to participate in the first price auction. The high types will randomize on $\big(\underline{\theta}, \overline{b}\big]$ where $\overline{b} = (1 - q^{n-1})\overline{\theta} + q^{n-1}r$ according to

$$G(b) = \frac{q}{1-q}\left[\left(\frac{\overline{\theta} - r}{\overline{\theta} - b}\right)^{\frac{1}{n-1}} - 1\right],$$

and will get the payoff of $q^{n-1}(\overline{\theta} - r)$, which leads to the ex ante equilibrium payoff of:

$$v^*_{\underline{\theta} < r < \overline{\theta}} = (1-q)q^{n-1}(\overline{\theta} - r).$$

The resulting revenue of the seller who chooses a reserve price $r \leq \underline{\theta}$:

$$\mathcal{R}^*_{\underline{\theta} < r < \overline{\theta}} = (1 - q^n)\overline{\theta} - nv^*_{\underline{\theta} < r < \overline{\theta}} = (1 - q^n)\overline{\theta} - n(1-q)q^{n-1}(\overline{\theta} - r).$$

Case iii: $r = \overline{\theta}$

In this case only high types are willing to participate, and they of course have no choice but to bid $\overline{\theta}$ in equilibrium, and the resulting revenue will be:

$$\mathcal{R}^*_{r=\overline{\theta}} = (1-q^n)\overline{\theta}.$$

Case iv: $r > \overline{\theta}$

In this case neither type wants to participate, so every buyer will choose to abstain and the seller will get zero revenue.

Revenue achieved in **Case ii** is clearly inferior to that achieved in **Case iii**, so setting $\underline{\theta} < r < \overline{\theta}$ cannot be part of any subgame-pefect equilibrium of the static auction game. The reserve prices $r \leq \underline{\theta}$ and $r = \overline{\theta}$ could however be optimal for the seller.

Case i: $r \leq \underline{\theta}$

In this case it is clear that both types will participate will be willing to participate. It can be easily shown that there is no Nash equilibrium in pure strategies. It is also immediately clear that the low types will never place a bid higher than their own valuation because winning with such a high bid would lead to a negative payoff. But low types should not place a bid that is lower than their valuation even if they have an opportunity to do so. Suppose low type bidders do place a bid $r < \underline{b} < \underline{\theta}$ in equilirium, then one of them could deviate to $\underline{b} + \epsilon$ and guarantee winning the auction for sure if his competitor is of low type as well, hence there is a profitable deviation.

Suppose $\Phi(b)$ is the unconditional distribution of equilibrium bids for every player. The expected payoff of a bidder with type $\overline{\theta}$ is given by:

$$\Phi^{n-1}(b)(\overline{\theta} - b). \tag{32}$$

Assuming that only low types bid $\underline{\theta}$ we must have $\Phi(\underline{\theta}) = q$ hence by indifference we have:

$$\Phi^{n-1}(b)(\overline{\theta} - b) = q^{n-1}(\overline{\theta} - \underline{\theta}). \tag{33}$$

hence $\Phi(b) = q\left(\frac{\overline{\theta}-\underline{\theta}}{\overline{\theta}-b}\right)^{\frac{1}{n-1}}$. To find the upper bound of the support we solve $q\left(\frac{\overline{\theta}-\underline{\theta}}{\overline{\theta}-\overline{b}}\right)^{\frac{1}{n-1}} = 1$, which leads to $\overline{b} = (1-q^{n-1})\overline{\theta} + q^{n-1}\underline{\theta}$. Hence the high type player randomizes over $\big(\underline{\theta}, \overline{b}]$. Since

$\Phi(b)$ is the unconditional distribution of equilibrium bids, the actual mixed strategy of the high type is:

$$G(b) \equiv \Phi(b|\theta_i = \overline{\theta}) = \frac{q}{1-q}\left[\left(\frac{\overline{\theta}-\underline{\theta}}{\overline{\theta}-b}\right)^{\frac{1}{n-1}} - 1\right]. \tag{34}$$

The above analysis naturally leads to the following lemma:

Lemma 8. *If $r \leq \underline{\theta}$,*

(i) the low type bids his own valuation in equilibrium: $\underline{b} = \underline{\theta}$,

(ii) the high type randomizes his bids on $\left(\underline{\theta}, (1-q^{n-1})\overline{\theta} + q^{n-1}\underline{\theta}\right]$ according to

$$G(b) = \frac{q}{1-q}\left[\left(\frac{\overline{\theta}-\underline{\theta}}{\overline{\theta}-b}\right)^{\frac{1}{n-1}} - 1\right].$$

The low type expected equilibrium payoff is 0, the high type expected equilibrium payoff is $q^{n-1}(\overline{\theta}-\underline{\theta})$, which leads to the ex ante equilibrium payoff of:

$$v^*_{r\leq\underline{\theta}} = (1-q)q^{n-1}(\overline{\theta}-\underline{\theta}).$$

The equilibrium in Lemma 8 is efficient, hence it leads to the total surplus given by: $(1-q^n)\overline{\theta} + q^n\underline{\theta}$. The resulting revenue of the seller who chooses a reserve price $r \leq \underline{\theta}$:

$$\begin{aligned}\mathcal{R}^*_{r\leq\underline{\theta}} &= (1-q^n)\overline{\theta} + q^n\underline{\theta} - nu^*_i \\ &= (1-q^n)\overline{\theta} + q^n\underline{\theta} - n(1-q)q^{n-1}(\overline{\theta}-\underline{\theta}).\end{aligned} \tag{35}$$

Case ii: $\underline{\theta} < r < \overline{\theta}$

In this case only the high types are willing to participate in the first price auction. It can also be shown that there is no equilibrium in pure strategies. Hence we will be looking for an equilibrium in mixed strategies. Suppose that a high type buyer randomizes his bids according to the distribution function $G(b)$. The payoff of a high type buyer who is bidding b is given by:

$$\begin{aligned}&\left(q^{n-1} + (n-1)(1-q)q^{n-2}G(b) + ... + (1-q)^{n-1}G^{n-1}(b)\right)(\overline{\theta}-b) \\ &= \left(q + (1-q)G(b)\right)^{n-1}(\overline{\theta}-b).\end{aligned} \tag{36}$$

Assuming that r is the lower bound of the support of $G(b)$ and that $G(b)$ has no mass points we get $G(r) = 0$. By indifference we get for every b in the support:

$$\big(q + (1-q)G(b)\big)^{n-1}(\overline{\theta} - b) = \big(q + (1-q)G(r)\big)^{n-1}(\overline{\theta} - r) = q^{n-1}(\overline{\theta} - r), \tag{37}$$

which immediately gives us:

$$G(b) = \frac{q}{1-q}\left[\left(\frac{\overline{\theta} - r}{\overline{\theta} - b}\right)^{\frac{1}{n-1}} - 1\right]. \tag{38}$$

To find the upper bound of the support $\overline{b}$ we solve $\frac{q}{1-q}\big[\big(\frac{\overline{\theta}-r}{\overline{\theta}-\overline{b}}\big)^{\frac{1}{n-1}} - 1\big] = 1$ which leads to $\overline{b} = (1-q^{n-1})\overline{\theta} + q^{n-1}r$. Hence the following lemma:

Lemma 9. *If $\underline{\theta} < r < \overline{\theta}$,*

(i) the low type chooses to abstain from participation $\underline{b} = \emptyset$,

(ii) the high type randomizes his bids on $\big[r, (1-q^{n-1})\overline{\theta} + q^{n-1}r\big]$ according to

$$G(b) = \frac{q}{1-q}\left[\left(\frac{\overline{\theta} - r}{\overline{\theta} - b}\right)^{\frac{1}{n-1}} - 1\right].$$

The low type expected equilibrium payoff is 0, the high type expected equilibrium payoff is $q^{n-1}(\overline{\theta} - r)$, which leads to the ex ante equilibrium payoff of:

$$v^*_{\underline{\theta}<r<\overline{\theta}} = (1-q)q^{n-1}(\overline{\theta} - r).$$

Since only the high types trade with the seller in the equilibrium in Lemma 9, the resulting total surplus is given by: $(1-q^n)\overline{\theta}$. The resulting revenue of the seller who chooses a reserve price $r \leq \underline{\theta}$:

$$\mathcal{R}^*_{\underline{\theta}<r<\overline{\theta}} = (1-q^n)\overline{\theta} - nu^*_i = (1-q^n)\overline{\theta} - n(1-q)q^{n-1}(\overline{\theta} - r). \tag{39}$$

Case iii: $r = \overline{\theta}$

In this case only high types are willing to participate, and they of course have no choice but to bid $b = \overline{\theta}$ in equilibrium, and the resulting revenue will be:

$$\mathcal{R}^*_{r=\overline{\theta}} = (1-q^n)\overline{\theta}. \tag{40}$$

Case iv: $r > \overline{\theta}$

In this case neither type wants to participate, so every buyer will choose to abstain and the seller will get zero revenue.

B Separating equilibrium payoffs

Suppose that in every period along the equilibrium path a low type buyer bids $\underline{b}$, and a high type buyer bids $\overline{b}$. Then a low type bidder wins with probability $1/n$ only if all his competitors are of low type as well, hence his equilibrium payoff is given by:

$$\frac{q^{n-1}}{n}(\underline{\theta} - \underline{b}). \tag{41}$$

A high type bidder may win in several different cases: whenever $k-1$ of his competitors are also high type buyers, he wins with probability $1/k$, hence his winning probability is equal to:

$$\begin{aligned}
&(1-q)^{n-1}\frac{1}{n} + (n-1)q(1-q)^{n-2}\frac{1}{n-1} + \frac{(n-1)(n-2)}{2}q^2(1-q)^{n-3}\frac{1}{n-2} + ... + q^{n-1}1 \\
&= (1-q)^{n-1}\frac{1}{n} + q(1-q)^{n-2} + \frac{(n-1)}{2}q^2(1-q)^{n-3} + ... + q^{n-1} \\
&= \frac{1}{n}\left[(1-q)^{n-1} + nq(1-q)^{n-2} + \frac{n(n-1)}{2}q^2(1-q)^{n-3} + ... + nq^{n-1}\right] \\
&= \frac{1}{n(1-q)}\left[(1-q)^{n} + nq(1-q)^{n-1} + \frac{n(n-1)}{2}q^2(1-q)^{n-2} + ... + nq^{n-1}(1-q)\right] \\
&= \frac{1}{n(1-q)}\Bigg[\underbrace{(1-q)^{n} + nq(1-q)^{n-1} + \frac{n(n-1)}{2}q^2(1-q)^{n-2} + ... + nq^{n-1}(1-q) + q^n}_{=(1-q+q)^n=1} - q^n\Bigg] \\
&= \frac{1}{n(1-q)}(1-q^n).
\end{aligned}$$

The expected payoff of a high type's buyer then is:

$$\frac{1-q^n}{n(1-q)}(\overline{\theta} - \overline{b}). \tag{42}$$

The resulting *ex ante* equilibrium payoff of each buyer is then

$$\begin{aligned}
v_i &= (1-q)\frac{1-q^n}{n(1-q)}(\overline{\theta} - \overline{b}) + q\frac{q^{n-1}}{n}(\underline{\theta} - \underline{b}) \\
&= \frac{1}{n}\left[(1-q^n)(\overline{\theta} - \overline{b}) + q^n(\underline{\theta} - \underline{b})\right].
\end{aligned} \tag{43}$$

The resulting revenue of the seller is:

$$\mathcal{R}_s = (1-q^n)\overline{b} + q^n\underline{b}. \tag{44}$$

C Proof of Proposition 4

Proof. [8] Note first that both in the low-revenue separating and zero-revenue pooling equilibrium, the buyer-game induced by the seller's equilibrium strategy is the repeated first-price auction game with zero reserve price. Denote $\mathcal{V}$ the set of strongly symmetric public perfect equilibrium payoffs of this buyer-game. Denote $\hat{v} = \sup \mathcal{V}$. We have to distinguish two classes of strongly symmetric public perfect equilibria: (i) equilibria in which a separating bidding profile is played *in the first period*, and (ii) equilibria in which a pooling bidding profile is played *in the first period*.

(i) *A separating bidding profile is played in the first period*

Suppose first that the optimal payoff $\hat{v}$ is achieved by a symmetric public perfect equilibrium in which the buyers separate in the first period. Suppose $b(\cdot)$ is the equilibrium action taken in the first period. Denote $\underline{b}$ and $\overline{b}$ the bids placed in the first period by a low-type buyer and a high-type buyer respectively. Suppose that the equilibrium continuation value after the first period is given by $v^* : \mathbb{R}_+^n \to \mathbb{R}$, then the equilibrium payoff of a high-type buyer i is given by:

$$(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}) + \delta\,\mathbb{E}\left(v^*(\overline{b}, b(\theta_{-i}))\right).$$

The equilibrium payoff of a low-type buyer i is given by:

$$(1-\delta)\frac{q^{n-1}}{n}(\underline{\theta}-\underline{b}) + \delta\,\mathbb{E}\left(v^*(\underline{b}, b(\theta_{-i}))\right).$$

The on-schedule incentive compatibility constraint of a low-type buyer is then given by:

$$(1-\delta)\frac{q^{n-1}}{n}(\underline{\theta}-\underline{b}) + \delta\,\mathbb{E}\left(v^*(\underline{b}, b(\theta_{-i}))\right) \geq (1-\delta)\frac{1-q^n}{n(1-q)}(\underline{\theta}-\overline{b}) + \delta\,\mathbb{E}\left(v^*(\overline{b}, b(\theta_{-i}))\right).$$

Subtract $\delta\hat{v}$ and divide both sides by $(1-\delta)$:

$$\frac{q^{n-1}}{n}(\underline{\theta}-\underline{b}) + \frac{\delta}{1-\delta}\,\mathbb{E}\left(v^*(\underline{b}, b(\theta_{-i})) - \hat{v}\right) \geq \frac{1-q^n}{n(1-q)}(\underline{\theta}-\overline{b}) + \frac{\delta}{1-\delta}\,\mathbb{E}\left(v^*(\overline{b}, b(\theta_{-i})) - \hat{v}\right),$$

[8] See a similar argument in Chapter 11.2 of Mailath and Samuelson (2006) in the context of a repeated price competition game with adverse selection.

and define $\overline{x} \equiv \frac{\delta}{1-\delta}\,\mathbb{E}\left(v^*(\overline{b}, b(\theta_{-i})) - \hat{v}\right)$ and $\underline{x} \equiv \frac{\delta}{1-\delta}\,\mathbb{E}\left(v^*(\underline{b}, b(\theta_{-i})) - \hat{v}\right)$. The incentive compatibility constraint of a low-type buyer can then be written as:

$$\frac{q^{n-1}}{n}(\underline{\theta} - \underline{b}) + \underline{x} \geq \frac{1-q^n}{n(1-q)}(\underline{\theta} - \overline{b}) + \overline{x}. \tag{45}$$

Recall that the continuation payoffs in any strongly symmetric public perfect equilibrium must be strongly symmetric public perfect equilibrium payoffs themselves, hence we must have $\overline{x} \leq 0$ and $\underline{x} \leq 0$ since $\hat{v} = \sup \mathcal{V}$.

The *ex ante* equilibrium payoff is given by:

$$\hat{v} = (1-\delta)\frac{1}{n}\left[(1-q^n)(\overline{\theta} - \overline{b}) + q^n(\underline{\theta} - \underline{b})\right] + (1-q)\delta\,\mathbb{E}\left(v^*(\overline{b}, b(\theta_{-i}))\right) + q\delta\,\mathbb{E}\left(v^*(\underline{b}, b(\theta_{-i}))\right).$$

Subtracting $\delta\hat{v}$ and dividing by $(1-\delta)$ on both sides, we obtain:

$$\frac{\hat{v} - \delta\hat{v}}{1-\delta} = \frac{1}{n}\left[(1-q^n)(\overline{\theta} - \overline{b}) + q^n(\underline{\theta} - \underline{b})\right] + (1-q)\frac{\delta}{1-\delta}\,\mathbb{E}\left(v^*(\overline{b}, b(\theta_{-i})) - \hat{v}\right) + q\frac{\delta}{1-\delta}\,\mathbb{E}\left(v^*(\underline{b}, b(\theta_{-i})) - \hat{v}\right),$$

which can be rewritten as:

$$\hat{v} = \frac{1}{n}\left[(1-q^n)(\overline{\theta} - \overline{b}) + q^n(\underline{\theta} - \underline{b})\right] + (1-q)\overline{x} + q\underline{x}.$$

Combining this expression with the low-type incentive compatibilty constraint in (45) and our observation that $\underline{x}, \overline{x} \leq 0$, we must conclude that[9]:

$$\hat{v} \leq \max_{\underline{b},\overline{b};\underline{x},\overline{x}} \frac{1}{n}\left[(1-q^n)(\overline{\theta} - \overline{b}) + q^n(\underline{\theta} - \underline{b})\right] + (1-q)\overline{x} + q\underline{x} \quad \text{subject to} \tag{46}$$

$$\text{(IC)} \quad \frac{q^{n-1}}{n}(\underline{\theta} - \underline{b}) + \underline{x} \geq \frac{1-q^n}{n(1-q)}(\underline{\theta} - \overline{b}) + \overline{x},$$

$$\text{(Feas)} \quad \underline{x}, \overline{x} \leq 0.$$

Let us consider the maximization problem in (46). Clearly the (IC) constraint must be binding at the optimum: suppose not, i.e. suppose $\frac{q^{n-1}}{n}(\underline{\theta} - \underline{b}) + \underline{x} > \frac{1-q^n}{n(1-q)}(\underline{\theta} - \overline{b}) + \overline{x}$,

[9] The solution to this maximization problem provides an upper bound on strongly symmetric equilibrium payoffs since all the other incentive compatibility constraints are ignored, and the constraint $\underline{x}, \overline{x} \leq 0$ is necessary for feasibility of continuation values but not sufficient.

then choose $\overline{b}' < \overline{b}$ such that the constraint is still satisfied, and this will clearly improve the value of the objective. Hence, at the optimum of (46), we must have

$$\frac{q^{n-1}}{n}(\underline{\theta} - \underline{b}) + \underline{x} = \frac{1 - q^n}{n(1-q)}(\underline{\theta} - \overline{b}) + \overline{x},$$

which we can solve for $(1 - q^n)(\underline{\theta} - \overline{b})$, to obtain:

$$(1 - q^n)(\underline{\theta} - \overline{b}) = (1 - q)q^{n-1}(\underline{\theta} - \underline{b}) + n(1 - q)(\underline{x} - \overline{x}),$$

which then implies:

$$(1 - q^n)(\overline{\theta} - \overline{b}) = (1 - q^n)(\overline{\theta} - \underline{\theta}) + (1 - q)q^{n-1}(\underline{\theta} - \underline{b}) + n(1 - q)(\underline{x} - \overline{x}). \tag{47}$$

Plugging (47) into the objective function in (46), we get:

$$\begin{aligned}
&\frac{1}{n}\left[(1 - q^n)(\overline{\theta} - \underline{\theta}) + (1 - q)q^{n-1}(\underline{\theta} - \underline{b}) + n(1 - q)(\underline{x} - \overline{x}) + q^n(\underline{\theta} - \underline{b})\right] + (1 - q)\overline{x} + q\underline{x} \\
&= \frac{1}{n}(1 - q^n)(\overline{\theta} - \underline{\theta}) + \frac{1}{n}\left[(1 - q)q^{n-1} + q^n\right](\underline{\theta} - \underline{b}) + (1 - q)(\underline{x} - \overline{x}) + (1 - q)\overline{x} + q\underline{x} \\
&= \frac{1}{n}(1 - q^n)(\overline{\theta} - \underline{\theta}) + \frac{1}{n}\left[(1 - q)q^{n-1} + q^n\right](\underline{\theta} - \underline{b}) + \underline{x} \\
&= \frac{1}{n}(1 - q^n)(\overline{\theta} - \underline{\theta}) + \frac{1}{n}q^{n-1}(\underline{\theta} - \underline{b}) + \underline{x},
\end{aligned}$$

which implies that:

$$\hat{v} \leq \max_{\underline{b},\underline{x}} \frac{1}{n}(1 - q^n)(\overline{\theta} - \underline{\theta}) + \frac{1}{n}q^{n-1}(\underline{\theta} - \underline{b}) + \underline{x} \;\text{ subject to } \overline{x} \leq 0.$$

The optimum is clearly achieved when $\underline{b} = 0$ and $\underline{x} = 0$, which means that:

$$\hat{v} \leq \frac{1}{n}(1 - q^n)(\overline{\theta} - \underline{\theta}) + \frac{1}{n}q^{n-1}\underline{\theta} = v^*_{\text{lrs}}.$$

Hence, if the buyers play a separating bidding profile in the first period in an optimal strongly symmetric equilibrium of this buyer-game, then the optimal equilibrium payoff cannot exceed the equilibrium payoff of the low-revenue separating equilibrium.

(ii) *A pooling bidding profile is played in the first period*

Consider now a class of strongly symmetric public perfect equilibria in which the buyers pool in the first period, and denote b the equilibrium action of both types in the first period. Suppose that the optimal payoff $\hat{v}$ is achieved by an equilibrium in this class. Suppose that $v^* : \mathbb{R}_+^n \to \mathbb{R}$ is the equilibrium continuation value after the first period. The *ex ante* equilibrium payoff is given by:

$$\hat{v} = (1-\delta)\frac{1}{n}\big(\,\mathbb{E}(\theta) - b\big) + \delta v^*(b,\ldots,b)$$

Subtracting $\delta\hat{v}$ and dividing by $(1-\delta)$ on both sides, we obtain:

$$\frac{\hat{v} - \delta\hat{v}}{1-\delta} = \frac{1}{n}\big(\,\mathbb{E}(\theta) - b\big) + \frac{\delta}{1-\delta}(v^*(b,\ldots,b) - \hat{v})$$

Denote $x = \frac{\delta}{1-\delta}(v^*(b,\ldots,b) - \hat{v})$ and rewrite the above expression as:

$$\hat{v} = \frac{1}{n}\big(\,\mathbb{E}(\theta) - b\big) + x$$

Since continuation values must be strongly symmetric equilibrium payoffs themselves, we have $x \leq 0$, and therefore:

$$\begin{aligned}\hat{v} &\leq \max_{b,x} \frac{1}{n}\big(\,\mathbb{E}(\theta) - b\big) + x \quad \text{subject to } x \leq 0 \\ &= \frac{1}{n}\,\mathbb{E}(\theta) = v^*_{\text{zrp}}\end{aligned}$$

Hence, if the buyers play a pooling bidding profile in the first period in an optimal strongly symmetric equilibrium of this buyer-game, then the optimal equilibrium payoff cannot exceed the equilibrium payoff of the zero-revenue pooling equilibrium.

We can now conclude that there are only two candidates for the optimal strongly symmetric public perfect equilibrium payoff of the buyer-game: the payoff from the low-revenue separating equilibrium and the payoff from the zero-revenue pooling equilibrium. The result then follows from the analysis in the main text.

□

D Proof of the Monotonicity lemma

Proof. Consider first the high reserve price state ω^h. Clearly in any public perfect equilibrium the payoff in this state must be zero, hence we can without loss of generality assume that $b_{\omega^h}(\overline{\theta}) = \overline{\theta}$ and $b_{\omega^h}(\underline{\theta}) = \emptyset$.

Consider now the low reserve price state ω^l, in which the buyer-game starts. Consider any strongly symmetric public perfect equilibrium of the buyer game. Pick any history that leads to state ω^l and suppose any high-type buyer bids according to $b_{\omega^l}(\overline{\theta}) = \overline{b}$ and any low-type buyer bids according to $b_{\omega^l}(\underline{\theta}) = \underline{b}$ after that history, and the equilibrium continuation value is given by $v^*_{\omega^l}(b) : A^n(\omega^l) \to \mathbb{R}$. The equilibrium payoff of a high-type buyer is given by:

$$(1-\delta)p(\overline{b})(\overline{\theta} - \overline{b}) + \delta\, \mathbb{E}\left(v^*_{\omega^l}(\overline{b}, b_{\omega^l}(\theta_{-i}))\right), \tag{48}$$

where $p(\overline{b})$ is the winning probability from bidding $\overline{b}$ in the current period. Analogously the equilibrium payoff of a low-type buyer i is equal to:

$$(1-\delta)p(\underline{b})(\underline{\theta} - \underline{b}) + \delta\, \mathbb{E}\left(v^*_{\omega^l}(\underline{b}, b_{\omega^l}(\theta_{-i}))\right), \tag{49}$$

where $p(\underline{b})$ is the winning probability from bidding $\underline{b}$ in the current period.

Since the above are assumed to be public perfect equilibrium payoffs, the following incentive compatibility must be satisfied, for a high type buyer:

$$(1-\delta)p(\overline{b})(\overline{\theta} - \overline{b}) + \delta\, \mathbb{E}\left(v^*_{\omega^l}(\overline{b}, b_{\omega^l}(\theta_{-i}))\right) \geq (1-\delta)p(\underline{b})(\overline{\theta} - \underline{b}) + \delta\, \mathbb{E}\left(v^*_{\omega^l}(\underline{b}, b_{\omega^l}(\theta_{-i}))\right), \tag{50}$$

and for a low type buyer:

$$(1-\delta)p(\underline{b})(\underline{\theta} - \underline{b}) + \delta\, \mathbb{E}\left(v^*_{\omega^l}(\underline{b}, b_{\omega^l}(\theta_{-i}))\right) \geq (1-\delta)p(\overline{b})(\underline{\theta} - \overline{b}) + \delta\, \mathbb{E}\left(v^*_{\omega^l}(\overline{b}, b_{\omega^l}(\theta_{-i}))\right). \tag{51}$$

Adding inequalities (50) and (51) together and canceling the continuation values on both sides, we obtain:

$$
\begin{aligned}
&(1-\delta)p(\bar{b})(\bar{\theta}-\bar{b}) + (1-\delta)p(\underline{b})(\underline{\theta}-\underline{b}) \geq (1-\delta)p(\underline{b})(\bar{\theta}-\underline{b}) + (1-\delta)p(\bar{b})(\underline{\theta}-\bar{b}) \\
&\Leftrightarrow p(\bar{b})(\bar{\theta}-\bar{b}) + p(\underline{b})(\underline{\theta}-\underline{b}) \geq p(\underline{b})(\bar{\theta}-\underline{b}) + p(\bar{b})(\underline{\theta}-\bar{b}) \\
&\Leftrightarrow p(\bar{b})\bar{\theta} + p(\underline{b})\underline{\theta} \geq p(\underline{b})\bar{\theta} + p(\bar{b})\underline{\theta} \\
&\Leftrightarrow p(\bar{b})(\bar{\theta}-\underline{\theta}) + p(\underline{b})(\underline{\theta}-\bar{\theta}) \geq 0 \\
&\Leftrightarrow \big(p(\bar{b}) - p(\underline{b})\big)(\bar{\theta}-\underline{\theta}) \geq 0 \\
&\Leftrightarrow p(\bar{b}) - p(\underline{b}) \geq 0,
\end{aligned}
$$

which implies that $\bar{b} \geq \underline{b}$. □

E Solutions of equilibrium conditions

E.1 Solution of Case 1

Recall that the equilibrium conditions in Case 1 are:

$$v^*_{\text{fse}} = \frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{b}^*)}{n(1-\delta(1-q)^n)}, \tag{52}$$

and:

$$(1-\delta)\frac{q^{n-1}}{n}(\underline{\theta}-\underline{b}^*) + \delta v^*_{\text{fse}} = 0, \tag{53}$$

where $v^*_{\text{fse}} = \frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}^*) + q^n(\underline{\theta}-\underline{b}^*)\big]$.

Combining the equations (52) and (53), we get

$$(1-\delta)\frac{q^{n-1}}{n}(\underline{\theta}-\underline{b}^*) + \delta\frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{b}^*)}{n(1-\delta(1-q)^n)} = 0, \tag{54}$$

which we can solve for the equilibrium value of $\underline{b}$:

$$\underline{b}^* = \frac{\delta q\big(1-q^n\big)\overline{\theta} + q^n\big(1-\delta(1-q)^n\big)\underline{\theta}}{\delta q\big(1-q^n\big) + q^n\big(1-\delta(1-q)^n\big)}, \tag{55}$$

which we can now use to compute the payoff of each type conditional upon winning with $\underline{b}^*$, for a low type buyer we have:

$$\begin{aligned}
\underline{\theta}-\underline{b}^* &= \underline{\theta} - \frac{\delta q\big(1-q^n\big)\overline{\theta} + q^n\big(1-\delta(1-q)^n\big)\underline{\theta}}{\delta q\big(1-q^n\big) + q^n\big(1-\delta(1-q)^n\big)} \\
&= \frac{\delta q\big(1-q^n\big)\underline{\theta} + q^n\big(1-\delta(1-q)^n\big)\underline{\theta} - \delta q\big(1-q^n\big)\overline{\theta} - q^n\big(1-\delta(1-q)^n\big)\underline{\theta}}{\delta q\big(1-q^n\big) + q^n\big(1-\delta(1-q)^n\big)} \\
&= \frac{-\delta q\big(1-q^n\big)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big) + q^n\big(1-\delta(1-q)^n\big)} < 0;
\end{aligned} \tag{56}$$

and for a high type buyer we have:

$$\begin{aligned}
\overline{\theta}-\underline{b}^* &= \overline{\theta} - \frac{\delta q\big(1-q^n\big)\overline{\theta} + q^n\big(1-\delta(1-q)^n\big)\underline{\theta}}{\delta q\big(1-q^n\big) + q^n\big(1-\delta(1-q)^n\big)} \\
&= \frac{\delta q\big(1-q^n\big)\overline{\theta} + q^n\big(1-\delta(1-q)^n\big)\overline{\theta} - \delta q\big(1-q^n\big)\overline{\theta} - q^n\big(1-\delta(1-q)^n\big)\underline{\theta}}{\delta q\big(1-q^n\big) + q^n\big(1-\delta(1-q)^n\big)} \\
&= \frac{q^n\big(1-\delta(1-q)^n\big)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big) + q^n\big(1-\delta(1-q)^n\big)} > 0,
\end{aligned} \tag{57}$$

which combined with (52) gives us the resulting equilibrium payoff:

$$v^*_{\text{fse}} = \frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{b}^*)}{n(1-\delta(1-q)^n)} = \tag{58}$$

$$= \frac{(1-\delta)(1-q^n)}{n(1-\delta(1-q)^n)} \times \frac{q^n\big(1-\delta(1-q)^n\big)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)}$$

$$= \frac{1}{n}\frac{(1-\delta)q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)}.$$

Recall that the *ex ante* equilibrium payoff in a separating equilibrium is equal to $\frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b})+q^n(\underline{\theta}-\underline{b})\big]$, we must therefore have:

$$\frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\underline{\theta}-\underline{b}^*)\big] = \frac{1}{n}\frac{(1-\delta)q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)},$$

which, knowing $\underline{\theta}-\underline{b}^*$ from (56), we can solve for $\overline{\theta}-\overline{b}^*$ to obtain:

$$\overline{\theta}-\overline{b}^* = \frac{(1-\delta)q^n(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)} - \frac{q^n(\underline{\theta}-\underline{b}^*)}{1-q^n} \tag{59}$$

$$= \frac{(1-\delta)q^n(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)} + \frac{q^n\delta q(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)}$$

$$= \frac{q^n(1-\delta(1-q))(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)},$$

from which we can now compute $\underline{\theta}-\overline{b}^*$:

$$\underline{\theta}-\overline{b}^* = \underline{\theta}-\overline{\theta}+\overline{\theta}-\overline{b}^* = \tag{60}$$

$$= \overline{\theta}-\overline{b}^* - (\overline{\theta}-\underline{\theta})$$

$$= \frac{q^n(1-\delta(1-q))(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)} - (\overline{\theta}-\underline{\theta})$$

$$= \frac{\delta q\big(2q^n-q^{n-1}-1+q^{n-1}(1-q)^n\big)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)}.$$

We can now use expression (59) to determine $\overline{b}^*$:

$$\overline{b}^* = \overline{\theta} - \frac{q^n(1-\delta(1-q))(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)}. \tag{61}$$

E.2 Solution of Case 2

Recall that in Case 2 the equilibrium conditions are given by:

$$v_{\text{fse}}^* = \frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{b}^*)}{n(1-\delta(1-q)^n)}, \tag{62}$$

and:

$$(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*) + \delta v_{\text{fse}}^* = (1-\delta)(\overline{\theta}-\overline{b}^*), \tag{63}$$

where $v_{\text{fse}}^* = \frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}^*) + q^n(\underline{\theta}-\underline{b}^*)\big]$.

The equilibrium condition in (62) implies that:

$$(1-q^n)(\overline{\theta}-\overline{b}^*) + q^n(\underline{\theta}-\underline{b}) = \frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{b}^*)}{1-\delta(1-q)^n}, \tag{64}$$

which can in turn be rewritten as:

$$(1-q^n)(\overline{\theta}-\overline{b}^*) + q^n(\underline{\theta}-\underline{b}^*) = \frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{\theta})}{1-\delta(1-q)^n} + \frac{(1-\delta)(1-q^n)(\underline{\theta}-\underline{b})}{1-\delta(1-q)^n}. \tag{65}$$

Collecting terms, we get:

$$(1-q^n)(\overline{\theta}-\overline{b}^*) + \left[\frac{q^n - \delta q^n(1-q)^n - (1-\delta)(1-q^n)}{1-\delta(1-q)^n}\right](\underline{\theta}-\underline{b}^*) = \frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{\theta})}{1-\delta(1-q)^n}. \tag{66}$$

Recall that the biding incentive compatibility constraint in (63) implies

$$(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*) + \frac{\delta}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}^*) + q^n(\underline{\theta}-\underline{b})\big] = (1-\delta)(\overline{\theta}-\overline{b}^*). \tag{67}$$

This condition can be rewritten as:

$$\frac{\delta q^n}{n}(\underline{\theta}-\underline{b}^*) = (1-\delta)(\overline{\theta}-\overline{b}^*) - (1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*) - \frac{\delta}{n}(1-q^n)(\overline{\theta}-\overline{b}^*) \tag{68}$$

$$= (1-\delta)(\overline{\theta}-\overline{b}^*) - \frac{1-q^n}{n}(\overline{\theta}-\overline{b}^*)\left(\frac{1-\delta}{1-q}+\delta\right) \tag{69}$$

$$= (\overline{\theta}-\overline{b}^*)\left[(1-\delta) - \frac{1-q^n}{n(1-q)}(1-\delta q)\right] \tag{70}$$

$$= \frac{\big[n(1-q)(1-\delta) - (1-q^n)(1-\delta q)\big](\overline{\theta}-\overline{b}^*)}{n(1-q)}. \tag{71}$$

Using equations (66) and (71), the system of equilibrium conditions can now be written as:

$$(1-q^n)(\overline{\theta}-\overline{b}^*)+\left[\frac{q^n-\delta q^n(1-q)^n-(1-\delta)(1-q^n)}{1-\delta(1-q)^n}\right](\underline{\theta}-\underline{b}^*)=\frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{\theta})}{1-\delta(1-q)^n},$$

$$\delta q^n(\underline{\theta}-\underline{b}^*)=\frac{\big[n(1-q)(1-\delta)-(1-q^n)(1-\delta q)\big](\overline{\theta}-\overline{b}^*)}{1-q},$$

which can be solved for optimal payoffs $\overline{\theta}-\overline{b}^*$ and $\underline{\theta}-\underline{b}^*$:

$$\underline{\theta}-\underline{b}^*=-\frac{1}{D(\delta)}\big[(1-q^n)(1-\delta q)-n(1-\delta)(1-q)\big](1-q^n)(\overline{\theta}-\underline{\theta}), \tag{72}$$

$$\overline{\theta}-\overline{b}^*=\frac{1}{D(\delta)}\delta q^n(1-q^n)(1-q)(\overline{\theta}-\underline{\theta}), \tag{73}$$

where $D(\delta)$ is given by:

$$D(\delta)=q^n\big(1-\delta(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]+(1-q^n)\big[(1-q^n)(1-\delta q)-n(1-\delta)(1-q)\big].$$

The *ex ante* equilibrium payoff can be found from:

$$nv^*_{\text{fse}}=(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\underline{\theta}-\underline{b}^*) \tag{74}$$

$$=\frac{q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{D(\delta)}\big[\delta(1-q^n)(1-q)-(1-q^n)(1-\delta q)+n(1-\delta)(1-q)\big] \tag{75}$$

$$=\frac{q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{D(\delta)}\big[(1-q^n)(\delta-\delta q-1+\delta q)+n(1-\delta)(1-q)\big] \tag{76}$$

$$=\frac{q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{D(\delta)}(1-\delta)\big[-(1-q^n)+n(1-q)\big]. \tag{77}$$

$$\tag{78}$$

Hence the *ex ante* equilibrium payoff is:

$$v^*_{\text{fse}}=\frac{1}{nD(\delta)}(1-\delta)q^n(1-q^n)\big[n(1-q)-(1-q^n)\big](\overline{\theta}-\underline{\theta}). \tag{79}$$

We can now determine the payoff of the high type who wins with a low bid, i.e. $\overline{\theta}-\underline{b}^*$. Combining the expression for the *ex ante* equilibrium payoff in (79) and the equilibrium condition in 62 we get

$$\frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{b}^*)}{n(1-\delta(1-q)^n)}=\frac{1}{nD(\delta)}(1-\delta)q^n(1-q^n)\big[n(1-q)-(1-q^n)\big](\overline{\theta}-\underline{\theta}), \tag{80}$$

which can be solved for $\overline{\theta}-\underline{b}^*$:

$$\overline{\theta}-\underline{b}^*=\frac{1}{D(\delta)}q^n(1-\delta(1-q)^n)\big[n(1-q)-(1-q^n)\big](\overline{\theta}-\underline{\theta}). \tag{81}$$

E.3 Solution of Case 3

Recall that in Case 2 the equilibrium conditions are given by:

$$(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*)+\delta v^*_{\text{fse}}=(1-\delta)(\overline{\theta}-\overline{b}), \tag{82}$$

and:

$$(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*)+\delta v^*_{\text{fse}}=(1-\delta)q^{n-1}(\overline{\theta}-\underline{b}), \tag{83}$$

where $v^*_{\text{fse}}=\frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b})+q^n(\underline{\theta}-\underline{b})\big]$.

Note that conditions (82) and (83) together imply $\overline{\theta}-\overline{b}^*=q^{n-1}(\overline{\theta}-\underline{b}^*)$. Hence the equilibrium payoff becomes:

$$\begin{aligned}
v^*_{\text{fse}} &= \frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\underline{\theta}-\underline{b})\big] && (84)\\
&= \frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\overline{\theta}-\overline{\theta}+\underline{\theta}-\underline{b}^*)\big] && (85)\\
&= \frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\overline{\theta}-\underline{b}^*)-q^n(\overline{\theta}-\underline{\theta})\big] && (86)\\
&= \frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}^*)+q(\overline{\theta}-\overline{b}^*)-q^n(\overline{\theta}-\underline{\theta})\big] && (87)\\
&= \frac{1}{n}\big[(1-q^n+q)(\overline{\theta}-\overline{b}^*)-q^n(\overline{\theta}-\underline{\theta})\big]. && (88)
\end{aligned}$$

The upward incentive compatibility constraint in (82) can then be written as:

$$(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*)+\delta\frac{1}{n}\big[(1-q^n+q)(\overline{\theta}-\overline{b}^*)-q^n(\overline{\theta}-\underline{\theta})\big]=(1-\delta)(\overline{\theta}-\overline{b}^*). \tag{89}$$

which can then be solved for $\overline{\theta}-\overline{b}^*$:

$$\overline{\theta}-\overline{b}^*=\frac{\delta q^n(1-q)(\overline{\theta}-\underline{\theta})}{(1-q^n)(1-\delta q)+\delta q(1-q)-n(1-\delta)(1-q)}. \tag{90}$$

We can now introduce shorthand notation for the denominator:

$$D(\delta)=(1-q^n)(1-\delta q)+\delta q(1-q)-n(1-\delta)(1-q). \tag{91}$$

The *ex ante* equilibrium payoff can now be calculated from (88):

$$
\begin{aligned}
nv^*_{\text{fse}} &= (1-q^n+q)(\overline{\theta}-\overline{b}^*) - q^n(\overline{\theta}-\underline{\theta}) \qquad (92)\\
&= (1-q^n+q)\frac{\delta q^n(1-q)(\overline{\theta}-\underline{\theta})}{(1-q^n)(1-\delta q)+\delta q(1-q)-n(1-\delta)(1-q)} - q^n(\overline{\theta}-\underline{\theta})\\
&= \frac{q^n(\overline{\theta}-\underline{\theta})}{D(\delta)}\big[(1-q^n+q)\delta(1-q)-(1-q^n)(1-\delta q)-\delta q(1-q)+n(1-\delta)(1-q)\big]\\
&= \frac{q^n(\overline{\theta}-\underline{\theta})}{D(\delta)}\big[(1-q^n)\big(\delta(1-q)-(1-\delta q)\big)+n(1-\delta)(1-q)\big]\\
&= \frac{(1-\delta)q^n(\overline{\theta}-\underline{\theta})}{D(\delta)}\big[n(1-q)-(1-q^n)\big]\\
&= \frac{1}{D(\delta)}(1-\delta)q^n\big[n(1-q)-(1-q^n)\big](\overline{\theta}-\underline{\theta}).
\end{aligned}
$$

The *ex ante* equilibrium payoff is then given by:

$$
v^*_{\text{fse}} = \frac{1}{nD(\delta)}(1-\delta)q^n\big[n(1-q)-(1-q^n)\big](\overline{\theta}-\underline{\theta}). \qquad (93)
$$

The payoff of a high type buyer who wins with the low bid can be calculated from 90 and the fact that $\overline{\theta}-\underline{b}^* = \frac{1}{q^{n-1}}(\overline{\theta}-\overline{b}^*)$, and is therefore given by:

$$
\begin{aligned}
\overline{\theta}-\underline{b}^* &= \frac{\delta q(1-q)(\overline{\theta}-\underline{\theta})}{(1-q^n)(1-\delta q)+\delta q(1-q)-n(1-\delta)(1-q)} \qquad (94)\\
&= \frac{1}{D(\delta)}\delta q(1-q)(\overline{\theta}-\underline{\theta}). \qquad (95)
\end{aligned}
$$

A low type buyer payoff can be calculated from $nv^*_i = (1-q^n)(\overline{\theta}-\overline{b}) + q^n(\underline{\theta}-\underline{b}^*)$:

$$
\begin{aligned}
q^n(\underline{\theta}-\underline{b}^*) &= nv^*_{\text{fse}} - (1-q^n)(\overline{\theta}-\overline{b}^*) \qquad (96)\\
&= \frac{1}{D(\delta)}(1-\delta)q^n\big[n(1-q)-(1-q^n)\big](\overline{\theta}-\underline{\theta}) - (1-q^n)\frac{1}{D(\delta)}\delta q^n(1-q)(\overline{\theta}-\underline{\theta}),
\end{aligned}
$$

which implies:

$$
\begin{aligned}
\underline{\theta}-\underline{b}^* &= \frac{1}{D(\delta)}\Big[(1-\delta)\big[n(1-q)-(1-q^n)\big]-(1-q^n)\delta(1-q)\Big](\overline{\theta}-\underline{\theta}) \qquad (97)\\
&= \frac{1}{D(\delta)}\big[n(1-q)(1-\delta)-(1-q^n)(1-\delta q)\big](\overline{\theta}-\underline{\theta}) \qquad (98)
\end{aligned}
$$

F Proof of Lemma 5

Proof. We have shown $\underline{\theta} < \underline{b}^*$ in the main text. To show $\underline{b}^* < \overline{b}^*$, consider the payoffs defined by (20) and (21). It suffices to show that $\overline{\theta} - \underline{b}^* > \overline{\theta} - \overline{b}^*$, which is equivalent to:

$$\frac{1}{D(\delta)} q^n \big(1 - \delta(1-q)^n\big)\big[n(1-q) - (1-q^n)\big](\overline{\theta} - \underline{\theta}) > \frac{1}{D(\delta)} \delta q^n (1-q^n)(1-q)(\overline{\theta} - \underline{\theta}) \tag{99}$$

$$\Leftrightarrow \big(1 - \delta(1-q)^n\big)\big[n(1-q) - (1-q^n)\big] > \delta(1-q^n)(1-q). \tag{100}$$

It is easy to see that the above inequality holds for all δ whenever:

$$\big(1 - (1-q)^n\big)\big[n(1-q) - (1-q^n)\big] > (1-q^n)(1-q) \tag{101}$$

since the left-hand side of the inequality is decreasing in δ and the right-hand side of the inequality is increasing in δ.

Recall now that we assume that $q \geq \frac{1-q^n}{n(1-q)}$ which is equivalent to:

$$n(1-q)^2 q \geq (1-q^n)(1-q), \tag{102}$$

and in particular implies that $n \geq 4$

I now show that 102 implies 101 by showing that:

$$\big(1 - (1-q)^n\big)\big[n(1-q) - (1-q^n)\big] > n(1-q)^2 q \tag{103}$$

for $n \geq 4$.

Observe first that $\big(1-(1-q)^n\big) > \big(1-(1-q)^2\big) = q(2-q)$ for $n \geq 4$. Since $n(1-q)-(1-q^n)$ is strictly positive it suffices to show that:

$$q(2-q)\big[n(1-q) - (1-q^n)\big] > n(1-q)^2 q, \tag{104}$$

which is equivalent to:

$$
\begin{aligned}
&(2-q)\big[n(1-q)-(1-q^n)\big] > n(1-q)^2 \qquad (105)\\
&(2-q)n(1-q)-n(1-q)^2 > (2-q)(1-q^n)\\
&n(1-q)(2-q-1+q) > (2-q)(1-q^n)\\
&n(1-q) > (2-q)(1-q^n)\\
&n(1-q) > (2-q)(1-q)\sum_{k=0}^{n-1} q^k\\
&n > (2-q)\sum_{k=0}^{n-1} q^k = (1-q)\sum_{k=0}^{n-1} q^k + \sum_{k=0}^{n-1} q^k = 1-q^n+\sum_{k=0}^{n-1} q^k.
\end{aligned}
$$

Consider the function $f(q) = 1 - q^n + \sum_{k=0}^{n-1} q^k$. Differentiating $f(q)$ with respect to q I get:

$$
\begin{aligned}
f'(q) &= -nq^{n-1} + \sum_{k=1}^{n-1} kq^{k-1} > -nq^{n-1} + \sum_{k=1}^{n-1} kq^{n-1}\\
&= -nq^{n-1} + q^{n-1}\sum_{k=1}^{n-1} k = q^{n-1}\left[\sum_{k=1}^{n-1} k - n\right]\\
&= q^{n-1}\left[\frac{(1+n-1)(n-1)}{2} - n\right]\\
&= q^{n-1}n\left[\frac{(n-1)}{2} - 1\right] = q^{n-1}n\frac{(n-3)}{2} > 0,
\end{aligned}
$$

where the last inequality is true since $n \geq 4$ by assumption.

Hence we can conclude that $f(q)$ is strictly increasing on $(0, 1)$. Computing $f(1)$ we obtain:

$$
f(1) = 1 - 1^n + \sum_{k=0}^{n-1} 1^k = n, \qquad (106)
$$

therefore $f(q) < n$ for all $q \in (0, 1)$. □

G Proofs of Propositions 6, 8, 12 (Full-surplus-extracting cPPE)

G.1 Proof of Proposition 6

Proof. Full surplus extraction and $\mathcal{R}^* \geq (1-q^n)\overline{\theta}$ are shown in the main text, hence by Lemma 2 it remains to check the incentive constraints and the no-collusion constraints. I start by checking incentive compatibility.

(I) *On-schedule incentive compatibility of the buyers*

Consider a high-type buyer, his equilibrium payoff must be higher than the payoff he could obtain by mimicking the behavior of a low type buyer:

$$\text{(HighIC-on-sch)} \qquad \frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*) \geq \frac{q^{n-1}}{n}(\overline{\theta}-\underline{b}^*). \tag{107}$$

Plugging the respective payoffs in, we obtain:

$$\frac{1-q^n}{1-q}\frac{q^n(1-\delta(1-q))(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)} \geq q^{n-1}\frac{q^n\big(1-\delta(1-q)^n\big)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)}, \tag{108}$$

which simplifies to:

$$\frac{1-q^n}{1-q}(1-\delta(1-q)) \geq q^{n-1}\big(1-\delta(1-q)^n\big) \tag{109}$$

$$\Leftrightarrow \frac{1-q^n}{1-q} - \delta(1-q^n) \geq q^{n-1} - q^{n-1}\delta(1-q)^n \tag{110}$$

$$\Leftrightarrow \frac{(1-q)\sum_{k=0}^{n-1} q^k}{1-q} - \delta(1-q^n) \geq q^{n-1} - q^{n-1}\delta(1-q)^n \tag{111}$$

$$\Leftrightarrow \sum_{k=0}^{n-2} q^k \geq \delta(1-q^n) - q^{n-1}\delta(1-q)^n \tag{112}$$

$$\Leftrightarrow \frac{1}{\delta}\sum_{k=0}^{n-2} q^k \geq (1-q^n) - q^{n-1}(1-q)^n. \tag{113}$$

Since $\frac{1}{\delta}\sum_{k=0}^{n-2} q^k > \sum_{k=0}^{n-2} q^k$, it is enough to show that:

$$\sum_{k=0}^{n-2} q^k \geq (1-q^n) - q^{n-1}(1-q)^n \tag{114}$$

$$\Leftrightarrow 1 + \sum_{k=1}^{n-2} q^k \geq 1 - q^n - q^{n-1}(1-q)^n \tag{115}$$

$$\Leftrightarrow \sum_{k=1}^{n-2} q^k + q^n \geq -q^{n-1}(1-q)^n, \tag{116}$$

which is clearly true since the left-hand side of the above inequality in (116) is strictly positive, and the right-hand side is strictly negative.

I now turn to off-schedule incentive compatibility for both types.

(II) *Off-schedule incentive compatibility of the buyers*

Consider first a high-type buyer who deviates to $\underline{b}^* + \epsilon$. The associated incentive compatibility constraint is given by:

$$\text{(HighIC-down)} \qquad (1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta} - \overline{b}^*) + \delta v^*_{\text{fse}} \geq (1-\delta)q^{n-1}(\overline{\theta} - \underline{b}^*). \tag{117}$$

Plugging the respective payoffs in, we obtain:

$$\frac{(1-\delta)(1-q^n)}{n(1-q)}\frac{q^n(1-\delta(1-q))(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)} + \frac{\delta}{n}\frac{(1-\delta)q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)}$$

$$\geq (1-\delta)q^{n-1}\frac{q^n\big(1-\delta(1-q)^n\big)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)},$$

which simplifies to:

$$\frac{1-q^n}{n(1-q)}(1-\delta(1-q)) + \frac{\delta}{n}(1-q^n) \geq q^{n-1}\big(1-\delta(1-q)^n\big) \tag{118}$$

$$\Leftrightarrow \frac{1-q^n}{1-q} - \delta(1-q^n) + \delta(1-q^n) \geq nq^{n-1}\big(1-\delta(1-q)^n\big) \tag{119}$$

$$\Leftrightarrow \frac{1-q^n}{1-q} \geq nq^{n-1}\big(1-\delta(1-q)^n\big) \tag{120}$$

$$\Leftrightarrow \frac{1-q^n}{1-q} - nq^{n-1} \geq -nq^{n-1}\delta(1-q)^n \tag{121}$$

$$\Leftrightarrow \sum_{k=0}^{n-1} q^k - nq^{n-1} \geq -nq^{n-1}\delta(1-q)^n. \tag{122}$$

which is true since the left-hand side of (122) is strictly positive and the right-hand side of (122) is strictly negative.

Now consider a high-type buyer who deviates to $\overline{b}^* + \epsilon$. The associated incentive compatibility constraint is given by:

$$\text{(HighIC-up)} \qquad (1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta} - \overline{b}^*) + \delta v^*_{\text{fse}} \geq (1-\delta)(\overline{\theta} - \overline{b}^*), \tag{123}$$

which is equivalent to:

$$\delta v^*_{\text{fse}} \geq (1-\delta)\left(1 - \frac{1-q^n}{n(1-q)}\right)(\overline{\theta} - \overline{b}^*). \tag{124}$$

Plugging the respective payoffs in, we get:

$$\frac{\delta}{n}\frac{(1-\delta)q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big) + q^n\big(1-\delta(1-q)^n\big)} \geq (1-\delta)\left(1 - \frac{1-q^n}{n(1-q)}\right)\frac{q^n(1-\delta(1-q))(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big) + q^n\big(1-\delta(1-q)^n\big)},$$

which is equivalent to:

$$\frac{\delta}{n}(1-q^n) \geq \left(1 - \frac{1-q^n}{n(1-q)}\right)(1-\delta(1-q)) \tag{125}$$

$$\Leftrightarrow \frac{\delta}{n}(1-q^n) \geq 1 - \delta(1-q) - \frac{1-q^n}{n(1-q)} + \frac{\delta}{n}(1-q^n) \tag{126}$$

$$\Leftrightarrow 0 \geq 1 - \delta(1-q) - \frac{1-q^n}{n(1-q)} \tag{127}$$

$$\Leftrightarrow \delta \geq \frac{1}{1-q} - \frac{1-q^n}{n(1-q)^2}. \tag{128}$$

The condition on δ identified in (128) can only be satisfied if:

$$\frac{1}{1-q} - \frac{1-q^n}{n(1-q)^2} < 1 \tag{129}$$

$$\Leftrightarrow 1 - \frac{1-q^n}{n(1-q)} < 1 - q \Leftrightarrow nq < \frac{1-q^n}{1-q}, \tag{130}$$

which is true by assumption.

(III) *No-collusion constraints*

Suppose $\overline{\theta}$ bids off schedule and $\underline{\theta}$ bids on schedule. The associated constraint is:

$$\text{(No-col-sep-2)} \qquad v^*_{\text{fse}} \geq v(\underline{b}^* + \epsilon, \underline{b}^*) = \frac{(1-\delta)\big[(1-q^n)(\overline{\theta} - \underline{b}^*) + q^n(\underline{\theta} - \underline{b}^*)\big]}{n(1-\delta q^n)}. \quad (131)$$

Computing $(1-q^n)(\overline{\theta} - \underline{b}^*) + q^n(\underline{\theta} - \underline{b}^*)$, we get

$$(1-q^n)(\overline{\theta} - \underline{b}^*) + q^n(\underline{\theta} - \underline{b}^*) = \quad (132)$$

$$= (1-q^n)\frac{q^n\big(1-\delta(1-q)^n\big)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big) + q^n\big(1-\delta(1-q)^n\big)} + q^n \frac{-\delta q\big(1-q^n\big)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big) + q^n\big(1-\delta(1-q)^n\big)}$$

$$= \frac{q^n(1-q^n)\big(1-\delta(1-q)^n - \delta q\big)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big) + q^n\big(1-\delta(1-q)^n\big)}, \quad (133)$$

which then implies that the payoff from this bidding profile is equal to:

$$v'(\underline{b}^* + \epsilon, \underline{b}^*) = \frac{(1-\delta)q^n(1-q^n)\big(1-\delta(1-q)^n - \delta q\big)(\overline{\theta}-\underline{\theta})}{n(1-\delta q^n)\big(\delta q\big(1-q^n\big) + q^n\big(1-\delta(1-q)^n\big)\big)}. \quad (134)$$

We need to establish that $v^*_{\text{fse}} \geq v'(\underline{b}^* + \epsilon, \underline{b}^*)$. i.e.

$$\frac{1}{n}\frac{(1-\delta)q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big) + q^n\big(1-\delta(1-q)^n\big)} \geq \frac{(1-\delta)q^n(1-q^n)\big(1-\delta(1-q)^n - \delta q\big)(\overline{\theta}-\underline{\theta})}{n(1-\delta q^n)\big(\delta q\big(1-q^n\big) + q^n\big(1-\delta(1-q)^n\big)\big)},$$

which simplifies to:

$$1 \geq \frac{1-\delta(1-q)^n - \delta q}{1-\delta q^n} \quad (135)$$

$$\Leftrightarrow 1 - \delta q^n \geq 1 - \delta(1-q)^n - \delta q \quad (136)$$

$$\Leftrightarrow -\delta q^n \geq -\delta(1-q)^n - \delta q \quad (137)$$

$$\Leftrightarrow -q^n \geq -(1-q)^n - q \quad (138)$$

$$\Leftrightarrow (1-q)^n \geq -q + q^n, \quad (139)$$

which is true since the right-hand side of 139 is strictly negative, and the left-hand side is strictly positive.

Suppose both types pool at $\underline{b}^*$. The associated constraint is:

$$\text{(No-col-pol)} \qquad v^*_{\text{fse}} \geq v(\underline{b}^*, \underline{b}^*) = \frac{1}{n}\big[(1-q)(\overline{\theta} - \underline{b}^*) + q(\underline{\theta} - \underline{b}^*)\big], \quad (140)$$

where

$$
\begin{aligned}
v(\underline{b}^*, \underline{b}^*) &= \frac{1}{n}\big[(1-q)(\overline{\theta}-\underline{b}^*) + q(\underline{\theta}-\underline{b}^*)\big] = \\
&= \frac{(1-q)}{n}\frac{q^n\big(1-\delta(1-q)^n\big)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)} + \frac{q}{n}\frac{-\delta q\big(1-q^n\big)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)} \\
&= \frac{\big((1-q)q^n\big(1-\delta(1-q)^n\big)-\delta q^2(1-q^n)\big)(\overline{\theta}-\underline{\theta})}{n\big(\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)\big)}. \qquad (141)
\end{aligned}
$$

Consider the numerator of (141) in the limit as δ goes to 1:

$$
\begin{aligned}
&(1-q)q^n\big(1-(1-q)^n\big) - q^2(1-q^n) && (142)\\
&= (1-q)\left[q^n\big(1-(1-q)^n\big) - q^2\sum_{k=0}^{n-1} q^k\right] && (143)\\
&= (1-q)\left[q^n - q^n(1-q)^n - q^2\sum_{k=0}^{n-3} q^k - q^n - q^{n+1}\right] && (144)\\
&= (1-q)\left[-q^n(1-q)^n - q^2\sum_{k=0}^{n-3} q^k - q^{n+1}\right] < 0. && (145)
\end{aligned}
$$

Recall that v^*_{fse} is weakly positive, whereas the payoff in 141 goes to a negative value. By continuity there is a δ^* in the neighborhood of 1 such that for all $\delta > \delta^*$ the equilibrium payoff v^*_{fse} exceeds the payoff in 141.

□

G.2 Proof of Proposition 8

Proof. Full surplus extraction and $\mathcal{R}^* \geq (1-q^n)\overline{\theta}$ are shown in the main text, hence by Lemma 2 it remains to check the incentive constraints and the no-collusion constraints. I start by checking incentive compatibility.

(I) *On-schedule incentive compatibility of the buyers*

Consider a high-type buyer on-schedule incentive compatibility condition:

$$
\text{(HighIC-on-sch)} \qquad \frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*) \geq \frac{q^{n-1}}{n}(\overline{\theta}-\underline{b}^*). \qquad (146)
$$

Plugging the payoffs defined in (20) and (21) I get:

$$\begin{aligned}\frac{1-q^n}{n(1-q)}\frac{1}{D(\delta)}\delta q^n(1-q^n)(1-q)(\overline{\theta}-\underline{\theta}) \\ \geq \frac{q^{n-1}}{n}\frac{1}{D(\delta)}q^n\big(1-\delta(1-q)^n\big)\big[n(1-q)-(1-q^n)\big](\overline{\theta}-\underline{\theta}),\end{aligned} \tag{147}$$

which is equivalent to:

$$\delta\big(1-q^n)\big)^2 \geq q^{n-1}\big(1-\delta(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]. \tag{148}$$

which is in particular true whenever

$$\delta(1-q^n)(1-q^n) \geq q^{n-1}\big[n(1-q)-(1-q^n)\big], \tag{149}$$

i.e. for all δ satisfying $\delta \geq \frac{q^{n-1}\big[n(1-q)-(1-q^n)\big]}{(1-q^n)(1-q^n)}$. Note that such δ exist in (0,1) since

$$\begin{aligned}&(1-q^n)(1-q^n) > q^{n-1}\big[n(1-q)-(1-q^n)\big] \\ &\Leftrightarrow (1-q^n)(1-q^n)+q^{n-1}(1-q^n) > nq^{n-1}(1-q) \\ &\Leftrightarrow (1-q^n)(1-q^n+q^{n-1}) > nq^{n-1}(1-q) \\ &\Leftrightarrow (1-q^n)(1+q^{n-1}(1-q)) > nq^{n-1}(1-q) \\ &\Leftrightarrow (1+q^{n-1}(1-q))\sum_{k=0}^{n-1}q^k > nq^{n-1},\end{aligned} \tag{150}$$

where the last inequality is true since $\sum_{k=0}^{n-1} q^k > nq^{n-1}$ and $1+q^{n-1}(1-q) > 1$. Thus the high type on-schedule incentive compatibility constraint is satisfied for a sufficiently high δ.

(II) *Off-schedule incentive compatibility of the buyers*

Let us now turn to the off-schedule incentive compatibility constraints of the buyers. Consider first a low-type buyer. He must be willing to participate in the bidding with the bid $\underline{b}^*$ as opposed to abstaining and getting a zero payoff:

$$\text{(LowIC)} \qquad (1-\delta)\frac{q^{n-1}}{n}(\underline{\theta}-\underline{b}^*)+\delta v^*_{\text{fse}} \geq 0. \tag{151}$$

Plugging the payoffs defined in (19) and (22), I obtain:

$$-(1-\delta)\frac{q^{n-1}}{n}\frac{1}{D(\delta)}\big[(1-q^n)(1-\delta q)-n(1-\delta)(1-q)\big](1-q^n)(\overline{\theta}-\underline{\theta}) \\ +\delta\frac{1-\delta}{nD(\delta)}q^n(1-q^n)\big[n(1-q)-(1-q^n)\big](\overline{\theta}-\underline{\theta})\geq 0, \tag{152}$$

which simplifies to:

$$\begin{aligned} &-\big[(1-q^n)(1-\delta q)-n(1-\delta)(1-q)\big]+\delta q\big[n(1-q)-(1-q^n)\big]\geq 0 \\ &\Leftrightarrow n(1-\delta)(1-q)-(1-q^n)(1-\delta q)+\delta qn(1-q)-\delta q(1-q^n)\geq 0 \\ &\Leftrightarrow n(1-\delta+\delta q)(1-q)-(1-q^n)\geq 0 \\ &\Leftrightarrow 1-\delta+\delta q\geq\frac{1-q^n}{n(1-q)}\Leftrightarrow\delta\leq\frac{1}{1-q}-\frac{1-q^n}{n(1-q)^2}, \end{aligned} \tag{153}$$

which is true since $\frac{1}{1-q}-\frac{1-q^n}{n(1-q)^2}\geq 1$ by assumption that $q\geq\frac{1-q^n}{n(1-q)}$.

Consider a high type buyer who attempts a downward deviation to $\underline{b}^*+\epsilon$. The associated incentive compatibility condition is given by:

$$\text{(HighIC-down)}\qquad (1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*)+\delta v^*_{\text{fse}}\geq(1-\delta)q^{n-1}(\overline{\theta}-\underline{b}^*). \tag{154}$$

Plugging the payoffs defined in (20), (21), and (22) into the above inequality, I obtain:

$$\begin{aligned} &(1-\delta)\frac{1-q^n}{n(1-q)}\frac{1}{D(\delta)}\delta q^n(1-q^n)(1-q)(\overline{\theta}-\underline{\theta}) \\ &\quad+\delta\frac{1-\delta}{nD(\delta)}q^n(1-q^n)\big[n(1-q)-(1-q^n)\big](\overline{\theta}-\underline{\theta}) \\ &\quad\geq(1-\delta)q^{n-1}\frac{1}{D(\delta)}q^n\big(1-\delta(1-q)^n\big)\big[n(1-q)-(1-q^n)\big](\overline{\theta}-\underline{\theta}), \end{aligned} \tag{155}$$

which simplifies to:

$$(1-q^n)\frac{1}{n}\delta(1-q^n)+\delta\frac{1}{n}(1-q^n)\big[n(1-q)-(1-q^n)\big] \\ \geq q^{n-1}\big(1-\delta(1-q)^n\big)\big[n(1-q)-(1-q^n)\big], \tag{156}$$

which can be further simplified to:

$$\delta(1-q^n)(1-q)\geq q^{n-1}\big(1-\delta(1-q)^n\big)\big[n(1-q)-(1-q^n)\big], \tag{157}$$

i.e. for all discount factors δ such that:

$$\delta \geq \frac{q^{n-1}\big[n(1-q)-(1-q^n)\big]}{(1-q^n)(1-q)+q^{n-1}(1-q)^n\big[n(1-q)-(1-q^n)\big]} \tag{158}$$

which can only be satisfied when:

$$\frac{q^{n-1}\big[n(1-q)-(1-q^n)\big]}{(1-q^n)(1-q)+q^{n-1}(1-q)^n\big[n(1-q)-(1-q^n)\big]} < 1, \tag{159}$$

or:

$$(1-q^n)(1-q) > q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big], \tag{160}$$

which is true by assumption.

(III) *No-collusion constraints*

Suppose $\overline{\theta}$ bids off schedule and $\underline{\theta}$ bids on schedule. The associated no-collusion constraint is given by:

$$\text{(No-col-sep-2)} \qquad v^*_{\text{fse}} \geq v(\underline{b}^*+\epsilon, \underline{b}^*) = \frac{(1-\delta)\big[(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\underline{\theta}-\underline{b}^*)\big]}{n(1-\delta q^n)}. \tag{161}$$

Plugging the payoffs in, we can rewrite the right-hand side as:

$$\begin{aligned}\frac{(1-\delta)}{n(1-\delta q^n)}\Bigg[&(1-q^n)\frac{1}{D(\delta)}q^n\big(1-\delta(1-q)^n\big)\big[n(1-q)-(1-q^n)\big](\overline{\theta}-\underline{\theta}) \\ &- q^n\frac{1}{D(\delta)}\big[(1-q^n)(1-\delta q)-n(1-\delta)(1-q)\big](1-q^n)(\overline{\theta}-\underline{\theta})\Bigg],\end{aligned} \tag{162}$$

which simplifies to:

$$\begin{aligned}\frac{(1-\delta)q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{n(1-\delta q^n)D(\delta)}\Bigg(&\big(1-\delta(1-q)^n\big)\big[n(1-q)-(1-q^n)\big] \\ &-\big[(1-q^n)(1-\delta q)-n(1-\delta)(1-q)\big]\Bigg).\end{aligned} \tag{163}$$

We now have to make sure that it is below v^*_{fse}, i.e.

$$\begin{aligned}&\frac{1-\delta}{nD(\delta)}q^n(1-q^n)\big[n(1-q)-(1-q^n)\big](\overline{\theta}-\underline{\theta}) \geq \\ &\geq \frac{(1-\delta)q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{n(1-\delta q^n)D(\delta)}\Bigg(\big(1-\delta(1-q)^n\big)\big[n(1-q)-(1-q^n)\big] \\ &\qquad -\big[(1-q^n)(1-\delta q)-n(1-\delta)(1-q)\big]\Bigg),\end{aligned} \tag{164}$$

which is equivalent to:

$$(1-\delta q^n)\big[n(1-q)-(1-q^n)\big] \geq \big(1-\delta(1-q)^n\big)\big[n(1-q)-(1-q^n)\big] \\ - \big[(1-q^n)(1-\delta q)-n(1-\delta)(1-q)\big], \tag{165}$$

which in turn simplifies to:

$$\big[(1-q^n)(1-\delta q)-n(1-\delta)(1-q)\big] \geq \delta\big(q^n-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big], \tag{166}$$

which can only be satisfied if:

$$\delta \geq \frac{n(1-q)-(1-q^n)}{n(1-q)-q(1-q^n)-\big(q^n-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]}, \tag{167}$$

which in turn can only be satisfied for a high enough $\delta \in (0,1)$ only if:

$$(1-q^n)(1-q) > \big(q^n-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]. \tag{168}$$

It is easy to show that the above inequality is implied by the parameter restriction of Case 2. Recall that the parameter restriction is given by:

$$(1-q^n)(1-q) > q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]. \tag{169}$$

Observe that $q^n-(1-q)^n < q^{n-1}\big(1-(1-q)^n\big)$, which establishes the result.

Suppose now the buyers pool at $\underline{b}^*$, the associated no-collusion constraint is:

$$\text{(No-col-pool)} \qquad v^*_{\text{fse}} \geq v(\underline{b}^*,\underline{b}^*) = \frac{1}{n}\big[(1-q)(\overline{\theta}-\underline{b}^*)+q(\underline{\theta}-\underline{b}^*)\big], \tag{170}$$

where

$$\begin{aligned} v(\underline{b}^*,\underline{b}^*) &= \frac{1}{n}\big[(1-q)(\overline{\theta}-\underline{b}^*)+q(\underline{\theta}-\underline{b}^*)\big] \\ &= \frac{1}{n}\Bigg[(1-q)\frac{1}{D(\delta)}q^n\big(1-\delta(1-q)^n\big)\big[n(1-q)-(1-q^n)\big](\overline{\theta}-\underline{\theta}) \\ &\quad - q\frac{1}{D(\delta)}\big[(1-q^n)(1-\delta q)-n(1-\delta)(1-q)\big](1-q^n)(\overline{\theta}-\underline{\theta})\Bigg]. \end{aligned} \tag{171}$$

I show that $v(\underline{b}^*,\underline{b}^*)$ converges to a strictly negative number as δ goes to 1. Indeed in the limit $v(\underline{b}^*,\underline{b}^*)$ is given by:

$$\frac{(1-q)(\overline{\theta}-\underline{\theta})}{nD(1)}\big[q^n\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]-q(1-q^n)(1-q^n)\big]. \tag{172}$$

I now verify that:

$$
\begin{aligned}
&\frac{(1-q)(\overline{\theta}-\underline{\theta})}{nD(1)}\left[q^n\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]-q(1-q^n)(1-q^n)\right]<0\\
&\Leftrightarrow q^n\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]-q(1-q^n)(1-q^n)<0\\
&\Leftrightarrow (1-q^n)(1-q^n)>q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big].
\end{aligned}
$$

It suffices to show that $(1-q^n)(1-q^n)>q^{n-1}\big[n(1-q)-(1-q^n)\big]$, which has already been established above. I repeat the argument here for completeness:

$$
\begin{aligned}
&(1-q^n)(1-q^n)>q^{n-1}\big[n(1-q)-(1-q^n)\big]\\
&\Leftrightarrow (1-q^n)(1-q^n)+q^{n-1}(1-q^n)>nq^{n-1}(1-q)\\
&\Leftrightarrow (1-q^n)(1-q^n+q^{n-1})>nq^{n-1}(1-q)\\
&\Leftrightarrow (1-q^n)(1+q^{n-1}(1-q))>nq^{n-1}(1-q)\\
&\Leftrightarrow (1+q^{n-1}(1-q))\sum_{k=0}^{n-1}q^k>nq^{n-1},
\end{aligned}
$$

where the last inequality is true since $\sum_{k=0}^{n-1}q^k>nq^{n-1}$ and $1+q^{n-1}(1-q)>1$.

□

G.3 Proof of Proposition 12

Proof. Full surplus extraction and $\mathcal{R}^*\geq(1-q^n)\overline{\theta}$ are shown in the main text, hence by Lemma 2 it remains to check the incentive constraints and the no-collusion constraints. Let us now check on-schedule incentive compatibility.

(I) *On-schedule incentive compatibility of the buyers*

Consider a high type buyer who contemplates an on-schedule deviation. The associated on-schedule incentive compatibility condition is given by:

$$
\text{(HighIC-on-sch)}\qquad \frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*)\geq\frac{q^{n-1}}{n}(\overline{\theta}-\underline{b}^*).
$$

Note that both $\overline{\theta} - \overline{b}^*$ and $\overline{\theta} - \underline{b}^*$ are strictly positive for δ high enough. Recall that by construction of this public perfect equilibrium $\overline{\theta} - \overline{b}^* = q^{n-1}(\overline{\theta} - \underline{b}^*)$ and therefore we obtain:

$$\frac{1-q^n}{n(1-q)}(\overline{\theta} - \overline{b}^*) > \frac{1}{n}(\overline{\theta} - \overline{b}^*) = \frac{q^{n-1}}{n}(\overline{\theta} - \underline{b}^*).$$

where the first inequality is true since $1 - q^n > 1 - q$ for $n \geq 2$ and $q \in (0,1)$, implying that the high-type on-schedule incentive compatibility is satisfied.

(II) *Off-schedule incentive compatibility of the buyers*

Having dealt with the on-schedule incentive compatibility constraint of the buyers, I now establish that the off-schedule incentive compatibility constraints of the buyers are satisfied. Consider first a low type buyer. A low type buyer must prefer participating in the auction with the bid $\underline{b}^*$ as opposed to abstaining and getting zero forever:

$$\text{(LowIC)} \qquad (1-\delta)\frac{q^{n-1}}{n}(\underline{\theta} - \underline{b}^*) + \delta v^*_{\text{fse}} \geq 0. \tag{173}$$

Plugging the payoffs from (26) and (29) I get:

$$-(1-\delta)\frac{q^{n-1}}{n}\frac{1}{D(\delta)}\big[(1-q^n)(1-\delta q) - n(1-\delta)(1-q)\big](\overline{\theta} - \underline{\theta}) \tag{174}$$
$$+\,\delta\frac{1}{nD(\delta)}(1-\delta)q^n\big[n(1-q) - (1-q^n)\big](\overline{\theta} - \underline{\theta}) \geq 0,$$

which is equivalent to:

$$-\big[(1-q^n)(1-\delta q) - n(1-\delta)(1-q)\big] + \delta q\big[n(1-q) - (1-q^n)\big] \geq 0 \tag{175}$$
$$\Leftrightarrow -(1-q^n)(1-\delta q) + n(1-\delta)(1-q) + \delta q n(1-q) - \delta q(1-q^n) \geq 0$$
$$\Leftrightarrow n(1-q)(1-\delta+\delta q) - (1-q^n) \geq 0$$
$$\Leftrightarrow 1-\delta+\delta q \geq \frac{1-q^n}{n(1-q)}$$
$$\Leftrightarrow 1 - \frac{1-q^n}{n(1-q)} \geq \delta - \delta q \Leftrightarrow \delta \leq \frac{1}{1-q} - \frac{1-q^n}{n(1-q)^2},$$

which is true since $\frac{1}{1-q} - \frac{1-q^n}{n(1-q)^2} \geq 1$ by assumption that $q \geq \frac{1-q^n}{n(1-q)}$.

(III) *No-collusion constraints*

Suppose $\overline{\theta}$ bids on schedule and $\underline{\theta}$ bids off schedule. The associated no-collusion constraint is is given by:

$$\text{(No-col-sep-1)} \qquad v^*_{\text{fse}} \geq v(\underline{b}^*, \emptyset) = \frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{b}^*)}{n\big(1-\delta(1-q)^n\big)}. \tag{176}$$

Recall the formula for $\overline{\theta} - \underline{b}^*$ in (28), plugging it into the payoff formula above, I get

$$v'(\underline{b}^*, \emptyset) = \frac{(1-\delta)(1-q^n)\delta q(1-q)(\overline{\theta}-\underline{\theta})}{nD(\delta)\big(1-\delta(1-q)^n\big)}. \tag{177}$$

The goal is to show that for δ sufficiently high $v^*_{\text{fse}} \geq v'(\underline{b}^*, \emptyset)$, i.e.

$$\frac{1}{nD(\delta)}(1-\delta)q^n\big[n(1-q)-(1-q^n)\big](\overline{\theta}-\underline{\theta}) \geq \frac{(1-\delta)(1-q^n)\delta q(1-q)(\overline{\theta}-\underline{\theta})}{nD(\delta)\big(1-\delta(1-q)^n\big)}, \tag{178}$$

which is equivalent to:

$$q^{n-1}\big[n(1-q)-(1-q^n)\big] \geq \frac{(1-q^n)\delta(1-q)}{\big(1-\delta(1-q)^n\big)} \tag{179}$$

$$\Leftrightarrow \big(1-\delta(1-q)^n\big)q^{n-1}\big[n(1-q)-(1-q^n)\big] \geq \delta(1-q^n)(1-q),$$

which can be satisfied for any $\delta \in (0,1)$ as long as it is true that[10]

$$\big(1-(1-q)^n\big)q^{n-1}\big[n(1-q)-(1-q^n)\big] \geq (1-q^n)(1-q),$$

which is assumed is Case 3.

Suppose $\overline{\theta}$ bids off schedule and $\underline{\theta}$ bids on schedule. The associated no-collusion constraint is

$$\text{(No-col-sep-2)} \qquad v^*_{\text{fse}} \geq v(\underline{b}^*+\epsilon, \underline{b}^*) = \frac{(1-\delta)\big[(1-q^n)(\overline{\theta}-\underline{b}^*)+q^n(\underline{\theta}-\underline{b}^*)\big]}{n\big(1-\delta q^n\big)}. \tag{180}$$

[10] Note that it is required that δ satisfy

$$\delta \leq \frac{q^{n-1}\big[n(1-q)-(1-q^n)\big]}{(1-q^n)(1-q)+q^{n-1}(1-q)^n\big[n(1-q)-(1-q^n)\big]}.$$

The restriction on the parameters assumed in Case 3 makes sure that the right-hand side of this inequality is weakly above 1.

Recall the formulas for $\underline{\theta} - \underline{b}^*$ and $\overline{\theta} - \underline{b}^*$ in (26) and (28) respectively, the above payoff can then be written as

$$v(\underline{b}^* + \epsilon, \underline{b}^*) = \frac{(1-\delta)(\overline{\theta} - \underline{\theta})}{nD(\delta)\big(1 - \delta q^n\big)} \Big[(1-q^n)\delta q(1-q) - q^n\big[(1-q^n)(1-\delta q) - n(1-\delta)(1-q)\big]\Big].$$

Our goal is to show that $v^*_{\text{fse}} \geq v'(\underline{b}^* + \epsilon, \underline{b}^*)$, i.e.

$$\begin{aligned} &\frac{1}{nD(\delta)}(1-\delta)q^n\big[n(1-q) - (1-q^n)\big](\overline{\theta} - \underline{\theta}) \qquad (181)\\ &\quad \geq \frac{(1-\delta)(\overline{\theta} - \underline{\theta})}{nD(\delta)\big(1 - \delta q^n\big)} \Big[(1-q^n)\delta q(1-q) - q^n\big[(1-q^n)(1-\delta q) - n(1-\delta)(1-q)\big]\Big], \end{aligned}$$

which is equivalent to:

$$\begin{aligned} &q^n\big[n(1-q) - (1-q^n)\big] \qquad (182)\\ &\quad \geq \frac{1}{\big(1 - \delta q^n\big)} \Big[(1-q^n)\delta q(1-q) - q^n\big[(1-q^n)(1-\delta q) - n(1-\delta)(1-q)\big]\Big], \end{aligned}$$

which holds for δ sufficiently close to 1 whenever it holds as a strict inequality at $\delta = 1$, i.e. whenever

$$\begin{aligned} &q^n\big[n(1-q) - (1-q^n)\big] > \frac{1}{\big(1 - q^n\big)}\big[(1-q^n)q(1-q) - q^n(1-q^n)(1-q)\big] \qquad (183)\\ &\Leftrightarrow q^n\big[n(1-q) - (1-q^n)\big] > q(1-q) - q^n(1-q)\\ &\Leftrightarrow q^{n-1}\big[n(1-q) - (1-q^n)\big] > (1-q)(1-q^{n-1}). \end{aligned}$$

Now the last line is true since:

$$(1-q)(1-q^{n-1}) < (1-q)(1-q^n) \leq q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q) - (1-q^n)\big].$$

where the first inequality is evidently true, and the second inequality holds true in Case 3 by assumption. The result follows by the fact that $q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big] < q^{n-1}\big[n(1-q) - (1-q^n)\big]$, which in turn is true because $1 - (1-q)^n < 1$.

Suppose both types pool at $\underline{b}^*$. The associated no-collusion constraint is:

$$\text{(No-col-pool)} \qquad v^*_{\text{fse}} \geq v(\underline{b}^*, \underline{b}^*) = \frac{1}{n}\big[(1-q)(\overline{\theta} - \underline{b}^*) + q(\underline{\theta} - \underline{b}^*)\big]. \qquad (184)$$

Recall again the formulas for $\underline{\theta} - \underline{b}^*$ and $\overline{\theta} - \underline{b}^*$ in (26) and (28) respectively, the pooling payoff is then given by:

$$v(\underline{b}^*, \underline{b}^*) = \frac{\overline{\theta} - \underline{\theta}}{nD(\delta)} \Big[(1-q)\delta q(1-q) - q\big[(1-q^n)(1-\delta q) - n(1-\delta)(1-q)\big] \Big]. \tag{185}$$

As in Cases 1 and 2, I show that $\lim_{\delta\to 1} v(\underline{b}^*, \underline{b}^*) < 0$ implying that $v(\underline{b}^*, \underline{b}^*) < 0$ for any δ sufficiently close to 1:

$$\begin{aligned}
\lim_{\delta\to 1} v'(\underline{b}^*, \underline{b}^*) &= \frac{\overline{\theta} - \underline{\theta}}{nD(1)} \big[(1-q)q(1-q) - q(1-q^n)(1-q)\big] \\
&= \frac{q(1-q)(\overline{\theta} - \underline{\theta})}{nD(1)} \big[(1-q) - (1-q^n)\big] \\
&= \frac{q(1-q)(\overline{\theta} - \underline{\theta})}{nD(1)} \big[q^n - q\big] < 0.
\end{aligned}$$

□

H Proofs of Propositions 7, 9, 10, 11, and Lemma 6 (Parameter regions)

H.1 Proof of Proposition 7

Proof. Both sides of the equation can be divided by $1-q$ to obtain: $\sum_{k=0}^{n-1} q^k - nq = 0$, which can again be divided by $1-q$ to obtain: $1 - \sum_{k=1}^{n-2}(n-1-k)q^k = 0$. Define the function:

$$g(q) = 1 - \sum_{k=1}^{n-2}(n-1-k)q^k.$$

Clearly $g(0) = 1$, and $g(1)$ is given by:

$$\begin{aligned} g(1) &= 1 - \sum_{k=1}^{n-2}(n-1-k) = 1 - (n-1)(n-2) + \sum_{k=1}^{n-2} k \\ &= 1 - (n-1)(n-2) + \frac{(n-1)(n-2)}{2} = 1 - \frac{(n-1)(n-2)}{2} = \frac{n}{2}(3-n) < 0. \end{aligned}$$

hence the equation has a solution on $(0,1)$ for every $n \geq 4$ by the Intermediate Value Theorem.

Consider now the derivative of $g(\cdot)$:

$$g'(q) = -\sum_{k=1}^{n-2}(n-1-k)kq^{k-1} < 0.$$

which implies that the solution q^* is unique and that $q < \frac{1-q^n}{n(1-q)}$ for all $q < q^*$ and vice versa. □

H.2 Proof of Proposition 9

Proof. Consider the equation:

$$\begin{aligned} &(1-q^n)(1-q) = q^{n-1}\big(1-(1-q)^n\big)\left[n(1-q) - (1-q^n)\right] \\ \Leftrightarrow\ &(1-q^n) = q^{n-1}\big(1-(1-q)^n\big)\left[n - \sum_{k=0}^{n-1} q^k\right] \\ \Leftrightarrow\ &(1-q)\sum_{k=0}^{n-1} q^k = q^{n-1}\big(1-(1-q)^n\big)(1-q)\sum_{k=0}^{n-2}(n-1-k)q^k \\ \Leftrightarrow\ &\sum_{k=0}^{n-1} q^k = q^{n-1}\big(1-(1-q)^n\big)\sum_{k=0}^{n-2}(n-1-k)q^k. \end{aligned}$$

and consider the function:

$$g(q) = q^{n-1}\big(1-(1-q)^n\big)\sum_{k=0}^{n-2}(n-1-k)q^k - \sum_{k=0}^{n-1} q^k.$$

Clearly $g(0) = -1$ and $g(1)$ is computed as:

$$\begin{aligned} g(1) &= \sum_{k=0}^{n-2}(n-1-k)1^k - \sum_{k=0}^{n-1} 1^k \\ &= (n-1)^2 - \sum_{k=0}^{n-2} k - n \\ &= (n-1)^2 - \frac{(n-1)(n-2)}{2} - n = n\frac{n-3}{2} > 0. \end{aligned}$$

The result follows by continuity of $g(q)$.

□

H.3 Proof of Proposition 10

Proof. Note that the expression can be rewritten as:

$$\underbrace{(1-q^n)(1-q)}_{\to 1-q \text{ as } n\to\infty} - nq^{n-1}\underbrace{(1-q)\big(1-(1-q)^n\big)}_{\to 1-q \text{ as } n\to\infty} + \underbrace{q^{n-1}(1-q^n)\big(1-(1-q)^n\big)}_{\to 0 \text{ as } n\to\infty}.$$

It thus remains to check that $\lim_{n\to\infty} nq^{n-1} = 0$. Taking logs, I get:

$$\begin{aligned} \log\left(nq^{n-1}\right) = \log(n) + (n-1)\log(q) &\leq \sqrt{n-1} + (n-1)\log(q) \\ &= (n-1)\left(\frac{1}{\sqrt{n-1}} + \log(q)\right). \end{aligned}$$

Note that since $\log(q)$ is strictly negative and $\frac{1}{\sqrt{n-1}}$ goes to 0 as n goes to infinity, we have for a large enough n:

$$(n-1)\left(\frac{1}{\sqrt{n-1}} + \log(q)\right) \leq (n-1)\frac{\log(q)}{2}.$$

Since $\log(q) < 0$ we have $\lim_{n\to\infty}(n-1)\frac{\log(q)}{2} = -\infty$, but then $\lim_{n\to\infty}\log\left(nq^{n-1}\right) = -\infty$, which establishes the claim.

□

H.4 Proof of Proposition 11

Proof. The parameter restriction can be rewritten as:

$$\frac{1-q^n}{1-(1-q)^n} > q^{n-1}\big[n - \sum_{k=0}^{n-1} q^k\big].$$

Observe that $\frac{1-q^n}{1-(1-q)^n} \geq 1$ for all $q \leq \frac{1}{2}$ since $1-q^n \geq 1-(1-q)^n$ is equivalent to $1-q \geq q$. It thus suffices to show that $1 \geq nq^{n-1}$ for all $q \in (0, \frac{1}{2}\big]$. Define the function $f(q) = nq^{n-1} - 1$. It is clearly strictly increasing in q since $f'(q) = n(n-1)q^{n-2}$. It thus suffices to check that the claim is true for $q = \frac{1}{2}$ or $1 \geq n\frac{1}{2^{n-1}}$ which is equivalent to $2^{n-1} \geq n$, which is true for all $n \geq 2$. □

H.5 Proof of Lemma 6

Proof. We can rewrite the two inequalities as:

$$q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big] - (1-q^n)(1-q), \geq 0. \tag{186}$$

$$n(1-q)q - (1-q^n) \geq 0 \tag{187}$$

Our goal is to show that the inequality in (186) implies the inequality in (187). It suffices to show that

$$n(1-q)q - (1-q^n) \geq q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big] - (1-q^n)(1-q), \tag{188}$$

which can be rewritten as:

$$n(1-q)q - (1-q^n) + (1-q^n)(1-q) \geq q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big] \tag{189}$$

$$\Leftrightarrow n(1-q)q - q(1-q^n) \geq q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big] \tag{190}$$

$$\Leftrightarrow q\big[n(1-q)-(1-q^n)\big] \geq q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big] \tag{191}$$

$$\Leftrightarrow 1 \geq q^{n-2}\big(1-(1-q)^n\big). \tag{192}$$

which is clearly true for any $n \geq 2$ and $q \in (0,1)$.

□