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# Asymmetric Platform Oligopoly

Martin Peitz[1]
Susumu Sato[2]

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[1]MaCCI, University of Mannheim, CEPR and CESifo, Email: martin.peitz@gmail.com
[2]Hitotsubashi University, Email: susumusato.econ@gmail.com

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# Asymmetric Platform Oligopy*

Martin Peitz[†] Susumu Sato[‡]

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**Abstract:** We propose a tractable model of asymmetric platform oligopoly with logit demand in which users from two distinct groups are subject to within-group and cross-group network effects and decide which platform to join. We characterize the equilibrium when platforms manage user access by setting participation fees for each user group. We explore the effects of platform entry, a change of incumbent platforms' quality under free entry, and the degree of compatibility. We show how the analysis can be extended to partial user participation.

**Keywords:** oligopoly theory; aggregative games; network effects; two-sided markets; two-sided single-homing; entry

**JEL Codes:** L13; L41; D43

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†Department of Economics and MaCCI, University of Mannheim, Germany. Also affiliated with CEPR and CESifo. E-mail: martin.peitz@gmail.com

‡Institute of Economic Research, Hitotsubashi University, Japan. E-mail: susumusato.econ@gmail.com

# 1. Introduction

Recent decades have seen the emergence of large digital platforms, such as Alphabet, Amazon, Apple, Meta, and Microsoft, that cater to two or more user groups. Some of their activities have been increasingly scrutinized by legislators, competition watchdogs, and regulators. The assessment of competition policy and regulatory interventions requires a framework of oligopolistic platform competition that accommodates platforms of different sizes. What is more, asymmetries are also a common feature in platform markets in which Big Tech is not present. Yet, as Jullien, Pavan and Rysman (2021, p. 522) note, "the literature still lacks a tractable model of platform competition in asymmetric [...] markets."[1] This paper aims to fill this gap by proposing a tractable yet flexible model of asymmetric oligopolistic platform competition.

We model two-sided platforms as firms that bring together users from two groups. Each user cares about the participation of other users in their own group and/or in the other group; for example, competing software packeages are made available to business and private users and each user benefits from improved functionality as the number of other users of the service increases. Every user in the same group obtains an average maximal utility (when network effects play out fully) that is adjusted by the realized network size plus a utility realization of their idiosyncratic taste. Then, each user makes a discrete choice between the different (asymmetric) platforms; in other words, each user single-homes.

We analyze a multinomial logit demand model augmented by within-group and cross-group network effects. While, for tractability reasons, most of the theoretical literature assumes linear network effects, we assume that user benefits depend on the logarithm of the sizes of the two user groups; this is a specification widely adopted in the empirical analysis of network effects and platforms (e.g. Ohashi, 2003; Rysman, 2004, 2007; Zhu and Iansiti, 2012). In line with our modelling choice, according to practitioners, the incremental benefit of additional users typically declines with the user level; for instance, Chen (2021, p. 256) writes: "... network effects become less incrementally powerful. In eBay's case, when you search something like 'Rolex vintage daytona,' the product experience (and associated conversion rate) improve dramatically as you add the first few listings. It might even continue with a first few dozen. But you don't need the search to return 1,000 or 5,000 listings ..."

Platform competition with single-homing by users of each group is of high theoretical interest because platforms directly compete for users in each group. It formalizes real-world markets when heterogeneous users make a discrete choice between different systems, standards, or applications, and the providers of such offers price discriminate between user groups. We gave the example of competing software packages with offers for business and private users. Another

[1] We removed the words "and/or partially covered" from this quote. In Section 6 we address partial coverage.

is competing cloud storage services that are offered to business and private users where network effects arise due to file-sharing possibilities. Yet another is enterprise resource planning softwares (e.g. by Oracle or SAP) that cater to large and small enterprises.

Platforms are heterogeneous with respect to their costs and the average value they offer to users (after controlling for network effects). They simultaneously set participation fees for both user groups to maximize own profit. A platform's profit function depends on the vector of all the platforms prices for both group; in our setting it can be rewritten as one that depends on two choice variables and their aggregates, which are the sum of the respective choice variables over all platforms. We show that there exists an equilibrium in the pricing game and provide several characterization results; in the case of multiple equilibria, these equilibria are ordered. In line with earlier work (Armstrong, 2006; Tan and Zhou, 2021), the fees set by each platform in each group feature a "discount" to attract users in the same or the other group, triggered by within- and cross-group network effects. New to the literature, we establish conditions under which the higher-quality platform sets higher fee for both user groups than a lower-quality platform and conditions under which it does not. We also explore when one subset of platforms subsidizes one user group, whereas another subset subsidizes the other group (and possibly a third subset subsidizes neither).

Exogenous platform entry necessarily increases user surplus if there are no cross-group network effects. In the presence of cross-group network effects, in our setting, one or both of the user groups benefits from entry; however, it is possible that one of the groups suffers. Platform entry can affect the price structures of incumbent platforms by influencing platform asymmetry and thereby lead to incumbent platforms subsidizing one user group because of entry. Furthermore, we show by example that platform entry may lead to higher profits of the incumbent platforms. Under endogenous entry, the number of fringe platforms depends on market conditions and the strategic choices of incumbent platforms, such as changes in the quality of their offers for at least one group of users. Under free entry such that some fringe platforms are active, we show that, after a change of quality offered to one or both user groups by one or several incumbent platforms, one of the two user groups is better off, while the other group is worse off – this presents a strong and novel see-saw property.

Turning to the analysis of partial compatibility, we show that better compatibility in some situations increases and in others decreases user surplus (assuming that there are no cross-group network effects). With asymmetric networks, better compatibility is more likely to benefit users by reducing the market power of a larger network. We also discuss how better compatibility tends to affect the two user groups when they are connected through cross-group network effects. Finally, we amend our framework in three different ways to allow for partial coverage – that is, some users in each group choose the outside option.

**Related literature** To tackle asymmetric firms, in our analysis we make use of the aggregative game property of our model. Platform competition with two-sided single-homing implies that we cannot resort to a single aggregate in contrast to the oligopoly models analyzed by Anderson, Erkal and Piccinin (2020) and Nocke and Schutz (2018) as well as the platform models in Anderson and Peitz (2020, 2023). Sato (2021b) uses our framework and shows that market share and profit are not necessarily positively correlated (which is in line with Belleflamme, Peitz and Toulemonde, 2022). Anderson and Peitz (2020) consider a competitive bottleneck model with logit demand that can be written as an aggregative game – compared to two-sided single-homing such a model is conceptually simpler since competition plays out on one side only and thus can resort to one aggregate. In our construction, profits can be written as a function of a platform's actions (such that there is a one-to-one relationship between actions and platform fees) and the corresponding aggregates as the sums of the actions over all platforms; thus, we work with a two-dimensional aggregate.

This paper contributes to the literature on (two-sided) platform competition. This literature has examined the importance of network effects in platform competition (see Jullien, Pavan and Rysman, 2021, for a review of the literature), typically under symmetry. Prominent works with two-sided single-homing include Armstrong (2006), Tan and Zhou (2021), and Jullien and Pavan (2019). Armstrong (2006, section 4) proposes a model with linear cross-group network effects and two symmetric platforms within a Hotelling setting on each side and examines the pricing implications of cross-group network effects;[2] Tan and Zhou (2021) examine the welfare property of free entry equilibria in a model with general network effects and symmetric platforms; Jullien and Pavan (2019) examine the pricing implications in duopoly with linear cross-group network effects when platforms and users face uncertainty about the distribution of users' tastes and derive insights regarding the platforms' information management policies.

Earlier literature focused on platforms catering to a single user group characterized by direct network effects. Contributions within the multinomial logit setting include Anderson, de Palma and Thisse (1992, chapter 7.8) and Starkweather (2003), both of which assumed linear direct network effects. In these settings, there is no explicit solution for the participation game with asymmetric platforms.[3] We also contribute to this literature and characterize the unique price equilibrium under asymmetric platform competition in the special case that cross-group network effects are absent.

[2]For an empirical application to the German magazine market, see Kaiser and Wright (2006). The model with asymmetric platforms is used to analyze platform taxation (Belleflamme and Toulemonde, 2018) and the relationship between profits and market shares (Belleflamme et al., 2022).

[3]The operations research literature has looked at monopoly pricing and assortment problems in the presence of direct network effects and multinomial logit demand; see e.g. Du, Cooper and Wang (2016) and Wang and Wang (2017). Wang and Wang (2017) include an explicit solution of the participation game when network effects are logarithmic.

The paper is organized as follows. In Section 2, we present the model. In Section 3.1, we characterize participation equilibria for any given platform fees and show that there is a unique interior participation equilibrium; we identify this as the unique asymptotically stable participation equilibrium and use this in the subsequent analysis. We then express profit functions as functions of two choice variables and their aggregates and express user welfare as a function of the aggregates (Section 3.2). In Section 3.3, we show that there exists an equilibrium of the platform pricing game; all equilibria can be ranked by the surplus of one of the two groups. We establish equilibrium uniqueness in two special cases: in the oligopoly with network goods (i.e., absent cross-group network effects) and under one-sided pricing (Section 3.4). In Section 4, we provide several characterization results. In Section 5, we provide comparative statics results with respect to the set of active platforms (exogenous platform "entry") and incumbent platforms' "quality" under free entry; we also mention results with respect to partial compatibility, which is analyzed in more detail in the Appendix. In Section 6, we extend our analysis to environments with partial coverage (details of this analysis are relegated to the Appendix). Section 7 concludes. All proofs are relegated to the Appendix.

# 2. The platform oligopoly model

Consider $M > 1$ platforms competing for users from two groups, $A$ and $B$. Each platform $i \in \{1, ..., M\}$ charges a membership or subscription fee $p_i^k \in \mathbb{R}$ to users from group $k \in \{A, B\}$. We consider the game in which, first, platforms simultaneously set participation fees $p_i^A, p_i^B$ and then a unit mass of users from both groups simultaneously decide which platform to join. We solve for subgame perfect Nash equilibria (applying the selection criterion detailed below). In the following, we describe the platforms' problem and the user demand model.

## 2.1. Platforms

Each platform $i$ incurs a constant marginal cost $c_i^k \geq 0$ for serving group-$k$ users. We denote platform $i$'s number of group-$k$ users by $n_i^k$ and the vector of prices for group $k$ by $p^k = (p_1^k, \ldots, p_M^k)$. Then, we can write platform $i$'s profit as $\pi_i(p^A, p^B) = (p_i^A - c_i^A)n_i^A(p^A, p^B) + (p_i^B - c_i^B)n_i^B(p^A, p^B)$, where $n_i^A$ and $n_i^B$ depend on the fees set by *all* platforms for *both* groups.

Our main focus is on two-sided pricing – that is, each platform $i$ charges fees $p_i^A$ and $p_i^B$ to each user group. We also consider one-sided pricing under which each platform $i$ has to set a fee of zero to one group (presuming that the marginal cost is zero for that group) or a fee equal to marginal costs (when allowing for positive marginal costs for that group).

## 2.2. Users

A unit mass of users from each group decide which platform to join. Each user's utility from joining a platform consists of a maximal value of the platform, network effects, and an idiosyncratic preference for the platform. Formally, the utility of a group-$k$ consumer from joining platform $i$ is given by

$$u_i^k = a_i^k - p_i^k + \alpha^k \log n_i^k + \beta^k \log n_i^l + \varepsilon_i^k. \tag{1}$$

The first term $a_i^k - p_i^k$ is the expected value of platform $i$ for group-$k$ users if all users from both groups joined this platform, where $a_i^k$ represents the "quality" of platform $i$ for group $k$. The second and third terms, $\alpha^k \log n_i^k$ and $\beta^k \log n_i^l$, capture within-group and cross-group network effects, where $\alpha^k \in [0,1)$ and $\beta^k \in [0,1)$ are the parameters that represent the importance of platform-specific within-group and cross-group network effects, and $n_i^k$ and $n_i^l$ are the number of group-$k$ and group-$l(\neq k)$ users who join platform $i$. We call $n_i^k$ group $k$'s *network size* of platform $i$. We note that the chosen logarithmic specification of network effects is broadly adopted in the empirical literature (e.g., Ohashi, 2003; Rysman, 2004, 2007; Zhu and Iansiti, 2012).[4]

The last term, $\varepsilon_i^k$, is an idiosyncratic taste shock from an i.i.d. type-I extreme value distribution. We assume that network effects are not too strong, that is, $\alpha^k + \beta^l < 1$ hold for any $k, l \in \{A, B\}$. Thus, $\max\{\alpha^A, \alpha^B\} + \max\{\beta^A, \beta^B\} < 1$. Table 1 summarizes the notation.

In e-commerce marketplaces, sellers and buyers constitute the two user groups and parameters $\beta^A$ and $\beta^B$ are positive, while, in the simplest version, $\alpha^A = \alpha^B = 0$. Here, there are mutual cross-group network effects since buyers are attracted to platforms with many sellers and sellers to platforms with many buyers. Similarly, for two-sided matching platforms such as heterosexual online dating platforms. On some social networks and media platforms, content providers (who monetize engagement themselves) and consumers constitute two user groups $A$ and $B$ and, in its simplest version when consumers only care about content and not their fellow consumers, $\beta^A$ and $\alpha^B$ are positive and $\alpha^A = \beta^B = 0$. For a discussion, see Belleflamme and Peitz (2021). Our model also nests the standard logit oligopoly model without an outside option (see e.g. Anderson, Erkal and Piccinin, 2020) – in this case, $\alpha^A = \alpha^B = \beta^A = \beta^B = 0$.

For given network sizes $\bar{n} = (\bar{n}_i^A, \bar{n}_i^B)_{i \in \{1,\ldots,M\}}$, group-$k$ consumer demand of platform $i$ can

[4] Most of the existing theoretical literature postulates linear network effects (e.g., Armstrong, 2006). However, in many real-world applications, a strictly concave function looks more plausible.

| Notation | Meaning |
|---|---|
| $k, l$ | indices for the two user groups |
| $a_i^k$ | group-$k$ quality of platform $i$ |
| $c_i^k$ | marginal cost for group-$k$ participation on platform $i$ |
| $p_i^k$ | group-$k$ fee of platform $i$ |
| $n_i^k$ | group-$k$ network size of platform $i$ |
| $\alpha^k$ | parameter for within-group network effect of group $k$ |
| $\beta^k$ | parameter for cross-group network effect enjoyed by group $k$ |

Table 1: Notation

be written as

$$
\begin{aligned}
n_i^k &= \Pr\left(u_i^k \geq u_j^k \text{ for all } j \neq i\right) \\
&= \frac{\exp(a_i^k - p_i^k)\left(\bar{n}_i^k\right)^{\alpha^k}\left(\bar{n}_i^l\right)^{\beta^k}}{\sum_{j=1}^M \exp(a_j^k - p_j^k)\left(\bar{n}_j^k\right)^{\alpha^k}\left(\bar{n}_j^l\right)^{\beta^k}} =: T_i^k(\bar{n}). \qquad (2)
\end{aligned}
$$

This is the multinomial demand structure with network sizes endogenously determining platform quality.

# 3. Equilibrium analysis

We first characterize the participation equilibrium at stage 2 for given platform fees. We then analyze subgame perfect Nash equilibria of the price-then-participation game.

## 3.1. Participation equilibrium

In a participation equilibrium, network sizes $n_i^k$ on the left-hand side are equal to $\bar{n}_i^k$ on the right-hand side of equation (2) for all $k \in \{A, B\}$ and $i \in \{1, ..., M\}$.

Due to complementarity in platform choices, there may be multiple participation equilibria, an issue pointed out by Anderson et al. (1992, chapter 7.8) and Tan and Zhou (2021), among others. In the present setting, equation (2) indicates that whenever users expect $\bar{n}_i^k = 0$, such an expectation will be self-fulfilling (for any platform prices). Therefore, there are several equilibria in which some platforms are chosen with probability zero.

We will first characterize the unique participation equilibrium for a given set, $\mathcal{M} \subseteq \{1, \ldots, M\}$, of *active* platforms (i.e., platforms with strictly positive demand for both groups). We call such an equilibrium an *interior participation equilibrium* when all platforms are active.

**Proposition 1.** *For any given prices* $p = (p_1^A, ..., p_M^A, p_1^B, ..., p_M^B)$*, there exists a unique participation equilibrium with the set of active platforms* $\mathcal{M} \subseteq \{1, \ldots, M\}$*. Equilibrium participation levels are given by*

$$n_i^k(p) = \frac{\exp[\Gamma^{kk}(a_i^k - p_i^k) + \Gamma^{kl}(a_i^l - p_i^l)]}{\sum_{j \in \mathcal{M}} \exp[\Gamma^{kk}(a_j^k - p_j^k) + \Gamma^{kl}(a_j^l - p_j^l)]}, \tag{3}$$

*for all* $i \in \mathcal{M}$ *and* $k, l \in \{A, B\}$ *with* $l \neq k$*, where* $\Gamma^{kk}$ *and* $\Gamma^{kl}$ *are given by*

$$\Gamma^{kk} = \frac{1 - \alpha^l}{(1 - \alpha^k)(1 - \alpha^l) - \beta^k \beta^l} \geq 1, \text{ and } \Gamma^{kl} = \frac{\beta^k}{(1 - \alpha^k)(1 - \alpha^l) - \beta^l \beta^k} \geq 0.$$

The demand system given by equation (3) is a logit demand system augmented by within-group and cross-group network effects. First, in the special case that $\alpha^k = \beta^k = 0$, equation (2) gives the standard logit choice probabilities

$$n_i^k = \frac{\exp(a_i^k - p_i^k)}{\sum_{j \in \mathcal{M}} \exp(a_j^k - p_j^k)}.$$

Second, consider the case of within-group network effects but no cross-group network effects ($\alpha^k > 0$, $\beta^k = 0$ for $k \in \{A, B\}$). Logit choice probabilities are then adjusted by those within-group network effects:

$$n_i^k = \frac{\exp\left(\frac{a_i^k - p_i^k}{1 - \alpha^k}\right)}{\sum_{j \in \mathcal{M}} \exp\left(\frac{a_j^k - p_j^k}{1 - \alpha^k}\right)}.$$

Third, consider the case of cross-group network effects but no within-group network effects ($\alpha^k = 0$, $\beta^k > 0$ for $k \in \{A, B\}$). Logit choice probabilities are then:

$$n_i^k = \frac{\exp\left(\frac{a_i^k - p_i^k + \beta^k(a_i^l - p_i^l)}{1 - \beta^k \beta^l}\right)}{\sum_{j \in \mathcal{M}} \exp\left(\frac{a_j^k - p_j^k + \beta^k(a_j^l - p_j^l)}{1 - \beta^k \beta^l}\right)}.$$

Finally, consider the case that $\alpha^k$ and $\beta^k$ are positive. In a participation equilibrium, each platform's maximal average value in group $k$, $a_i^k - p_i^k$, is amplified by within-group and cross-group network effects represented by $\Gamma^{kk}$ and $\Gamma^{kl}$, respectively. These amplifiers translate the base values of platform $i$ in the two groups into the externality-adjusted group-$k$ values of platform $i$, which is given by $\Gamma^{kk}(a_i^k - p_i^k) + \Gamma^{kl}(a_i^l - p_i^l)$. In the participation equilibrium, it turns out that users make a choice based on this externality-adjusted value rather than the original values, leading to expression (3).

To summarize, we obtain a tractable closed-form expression of user participation with network effects because network effects are logarithmic in network size and demand takes the logit form.[5] The multiplicity of participation equilibria arises from the logarithmic specification of the network effects, which makes an empty platform worthless for users; and thus the set of active platforms is not pinned down. There are two ways to address this multiplicity. One possibility is to postulate that for reasons outside the model there is a given set of active platforms. Proposition 1 then characterizes equilibrium participation decisions for any set of prices of these platforms.

The other possibility to address the multiplicity of participation equilibria is to propose a particular selection criterion. We do so in the analysis that follows and provide a selection criterion according to which all available platforms are active in equilibrium.

**Equilibrium selection.** We impose *asymptotic stability* of best-response dynamics as our selection criterion and show that the only equilibrium that meets the selection criterion is the interior participation equilibrium.[6] The notion of best-response dynamics corresponds to that used in the literature of population games (Sandholm, 2010, Chapter 6.2), and the notion of asymptotic stability is used to capture the stability of dynamic systems (Luenberger, 1979, Chapter 5.9).

**Definition 1.** *Define the best-response dynamics and asymptotic stability of network sizes as follows:*

1. *A best-response dynamics* $\{n_t\}_{t=0}^{\infty}$ *from the initial network sizes* $n_0 = \left(n_{i,0}^A, n_{i,0}^B\right)_{i \in \{1,\dots,M\}}$ *is defined by a sequence of network sizes* $n_t = \left(n_{i,t}^A, n_{i,t}^B\right)_{i \in \{1,\dots,M\}}$ *such that* $n_{i,t}^k = T_i^k\left(n_{t-1}\right)$ *according to the best-response functions* $T_i^k$ *defined in equation* (2) *for all* $t \in \{1, 2, \dots\}$, $i \in \{1, \dots, M\}$ *and* $k \in \{A, B\}$.

2. *A network size vector* $n = \left(n_i^A, n_i^B\right)_{i \in \{1,\dots,M\}}$ *is the limit of the best-response dynamics* $\{n_t\}_{t=0}^{\infty}$ *from the initial network size* $n_0$ *if* $n = \lim_{t \to \infty} n_t$.

[5] Linear demand models with linear network effects also give rise to closed-form demand functions (e.g., Armstrong, 2006). In a linear demand model with linear network effects, the choice probability can be written as a linear function of expected network sizes, which makes it possible to use linear algebra to obtain the closed-form solution for network sizes.

[6] Other selection criteria used in the literature on network effects in industrial organization include: Pareto dominance (Katz and Shapiro, 1986; Fudenberg and Tirole, 2000), coalitional rationalizability or coalition proofness (Ambrus and Argenziano, 2009; Karle, Peitz and Reisinger, 2020), potential maximization (Chan, 2021), and focality advantage or attached consumers (Caillaud and Jullien, 2003; Halaburda, Jullien and Yehezkel, 2020; Biglaiser and Crémer, 2020). This includes dynamic consideration leading to incumbency advantages in the cases of focality and attached consumers. In our model, for any cost-adjusted quality and any prices, a platform facing unfavorable beliefs – in the sense that each user expects the smallest number to join that is compatible with equilibrium – will not become active. If such unfavorable beliefs are associated with the status of being an entrant, entry will not be possible. For further work on incumbency advantage as a result of dynamic user choice in the presence of network effects, see Biglaiser, Crémer and Veiga (2022).

*3. A participation equilibrium with the equilibrium network sizes $n$ is asymptotically stable if for any strictly positive $n_0$, $n$ is the limit of the best-response dynamics from the initial network sizes $n_0$.*

Definition 1 requires that the equilibrium network sizes are the result of best-response dynamics starting from any interior starting point. We call a participation equilibrium with asymptotically stable network sizes an *asymptotically stable participation equilibrium.*

The following remark establishes that the interior participation equilibrium is the only equilibrium that is asymptotically stable.

**Remark 1.** *For any given prices $p = (p_1^A, ..., p_M^A, p_1^B, ..., p_M^B)$, the interior participation equilibrium, characterized by equations* (3) *with $\mathcal{M} = \{1, \ldots, M\}$, is the unique asymptotically stable participation equilibrium.*

## 3.2. Aggregates, profit functions, and user surplus

We will write platform profits as functions of own actions and corresponding aggregates. Furthermore, we will write user surplus of the two groups as functions of these aggregates. To do so, we define a platform's own actions as

$$h_i^A := \exp\left[\Gamma^{AA}(a_i^A - p_i^A) + \Gamma^{AB}(a_i^B - p_i^B)\right],$$
$$h_i^B := \exp\left[\Gamma^{BB}(a_i^B - p_i^B) + \Gamma^{BA}(a_i^A - p_i^A)\right],$$

and the corresponding aggregates $H^A := \sum_{j=1}^M h_j^A$ and $H^B := \sum_{j=1}^M h_j^B$. Thus, group-$k$ demand on platform $i$ is $n_i^k = h_i^k / H^k$.

There is a one-to-one mapping between $(p_i^A, p_i^B)$ and $(h_i^A, h_i^B)$. As we show in the following lemma, any $(h_i^A, h_i^B)$ induce prices $(p_i^A(h_i^A, h_i^B), p_i^B(h_i^A, h_i^B))$.

**Lemma 1.** *Platform fees can be written as functions of $(h_i^A, h_i^B)$:*

$$p_i^A(h_i^A, h_i^B) = a_i^A - (1 - \alpha^A)\log h_i^A + \beta^A \log h_i^B, \tag{4}$$
$$p_i^B(h_i^A, h_i^B) = a_i^B - (1 - \alpha^B)\log h_i^B + \beta^B \log h_i^A. \tag{5}$$

Recall that platform $i$'s profit as a function of platform fees is $(p_i^A - c_i^A)n_i^A + (p_i^B - c_i^B)n_i^B$. Since $n_i^k = h_i^k / H^k$ and there is a one-to-one mapping between $(p_i^A, p_i^B)$ and $(h_i^A, h_i^B)$, the profit of platform $i$ can be written as the function of the two action variables $h_i^A$ and $h_i^B$ and their aggregates $H^A$ and $H^B$:

$$\begin{aligned}\Pi_i(h_i^A, h_i^B, H^A, H^B) &= \Pi_i^A(h_i^A, h_i^B, H^A) + \Pi_i^B(h_i^A, h_i^B, H^B) \\ &= \frac{h_i^A}{H^A}[p_i^A(h_i^A, h_i^B) - c_i^A] + \frac{h_i^B}{H^B}[p_i^B(h_i^A, h_i^B) - c_i^B],\end{aligned}$$

where we defined $\Pi_i^k = \frac{h_i^k}{H^k}[p_i^k(h_i^k, h_i^l) - c_i^k]$, $k, l \in \{A, B\}$, $l \neq k$.

Group-$k$ user surplus $CS^k$ is given by the expected indirect utility of users, and the aggregate user surplus $CS$ is given by the sum of the user surplus in both groups:

$$\begin{aligned}CS^k &:= \log\left[\sum_{i=1}^{M} \exp(a_i^k - p_i^k)(n_i^k)^{\alpha^k}(n_i^l)^{\beta^k}\right] \\ &= (1-\alpha^k)\log H^k - \beta^k \log H^l, \\ CS &:= CS^A + CS^B \\ &= (1-\alpha^A-\beta^B)\log H^A + (1-\alpha^B-\beta^A)\log H^B.\end{aligned}$$

We observe that user surplus of group $k$, $CS^k$, is increasing in the aggregate of this group, $H^k$, and weakly decreasing in the aggregate of the other user group, $H^l$; it is strictly decreasing if and only if group $l$ exerts a cross-group network effect. Total user surplus $CS = CS^A + CS^B$ increases in each of the two aggregates $H^A$ and $H^B$.

## 3.3. Price equilibrium in asymmetric platform oligopoly

Using the demand system obtained from the participation equilibrium, we analyze price competition between platforms using the continuation profits from the participation equilibrium at stage 2.

We establish the following lemma that guarantees that we can restrict attention to the first-order conditions of profit maximization when analyzing platform pricing.

**Lemma 2.** *For any given $H_{-i}^A = \sum_{j\neq i} h_j^A$ and $H_{-i}^B = \sum_{j\neq i} h_j^B$, there is a unique solution to the first-order conditions of profit maximization of $\Pi_i(h_i^A, h_i^B, h_i^A + H_{-i}^A, h_i^B + H_{-i}^B)$ with respect to $h_i^A$, $h_i^B$, and this solution is a global maximizer of platform i's pricing problem.*

The derivative of $\Pi_i$ with respect to $h_i^A$ is

$$\begin{aligned}\frac{\partial \Pi_i}{\partial h_i^A} &= \left(\frac{1}{H^A} - \frac{\partial H^A}{\partial h_i^A}\frac{h_i^A}{(H^A)^2}\right)[p_i^A(h_i^A, h_i^B) - c_i^A] + \frac{h_i^A}{H_i^A}\frac{\partial p_i^A}{\partial h_i^A} + \frac{h_i^B}{H^B}\frac{\partial p_i^B}{\partial h_i^A} \\ &= \frac{1}{h_i^A}\left[\frac{h_i^A}{H^A}\left(1 - \frac{h_i^A}{H^A}\right)[p_i^A(h_i^A, h_i^B) - c_i^A] - (1-\alpha^A)\frac{h_i^A}{H^A} + \beta^B\frac{h_i^B}{H^B}\right].\end{aligned}$$

Therefore, from $\partial\Pi_i/\partial h_i^A = 0$, we have the characterization of the price-cost margins:

$$p_i^A(h_i^A, h_i^B) - c_i^A = \frac{1}{1-\frac{h_i^A}{H^A}}\left(1-\alpha^A - \beta^B \frac{h_i^B}{H^B}\frac{H^A}{h_i^A}\right) = \frac{1}{1-n_i^A}\left(1-\alpha^A - \beta^B \frac{n_i^B}{n_i^A}\right).$$

In the standard multinomial logit model without network effects ($\alpha^k = \beta^k = 0$, for all $k \in \{A, B\}$), the price-cost margin is equal to $1/(1-n_i^k)$. In the presence of within-group network effects $\alpha^k > 0$, the price-cost margin is reduced by $\alpha^k$. The lower price-cost margin is due to the larger price elasticity of demand arising from within-group network effects. In the presence of cross-group network effect $\beta^l > 0$, the price-cost margin for group $k$ is reduced by the amount $\beta^l n_i^l/n_i^k$. Here, the lower price-cost margin is due to the cross-subsidization incentive of the platform: it expands participation of group $k$ to attract users in group $l$; this is in line with the formulas for price-cost margins in symmetric platform oligopoly reported in Armstrong (2006) and Tan and Zhou (2021).

In an equilibrium, the system of first-order conditions

$$\begin{aligned} a_i^A - c_i^A - (1-\alpha^A)\log h_i^A + \beta^A \log h_i^B &= \frac{1}{1-\frac{h_i^A}{H^A}}\left(1-\alpha^A - \beta^B \frac{h_i^B}{H^B}\frac{H^A}{h_i^A}\right) \\ a_i^B - c_i^B - (1-\alpha^B)\log h_i^B + \beta^B \log h_i^A &= \frac{1}{1-\frac{h_i^B}{H^B}}\left(1-\alpha^B - \beta^A \frac{h_i^A}{H^A}\frac{H^B}{h_i^B}\right) \end{aligned}$$

must be satisfied for all $i \in \{1, 2, \ldots, M\}$. As shown in the following lemma, for each $i$, this defines implicit best replies $(h_i^A(H^A, H^B), h_i^B(H^A, H^B))$.

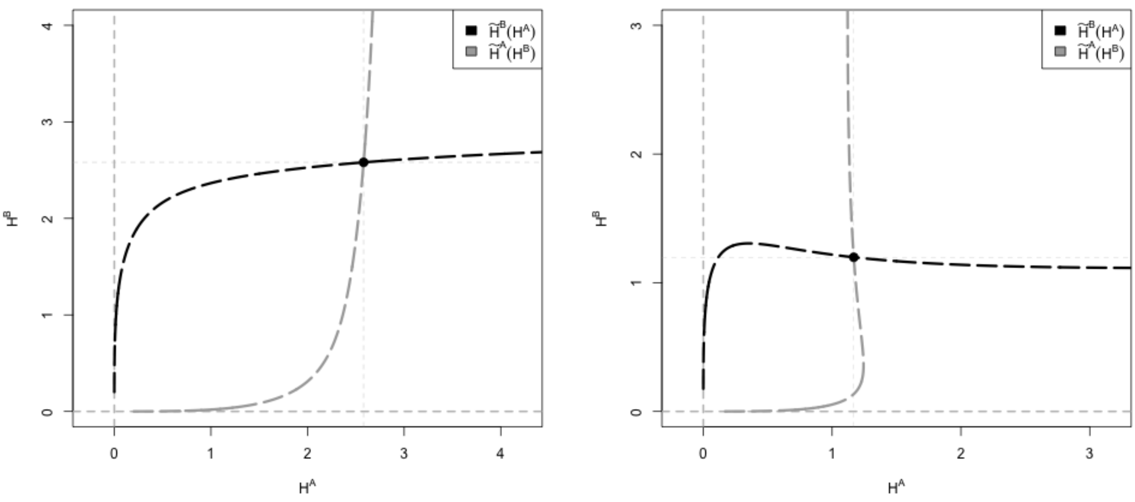
**Lemma 3.** *For any $(H^A, H^B)$, the system of first-order conditions defines implicit best replies $(h_i^A(H^A, H^B), h_i^B(H^A, H^B))$ for each platform $i \in \{1, \ldots, M\}$.*

Summing over all $i$, an equilibrium satisfies

$$\sum_{i=1}^{M} h_i^k(H^A, H^B) = H^k, \tag{6}$$

for $k, l \in \{A, B\}, l \neq k$. With the following proposition, we establish that there exists a price equilibrium and that, whenever multiple equilibria exist, these are ordered in terms of surplus of one of the two user groups: if one equilibrium features higher surplus for one group, the other equilibrium features a higher surplus for the other group.

**Proposition 2.** *There exists a price equilibrium pinned down by aggregates $(H^{A*}, H^{B*})$. When there are multiple price equilibria for a given set of active platforms, we obtain the ranking for*




(a) (b)

Figure 1: Shapes of $\tilde{H}^A(H^B)$ and $\tilde{H}^B(H^A)$.

*any pair of equilibrium aggregates given by* $(H_1^{A*}, H_1^{B*})$ *and* $(H_2^{A*}, H_2^{B*})$ *with associated user surpluses* $(CS_1^{A*}, CS_1^{B*})$ *and* $(CS_2^{A*}, CS_2^{B*})$*:* $CS_1^{A*} > CS_2^{A*}$ *holds if and only if* $CS_1^{B*} < CS_2^{B*}$*.*

We note that the equilibrium is always unique if platforms are symmetric.[7] A price equilibrium is characterized by the pair of aggregates $(H^{A*}, H^{B*})$ that satisfy the system of equations (6), which implicitly defines functions $\tilde{H}^k(H^l)$. An intersection of these two functions constitutes an equilibrium, as illustrated by the two numerical examples in Figure 1.[8]

Since the surplus of group-$k$ users, $CS^k = (1-\alpha^k)\log H^k - \beta^k \log H^l$, depends only on aggregates $(H^A, H^B)$, the characterization of equilibrium aggregates directly characterizes user surplus in equilibrium. We note that in the aggregative-games frameworks of price competition in standard oligopoly (Anderson, Erkal and Piccinin, 2020) and platform competition with competitive bottlenecks (Anderson and Peitz, 2020) consumer surplus (i.e., user surplus on the single-homing side) depends on a one-dimensional aggregate. Under two-sided single-homing, user surplus depends positively on the aggregate of this group and negatively on the aggregate of the other group.

A technical issue in the equilibrium existence results with price competition is how to obtain a compact strategy space. In a standard logit model without network effects, Nocke and Schutz (2018, forthcoming) directly show that setting too high prices is always unprofitable, thereby

[7]While we could not rule out multiple price equilibria with asymmetric platforms, all the numerical examples that we looked at have a unique equilibrium.

[8]The figures illustrate the shape of $(\tilde{H}^A, \tilde{H}^B)$ for parameter values $\alpha^A = \alpha^B = 0.1$ and $\beta^A = \beta^B = 0.3$. Panel (a) does so with $M = 2$, $(v_1^A, v_1^B) = (3, 0)$, and $(v_2^A, v_2^B) = (0, 3)$, where $v_i^k = a_i^k - c_i^k$, panel (b) with $M = 3$, $(v_1^A, v_1^B) = (0, 1)$ and $(v_2^A, v_2^B) = (v_3^A, v_3^B) = (0, 0.5)$.

obtaining upper bounds on the strategy space. In the logit demand with within-group network effects (i.e., $\alpha^A, \alpha^B > 0$ but $\beta^A = \beta^B = 0$), we obtain an upper bound on prices in the same way as Nocke and Schutz (2018, forthcoming). With cross-group network effects (i.e., $\beta^A > 0$ or $\beta^B > 0$), we also have to worry about a *lower bound* on prices because, in theory, platforms could choose to turn towards negative infinite fees for one group and positive infinite fees for the other group at the same time. In the proof, we show that this strategy is always dominated as long as $\alpha^k + \beta^l < 1$ for $k, l \in \{A, B\}$ with $l \neq k$.

We postulated that within- and cross-group network effects are non-negative. However, in some real-world environments, some network effects are arguably negative. We note that all of our analysis is applicable to the case with negative within-group network effects (i.e., $\alpha^k < 0$).[9] However, our analysis fails to apply with negative cross-group network effects (i.e, $\beta^k < 0$) due to our logarithmic specification. With negative cross-group network effects experienced by one group – for instance, group $A$ – a platform can charge an unboundedly high fee to group-$B$ users to increase $\beta^A \log n_i^B$ without bounds and then enjoy a monopoly profit from group-$A$ users.

In Section 4, for a given set of active platforms, we provide equilibrium characterization results – they hold for any price equilibrium, no matter how it is selected. In Section 5, we derive comparative statics results under a given selection rule at the pricing stage (e.g., always selecting the price equilibrium with maximal surplus for group $k$ with $k \in \{A, B\}$).

## 3.4. Special cases: Network goods and platforms with one-sided pricing

It is insightful to consider the special case of only within-group network effects (i.e., $\beta^A = \beta^B = 0$). In other words, we analyze the asymmetric logit model with network effects. Users in one group do not care about user participation in the other group and it is sufficient to consider group $A$. The pricing equation for platform $i$ becomes $p_i^A - c_i^A = a_i^A - c_i^A - (1 - \alpha^A) \log h_i^A$. Thus, the first-order condition of profit maximization for group $A$ can be written as

$$(1 - \alpha^A) \frac{H^A}{H^A - h_i^A} = (a_i^A - c_i^A) - (1 - \alpha^A) \log h_i^A. \tag{7}$$

Note that the right-hand side is decreasing in $h_i^A$, while the left-hand side is increasing in $h_i^A$. Thus, for any $H^A$ there is a unique $h_i^A(H^A)$. Note also that the right-hand side does not depend on $H^A$, while the left-hand side is shifted downward after an increase in $H^A$. Hence, $h_i^A(H^A)$ is increasing in $H^A$.

[9] In this case, we strengthen our assumption that $\alpha^k + \beta^l < 1$ to $|\alpha^k| + \beta^l < 1$ for all $k, l \in \{A, B\}$, which implies that $|\alpha^k| < 1$, $k \in \{A, B\}$, and ensures the asymptotic stability of the interior participation equilibria. See footnote 24 in Appendix A.1 for detail.

**Remark 2.** *There exists a unique price equilibrium when* $\beta^A = \beta^B = 0$.

Participation may be free for one user group. For example, shopping malls and flea markets typically charge retailers but often not end users. This may be because platforms would charge negative fees (or fees below costs) and such fees are not feasible. Alternatively, platforms would like to charge end user fees but such positive fees would go hand-in-hand with high transaction costs or are simply not possible (as in traditional free-to-air radio or television broadcasting).

As mentioned in Section 2.1, we assume that marginal costs are zero for users in the group with a zero fee. If we were to allow positive symmetric marginal costs also for this group, our analysis applies if instead of a zero fee we were to consider a fee equal to marginal costs.

Suppose that group $B$ is the zero-fee group. Using the equations from Lemma 1, we then must have

$$\begin{aligned} p_i^A(h_i^A, h_i^B) &= a_i^A - (1-\alpha^A)\log h_i^A + \beta^A \log h_i^B, \\ 0 &= a_i^B - (1-\alpha^B)\log h_i^B + \beta^B \log h_i^A. \end{aligned}$$

We rewrite the second equation as $\log h_i^B = a_i^B/(1-\alpha^B) + (\beta^B/(1-\alpha^B))\log h_i^A$ and substitute into the first equation to obtain (with an abuse of notation, we write $p_i^A$ as a function of $h_i^A$)

$$\begin{aligned} p_i^A(h_i^A) - c_i^A &= a_i^A - c_i^A - (1-\alpha^A)\log h_i^A + \beta^A \left[\frac{a_i^B}{1-\alpha^B} + \frac{\beta^B}{1-\alpha^B}\log h_i^A\right] \\ &= \tilde{a}_i^A - c_i^A - (1-\tilde{\alpha}^A)\log h_i^A \qquad (8) \end{aligned}$$

where $\tilde{a}_i^A := a_i^A + \frac{\beta^A}{1-\alpha^B} a_i^B$ and $\tilde{\alpha}^A := \alpha^A + \frac{\beta^A\beta^B}{1-\alpha^B}$. In the special case that users in the group with monetization (group $A$) do not care about the participation of the other group (i.e., $\beta = 0$), prices are the same as in the model with network goods.

Platform $i$'s profit as a function of $h_i^A$ and its aggregate is

$$\Pi_i(h_i^A, H^A) = \frac{h_i^A}{H^A}[p_i^A(h_i^A) - c_i^A] = \frac{h_i^A}{H^A}[\tilde{a}_i^A - c_i^A - (1-\tilde{\alpha}^A)\log h_i^A].$$

Then the analysis for platforms with only direct network effects for group $A$ applies after a change of variables from $(a_i^A, \alpha^A)$ to $(\tilde{a}_i^A, \tilde{\alpha}^A)$, where $\tilde{\alpha}^A < 1$ holds because this is equivalent to $\beta^A\beta^B < (1-\alpha^A)(1-\alpha^B)$ and implied by our assumption $\alpha^k + \beta^l < 1$ for $k, l \in \{A, B\}, l \neq k$. Thus, with the change of variables, Remark 2 applies and a unique price equilibrium exists.

**Remark 3.** *There exists a unique price equilibrium under one-sided pricing.*

The equivalence between the pricing of network goods and one-sided pricing is reminiscent of and extends the equivalence between direct and "indirect" network effects in the literature on network effects (e.g., Katz and Shapiro, 1985, and Church and Gandal, 1992, in oligopoly models different from ours), where positive indirect network effects would be the case that $\alpha^A = \alpha^B = 0$ and $\beta^A > 0, \beta^B > 0$.

Note that, in contrast to the setting with two-sided pricing, our analysis under one-sided pricing carries over to the case with negative cross-group network effects (as long as they are not too large), because the model can be translated into the model of network goods. This means that our framework can cover purely ad-funded media platforms under two-sided single-homing.

# 4. Equilibrium characterization results

**Platform type and market share** The relative position of a platform with respect to the size of its user groups is determined by its "type" $(v_i^A, v_i^B)$ where $v_i^k = a_i^k - c_i^k$ is the cost-adjusted quality that platform $i$ offers to group-$k$ users. Thus, $v_i^k$ stands for the platform's ability to provide value to group-$k$ users. Proposition 2 allows us to conduct an equilibrium analysis of platform oligopoly with arbitrary heterogeneity of platforms with respect to their cost-adjusted quality on each side. We first take a look at an individual platform (we make use of this lemma in the proofs of several of the following propositions).

**Lemma 4.**

1. *For any given aggregates $(H^A, H^B)$ and network size $(n_i^A, n_i^B) \in (0,1)^2$, there exists a type $(v_i^A, v_i^B)$ such that $h_i^k(H^A, H^B)/H^k = n_i^k$ for both $k \in \{A, B\}$.*

2. *For any given type $(v_i^A, v_i^B)$ and network size $(n_i^A, n_i^B) \in (0,1)^2$ of platform $i$, there exists a unique pair of aggregates $(H^A, H^B)$ such that $h_i^k(H^A, H^B)/H^k = n_i^k$ for both $k \in \{A, B\}$.*

Furthermore, for any market structure, we can find a profile of cost-adjusted qualities that decentralizes any market share allocation as an equilibrium outcome, as we formally establish in the following remark, where we define the aggregate type for group-$k$ users with $\bar{v}^k := \log \sum_{j=1}^M \exp\{v_i^k\}$ for $k \in \{A, B\}$.

**Remark 4.** *Pick any profile of network sizes $(n_i^A, n_i^B)_{i\in\{1,\dots,M\}}$ such that $\sum_{j=1}^M n_j^k = 1$ for $k \in \{A, B\}$.*

1. *In addition, pick any aggregates $(H^A, H^B) \in \mathbb{R}^2_{++}$. There exists a unique type profile $(v_i^A, v_i^B)_{i\in\{1,\dots,M\}}$ such that the equilibrium network sizes and aggregates in the price equilibrium are $(n_i^A, n_i^B)_{i\in\{1,\dots,M\}}$ and $(H^A, H^B)$, respectively.*

2. *In addition, pick any aggregate type* $(\bar{v}^A, \bar{v}^B) \in \mathbb{R}^2$. *There exists a unique type profile* $(v_i^A, v_i^B)_{i\in\{1,\dots,M\}}$ *generating aggregate type* $(\bar{v}^A, \bar{v}^B) \in \mathbb{R}^2$ *such that the equilibrium network sizes are* $(n_i^A, n_i^B)_{i\in\{1,\dots,M\}}$.

In the following, we address the question of how market shares, price-cost margins, and profits differ across different platforms when they are asymmetric with respect to what they offer to users in *one* group. We start with market shares.

**Comparison of market shares** In the following result we establish that the platform with higher cost-adjusted quality for one user group has a strictly larger market share for this user group and a weakly larger market share for the other user group – it is strictly larger if at least one of the cross-group network effects is positive ($\beta^A > 0$ or $\beta^B > 0$).

**Proposition 3.** *Take any two platforms i and j with* $v_i^A > v_j^A$ *and* $v_i^B = v_j^B$. *Then, in equilibrium,* $n_i^A > n_j^A$ *and* $n_i^B \geq n_j^B$. *Furthermore,* $n_i^B > n_j^B$ *if and only if* $\beta^A > 0$ *or* $\beta^B > 0$.

We also note that if a platform is of higher type for both groups (i.e., $v_i^A > v_j^A$ and $v_i^B > v_j^B$), then $n_i^A > n_j^A$ and $n_i^B > n_j^B$ for any network effects.

Our findings under two-sided single-homing can be contrasted to what happens under competitive bottleneck in Anderson and Peitz (2020). In that setting, platforms are asymmetric regarding the quality offered to single-homing users (say group $A$). When platforms set the participation level for multi-homing users (group $B$) and participation fees for single-homing users,[10] the higher-quality platform admits fewer multi-homing users.[11] In that setting, the higher-quality platform does not admit as many group-$B$ users as its lower-quality competitor and still attracts more single-homing users in equilibrium. Such a reduced number of group-$B$ users is attractive for the higher-quality platform because this raises revenues from the multi-homing group.

**Comparison of price-cost margins** We next look at the pricing implications for users in one user group (group $B$ in the proposition below) when platforms are asymmetric with respect to the other group. To do so, we consider two polar cases: (i) only the other group benefits from cross-group network effects and (ii) the reverse; that is, the group for which platforms are symmetric with respect to the cost-adjusted quality they offer to that group benefits from cross-group network effects.

[10] We take note that the models differ not only with respect to the homing assumption. Most importantly, Anderson and Peitz (2020) do not allow for setting the participation fee on the multi-homing side, which would complicate their analysis because of feedback loops, but instead assume that platforms set participation levels.

[11] See Proposition 11 in the online appendix of Anderson and Peitz, 2020. The relevant case for comparison is the one with positive cross-group network effects, which means that $\gamma < 0$ according to their notation.

**Proposition 4.** *Take any two platforms $i$ and $j$ with $v_i^A > v_j^A$ and $v_i^B = v_j^B$. (i) Suppose that $\beta^A > 0$ and $\beta^B = 0$. Then, the price-cost margin from group-$B$ users is smaller on the higher-quality platform $i$ than on $j$. (ii) Suppose that $\beta^A = 0$ and $\beta^B > 0$. Then, the price-cost margin from group-$B$ users is larger on the higher-quality platform $i$ than on $j$.*

Thus, it depends on the direction of cross-group network effects whether the user group that considers two platforms to be symmetric in their cost-adjusted quality (say group $B$) faces a higher or lower price-cost margin on the platform with higher cost-adjusted quality for the other user group (say group $A$). If only group $A$ benefits from cross-group network effects ($\beta^A > 0$, $\beta^B = 0$), the platform with the higher cost-adjusted quality for group $A$ sets a lower price-cost margin for group $B$ than the competing platform. This lower price-cost margin for group-$B$ users fosters the participation of those users. Since $\beta^A > 0$, this gives an extra push to group-$A$ users to join this platform. The platform with the higher (ex ante) cost-adjusted quality benefits more from this. This implies that the asymmetry between platforms for group $A$ is amplified.

In the opposite case, in which only group $B$ benefits from cross-group network effects ($\beta^A = 0, \beta^B > 0$), the platform with the higher cost-adjusted quality for group $A$ will have more group-$A$ participation, which translates into an endogenous quality advantage for group $B$, $\beta^B(\log n_i^A - \log n_j^A)$. In equilibrium, this results a higher price-cost margin to group-$B$ users than the one charged by the competing platform.

Next, we take a look at the user group that experiences different cost-adjusted qualities across platforms. One might expect that the platform that offers the higher cost-adjusted quality always has a higher price-cost margin for the same group (as would happen absent network effects, as shown in Proposition 1 in Anderson and de Palma, 2001 and Proposition 4 in Anderson et al., 2020). While this is correct under a number of conditions (parameter conditions or outcome variables), we show by example that this is not always the case.

**Remark 5.** *Take any two platforms $i$ and $j$ with $v_i^B = v_j^B$. Then, the price-cost margin for group-$A$ users is larger on platform $i$ with the higher cost-adjusted quality $v_i^A > v_j^A$ if cross-group network effects are not mutual, that is, (1) $\beta^A = 0$ or (2) $\beta^B = 0$, (3) platforms $i$ and $j$ attract weakly more users from group $A$ than $B$, or (4) platforms set fees above costs for both user groups. However, there are environments in which the platform with the lower quality has a higher price-cost margin for group-$A$ users; this can only happen if $\beta^A > 0$, $\beta^B > 0$ and, in equilibrium, $n_i^B > n_i^A$.*

We conclude that cost-adjusted quality differences between platforms for one user group (when cost-adjusted quality for the other user group is the same across platforms) give rise to non-trivial differences in user participation across platforms in the presence of cross-group network effects. This hints at platform asymmetry shaping the pricing structures of two-sided platforms. We can write the relation between market shares and price-cost margins as

$$\mu_i^A = \frac{1}{1-n_i^A}\left(1-\alpha^A-\beta^B\frac{n_i^B}{n_i^A}\right), \tag{9}$$

$$\mu_i^B = \frac{1}{1-n_i^B}\left(1-\alpha^B-\beta^A\frac{n_i^A}{n_i^B}\right), \tag{10}$$

where $\mu_i^k := p_i^k - c_i^k$, $k \in \{A, B\}$. Equations (9) and (10) show that it depends on the relative size $n_i^A/n_i^B$ of platform $i$ on the two sides whether the price-cost margin is positive or negative.

These price-cost margins can be related to cost-adjusted qualities, which are the primitives of our model. We introduce subsets of platforms $\mathcal{M}^A, \mathcal{M}^B, \mathcal{M}^{AB} \subseteq \{1, 2, \ldots, M\}$, where the superscript indicates the user group(s) for which the platform charges positive price-cost margins; that is $\mu_i^A \geq 0$ $\mu_i^B < 0$ for all $i \in \mathcal{M}^A$, $\mu_i^A \geq 0$ and $\mu_i^B \geq 0$ for all $i \in \mathcal{M}^{AB}$, and $\mu_i^A \geq 0$ and $\mu_i^B < 0$ for all $i \in \mathcal{M}^B$.

**Proposition 5.** *Suppose that $\beta^B \geq \beta^A$, $v_i^B = v^B$ for all $i = 1, \ldots, M$, and $v_1^A \geq v_2^A \geq \cdots \geq v_M^A$. Suppose also that $\beta^B > 0$. Consider any two platforms $i$ and $j$ belonging to different subsets $\mathcal{M}^A$, $\mathcal{M}^B$, and $\mathcal{M}^{AB}$.*

*(i) If $i \in \mathcal{M}^A$ and $j \in \mathcal{M}^{AB}$, then $i < j$.*

*(ii) If $i \in \mathcal{M}^{AB}$ and $j \in \mathcal{M}^B$, then $i < j$.*

*(iii) If $i \in \mathcal{M}^A$ and $j \in \mathcal{M}^B$, then $i < j$.*

*Furthermore, $\mathcal{M}^A \cup \mathcal{M}^B \cup \mathcal{M}^{AB} = \{1, 2, \ldots, M\}$ and for any $M \geq 3$ and $\beta^A > 0$, there exist cost-adjusted qualities such that none of the subsets is empty.*

In Proposition 5 it is postulated that platforms are asymmetric only with respect to the cost-adjusted quality that each platform offers to group-$A$ users. In such a case, platforms with high group-$A$ net quality regard group $A$ as the money side and group $B$ as the subsidy side because they earn more from the group-$A$ users. By contrast, platforms with low group-$A$ net quality regard the other group (group $B$) as the money side and group $A$ as the subsidy side. Those with intermediate group-$A$ net quality charge positive price-cost margins to both groups.

In monopoly settings, it has been shown that platforms tend to set a high price to the group that is less price sensitive (e.g., Armstrong, 2006; Weyl, 2010). Our novel result in asymmetric oligopoly is that different price sensitivities endogenously arise due to quality differences, thereby endogenously leading to opposing price structures across platforms.

**Profit comparison** Our framework also allows for a simple and intuitive characterization of profit rankings: the larger the net quality of a platform on one side, the larger is the equilibrium profit of the platform.

**Proposition 6.** *Consider two platforms $i$ and $j$ with $v_i^A > v_j^A$ and $v_i^B = v_j^B$. Then, in equilibrium, platform $i$ obtains higher profit than platform $j$.*

By Proposition 3, the higher-quality platform has larger market shares for both groups and, by Proposition 6, larger profits. The finding that market shares and profits are positively associated has been obtained in oligopoly with price competition and differentiated products absent network effects (Proposition 1 in Anderson and de Palma, 2001 and Proposition 4 in Anderson et al., 2020). It thus extends to platform oligopoly with two-sided single-homing in which user decisions in the two groups are interdependent through cross-group network effects, when platforms are ranked by cost-adjusted quality on one side. More generally, two platforms may be asymmetric with respect to both groups. Then, it might well be the case that a platform with lower market shares in both groups makes a higher profit than a rival (as was mentioned in the introduction and shown in Sato, 2021b confirming Belleflamme et al., 2022, the latter analyzing a linear duopoly setting).

In the proof of Proposition 6, we also establish that platform profit is decreasing in the equilibrium user surpluses $(CS^A, CS^B)$. This fact is used in the proofs of some of the results in Section 5.

**Network goods and two-sided platforms with one-sided pricing** In this subsection, we ask which of the results in the previous section depend on the platform's ability to charge both user groups. We continue to assume that platforms are asymmetric regarding the primitives of the model with respect to one of the two user groups. We then have to make the case distinction whether or not the asymmetry is on the zero-pricing side.

As a backdrop, let us study asymmetries in the model with a network good.

**Remark 6.** *In the model with network goods, take any two platforms with $v_i > v_j$ for some $i, j$. Then, in the unique equilibrium, $n_i > n_j$, $p_i - c_i > p_j - c_j > 0$, and $\Pi_i > \Pi_j > 0$.*

Thus, within-group network effects do not overturn the cross-section result in oligopoly models without any network effects. We now turn to the model with two user groups and one-sided pricing (i.e., $\max\{\beta^A, \beta^B\} > 0$) under the assumption that marginal costs are zero for the non-paying user group (i.e., the zero pricing side).

First, suppose that the zero pricing side is asymmetric (group $A$ according to the condition in the previous subsection) and symmetric for users who are charged (group-$B$ users, which implies that $a_i^B - c_i^B = a_j^B - c_j^B \equiv v^B$ for all $i, j$). We recall that the model is equivalent to the model with network effects in group $B$ after a change of variables to $\tilde{a}_i^B = a_i^B + (\beta^B/(1-\alpha^A))a_i^A$ and $\tilde{\alpha}^B = \alpha^B + \beta^A\beta^B/(1-\alpha^A)$. We write the new cost-adjusted quality as $\tilde{v}_i^B := v^B + (\beta^B/(1-\alpha^A))a_i^A$.

Since marginal costs for group-$A$ users are zero, the platform asymmetry is due to differences in $a^A$ only. Thus, as long as users in group $B$ care about the participation of group-$A$ users ($\beta^B > 0$), any asymmetry of the primitives in group $A$ gives rise to an induced asymmetry in group $B$ – that is, $a_i^A > a_j^A$ implies that $\tilde{v}_i^B > \tilde{v}_j^B$ if and only if $\beta^B > 0$. Then, we obtain from Proposition 6 that the platform with the larger $a^A$ has a larger market share of group-$B$ users, a higher price-cost margin on group-$B$ users, and larger profits. In particular, if a platform offers a larger quality than its rival on the zero-pricing side, this will translate into a larger market share for the corresponding user group and, thus, also lead to a larger market share in the market for group-$B$ users. Otherwise, if $\beta^B = 0$, group $B$ is isolated from the group-$A$ asymmetry and the outcome for group $A$ will be symmetric: all platforms have the same price-cost margin, the same market share, and the same profit, irrespective of the fact that $n_i^A > n_j^A$. This fact follows from the exogenous asymmetry $v_i^A > v_j^A$, which always enters the equation, and the endogenous asymmetry $n_i^B > n_j^B$ that reinforces the original asymmetry if and only if $\beta^A > 0$ and $\beta^B > 0$.

Second, suppose that the zero pricing side (group $A$) is symmetric (i.e., $a_i^A = a_j^A \equiv a^A$). The change of variable in quality then is $\tilde{v}_i^B := v_i^B + (\beta^B/(1-\alpha^A))a^A$, which leads to a parallel shift of $\tilde{v}_i^B$ compared to the setting with $\beta^B = 0$. Thus, as directly follows from Proposition 6, $v_i^B > v_j^B$ implies that $n_i^B > n_j^B$, $p_i^B - c_i^B > p_j^B - c_j^B > 0$, and $\Pi_i > \Pi_j$. For $\beta^A > 0$, the asymmetry of primitives regarding group $B$ also leads to an asymmetric outcome for group $A$; that is, $v_i^B > v_j^B$ implies that $n_i^A > n_j^A$. By contrast, for $\beta^A = 0$, market shares regarding group-$A$ users must be symmetric because the platform does not have any price instrument for this group of users.

It is obvious that negative price-cost margins for group-$B$ users can not be an equilibrium outcome under one-sided pricing because this necessarily leads to losses of the platform. This implies that the possibility of heterogenous cross-subsidization strategies in equilibrium (as shown in Proposition 5) can not arise under one-sided pricing.

Qualitative results on market shares under one-sided pricing are broadly in line with those under two-sided pricing. However, there are some differences: As follows from Proposition 3, if cost-adjusted qualities are asymmetric in group $B$ only, then under two-sided pricing, $v_i^B > v_j^B$ implies that $n_i^A > n_j^A$ if and only if $\beta^A > 0$ or $\beta^B > 0$. Under one-sided pricing, we have that $n_i^A > n_j^A$ if and only if $\beta^A > 0$, since platforms lack a pricing instrument for group $A$. Thus, for $\beta^B = 0$ and $\beta^A > 0$ we have that $v_i^B > v_j^B$ implies that $n_i^A > n_j^A$ under two-sided pricing but $n_i^A = n_j^A$ under one-sided pricing.

The restriction to one-sided pricing does not affect the profit ranking with one exception: A platform that provides a higher cost-adjusted quality for one user group (and the same for the other group) makes higher profits under one-sided as well as under two-sided pricing, unless the asymmetry applies to the non-paying group and this group does not exert a positive cross-group effect on the paying group in which case each platform makes the same profit despite the asymmetry.

# 5. Active platforms, platform quality, and compatibility

In this section, we investigate comparative statics properties of three shocks or interventions: changes to the set of active platforms, changes to the incumbent platforms' characteristics under free entry, and partial compatibility.

## 5.1. Active platforms

We provide comparative statics results about the effects of an additional platform becoming active. In other words, the number of active platforms increases from $M \geq 2$ to $M+1$ platforms. Under the selection criterion of asymptotic stability at the participation stage, this is equivalent to exogenous entry of a new platform. Using an alternative selection criterion, adding a platform to the set of active platforms amounts to an additional platform overcoming the curse of unfavorable user beliefs.[12] For ease of exposition, in what follows we speak of platform entry when an additional platform becomes element of the set of active platforms.

It is instructive to first consider the case in which all platforms are symmetric. In the symmetric setting with $M$ platforms, we must have $n_i^A = n_i^B = 1/M$ for all $i \in \{1, \dots, M\}$ in any equilibrium. Profit maximization requires that $\partial \Pi_i / \partial h_i^k = 0$ for all $i \in \{1, \dots, M\}$ and $k \in \{A, B\}$. Using symmetry ($n_i^A = n_i^B = 1/M$), first-order conditions for group $A$ become

$$\left(1 - \frac{1}{M}\right) \left[ v^A - (1-\alpha^A) \log\left(\frac{1}{M}\right) + \beta^A \log\left(\frac{1}{M}\right) - CS^A \right] - 1 + \alpha^A + \beta^B = 0.$$

These first-order conditions can be rewritten as price-cost margins

$$p^{k*} - c^{k*} = \frac{M}{M-1} \left(1 - \alpha^k - \beta^l\right),$$

[12]See footnote 6. If an entrant can not overcome this curse, it can not successfully use divide-and-conquer strategies according to which it would subsidize one group to make sure that some users from this group join and monetize through the other group (on the use of divide-and-conquer strategies with homogeneous platforms, see e.g. Caillaud and Jullien, 2003). Under our logarithmic network effect functions even extreme subsidization does not achieve this (despite the fact that platforms are horizontally differentiated) and active platforms do not adjust their prices in response to entry by an entrant facing unfavorable beliefs.

for $k, l \in \{A, B\}, l \neq k$. This leads to the equilibrium profit and group-$k$ user surplus:

$$\Pi^* = \frac{1}{M-1}\left(2 - \alpha^A - \alpha^B - \beta^A - \beta^B\right)$$
$$CS^{k*} = v^k + (1 - \alpha^k - \beta^k)\log M - (1 - \alpha^k - \beta^l)\frac{M}{M-1}.$$

Taking derivatives of the equilibrium outcomes with respect to $M$, we obtain

$$\frac{\partial p^{k*}}{\partial M} = -\frac{(1 - \alpha^k - \beta^l)}{(M-1)^2} < 0,$$
$$\frac{\partial \Pi^*}{\partial M} = -\frac{(2 - \alpha^A - \alpha^B - \beta^A - \beta^B)}{(M-1)^2} < 0,$$
$$\frac{\partial CS^{k*}}{\partial M} = \frac{1 - \alpha^k - \beta^k}{M} + \frac{1 - \alpha^k - \beta^l}{(M-1)^2} > 0.$$

Thus, exogenous platform entry always leads to lower prices, lower platform profits, and higher user benefits for each group in symmetric environments, which is in line with findings in standard oligopoly. By contrast, Tan and Zhou (2021) provide an example in a symmetric setting such that exogenous entry can lead to higher prices, higher platform profits, and lower user benefits.

To understand the difference between our finding and the one in Tan and Zhou (2021) of the effect of entry on prices, consider the special case that $c_i^k = 0$, $k \in \{A, B\}$. However, suppose that the network effect takes the more general form $\gamma^{kl}(n_i^l)$ – in our model, $\gamma^{kk}(n_i^k) = \alpha^k \log n_i^k$ and $\gamma^{kl}(n_i^l) = \beta^k \log n_i^l$ for $l \neq k$. As Tan and Zhou (2021) show, the symmetric equilibrium price $p^{k*}$ can be written as

$$p^{k*} = \frac{M}{M-1} - \frac{1}{M-1}\sum_{k' \in \{A,B\}}\left(\left.\frac{\partial \gamma^{k'k}(n_i^{k*})}{\partial n_i^k}\right|_{n_i^k = 1/M}\right). \tag{11}$$

The first term is the standard market power term, which is decreasing in $M$ but the second term may be increasing in $M$ depending on the shape of $\gamma^{k'k}(\cdot)$. The second term reflects the fact that network effects drive pricing incentives, which depend on the number of active firms. Entry reduces the relative size advantage of a platform that attracts an additional unit mass of group-$k$ users since each of the $M-1$ competitors loses $1/(M-1)$. Holding marginal network benefits constant, entry lowers the incentives to reduce price. When marginal network benefits are not constant, the extent to which size advantage matters depends on the marginal network benefit functions $(\partial \gamma^{k'k}/\partial n_i^k)_{n_i^k = 1/M}$. Tan and Zhou (2021) use an example with linear network benefit functions (i.e., $\gamma^{kk}(x) = \bar{\alpha}^k x$ for $\bar{\alpha}^k \geq 0$ and $\gamma^{kl}(x) = \bar{\beta}^k x$ for $l \neq k$) and show that $p^{k*}$ and $\Pi^*$ are increasing in $M$ for $M$ sufficiently small and $\bar{\alpha}^k + \bar{\beta}^l$ sufficiently large (see their

Example 4). While the second term continues to be increasing in entry in our setting (as in the linear case), for any parameter values satisfying our assumption $\alpha^k + \beta^l < 1$, it is always dominated by the decrease of the first term, leading to price-decreasing entry.[13] Intuitively, with a strictly concave network benefit function, platforms have a stronger incentive to reduce price after entry than with a linear network benefit function since the marginal network benefit increases with entry.

We turn to the case in which platforms are asymmetric. We establish below that in that case, one user group may be worse off after a new platform enters (while the other group is better off).

**Proposition 7.** *Consider the effect of entry of platform $E$ on user surplus.*

1. *For any given entry of a platform with $(a_E^A, c_E^A, a_E^B, c_E^B)$, there exists a value $\underline{\beta}$ such that entry increases user surplus for both groups if $\beta^A < \underline{\beta}$ and $\beta^B < \underline{\beta}$.*

2. *For any given $\beta^A > 0$, there exists a type of platform with $(a_E^A, c_E^A, a_E^B, c_E^B)$ such that the minimal user surplus of group $A$ or group $B$ decreases after entry.*

3. *Entry increases the minimal or maximal user surplus of at least one user group.*

Proposition 7-1 shows that in the absence of cross-group network effects ($\beta^A = \beta^B = 0$), $H^k$ increases with entry and, thus, user surplus must go up. While this property is satisfied in standard oligopoly models without network effects, it is a priori not obvious that this result carries over to a model with network effects. The reason is that, under full participation, the entering platforms attract users from the incumbent platforms reducing the network benefits of the users active on incumbent platforms due to reduced participation on those platforms. Nonetheless, in our setting, entry of a new platform always benefits users if cross-group network effects are zero (or sufficiently weak). Proposition 7-1 establishes this result.

In the presence of cross-group network effects, entry of a platform may hurt one of the user groups, as established in Proposition 7-2. The proof of Proposition 7-2 indicates that an instance of entry that lowers the user surplus for one group (group $A$) is the entry of a platform that primarily caters to the needs of the other user group ($a_E^B - c_E^B$ large, and $a_E^A - c_E^A$ small and possibly negative); one may call such a platform "highly focused" on one user group. In such a case, entry will not add surplus to group-$A$ users, but reduces the market shares of the incumbent platforms. This reduces the network benefits that group-$A$ users enjoy from joining

[13] With a logarithmic network benefit function, our assumption $\alpha^k + \beta^l < 1$ is needed to obtain compact strategy sets and prevent a platform from setting infinitely low price on one side to enjoy monopoly power on the other side.

existing platforms or the incumbent platforms' incentive to attract group-$A$ users. Then, such entry lowers group-$A$ user surplus.

Although entry may harm one user group, Proposition 7-3 establishes that at least one user group benefits from entry. Our results indicate that the welfare effects of entry of a two-sided platform crucially depend on the characteristics of the entrant. Entry of a highly focused platform may hurt the users in the group that the entrant is not focused on.

According to the existing literature, entry may hurt users even under platform symmetry (Tan and Zhou, 2021, Example 4). Other works address welfare effects of platform entry in different market environments. Correia-da-Silva, Jullien, Lefouili and Pinho (2019) consider homogeneous-product Cournot platform models and examine the welfare effects of exogenous entry. They find that platform entry may reduce consumer surplus of all groups due to the fragmentation of network benefits; Gama, Lahmandi-Ayed and Pereira (2020) find such a result when the platform caters to a single user group and this group experiences network effects. Anderson and Peitz (2020) consider an asymmetric platform oligopoly in which one user group multi-homes and the other single-homes (competitive bottleneck) and study the consumer welfare effect of platform entry (see footnote 14).

With mutual cross-group network effects and asymmetric platforms, entry might affect the platforms' price structure qualitatively differently than for networks (i.e., settings with within-group network effects only). The following example illustrates such a possibility (we provide a proof in the Appendix).

**Example 1** (Entry into a previously symmetric market)**.** *Suppose that two symmetric platforms with the cost-adjusted quality $(v_I^A, v_I^B)$ were active before entry. Then, in the pre-entry equilibrium, price-cost margins are positive for both groups. If $\beta^A > 0$ and $\beta^B > 0$, there exist entrant types $(v_E^A, v_E^B)$ such that there is a post-entry equilibrium in which $\mu_I^B < 0$ and $\mu_E^A < 0$ and other entrant types such that $\mu_I^A < 0$ and $\mu_E^B < 0$.*

Hence, the asymmetry induced by entry or exit qualitatively affects the price structure of competing platforms. Platform entry may lead to negative price-cost margins of incumbent platforms for one user group in situations in which their margins were positive absent entry. Here subsidization of one group is a response to entry of a platform that is more attractive to that group.

Next, we turn to the effect of platform entry on profit. As we show in the following proposition, entry may increase the profit of incumbent platforms due to the asymmetry it introduces in the market. This also implies that industry profits increase.

**Proposition 8.** *There exist pairs of a pre-entry conditions, entrant types, and a post-entry equilibrium such that entry increases the profit of incumbent platforms. In such a case, entry* *necessarily reduces the user surplus of one group.*

Proposition 8 establishes that incumbent platforms may benefit from entry of another platform. For example, when incumbents are symmetric before the entry, they compete rather fiercely with each other for both user groups. After entry, the incumbent platforms sacrifice a large share of one user group (say group $B$) even though they now subsidize that group because they offer much lower cost-adjusted quality than the entrant to this group. At the same time, because the incumbent platforms have become less inclined to compete for group-$B$ users through an increase of their group-$A$ user base, they increase their margins for group $A$ but lose rather few group-$A$ users to the entrant because the entrant offers a low cost-adjusted quality to that group. This softening of competition for group-$A$ users increases incumbents' profits from that group, which may dominate the profit loss in the market for group-$B$ users.

We present a numerical example to illustrate this finding. Prior to entry there are two symmetric incumbents. Suppose that $\alpha^A = \alpha^B = 0$, $\beta^A = \beta^B = 0.95$, and the incumbents' type $(v_I^A, v_I^B) = (0, 0)$. Since platforms are symmetric, we have that $n_I^A = n_I^B = 0.5$. Furthermore, the two groups are symmetric, and therefore, price-cost margins are the same for the two groups. The pre-entry outcome is reported in Table 2. Suppose now that an entrant of type $(v_E^A, v_E^B) \simeq (-7.43, 9.66)$ enters. Thus, the entrant is more attractive for group-$B$ users and less attractive for group-$A$ users. The equilibrium reflects these differences: The entrant makes large inroads in the market for group-$B$ users (market share of 80 %) and obtains a smaller market share than the incumbents in the market for group-$A$ users (market share of 20 %) even though it subsidizes group-$A$ users and affords a high price-cost margin for group-$B$ users (we find that $\mu_E^A \simeq -3.5$ and $\mu_E^B \simeq 3.812$). This leads to losses in the market for group-$A$ users and profits in the market for group-$B$ users ($\Pi_E^A \simeq -0.7$ and $\Pi_E^B \simeq 3.05$). Entry is profitable, but also raises profits of the incumbents' profits, as can be seen from the last column in Table 2. The striking feature is that the incumbents can double their profits even though they face an entrant with a far superior cost-adjusted quality for one group such that it makes much higher profits than each of the two incumbents. Furthermore, such entry lowers the user surplus for one group: Before entry, the equilibrium user surplus of each of the two groups is $CS^{A*} = CS^{B*} \simeq -0.065$. After entry, we have $CS^{A**} \simeq -2.54$ and $CS^{B**} \simeq 4.54$. Hence, platform entry in this example benefits group-$B$ users but hurts group-$A$ users.

This result stands in stark contrast to results in standard oligopoly: Entry increases the competitive pressure and therefore reduces incumbents' price-cost margins and profits. It also stands in contrast to the setting with within-group network effects only, where entry always increases the equilibrium aggregate, as follows from Proposition 7-1. As a result, price-cost margins and platform profits are necessarily lower after entry in the model with network goods.

|  | $n_I^A$ | $n_I^B$ | $\mu_I^A$ | $\mu_I^B$ | $\Pi_I^A$ | $\Pi_I^B$ | $\Pi_I$ |
|---|---|---|---|---|---|---|---|
| Pre-entry | 0.5 | 0.5 | 0.10 | 0.10 | 0.05 | 0.05 | 0.10 |
| Post-entry | 0.4 | 0.1 | 1.27 | -3.11 | 0.51 | -0.31 | 0.20 |
| Difference | -0.1 | -0.4 | 1.17 | -3.01 | 0.46 | -0.36 | 0.10 |

Table 2: A numerical example illustrating Proposition 8 with $\alpha^A = \alpha^B = 0$, $\beta^A = \beta^B = 0.95$, and $(n_I^A, n_I^B) = (0.4, 0.1)$.

Furthermore, since the setting with cross-group network effects and one-sided pricing presented in Section 3.4 only requires a change of variable in the setting with within-group network effects only, price-cost margins and platform profits are also lower after entry under one-sided pricing. Our finding in Proposition relates to the finding by Tan and Zhou (2021) that in their more flexible but symmetric setting, platform entry can increase incumbent platforms' profits – in our model, such an increase can not happen under platform symmetry.

## 5.2. Shocks to incumbent platforms under free entry

To study long-run competition, we consider platform competition under free entry of "fringe" platforms. To this end, we extend the baseline framework by incorporating symmetric entrants as in Anderson, Erkal and Piccinin (2013).

Suppose that, along with $M_I \geq 1$ incumbents $\{1, \ldots, M_I\}$, $\bar{M}_E \geq 1$ (potential) entrants $\mathcal{E} := \{M_I + 1, \ldots, M_I + \bar{M}_E\}$ choose whether to enter. Entrants $e \in \mathcal{E}$ all have the same characteristics $\left(a_E^A, a_E^B, c_E^A, c_E^B\right)$ and incur entry cost $K > 0$ to become active. Incumbent platform $i \in \{1, \ldots, M_I\}$ has characteristics $(a_i^A, a_i^B, c_i^A, c_i^B)$ that may differ from those of other platforms. We assume that entry costs are such that some of the potential entrants become active and the number of potential entrants $\bar{M}_E$ is sufficiently large to ensure that the number of active entrants $M_E$ is less than $\bar{M}_E$. In our analysis we ignore integer constraints.

Let $\pi_E\left(H^A, H^B\right)$ be the post-entry profit of an entrant when it optimally chooses the action variables $(h_E^A, h_E^B)$ and the values of the aggregates are given by $(H^A, H^B)$. Specifically, the post-entry profit with aggregates $(H^A, H^B)$, $\pi_E\left(H^A, H^B\right)$, is given by

$$\pi_E(H^A, H^B) := \Pi_E(h_E^A(H^A, H^B), h_E^B(H^A, H^B), H^A, H^B)$$

Using this notation, we define the free-entry equilibrium as follows.

**Definition 2.** *The number of active entrants $M_E$ constitutes a free-entry equilibrium if the* *triple* $\left(H^A, H^B, M_E\right)$ *satisfies the following conditions:*

$$\begin{aligned} \pi_E\left(H^A, H^B\right) - K &= 0, \quad (12) \\ \sum_{i=1}^{M_I} h_i^A\left(H^A, H^B\right) + M_E h_E^A\left(H^A, H^B\right) &= H^A, \\ \sum_{i=1}^{M_I} h_i^B\left(H^B, H^A\right) + M_E h_E^B\left(H^B, H^A\right) &= H^B. \end{aligned}$$

The definition of free-entry equilibrium endogenizes the number of active entrants $M_E$ through the zero-profit condition (12). Entrants sequentially enter as long as the post-entry profit exceeds the entry cost, and the entry stops once additional entry becomes unprofitable. Using Definition 2, we examine the welfare effects of a shock to the incumbent platforms' characteristics, which is captured by a change in $(a_i^A, a_i^B, c_i^A, c_i^B)$ for $i \in \{1, \ldots, M_I\}$.

In the aggregative game analysis of standard oligopoly, the zero-profit condition of entrants uniquely pins down the value of single aggregate (e.g., Davidson and Mukherjee, 2007; Ino and Matsumura, 2012; Anderson et al., 2013, 2020). Because consumer surplus is determined solely by the value of the aggregate, any change in the competitive environment, such as incumbents' investment and platform mergers does not affect consumer surplus, as long as there is at least some entry. By contrast, with two-sided platforms, the zero profit condition (12) only pins down the relation between the two aggregates $(H^A, H^B)$. Therefore, the competitive environments are no longer necessarily neutral to the user surplus in each group and the total user surplus. In a particular setting, we establish a *strong see-saw property*: any change in the competitive environment that increases user surplus of one group reduces user surplus of the other group.

For instance, suppose that an incumbent invests in group-$A$ benefit $a_i^A$ so that entrants' network size on group $A$ decreases. In a standard oligopoly, competition for group-$A$ users becomes more intense due to the incumbent's investment. As an equilibrium response, fewer entrants will join, so the competition for group-$A$ users becomes weaker. In two-sided markets, a more subtle strategic interaction may exist due to network effects and implied changes in the two-sided pricing structure.

**Proposition 9.** *Consider a free-entry equilibrium with a non-empty set of entrants.Then, any change in competitive environments that increases the surplus of one user group decreases the surplus of the other user group. Formally, holding the parameters* $(\alpha^A, \alpha^B, \beta^A, \beta^B, a_E^A, a_E^B, c_E^A, c_E^B, K)$ *fixed, compare two free-entry equilibria that differ in other parameters. Denoting the equilibrium surplus of the two user groups under the two settings by* $(CS^{A*}, CS^{B*})$ *and* $(CS^{A**}, CS^{B**})$*, we have that*

$$\left(CS^{A*} - CS^{A**}\right)\left(CS^{B*} - CS^{B**}\right) < 0.$$

Because the post-entry profit of $\Pi_E$ is decreasing in user surpluses $(CS^A, CS^B)$, to keep $\Pi_E$ constant, any increase in $CS^A$ must be compensated by a corresponding decrease in $CS^B$. Hence, Proposition 6 establishes a strong see-saw property in user surplus.

The strong see-saw property poses a challenge to competition authorities evaluating business practices of large incumbent platforms in an environment with fringe platforms. Because an incumbent platform's practice generically benefits users in one group at the expense of those in the other group, the competition authority must decide which group to protect (or which weights to give them in an overall consumer welfare ranking). In the context of e-commerce, some authorities focus on private consumers, which is in line with a narrow interpretation of the consumer welfare standard. For instance, Khan (2017) argues that such an approach fails to recognize other harms of incumbent platforms' practices, including the harm to third-party sellers, which can be included under a broader interpretation of the consumer welfare standard. Proposition 9 establishes that there is a conflict between what benefits users of one group and what benefits the other. This conflict is inevitable in two-sided platform competition with free entry of the type studied in this paper.[14]

Regarding the welfare property of free entry, note that Tan and Zhou (2021) show the following: When taste shocks follow the type-I extreme value distribution, platform entry is socially excessive (see their Lemma 2 and the following paragraph). Thus, platform entry is socially excessive in our model when platforms are symmetric.[15]

## 5.3. Partial compatibility

In this section we address how an increase of the degree of compatibility affects market shares, prices, and user surplus. Suppose that there are only within-group network effects and, thus, each user group can be analyzed in isolation. Partial compatibility implies that a fraction of network effects are industry-wide. It is gained if some of the functionalities are available to all users, not only those on the same platform, but also those on competing platforms. The fraction of functionalities available to all users is denoted by $\lambda$, the degree of compatibility. An example of a regulatory intervention with the goal to increase compatibility is Article 7 in the Digital Markets Act (DMA) in the European Union. According to this regulation, a gatekeeper of a number-independent interpersonal communications service must "make the basic function-

[14] Anderson and Peitz (2020) establish a strong see-saw property of exogenous entry for purely ad-funded media platforms in competitive bottleneck with advertisers exerting a negative effect on consumers. As they show, their competitive bottleneck model fails to give rise to the see-saw property if consumers benefit from more advertisers (i.e., advertisers exert a positive cross-group network effect on consumers) or if media platforms can set a fee on the consumer side.

[15] For conditions on the distribution function of the taste shocks that lead to excessive entry, see also Tan and Zhou (2024, Proposition 2).

alities of its number-independent interpersonal communications services interoperable with the number-independent interpersonal communications services of another provider."[16]

Partial compatibility allows users to benefit from the presence of users on different platforms.

$$
\begin{aligned}
u_i^k &= a_i^k - p_i^k + \lambda \alpha^k \log \sum_{j=1}^{M} n_j^k + (1-\lambda)\alpha^k \log n_i^k + \varepsilon_i^k \\
&= a_i^k - p_i^k + (1-\lambda)\alpha^k \log n_i^k + \varepsilon_i^k.
\end{aligned}
$$

By Remark 2 there exists a unique price equilibrium for any value of $\lambda \in [0,1]$.

How does the equilibrium depend on the degree of compatibility $\lambda$? The general answer is the following and has been formalized by Crémer, Rey and Tirole (2000) in the Katz-Shapiro model: A decrease in compatibility increases the quality differentiation between two asymmetric platforms. The larger platform, which relies relatively less on access to the other platform's users, gains a competitive advantage, and competition between the two platforms is softened.[17]

Our framework provides related insights (for more details and additional insights see Appendix A.2).[18] When the degree of compatibility is increased, lower-quality platforms gain market share while higher-quality platforms lose and, thus, industry concentration (e.g., measured by the HHI) goes down (Proposition A.1 in Appendix A.2). If the asymmetry between platforms is sufficiently small, increased compatibility reduces the intensity of price competition and platforms set higher prices. Nevertheless, users benefit from increased compatibility since the direct effect dominates the effect on prices. As we establish in the duopoly case, if the asymmetry is sufficiently large, the platform with the higher cost-adjusted quality sets a lower price after an increase in compatibility (Proposition A.2 in Appendix A.2).

Restricting attention to the duopoly case, we also address the effect of compatibility on industry concentration under cross-group network effects, where we consider the case that partial compatibility applies to both user groups. Confirming the result derived under within-group network effects only, we show that compatibility mitigates industry concentration – that is, for each user group $k \in \{A, B\}$, an increase of the degree of compatibility decreases the

[16] The provision applies only to gatekeeper platforms and interoperability has to be offered upon the request of another provider. As a caveat, our model does not accommodate the situation that some but not all of the competing providers ask for interoperability.

[17] Crémer, Rey and Tirole (2000) study connectivity between asymmetric internet backbone providers in the Katz-Shapiro model (Katz and Shapiro, 1985). Our statement is a minor rephrasing of their explanation that "connectivity creates a quality differentiation between the two networks. The larger backbone, which relies relatively less on access to the other backbone's customers, gains a competitive advantage, and competition between the two backbones is softened." (Crémer, Rey and Tirole, 2000, p. 435)

[18] There is an important difference in the nature of the asymmetry in our framework compared to the one considered by Crémer, Rey and Tirole (2000). In the latter, full compatibility makes platforms symmetric, as the asymmetry between platforms is due to size differences in the installed base. By contrast, in our framework, even under full compatibility, platforms are asymmetric.

group-$k$ market share of the platform that is of higher cost-adjusted quality for each user group (see Proposition A.3 in Appendix A.2).

Multi-homing is alternative way for each user to "better" interact with other users. The more users multi-home, the larger is the number of users any single-homing user has access to (for given relative market shares of platforms among single-homing users). However, our model of partial compatibility does not translate into a model in which a fraction $\lambda$ of users multi-home.[19] A single-homing user has then access to all single-homing users on the same platform and all multi-homing users and the network benefit function becomes $\alpha^k \log(\lambda + (1-\lambda)n_i) = \alpha^k \log(\lambda \sum_{j=1}^M n_j^k + (1-\lambda)n_i)$, which is different from the function under partial compatibility, $\lambda\alpha^k \log \sum_{j=1}^M n_j^k + (1-\lambda)\alpha^k \log n_i = (1-\lambda)\alpha^k \log n_i$. Furthermore, we can not use aggregative games tools in such a model. We note that under linear network effects, the two functions would be the same.[20]

# 6. Partially covered markets

One version to analyze partial coverage is to assume that the outside option is also subject to the same network effects and idiosyncratic taste shocks as the for-profit platforms (for details, see Appendix A.3.1). This applies if choosing the outside option consists in choosing a non-commercial offer that is free of charge. For example, this could be an open-source software platform or programming language that is provided free of charge and brings together users and developers. Our model in Section 2 can easily be generalized and accommodate such a free platform by adding platform 0 that offers quality $a_0^k$ to side $k \in \{A, B\}$ at zero price, $p_0^k = 0$. Following our change of variables, platform 0 then offers $(h_0^A, h_0^B)$, which is independent of the choices offered by the for-profit platforms, and we write $H^k = \sum_{i=0}^M h_i^k$. The equilibrium characterization of the participation game (Remark 1) and the existence of a non-empty ordered set of price equilibria (Proposition 2) generalize to the introduction of such an outside option. Also, the characterization results of a price equilibrium in Section 4 continue to hold. In the presence of outside options for each user group, it is of interest to consider comparative statics in the attractiveness of the outside options: As the outside option becomes more attractive for group-$k$ users, user surplus of this group will increase, whereas user surplus of the other group will (weakly) decrease.

[19] For example, such partial multi-homing may be the result of users having installed a multi-homing device such a meta search engine that allows them to access all platforms or may reflect an environment in which a fraction of users has chosen the option to be visible on different messaging services, while others declined the offer.

[20] Even with linear network effects, there may be an interesting interplay between multi-homing and compatibility; for an analysis in symmetric Hotelling duopoly with linear network effects, see Doganoglu and Wright (2006).

Another version to analyze partial coverage is to assume that users make an opt-in decision upfront – that is, before observing their taste realization and before observing the prices set by the platforms (for details, see Appendix A.3.2). The idiosyncratic outside option is given by $a_0^k + \theta\varepsilon_0^k$, where, for each group-$k$ user, $\varepsilon_0^k$ is an i.i.d. draw from some distribution function. We adopt the concept of fulfilled expectation equilibrium in the spirit of Katz and Shapiro (1985), where users make opt-in decisions by forming an expectation over preferences and prices, and platforms take the aggregate user base $(N^A, N^B)$ as given when they set prices. We characterize the unique interior participation equilibrium for given platform prices (Proposition A.4) and illustrate for the case that outside options are exponentially distributed. We also show the existence of a non-empty ordered set of price equilibria (Proposition A.4) and that (under some condition) price equilibria are ordered by group-$k$ user surplus such that the $CS^k$-maximal equilibrium is the $CS^l$-minimal equilibrium for $k, l \in \{A, B\}$, $l \neq k$ (Proposition A.5). The characterization results of a price equilibrium in Section 4 continue to hold, but comparative statics analysis in this setting is generally complicated. Nevertheless, if users enjoy within-group network effects alone, we provide a comparative statics result with respect to the base attractiveness of the outside option $a_0^k$ for group-$k$ users. A higher $a_0^k$ reduces user participation and thereby the network benefits that can be obtained. As we show for the case in which the idiosyncratic component of the outside option is exponentially distributed, given $a_0^k$ sufficiently larger, this may end up hurting users overall (Remark A.3) because users who decide not to opt in exert a negative externality on users who opt in. We note that this result is not driven by platform asymmetry and can also be obtained under symmetry.

A third version to analyze partial coverage is to assume that users simultaneously decide whether and which platform to join, after observing platform prices (for details, see Appendix A.3.3). We characterize the interior participation equilibrium in this setting, in which the aggregates from the setting with full coverage are replaced by augmented aggregates that account for the fact that some users abstain from joining a platform (Proposition A.6). The existence of a price equilibrium holds for sufficiently unattractive outside options. While this version appears to be a natural way to introduce outside options, we did not find it tractable.

# 7. Discussion and conclusion

We propose a two-sided single-homing model of platform competition that features differences between platforms with respect to (i) marginal costs incurred for users of the two groups and (ii) the utility that platforms offer to their users (for given participation rates by both groups). Incorporating platform asymmetries provides a rich setting that allows us to explore the relative outcomes of platforms in equilibrium and the impact of exogenous shocks on the performance of different platforms. After establishing the existence and uniqueness of the participation equilibrium for a given set of active platforms, we characterize the equilibrium outcome under price competition and obtain insights with respect to exogenous platform entry, incumbent platform investments under free entry, and mandated partial compatibility. Our analysis makes use of the IIA structure of the demand systems of both groups. Platform profits can be written as functions of two action variables and their aggregates (as the sum of action variables across platforms).

We follow the seminal work on platform competition and focus on the platform's pricing decisions. Our analysis can be extended to cover other design decisions if these decisions are taken concurrently with the pricing decision.[21] It is also interesting to extend the analysis to environments in which platforms do not charge any fees to one user group, but can use non-price strategies that directly affect the attractiveness of the platform for that group. For example, social media platforms typically charge advertisers but do not charge end users and devise non-price strategies to attract end users. We leave extensions in this direction for future work, as they are outside the canonical platform competition model.

We make the functional form assumption that network effects enter as logarithmic functions of participation numbers of each group into user utility and that users experience taste shocks that lead to a logit structure. This specification can be seen as a special case of the model of Tan and Zhou (2021). While such a logarithmic specification of network effects is popular in empirical work, most previous theoretical work assumed linear network effects and few theoretical studies allow for more general forms of network effects (Hagiu, 2009; Weyl, 2010; Belleflamme and Peitz, 2019; Tan and Zhou, 2021). Within the logit demand setting, any generalization beyond logarithmic network effects would make it impossible to obtain closed-form solutions for the participation equilibrium and to subsequently write the platforms' profit functions as a function of their action variables and the aggregates thereof.

In our framework, users draw idiosyncratic taste shocks that enter their utility function as a stand-alone value. Users may also be heterogeneous regarding their sensitivity to network size (i.e., group-$k$ users may differ in their network effect parameters $\alpha^k$ and $\beta^k$). Unfortunately, the aggregative game framework is not sufficiently malleable to accommodate such a heterogeneity. We can think of our analysis as analyzing the model in which all users are of the "average" type $(\mathbb{E}[\alpha^k], \mathbb{E}[\beta^k])$. Presuming that there is an equilibrium also with heterogeneous network effects, we conjecture that our characterization results hold by continuity in a setting close to the limit when the heterogeneity disappears. We also conjecture that introducing heterogeneous

[21] In this case, platforms compete in utilities $\bar{u}_i^k = a_i^k - p_i^k$ for users and platforms may increase value $a_i^k$. In particular, suppose that there is a one-to-one relationship between value $a_i^k$ and per-user cost $c_i^k$ that depends on the user group and the identity of the platform. Thus, we can write $c_i^k(a_i^k)$, and platforms set $a_i^k$ such that $c_i^k(a_i^k)' = 1$.

network effects would lead to composition effects: In the case of heterogeneous cross-group network effects, a platform that has a higher share of group-$A$ users than another platform will attract relatively more group-$B$ users that are particularly sensitive to network effects.[22]

Arguably, the canonical model of platform competition features two-sided single-homing. This specification is widely adopted by the literature, including by Armstrong (2006), Jullien and Pavan (2019), and Tan and Zhou (2021). In various real-world environments, however, some users in one or both groups can multi-home (see e.g. Armstrong, 2006, section 5, and Anderson and Peitz, 2020, section 6, for the former and Bakos and Halaburda, 2020, Adachi, Sato and Tremblay, 2023, and Teh, Liu, Wright and Zhou, 2023, for the latter).[23] As pointed out in Section 5.3, when a fraction of users multi-homes, our model loses the aggregative game property and our analysis does not extend to such more-complex homing decisions.

[22]In a model in which users are only heterogeneous with respect to their sensitivity to network size, Ambrus and Argenziano (2009) show in a symmetric duopoly setting the emergence of asymmetric equilibria: Platform 1 sets a lower price for group-$k$ users than platform 2 and a higher price for uses of the other group $l$. Less-sensitive group-$k$ then buy from platform 1, whereas more-sensitive group-$k$ users from platform 2. Because of the heterogeneity, there is endogenous differentiation between platforms, which allows them to make positive profits in equilibrium.

[23]Work on ad-funded media platforms has also looked at the effects of viewer multi-homing; see, e.g., Ambrus, Calvano and Reisinger (2016) and Anderson, Foros and Kind (2019).

# A. Appendix

## A.1. Relegated proofs

*Proof of Proposition 1.* Denote $y_i^k = \exp(a_i^k - p_i^k)$. Since, in equibrium, $n_i^k = \bar{n}_i^k$, equations (2) can be written as

$$n_i^k = \frac{y_i^k \left(n_i^k\right)^{\alpha^k} \left(n_i^l\right)^{\beta^k}}{\sum_{j=1}^M y_j^k \left(n_j^k\right)^{\alpha^k} \left(n_j^l\right)^{\beta^k}}, \tag{A.1}$$

$$n_i^l = \frac{y_i^l \left(n_i^l\right)^{\alpha^l} \left(n_i^k\right)^{\beta^l}}{\sum_{j=1}^M y_j^l \left(n_j^l\right)^{\alpha^l} \left(n_j^k\right)^{\beta^l}}. \tag{A.2}$$

Using the above conditions, for each $j$ and $i$ in $\mathcal{M}$, we have

$$\begin{aligned}
\frac{n_i^k}{n_j^k} &= \left(\frac{y_i^k}{y_j^k}\right)\left(\frac{n_i^k}{n_j^k}\right)^{\alpha^k}\left(\frac{n_i^l}{n_j^l}\right)^{\beta^k} \\
\iff \frac{n_i^k}{n_j^k} &= \left(\frac{y_i^k}{y_j^k}\right)^{\frac{1}{1-\alpha^k}}\left(\frac{n_i^l}{n_j^l}\right)^{\frac{\beta^k}{1-\alpha^k}} \\
&= \left(\frac{y_i^k}{y_j^k}\right)^{\frac{1}{1-\alpha^k}}\left[\left(\frac{y_i^l}{y_j^l}\right)^{\frac{1}{1-\alpha^l}}\left(\frac{n_i^k}{n_j^k}\right)^{\frac{\beta^l}{1-\alpha^l}}\right]^{\frac{\beta^k}{1-\alpha^k}} \\
\iff \left(\frac{n_i^k}{n_j^k}\right)^{\frac{(1-\alpha^k)(1-\alpha^l)-\beta^k\beta^l}{(1-\alpha^k)(1-\alpha^l)}} &= \left(\frac{y_i^k}{y_j^k}\right)^{\frac{1}{1-\alpha^k}}\left(\frac{y_i^l}{y_j^l}\right)^{\frac{\beta^k}{(1-\alpha^k)(1-\alpha^l)}} \\
\iff \frac{n_i^k}{n_j^k} &= \left(\frac{y_i^k}{y_j^k}\right)^{\Gamma^{kk}}\left(\frac{y_i^l}{y_j^l}\right)^{\Gamma^{kl}}.
\end{aligned}$$

By substituting the last equation into equation (A.1), we obtain the equation

$$n_i^k = \frac{\left(y_i^k\right)^{1+\alpha^k\Gamma^{kk}+\beta^k\Gamma^{lk}}\left(y_i^l\right)^{\alpha^k\Gamma^{kl}+\beta^k\Gamma^{ll}}}{\sum_{j\in\mathcal{M}}\left(y_j^k\right)^{1+\alpha^k\Gamma^{kk}+\beta^k\Gamma^{lk}}\left(y_j^l\right)^{\alpha^k\Gamma^{kl}+\beta^k\Gamma^{ll}}}. \tag{A.3}$$

Noting that

$$\begin{aligned}
1+\alpha^k\Gamma^{kk}+\beta^k\Gamma^{lk} &= \frac{(1-\alpha^k)(1-\alpha^l)-\beta^k\beta^l+\alpha^k(1-\alpha^l)+\beta^k\beta^l}{(1-\alpha^k)(1-\alpha^l)-\beta^k\beta^l} = \Gamma^{kk}, \\
\alpha^k\Gamma^{kl}+\beta^k\Gamma^{ll} &= \frac{\alpha^k\beta^k+\beta^k(1-\alpha^k)}{(1-\alpha^k)(1-\alpha^l)-\beta^k\beta^l} = \Gamma^{kl},
\end{aligned}$$

equation (A.3) can be written as

$$n_i^k = \frac{\left(y_i^k\right)^{\Gamma^{kk}} \left(y_i^l\right)^{\Gamma^{kl}}}{\sum_{j \in \mathcal{M}} \left(y_j^k\right)^{\Gamma^{kk}} \left(y_j^l\right)^{\Gamma^{kl}}}$$

Finally, noting that

$$\left(y_i^k\right)^{\Gamma^{kk}} \left(y_i^l\right)^{\Gamma^{kl}} = \exp\left(\Gamma^{kk}(a_i^k - p_i^k) + \Gamma^{kl}(a_i^l - p_i^l)\right),$$

we obtain equation (3). □

*Proof of Remark 1.* Start with an initial value of the vector of network sizes $(n_{i,0}^A, n_{i,0}^B)_{i=1,\dots,M}$ such that $n_{i,0}^k > 0$ for all $i \in \{1, \dots, M\}$ and $k \in \{A, B\}$. For each $t > 0$, update the network sizes based on the value of network sizes in the previous iteration $t-1$. Then, the sequence of network sizes $\{(n_i^t)_{i=1,\dots,M}\}_{t=0,\dots}$ is obtained. Here, for any $t > 0$, we have

$$\frac{n_{i,t}^k}{n_{j,t}^k} = \frac{y_i^k}{y_j^k} \left(\frac{n_{i,t-1}^k}{n_{j,t-1}^k}\right)^{\alpha^k} \left(\frac{n_{i,t-1}^l}{n_{j,t-1}^l}\right)^{\beta^k}$$

By taking the logarithm and letting $x_t^k := \log(n_{i,t}^k / n_{j,t}^k)$ and $\sigma^k := \log(y_i^k / y_j^k)$, we have

$$\begin{pmatrix} x_t^A \\ x_t^B \end{pmatrix} = J \begin{pmatrix} x_{t-1}^A \\ x_{t-1}^B \end{pmatrix} + \begin{pmatrix} \sigma^A \\ \sigma^B \end{pmatrix},$$

where

$$J = \begin{bmatrix} \alpha^A & \beta^A \\ \beta^B & \alpha^B \end{bmatrix}.$$

If any eigenvalue of $J$ has an absolute value less than 1, $(x_t^A, x_t^B)$ converges to a unique value $(x^A, x^B)$ regardless of the initial value $(x_0^A, x_0^B)$ (see Luenberger, 1979, Chapter 5.9). At such value, we must satisfy $x_t^k = x_{t-1}^k = x^k$. Solving for $x^k$, we have

$$x^k = \frac{(1-\alpha^l)\sigma^k + \beta^k \sigma^l}{(1-\alpha^k)(1-\alpha^l) - \beta^k \beta^l}.$$

Then, using the relation $\lim_{t\to\infty}(n_{i,t}^k / n_{j,t}^k) = n_i^k / n_j^k = \exp(x^k)$, we obtain the relation (A.1). Therefore, from any starting value of positive network sizes, the best-response dynamics converges to the interior participation equilibrium.

Lastly, we show that any eigenvalue of $J$ has an absolute value less than 1. A scalar $b$ is an eigenvalue of $J$ if and only if it is the solution to the quadratic equation

$$\xi(b) = b^2 - (\alpha^A + \alpha^B)b + (\alpha^A\alpha^B - \beta^A\beta^B) = 0.$$

Because $\xi(b)$ is quadratic, $\xi(b) = 0$ has at most two solutions. Furthermore, because

$$\begin{aligned} \xi(-1) &= (1+\alpha^A)(1+\alpha^B) - \beta^A\beta^B > 0, \\ \xi\left(\frac{\alpha^A+\alpha^B}{2}\right) &= -\frac{(\alpha^A-\alpha^B)^2}{4} - \beta^A\beta^B < 0, \\ \xi(1) &= (1-\alpha^A)(1-\alpha^B) - \beta^A\beta^B > 0, \end{aligned}$$

There are two solutions to $\xi(b) = 0$ that lie in $(-1, 1)$, which completes the proof.[24]

Thus, the demand for platform in group $k$ is the group-$k$ network size of platform $i$ given by equation (3) with $\mathcal{M} = \{1, \ldots, M\}$. □

*Proof of Lemma 1.* The expressions for $h_i^A$ and $h_i^B$ can be rewritten as

$$\begin{aligned} \log h_i^A &= \Gamma^{AA}(a_i^A - p_i^A) + \Gamma^{AB}(a_i^B - p_i^B), \\ \log h_i^B &= \Gamma^{BB}(a_i^B - p_i^B) + \Gamma^{BA}(a_i^A - p_i^A). \end{aligned}$$

Rewriting the second equation as

$$a_i^B - p_i^B = \frac{1}{\Gamma^{BB}} \log h_i^B - \frac{\Gamma^{BA}}{\Gamma^{BB}}(a_i^A - p_i^A),$$

the first equation can be rewritten as

$$\begin{aligned} \log h_i^A &= \left[\Gamma^{AA} - \frac{\Gamma^{AB}\Gamma^{BA}}{\Gamma^{BB}}\right](a_i^A - p_i^A) + \frac{\Gamma^{AB}}{\Gamma^{BB}} \log h_i^B \\ &= \frac{\Gamma^{AA}\Gamma^{BB} - \Gamma^{AB}\Gamma^{BA}}{\Gamma^{BB}}(a_i^A - p_i^A) + \frac{\Gamma^{AB}}{\Gamma^{BB}} \log h_i^B \\ &= \frac{1}{(1-\alpha^A)(1-\alpha^B) - \beta^A\beta^B} \frac{1}{\Gamma^{BB}}(a_i^A - p_i^A) + \frac{\beta^A}{1-\alpha^A} \log h_i^B. \\ &= \frac{1}{1-\alpha^A}(a_i^A - p_i^A) + \frac{\beta^A}{1-\alpha^A} \log h_i^B. \end{aligned}$$

Therefore, we obtain the values of $(p_i^A, p_i^B)$ as a function of $(h_i^A, h_i^B)$, given by equations (4) and (5). □

*Proof of Lemma 2.* In the first part of the proof, we show that any solution to the first-order

[24]For $\alpha^k < 0$, this argument goes through if $|\alpha^k| < 1$ for $k \in \{A, B\}$.

conditions of profit maximization is a global maximizer. Define

$$
\begin{aligned}
f_i^A(h_i^A, h_i^B) &:= \left(1 - \frac{h_i^A}{H^A}\right)\left[p_i^A(h_i^A, h_i^B) - c_i^A\right] - 1 + \alpha^A + \beta^B \frac{h_i^B}{H^B}\frac{H^A}{h_i^A}, \\
f_i^B(h_i^A, h_i^B) &:= \left(1 - \frac{h_i^B}{H^B}\right)\left[p_i^B(h_i^A, h_i^B) - c_i^B\right] - 1 + \alpha^B + \beta^A \frac{h_i^A}{H^A}\frac{H^B}{h_i^B}
\end{aligned}
$$

and, thus, $\partial \Pi_i / \partial h_i^k = f_i^k(h_i^A, h_i^B)/H^k$, for $k \in \{A, B\}$. Hence, $\partial^2 \Pi_i / (\partial h_i^k)^2 = -\frac{f_i^k(h_i^A, h_i^B)}{(H^k)^2} + \frac{1}{H^k}\frac{\partial f_i^k}{\partial h_i^k}$. When the first-order conditions of profit maximization hold, the first term on the right-hand side is zero. Then, $\Pi_i(h_i^A, h_i^B, h_i^A + H_{-i}^A, h_i^B + H_{-i}^B)$ is a local maximizer in $(h_i^A, h_i^B)$ at any point at which the first-order conditions of profit maximization hold if $\partial f_i^A / \partial h_i^A < 0$, $\partial f_i^B / \partial h_i^B < 0$, and $(\partial f_i^A / \partial h_i^A)(\partial f_i^B / \partial h_i^B) - (\partial f_i^A / \partial h_i^B)(\partial f_i^B / \partial h_i^A) > 0$. Furthermore, this establishes that the Jacobian of $(f_i^A, f_i^B))$ is a $P$-matrix. This implies that $(f_i^A, f_i^B)$ is injective on $(0, \infty)^2$ (Gale and Nikaido, 1965) and, therefore, a solution to the first-order conditions of profit maximization is a global maximizer, provided that such a solution exists.

To see that the three inequalities hold, first note that

$$
\begin{aligned}
\frac{\partial f_i^A}{\partial h_i^A} &= \frac{1}{h_i^A}\left[-n_i^A(1 - n_i^A)[p_i^A - c_i^A] - (1 - \alpha^A)(1 - n_i^A) - \beta^B(1 - n_i^A)\frac{n_i^B}{n_i^A}\right] \\
&= \frac{1}{h_i^A}\left[-n_i^A\left(1 - \alpha^A - \beta^B\frac{n_i^B}{n_i^A}\right) - (1 - n_i^A)\left(1 - \alpha^A + \beta^B\frac{n_i^B}{n_i^A}\right)\right] \\
&= -\frac{1}{h_i^A}\left[1 - \alpha^A - \beta^B n_i^B + \beta^B(1 - n_i^A)\frac{n_i^B}{n_i^A}\right] < 0,
\end{aligned}
$$

which establishes the first inequality above. Correspondingly, the second inequality holds. Third, we establish $(\partial f_i^A / \partial h_i^A)(\partial f_i^B / \partial h_i^B) - (\partial f_i^A / \partial h_i^B)(\partial f_i^B / \partial h_i^A) > 0$. To do so, note that

$$
\frac{\partial f_i^A}{\partial h_i^B} = \frac{1}{h_i^B}\left[\beta^A(1 - n_i^A) + \beta^B(1 - n_i^B)\frac{n_i^B}{n_i^A}\right] > 0.
$$

Without loss of generality, assume that $\beta^A \geq \beta^B$. Recall that $1 - \max\{\alpha^A, \alpha^B\} > \max\{\beta^A, \beta^B\}$.

Therefore, $1-\alpha^A > \beta^A$ and $1-\alpha^B > \beta^A$. Then, we have

$$
\begin{aligned}
& h_i^A h_i^B \left( \frac{\partial f_i^A}{\partial h_i^A} \frac{\partial f_i^B}{\partial h_i^B} - \frac{\partial f_i^A}{\partial h_i^B} \frac{\partial f_i^B}{\partial h_i^A} \right) \\
= \; & \left(1-\alpha^A-\beta^B n_i^B\right)\left(1-\alpha^B-\beta^A n_i^A\right) \\
& + \left(1-\alpha^A-\beta^B n_i^B\right)\beta^A(1-n_i^B)\frac{n_i^A}{n_i^B} + \left(1-\alpha^B-\beta^A n_i^A\right)\beta^B(1-n_i^A)\frac{n_i^B}{n_i^A} \\
& -(\beta^A)^2(1-n_i^A)^2\frac{n_i^A}{n_i^B} - (\beta^B)^2(1-n_i^B)^2\frac{n_i^B}{n_i^A} - \beta^A\beta^B(1-n_i^A)(1-n_i^B) \\
> \; & \underbrace{(\beta^A-\beta^B n_i^B)\beta^A(1-n_i^A)}_{\text{(i)}} + \underbrace{(\beta^A-\beta^B n_i^B)\beta^A(1-n_i^B)\frac{n_i^A}{n_i^B}}_{\text{(ii)}} + \underbrace{\beta^A\beta^B(1-n_i^A)^2\frac{n_i^B}{n_i^A}}_{\text{(iii)}} \\
& - \underbrace{(\beta^A)^2(1-n_i^A)^2\frac{n_i^A}{n_i^B}}_{\text{(iv)}} - \underbrace{(\beta^B)^2(1-n_i^B)^2\frac{n_i^B}{n_i^A}}_{\text{(v)}} - \underbrace{\beta^A\beta^B(1-n_i^A)(1-n_i^B)}_{\text{(vi)}}.
\end{aligned}
$$

Every pair $(n_i^A, n_i^B)$ belongs to one of three cases, and we show that the above expression is positive in each case.

1. First, consider the case with $n_i^A \geq n_i^B$. In this case, (ii) $>$ (iv), (iii) $>$ (v), and (i) $>$ (vi), so the expression under consideration is positive.

2. Next, consider the case with $n_i^B \in (n_i^A, n_i^A\beta^A/\beta^B]$. In this case, we have $\beta^B n_i^B < \beta^A n_i^A$, so (i) $>$ (iv), (iii) $>$ (vi), and (ii) $>$ (v), so the expression under consideration is positive.

3. Finally, consider the case with $n_i^B > n_i^A\beta^A/\beta^B$. Because $\beta^A \geq \beta^B$, we have (i) $\geq$ (vi). Next we show that (ii) + (iii) $>$ (iv) + (v). Noting that (ii) $-$ (iv) $\geq -(\beta^A)^2(n_i^A/n_i^B)[(1-n_i^A)^2-(1-n_i^B)^2]$ and (iii) $-$ (v) $\geq (\beta^A)^2[(1-n_i^A)^2-(1-n_i^B)^2]$ when $n_i^B > n_i^A\beta^A/\beta^B$, we have

$$\text{(ii)} + \text{(iii)} - [\text{(iv)} + \text{(v)}] \geq (\beta^A)^2[(1-n_i^A)^2-(1-n_i^B)^2]\left(1-\frac{n_i^A}{n_i^B}\right) > 0,$$

   which shows that the expression under consideration is positive. This completes the first part of the proof.

In the second part of the proof, we show that there always exists a solution to the system of equations

$$\begin{pmatrix} f_i^A(h_i^A, h_i^B) \\ f_i^B(h_i^A, h_i^B) \end{pmatrix} = \begin{pmatrix} 0, \\ 0. \end{pmatrix} \tag{A.4}$$

Step 1: existence of a solution $\tilde{h}_i^B(h_i^A)$ to $f_i^B(h_i^A, h_i^B) = 0$ given $h_i^A$. Fix $h_i^A$ and consider the solution to the equation $f_i^B(h_i^A, h_i^B) = 0$ given $h_i^A$, denoted by $\tilde{h}_i^B(h_i^A)$. We show that $\tilde{h}_i^B(h_i^A)$ exists in $(0, \infty)$, for any given $h_i^A \in (0, \infty)$. To see this, note that we have

$$\begin{aligned}
&\lim_{h_i^B \to 0} f_i^B(h_i^A, h_i^B) \\
&= \lim_{h_i^B \to 0} \left[a_i^B - c_i^B - (1-\alpha^B)\log h_i^B + \beta^B \log h_i^A\right] - 1 + \alpha^B + \lim_{h_i^B \to 0} \left(\beta^A \frac{h_i^A}{h_i^A + H_{-i}^A} \frac{h_i^B + H_{-i}^B}{h_i^B}\right) \\
&= \infty > 0, \\
&\lim_{h_i^B \to \infty} f_i^B(h_i^A, h_i^B) \\
&= \lim_{h_i^B \to \infty} \left[\frac{H_{-i}^B}{h_i^B + H_{-i}^B} \left[a_i^B - c_i^B - (1-\alpha^B)\log h_i^B + \beta^B \log h_i^A\right]\right] - 1 + \alpha^A + \beta^B \frac{h_i^A}{H^A} \\
&= -1 + \alpha^A + \beta^B \frac{h_i^A}{h_i^A + H_{-i}^A} < 0.
\end{aligned}$$

Hence, by the intermediate value theorem, the solution $\tilde{h}_i^B(h_i^A) \in (0, \infty)$ exists for any given $h_i^A \in [0, \infty]$. Note that such a solution is unique and continuous in $h_i^A$. To see this, note that we already established that we have $\partial f_i^B / \partial h_i^B < 0$ whenever $f_i^B = 0$ holds. Hence, $\tilde{h}_i^B(h_i^A)$ is unique and, from the implicit function theorem, continuous.

Step 2: preliminaries on the existence of solution to equation $f_i^A(h_i^A, \tilde{h}_i^B(h_i^A)) = 0$. We show that there exists a solution to the equation $f_i^A(h_i^A, \tilde{h}_i^B(h_i^A)) = 0$. As preliminaries, we show the following four limit results: $\lim_{h_i^A \to 0} \tilde{h}_i^B(h_i^B) = 0$, $\lim_{h_i^A \to \infty} \tilde{h}_i^B(h_i^A) = \infty$, $\lim_{h_i^A \to 0} \frac{h_i^A}{\tilde{h}_i^B(h_i^A)} = 0$, $\lim_{h_i^A \to \infty} \frac{h_i^A}{\tilde{h}_i^B(h_i^A)} = \infty$.

First, we show that $\lim_{h_i^A \to 0} \tilde{h}_i^B(h_i^A) = 0$. Suppose to the contrary that $\lim_{h_i^A \to 0} \tilde{h}_i^B(h_i^A) > 0$. Then, there exists $\underline{h}_i^B > 0$ such that $\lim_{h_i^A \to \infty} \tilde{h}_i^B(h_i^A) = \underline{h}_i^B$, and we would have

$$\begin{aligned}
&\lim_{h_i^A \to 0} f_i^B(h_i^A, \tilde{h}_i^B(h_i^A)) \\
=\ & \frac{H_{-i}^B}{\underline{h}_i^B + H_{-i}^B} \left[a_i^B - c_i^B - (1-\alpha^B)\log \underline{h}_i^B + \beta^B \lim_{h_i^A \to 0} \left(\log h_i^A\right)\right] - 1 + \alpha^B \\
=\ & -\infty \\
<\ & 0,
\end{aligned}$$

contradicting the definition of $\tilde{h}_i^B(h_i^A)$. Hence, $\lim_{h_i^A \to 0} \tilde{h}_i^B(h_i^A) = 0$.

Second, we show that $\lim_{h_i^A \to \infty} \tilde{h}_i^B(h_i^A) = \infty$. Suppose to the contrary that $\lim_{h_i^A \to \infty} \tilde{h}_i^B(h_i^A) <$

$\infty$. Then, there exists $\overline{h}^B < \infty$ such that $\lim_{h_i^A \to \infty} \tilde{h}_i^B(h_i^A) = \overline{h}_i^B$, and we would have

$$\begin{aligned}
& \lim_{h_i^A \to \infty} f_i^B(h_i^A, \tilde{h}_i^B(h_i^A)) \\
= & \frac{H_{-i}^B}{\overline{h}_i^B + H_{-i}^B} \left[ a_i^B - c_i^B - (1-\alpha^B) \log \overline{h}_i^B + \beta^B \lim_{h_i^A \to \infty} \log h_i^A \right] - 1 + \alpha^B + \beta^A \frac{H^B}{\overline{h}_i^B} \\
= & \infty > 0,
\end{aligned}$$

contradicting the definition of $\tilde{h}_i^B(h_i^A)$. Hence, $\lim_{h_i^A \to \infty} \tilde{h}_i^B(h_i^A) = \infty$.

Third, we show that $\lim_{h_i^A \to 0}[h_i^A / \tilde{h}_i^B(h_i^A)] = 0$. Otherwise, there exists a constant $\underline{\kappa} > 0$ such that $\lim_{h_i^A \to 0}[h_i^A / \tilde{h}_i^B(h_i^A)] = \underline{\kappa}$, and

$$\begin{aligned}
& \lim_{h_i^A \to 0} f_i^B(h_i^A, \tilde{h}_i^B(h_i^A)) \\
= & a_i^B - c_i^B - (1 - \alpha^B - \beta^B) \lim_{h_i^A \to 0} \log \tilde{h}_i^B(h_i^A) + \beta^B \log \underline{\kappa} - 1 + \alpha^B + \beta^A \underline{\kappa} \frac{H_{-i}^B}{H_{-i}^A} \\
= & \infty
\end{aligned}$$

Hence, we have $\lim_{h_i^A \to 0}[h_i^A / \tilde{h}_i^B(h_i^A)] = 0$.

Fourth, we show that $\lim_{h_i^A \to \infty}[h_i^A / \tilde{h}_i^B(h_i^A)] = \infty$. Otherwise, there exists $\overline{\kappa} < \infty$ such that $\lim_{h_i^A \to \infty}[h_i^A / \tilde{h}_i^B(h_i^A)] = \overline{\kappa}$. Then, we would have

$$\begin{aligned}
& \lim_{h_i^A \to \infty} f_i^B(h_i^A, \tilde{h}_i^B(h_i^A)) \\
= & \lim_{h_i^A \to \infty} \left[ \frac{H_{-i}^B}{\tilde{h}_i^B(h_i^A) + H_{-i}^B} \left( a_i^B - c_i^B - (1 - \alpha^B - \beta^B) \log \tilde{h}_i^B(h_i^A) + \beta^B \log \overline{\kappa} \right) \right] - 1 + \alpha^B + \beta^A \\
= & -1 + \alpha^B + \beta^A < 0,
\end{aligned}$$

contradicting the definition of $\tilde{h}_i^B(h_i^A)$. Hence, we have $\lim_{h_i^A \to \infty}[h_i^A / \tilde{h}_i^B(h_i^A)] = \infty$.

Step 3: proof of the existence of a solution to equation $f_i^A(h_i^A, \tilde{h}_i^B(h_i^A)) = 0$. To show the existence of the solution to the equation $f_i^A(h_i^A, \tilde{h}_i^B(h_i^A)) = 0$, we show that

$$\begin{aligned}
\lim_{h_i^A \to 0} f_i^A(h_i^A, \tilde{h}_i^B(h_i^A)) & > 0, \\
\lim_{h_i^A \to \infty} f_i^A(h_i^A, \tilde{h}_i^B(h_i^A)) & < 0.
\end{aligned}$$

Then, the intermediate value theorem implies that there exists a solution to the equation.

We first show that $\lim_{h_i^A \to 0} f_i^A(h_i^A, \tilde{h}_i^B(h_i^A)) > 0$. To see this, note that we can write $f_i^A$ as

$$\begin{aligned} f_i^A(h_i^A, h_i^B) &= \frac{H_{-i}^A}{h_i^A + H_{-i}^A}\left[a_i^A - c_i^A - (1-\alpha^A)\log h_i^A + \beta^A \log h_i^B\right] - 1 + \alpha^A + \beta^B \frac{h_i^B}{h_i^A}\frac{h_i^A + H_{-i}^A}{h_i^B + H_{-i}^B} \\ &= \frac{H_{-i}^A}{h_i^A + H_{-i}^A}\left[a_i^A - c_i^A - (1-\alpha^A-\beta^A)\log h_i^A - \beta^A \log\left(\frac{h_i^A}{h_i^B}\right)\right] \\ &\quad -1 + \alpha^A + \beta^B \frac{h_i^B}{h_i^A}\frac{h_i^A + H_{-i}^A}{h_i^B + H_{-i}^B}. \end{aligned}$$

Hence, because $\lim_{h_i^A \to 0} \tilde{h}_i^B(h_i^A) = 0$ and $\lim_{h_i^A \to 0}[h_i^A / \tilde{h}_i^B(h_i^A)] = 0$, we have

$$\begin{aligned} &\lim_{h_i^A \to 0} f_i^A(h_i^A, \tilde{h}_i^B(h_i^A)) \\ &= a_i^A - c_i^A + \lim_{h_i^A \to 0}\left[-(1-\alpha^A-\beta^A)\log h_i^A - \beta^A \log\left(\frac{h_i^A}{\tilde{h}_i^B(h_i^A)}\right) - 1 + \alpha^A + \beta^B \frac{\tilde{h}_i^B(h_i^A)}{h_i^A}\right] \\ &= \infty. \end{aligned}$$

Next, we show that $\lim_{h_i^A \to \infty} f_i^A(h_i^A, \tilde{h}_i^B(h_i^A)) < 0$. To see this, note that

$$\begin{aligned} &f_i^A(h_i^A, h_i^B) \\ &= \frac{h_i^A}{h_i^A + H_{-i}^A}\frac{H_{-i}^A}{h_i^A}\left[a_i^A - a_i^A - (1-\alpha^A)\log h_i^A + \beta^A \log h_i^B\right] - 1 + \alpha^A + \beta^B \frac{h_i^B}{h_i^B + H_{-i}^B}\frac{h_i^A + H_{-i}^A}{h_i^A}. \end{aligned}$$

Hence, because $\lim_{h_i^A \to \infty} \tilde{h}_i^B(h_i^A) = \infty$ and $\lim_{h_i^A \to \infty}[h_i^A / \tilde{h}_i^B(h_i^A)] = \infty$, we have

$$\begin{aligned} &\lim_{h_i^A \to \infty} f_i^A(h_i^A, \tilde{h}_i^B(h_i^A)) \\ &= \lim_{h_i^A \to \infty}\left(\frac{\log h_i^A}{h_i^A}\right) + \lim_{h_i^A \to \infty}\left(\frac{\log \tilde{h}_i^B(h_i^A)}{\tilde{h}_i^B(h_i^A)}\frac{\tilde{h}_i^B(h_i^A)}{h_i^A}\right) - 1 + \alpha^A + \beta^B \\ &= -1 + \alpha^A + \beta^B < 0. \end{aligned}$$

Put together, there exists a solution to the equation $f_i^A(h_i^A, \tilde{h}_i^B(h_i^A)) = 0$. Letting $h_i^{A*}$ be a solution and $h_i^{B*} := \tilde{h}_i^B(h_i^{A*})$, the pair $(h_i^{A*}, h_i^{B*})$ is a solution to the system of equations (A.4). □

*Proof of Lemma 3.* Let

$$\begin{aligned} \tilde{f}_i^A(h_i^A, h_i^B, H^A, H^B) &= \left(1 - \frac{h_i^A}{H^A}\right)\left[p_i^A(h_i^A, h_i^B) - c_i^A\right] - 1 + \alpha^A + \beta^B \frac{h_i^B}{H^B}\frac{H^A}{h_i^A}, \\ \tilde{f}_i^B(h_i^A, h_i^B, H^A, H^B) &= \left(1 - \frac{h_i^B}{H^B}\right)\left[p_i^B(h_i^A, h_i^B) - c_i^B\right] - 1 + \alpha^B + \beta^A \frac{h_i^A}{H^A}\frac{H^B}{h_i^B} \end{aligned}$$

By the implicit function theorem, implicit best replies are well-defined, if the matrix

$$\begin{pmatrix} \frac{\partial \widetilde{f}_i^A}{\partial h_i^A} & \frac{\partial \widetilde{f}_i^A}{\partial h_i^B} \\ \frac{\partial \widetilde{f}_i^B}{\partial h_i^A} & \frac{\partial \widetilde{f}_i^B}{\partial h_i^B} \end{pmatrix}$$

has a determinant different from zero. Then, taking $(H^A, H^B)$ as given, we have

$$\begin{aligned} \frac{\partial \widetilde{f}_i^A}{\partial h_i^A} &= \frac{1}{h_i^A}\left[-n_i^A(p_i^A - c_i^A) - (1-\alpha^A)(1-n_i^A) - \beta^B \frac{n_i^B}{n_i^A}\right] \\ &= \frac{1}{h_i^A}\left[-\frac{n_i^A}{1-n_i^A}\left(1-\alpha^A - \beta^B\frac{n_i^B}{n_i^A}\right) - (1-\alpha^A)(1-n_i^A) - \beta^B\frac{n_i^B}{n_i^A}\right] \\ &= -\frac{1}{h_i^A}\frac{1}{1-n_i^A}\left\{\left[(1-n_i^A)^2 + n_i^A\right](1-\alpha^A) - \beta^B n^B\left(1 - \frac{1-n_i^A}{n_i^A}\right)\right\} < 0, \end{aligned}$$

which can be shown as follows: $(1-n_i^A)^2 + n_i^A$ takes positive value, is minimized at $n_i^A = 1/2$, and increasing in $n_i^A > 1/2$, while $1-(1-n_i^A)/n_i^A$ is increasing, takes value zero at $n_i^A = 1$ and is maximized at $n_i^A = 1$. Thus, for $n_i^A \leq 1/2$ the derivative must be negative. For $n_i^A > 1/2$, since $1-\alpha^A > \beta^B$ by assumption, it is sufficient to show that $(1-n_i^A)^2 + n_i^A \geq 1 - \frac{1-n_i^A}{n_i^A}$ which is equivalent to $(1-(n_i^A)^2)(1-n_i^A) \geq 0$ and, thus, always holds. Note that we can write

$$\frac{\partial \widetilde{f}_i^A}{\partial h_i^A} = -\frac{1}{h_i^A}\left\{\frac{(1-n_i^A)^2 + n_i^A}{1-n_i^A}(1-\alpha^A) - \beta^B\frac{n_i^B}{1-n_i^A} + \beta^B\frac{n_i^B}{n_i^A}\right\}.$$

Also, we have

$$\frac{\partial \widetilde{f}_i^A}{\partial h_i^B} = \frac{1}{h_i^B}\left[\beta^A(1-n_i^A) + \beta^B\frac{n_i^B}{n_i^A}\right].$$

Therefore,

$$\begin{aligned} & h_i^A h_i^B\left(\frac{\partial \widetilde{f}_i^A}{\partial h_i^A}\frac{\partial \widetilde{f}_i^B}{\partial h_i^B} - \frac{\partial \widetilde{f}_i^A}{\partial h_i^B}\frac{\partial \widetilde{f}_i^B}{\partial h_i^A}\right) \\ =& \left\{\left[1-n_i^A + \frac{n_i^A}{1-n_i^A}\right](1-\alpha^A) + \frac{n_i^B}{n_i^A}\beta^B - \frac{n_i^B}{1-n_i^A}\beta^B\right\} \\ &\times\left\{\left[1-n_i^B + \frac{n_i^B}{1-n_i^B}\right](1-\alpha^B) + \frac{n_i^A}{n_i^B}\beta^A - \frac{n_i^A}{1-n_i^B}\beta^A\right\} \\ &-\left(\beta^A(1-n_i^A) + \beta^B\frac{n_i^B}{n_i^A}\right)\left(\beta^B(1-n_i^B) + \beta^A\frac{n_i^A}{n_i^B}\right). \end{aligned}$$

Suppose without loss of generality that $\beta^A \geq \beta^B$. Then, because $\min\{1-\alpha^A, 1-\alpha^B\} \geq \beta^A$, the last expression is greater than

$$
\begin{aligned}
&\left\{\left[1-n_i^A+\frac{n_i^A}{1-n_i^A}\right]\beta^A+\left(\frac{n_i^B}{n_i^A}-\frac{n^B}{1-n_i^A}\right)\beta^B\right\}\beta^A\left[1-n_i^B+\frac{n_i^B}{1-n_i^B}-\frac{n_i^A}{1-n_i^B}+\frac{n_i^A}{n_i^B}\right]\\
&-\left(\beta^A(1-n_i^A)+\beta^B\frac{n_i^B}{n_i^A}\right)\left(\beta^B(1-n_i^B)+\beta^A\frac{n_i^A}{n_i^B}\right),
\end{aligned}
$$

which, by dividing by $(\beta^A)^2$, has the same sign as

$$
\begin{aligned}
&\left\{\left[1-n_i^A+\frac{n_i^A}{1-n_i^A}\right]+\left(\frac{n_i^B}{n_i^A}-\frac{n^B}{1-n_i^A}\right)\frac{\beta^B}{\beta^A}\right\}\left[1-n_i^B+\frac{n_i^B}{1-n_i^B}-\frac{n_i^A}{1-n_i^B}+\frac{n_i^A}{n_i^B}\right]\\
&-\left((1-n_i^A)+\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)\left(\frac{\beta^B}{\beta^A}(1-n_i^B)+\frac{n_i^A}{n_i^B}\right)\\
&=\left(1-n_i^A+\frac{n_i^B}{n_i^A}\frac{\beta^A}{\beta^B}\right)\left(1-n_i^B+\frac{n_i^A}{n_i^B}\right)+\frac{n_i^A}{1-n_i^A}\left(1-\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)\left(1-n_i^B+\frac{n_i^A}{n_i^B}\right)\\
&\quad+\frac{n_i^B}{1-n_i^B}\left(1-\frac{n_i^A}{n_i^B}\right)\left(1-n_i^A+\frac{n_i^B}{n_i^A}\frac{\beta^B}{\beta^A}\right)+\frac{(n_i^A)^2}{(1-n_i^A)(1-n_i^B)}\left(1-\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)\left(\frac{n_i^B}{n_i^A}-1\right)\\
&\quad-\left(1-n_i^A+\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)\left[\frac{\beta^B}{\beta^A}(1-n_i^B)+\frac{n_i^A}{n_i^B}\right]\\
&=\left(1-n_i^A+\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)\left(1-\frac{\beta^B}{\beta^A}\right)(1-n_i^B)\\
&\quad+\frac{n_i^A}{1-n_i^B}\left(1-n_i^A+\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)\left(\frac{n_i^B}{n_i^A}-1\right)\\
&\quad+\frac{n_i^A}{1-n_i^A}\left(1-n_i^B+\frac{n_i^A}{n_i^B}\right)\left(1-\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)\\
&\quad+\frac{(n_i^A)^2}{(1-n_i^A)(1-n_i^B)}\left(\frac{n_i^B}{n_i^A}-1\right)\left(1-\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)\\
&=\frac{(1-n_i^A)(1-n_i^B)^2}{(1-n_i^A)(1-n_i^B)}\left[1-n_i^A+\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right]\left(1-\frac{\beta^B}{\beta^A}\right)\\
&\quad+\frac{n_i^A}{(1-n_i^A)(1-n_i^B)}(1-n_i^A)\left(1-n_i^A+\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)\left(\frac{n_i^B}{n_i^A}-1\right)\\
&\quad+\frac{n_i^A}{(1-n_i^A)(1-n_i^B)}(1-n_i^B)\left(1-n_i^B+\frac{n_i^A}{n_i^B}\right)\left(1-\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)\\
&\quad+\frac{n_i^A}{(1-n_i^A)(1-n_i^B)}n_i^A\left(\frac{n_i^B}{n_i^A}-1\right)\left(1-\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right),
\end{aligned}
$$

which is positive if

$$\underbrace{(1-n_i^A)\left(1-n_i^A+\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)\left(\frac{n_i^B}{n_i^A}-1\right)}_{\text{(i)}}$$
$$+\underbrace{(1-n_i^B)\left(1-n_i^B+\frac{n_i^A}{n_i^B}\right)\left(1-\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)}_{\text{(ii)}}$$
$$+\underbrace{n_i^A\left(\frac{n_i^B}{n_i^A}-1\right)\left(1-\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)}_{\text{(iii)}}$$

is positive. Any value of $n_i^A$ belongs to one of three cases, and we show that (i) + (ii) + (iii) $> 0$ for each case.

1. The first case we consider is $n_i^A \geq n_i^B$. In this case, we have $n_i^B/n_i^A - 1 \leq 0$ and

$$\begin{aligned}
&\text{(i)} + \text{(ii)} + \text{(iii)} \\
=&\left(1-\frac{n_i^B}{n_i^A}\right)\left[(1-n_i^B)^2-(1-n_i^A)^2+(1-n_i^B)\frac{n_i^A}{n_i^B}-(1-n_i^A)\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}-\left(n_i^A-\frac{\beta^B}{\beta^A}n_i^B\right)\right] \\
&+\frac{n_i^B}{n_i^A}\left(1-\frac{\beta^B}{\beta^A}\right)\left[(1-n_i^B)^2+(1-n_i^B)\frac{n_i^A}{n_i^B}\right] \\
=&\left(1-\frac{n_i^B}{n_i^A}\right)\left[(n_i^A-n_i^B)(2-n_i^A-n_i^B)+(1-n_i^B)\frac{n_i^A}{n_i^B}-(1-n_i^A)\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}-\left(n_i^A-\frac{\beta^B}{\beta^A}n_i^B\right)\right] \\
&+\frac{n_i^B}{n_i^A}\left(1-\frac{\beta^B}{\beta^A}\right)\left[(1-n_i^B)^2+(1-n_i^B)\frac{n_i^A}{n_i^B}\right] \\
=&\left(1-\frac{n_i^B}{n_i^A}\right)\left[(n_i^A-n_i^B)(2-n_i^A-n_i^B)+(1-n_i^B)\frac{n_i^A}{n_i^B}-n_i^A+\frac{\beta^B}{\beta^A}n_i^B\right] \\
&+\frac{n_i^B}{n_i^A}(1-n_i^B)^2\left(1-\frac{\beta^B}{\beta^A}\right)+(1-n_i^B)\left(1-\frac{\beta^B}{\beta^A}\right) \\
\geq&\left(1-\frac{n_i^B}{n_i^A}\right)\left[(n_i^A-n_i^B)(2-n_i^A-n_i^B)+(1-n_i^B)\frac{n_i^A}{n_i^B}-\frac{\beta^B}{\beta^A}(1-n_i^B)\right] \\
>&0
\end{aligned}$$

where, for the third equation, we used

$$
\begin{aligned}
(1-n_i^B)\frac{n_i^A}{n_i^B} &= (1-n_i^A)\frac{n_i^A}{n_i^B} + (n_i^A - n_i^B)\frac{n_i^A}{n_i^B} \\
&= (1-n_i^A)\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} + (1-n_i^A)\left(\frac{n_i^A}{n_i^B} - \frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right) + (n_i^A - n_i^B)\frac{n_i^A}{n_i^B} \\
&= (1-n_i^A)\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} + \frac{n_i^A}{n_i^B}n_i^A - \frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} + (1-n_i^A)\frac{n_i^A}{n_i^B} - n_i^A + \frac{\beta^B}{\beta^A}n_i^B \\
&= (1-n_i^A)\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} + n_i^A - \frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} + \left(\frac{n_i^A}{n_i^B} - 1\right)n_i^A + (1-n_i^A)\frac{n_i^A}{n_i^B} - n_i^A + \frac{\beta^B}{\beta^A}n_i^B \\
&= (1-n_i^A)\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} + n_i^A - \frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} + (1-n_i^B)\frac{n_i^A}{n_i^B} - n_i^A + \frac{\beta^B}{\beta^A}n_i^B
\end{aligned}
$$

and, for the inequality

$$
\begin{aligned}
1 - n_i^B &\geq 1 - \frac{n_i^B}{n_i^A}, \\
1 - \frac{\beta^B}{\beta^A} - n_i^A &\geq -\frac{\beta^B}{\beta^A}.
\end{aligned}
$$

2. The second case is $n_i^A \in [n_i^B \beta^B/\beta^A, n_i^B)$. In this case, (i) $> 0$, (ii) $\geq 0$, and (iii) $\geq 0$.

3. The third case is $n_i^A < n_i^B \beta^B/\beta^A$. In this case, we have $n_i^B/n_i^A - 1 > (\beta^B n_i^B)/(\beta^A n_i^A) - 1 > 0$. Therefore,

$$
\begin{aligned}
&\text{(i)} + \text{(ii)} + \text{(iii)} \\
=&\left(\frac{n_i^B}{n_i^A} - 1\right)\left\{(1-n_i^A)^2 - (1-n_i^B)^2 + (1-n_i^A)\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} - (1-n_i^B)\frac{n_i^A}{n_i^B} - n_i^A\left(\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} - 1\right)\right\},
\end{aligned}
$$

which is positive if and only if

$$
\begin{aligned}
&(1-n_i^A)^2 - (1-n_i^B)^2 + (1-n_i^A)\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} - (1-n_i^B)\frac{n_i^A}{n_i^B} - n_i^A\left(\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} - 1\right) \\
=&(n_i^B - n_i^A)\left(2 - n_i^B - n_i^A + \frac{n_i^A}{n_i^B}\right) + (1-n_i^A)\left(\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} - \frac{n_i^A}{n_i^B}\right) - n_i^A\left(\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} - 1\right)
\end{aligned}
$$

is positive. Note that we have

$$
\begin{aligned}
&(n_i^B - n_i^A)\left(2 - n_i^B - n_i^A + \frac{n_i^A}{n_i^B}\right) + (1 - n_i^A)\left(\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} - \frac{n_i^A}{n_i^B}\right) - n_i^A\left(\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} - 1\right) \\
\geq &(n_i^B - n_i^A)\frac{n_i^A}{n_i^B} + (1 - n_i^A)\left(\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} - \frac{n_i^A}{n_i^B}\right) - n_i^A\left(\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} - 1\right) \\
= &\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} - \frac{n_i^A}{n_i^B} - \frac{\beta^B}{\beta^A}n_i^B + n_i^A - n_i^A\left(\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} - 1\right) \\
= &\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} - \frac{n_i^A}{n_i^B} + 2n_i^A - 2\frac{\beta^B}{\beta^A}n_i^B.
\end{aligned}
$$

At $n_i^A = \beta^B n_i^B / \beta^A$, the last expression above is

$$
1 - \frac{\beta^B}{\beta^A} \geq 0.
$$

Furthermore, for any region where $n_i^A \leq \beta^B n_i^B / \beta^A$, the expression under consideration has the following derivative with respect to $n_i^A$:

$$
-\frac{1}{n_i^A}\frac{n_i^B}{n_i^A}\frac{\beta^B}{\beta^A} - \frac{1}{n_i^B} + 2 < 0.
$$

Therefore, for any given $n_i^B$ and any $n_i^A < \beta^B n_i^B / \beta^A$, the expression under consideration is positive.

□

*Proof of Proposition 2.* To show the existence of the equilibrium, recall from Section 3.2 that $CS^A = (1 - \alpha^A)\log H^A - \beta^A \log H^B$ and $CS^B = (1 - \alpha^B)\log H^B - \beta^B \log H^A$.

Denote market share for group $k$ as a function of the aggregates by

$$
n_i^k(H^A, H^B) = \frac{h_i^k(H^A, H^B)}{H^k}.
$$

Noting that $\log h_i^A = \log n_i^A + \log H^A$, we can rewrite the first-order condition for $(h_i^A, h_i^B)$ as the condition for $(n_i^A, n_i^B)$ in the following way:

$$
g_i^A = (1 - n_i^A)[a_i^A - c_i^A - (1 - \alpha^A)\log n_i^A + \beta^A \log n_i^B - CS^A] - 1 + \alpha^A + \beta^B \frac{n_i^B}{n_i^A} = 0, \quad \text{(A.5)}
$$

$$
g_i^B = (1 - n_i^B)[a_i^B - c_i^B - (1 - \alpha^B)\log n_i^B + \beta^B \log n_i^A - CS^B] - 1 + \alpha^B + \beta^A \frac{n_i^A}{n_i^B} = 0. \quad \text{(A.6)}
$$

The solution can be written as $(\widetilde{n}_i^A(CS^A, CS^B), \widetilde{n}_i^B(CS^A, CS^B))$. By the implicit function theorem, we have

$$\text{Sign}\left[\frac{\widetilde{n}_i^k}{\partial CS^l}(CS^A, CS^B)\right]_{k,l\in\{A,B\}} = -\text{Sign}\begin{pmatrix} \frac{\partial g_i^B}{\partial n_i^B} & -\frac{\partial g_i^A}{\partial n_i^B} \\ -\frac{\partial g_i^B}{\partial n_i^A} & \frac{\partial g_i^A}{\partial n_i^A} \end{pmatrix}\begin{pmatrix} \frac{\partial g_i^A}{\partial CS^A} & 0 \\ 0 & \frac{\partial g_i^B}{\partial CS^B} \end{pmatrix}$$

Thus,

$$\text{Sign}\left(\frac{\partial \widetilde{n}_i^A}{\partial CS^A}\right) = \text{Sign}\left(-\frac{\partial g_i^A}{\partial CS^A}\frac{\partial g_i^B}{\partial n_i^B}\right) < 0,$$

and

$$\text{Sign}\left(\frac{\partial \widetilde{n}_i^B}{\partial CS^A}\right) = \text{Sign}\left(\frac{\partial g_i^A}{\partial CS^A}\frac{\partial g_i^B}{\partial n_i^A}\right) \leq 0,$$

where

$$\begin{aligned}\frac{\partial g_i^A}{\partial n_i^A} &= -\frac{1}{n_i^A}\left[n_i^A(p_i^A - c_i^A) + (1-\alpha^A)(1-n_i^A) + \beta^B\frac{n_i^B}{n_i^A}\right] < 0, \\ \frac{\partial g_i^A}{\partial n_i^B} &= \frac{1}{n_i^B}\left[\beta^A(1-n_i^A) + \beta^B\frac{n_i^B}{n_i^A}\right] \geq 0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial g_i^A}{\partial CS^A} &= -(1-n_i^A) < 0, \quad & \frac{\partial g_i^A}{\partial CS^B} &= 0, \\ \frac{\partial g_i^B}{\partial CS^B} &= -(1-n_i^B) < 0, \quad & \frac{\partial g_i^B}{\partial CS^A} &= 0.\end{aligned}$$

Fix $CS^A$ and let $\widehat{CS}^B(CS^A)$ be the solution to the equation

$$\sum_{i=1,\ldots,M} \widetilde{n}_i^B(CS^A, CS^B) = 1.$$

Because $\widetilde{n}_i^B$ is decreasing in $CS^B$, there exists a unique solution to the above equation, provided that it exists. To show existence, we establish that (1) $\lim_{CS^B\to\infty}\widetilde{n}_i^B(CS^A, CS^B) = 0$ and (2) $\lim_{CS^B\to-\infty}\widetilde{n}_i^B(CS^A, CS^B) = 1$.

On (1): to satisfy $g_i^B = 0$ while letting $CS^B \to \infty$, we must have $(1-\alpha^B)\log n_i^B - \beta^B \log n_i^A \to \infty$ or $n_i^A/n_i^B \to \infty$. In the former case, we must have $(1-\alpha^A)\log n_i^A - \beta^A \log n_i^B \to -\infty$ and thus $g_i^A \to \infty$, violating the requirement that $g_i^B = 0$. Hence, we must have $n_i^A/n_i^B \to \infty$, implying that $\lim_{CS^B\to\infty}\widetilde{n}_i^B(CS^A, CS^B) = 0$.

On (2): suppose that $CS^B \to -\infty$. In this case, we must have $\widetilde{n}_i^B(CS^A, CS^B) \to 1$, because

$g_i^B \to \infty$ otherwise.

Thus, we have shown that

$$\lim_{CS^B \to \infty} \sum_{i=1}^{M} \widetilde{n}_i^B(CS^A, CS^B) = 0 < 1,$$
$$\lim_{CS^B \to -\infty} \sum_{i=1}^{M} \widetilde{n}_i^B(CS^A, CS^B) = M > 1.$$

By the intermediate value theorem and the monotonicity of $\widetilde{n}_i^B$ in $CS^B$, there exists a unique value $\widehat{CS}^B(H^A)$ that satisfies the equation $\sum_{i=1}^{M} \widetilde{n}_i^B(CS^A, CS^B) = 1$ given any $CS^A$.

Next, let $CS^A$ vary while requiring that $CS^B = \widehat{CS}^B(CS^A)$. Let $CS^A \to \infty$. We must have $(1-\alpha^A)\log n_i^A - \beta^A \log n_i^B \to -\infty$ or $n_i^B/n_i^A \to \infty$. Both of these conditions require that $n_i^A$ converges to 0. As $CS^A \to -\infty$, we must have that for each $i$,

$$(1-n_i^A)\left[(1-\alpha^A)\log n_i^A - \beta^A \log n_i^B + CS^A\right]$$

is finite, which requires that either $n_i^B \to 0$ or $n_i^A \to 1$. Suppose that there exists a platform with $n_i^B \to 0$. This implies that $\widehat{CS}^B(CS^A) \to \infty$, because $g_i^B \to \infty$ otherwise. However, then we must have $\widetilde{n}_j^B \to 0$ for all $j$, which contradicts the condition $\sum_{i=1}^{M} \widetilde{n}_j^B(CS^A, \widehat{CS}^B(CS^A)) = 1$. Thus, there can be no $i$ such that $n_i^B \to 0$ as $CS^A \to -\infty$. Therefore,

$$\lim_{CS^A \to -\infty} \widetilde{n}_i^A(CS^A, \widehat{CS}^B(CS^A)) = 1$$

for all $i \in \{1, \ldots, M\}$. Hence, we have

$$\lim_{CS^A \to \infty} \sum_{i=1}^{M} \widetilde{n}_i^A(CS^A, \widehat{CS}^B(CS^A)) = 0 < 1,$$
$$\lim_{CS^A \to -\infty} \sum_{i=1}^{M} \widetilde{n}_i^A(CS^A, \widehat{CS}^B(CS^A)) = M > 1.$$

As the last step to establish equilibrium existence, the intermediate value theorem implies that there exists a solution to the equilibrium condition

$$\sum_{i=1}^{M} \widetilde{n}_i^A(CS^A, \widehat{CS}^B(CS^A)) = 1.$$

To establish that equilibria are ordered in terms of user surplus, we note that

$$\frac{\partial \widehat{CS}^B(CS^A)}{\partial CS^A} = -\frac{\sum_{i=1}^M \frac{\partial \widetilde{n}_i^B(CS^A, CS^B)}{\partial CS^A}}{\sum_{i=1}^M \frac{\partial \widetilde{n}_i^B(CS^A, CS^B)}{\partial CS^B}} \leq 0.$$

For any equilibrium values of $CS^A$, $CS_1^{A*}$ and $CS_2^{A*}$ such that $CS_1^{A*} > CS_2^{A*}$, we have

$$CS_1^{B*} = \widehat{CS}^B(CS_1^{A*}) < \widehat{CS}^B(CS_2^{A*}) = CS_2^{B*}.$$

Therefore, equilibria are ranked in terms of group-$A$ or group-$B$ user surplus.

There exists $CS^A$-maximal and $CS^A$-minimal equilibria, the former of which minimizes $CS^B$, and the latter maximizes it. □

*Proof of Remark 2.* We rewrite the first-order condition as

$$F(h_i^A, H^A) := (1-\alpha^A)\frac{H^A}{H^A - h_i^A} - (a_i^A - c_i^A) + (1-\alpha^A)\log h_i^A = 0.$$

By the implicit function theorem,

$$\frac{dh_i^A}{dH^A} = -\frac{\frac{\partial F(h_i^A, H^A)}{\partial H^A}}{\frac{\partial F(h_i^A, H^A)}{\partial h_i^A}}.$$

Since $\frac{\partial F(h_i^A, H^A)}{\partial h_i^A} = (1-\alpha^A)\frac{H^A}{(H^A - h_i^A)^2} + (1-\alpha^A)\frac{1}{h_i^A} = (1-\alpha^A)\frac{h_i^A H^A + (H^A - h_i^A)^2}{h_i^A (H^A - h_i^A)^2}$ and $\frac{\partial F(h_i^A, H^A)}{\partial H^A} = -(1-\alpha^A)\frac{h_i^A}{(H^A - h_i^A)^2}$, we have that

$$\frac{dh_i^A}{dH^A} = \frac{\frac{h_i^A}{(H^A - h_i^A)^2}}{\frac{h_i^A H^A + (H^A - h_i^A)^2}{h_i^A (H^A - h_i^A)^2}} = \frac{(h_i^A)^2}{h_i^A H^A + (H^A - h_i^A)^2} > 0$$

The equilibrium is unique if $\sum_i \frac{dh_i^A}{dH^A} < 1$. Hence, uniqueness is implied by inequalities

$$\frac{(h_i^A)^2}{h_i^A H^A + (H^A - h_i^A)^2} < \frac{h_i^A}{H^A}$$

for all $i \in \{1, \ldots, M\}$, which is always satisfied. To see this, we rewrite inequalities as $h_i^A H^A < h_i^A H^A + (H^A - h_i^A)^2$, which is equivalent to $0 < (H^A - h_i^A)^2$. □

*Proof of Lemma 4.* Consider the type $(\tau_i^A, \tau_i^B)$ with $\tau_i^k := \exp\{v_i^k\}$ of a platform that is consistent with the network sizes $(n_i^A, n_i^B)$ and aggregates $(H^A, H^B)$. Solving explicitly for $(\tau_i^A, \tau_i^B)$

that is consistent with $(n_i^A, n_i^B)$ and $(H^A, H^B)$, we obtain the solution

$$\tau_i^k = \frac{\left(n_i^k H^k\right)^{1-\alpha^k}}{\left(n_i^l H^l\right)^{\beta^k}} \exp\left[\frac{1}{1-n_i^k}\left(1-\alpha^k-\beta^l\frac{n_i^l}{n_i^k}\right)\right].$$

This completes the first part of the proof.

By rewriting the above equation, we obtain

$$H^k = \frac{\left(\tau_i^k\right)^{\Gamma^{kk}}\left(\tau_i^l\right)^{\Gamma^{kl}}}{n_i^k} \exp\left[-\frac{\Gamma^{kk}}{1-n_i^k}\left(1-\alpha^k-\beta^l\frac{n_i^l}{n_i^k}\right)-\frac{\Gamma^{kl}}{1-n_i^l}\left(1-\alpha^l-\beta^k\frac{n_i^k}{n_i^l}\right)\right],$$

which completes the second part of the proof. □

*Proof of Remark 4.* Again, we rewrite platform types as $(\tau_i^A, \tau_i^B)$ with $\tau_i^k = \exp\{v_i^k\}$. The first part of the proposition immediately follows from Lemma 4-1.

Letting $n = (n_i^A, n_i^B)_{i\in\{1,\ldots,M\}}$, we can write

$$\sum_{j=1}^M \tau_j^k = \mathcal{T}^k(n)\left(H^k\right)^{1-\alpha^k}\left(H^l\right)^{-\beta^k},$$

where

$$\mathcal{T}^k(n) = \sum_{j=1}^M \left(n_j^k\right)^{1-\alpha^k}\left(n_j^l\right)^{-\beta^k} \exp\left[\frac{1}{1-n_j^k}\left(1-\alpha^k-\beta^l\frac{n_j^l}{n_j^k}\right)\right]$$

For the chosen $(\bar{\tau}^A, \bar{\tau}^B)$, set

$$H^k = \left(\frac{\bar{\tau}^k}{\mathcal{T}^k(n)}\right)^{\Gamma^{kk}}\left(\frac{\bar{\tau}^l}{\mathcal{T}^l(n)}\right)^{\Gamma^{kl}},$$

we obtain the type profiles $(\tau_i^A, \tau_i^B)_{i\in\{1,\ldots,M\}}$ such that $\sum_{j=1}^M \tau_j^k = \bar{\tau}^k$, and the network sizes are consistent with the aggregates $(H^A, H^B)$, which completes the second part of the proof, with $\bar{v}^k = \log\bar{\tau}^k$. □

*Proof of Proposition 3.* Using the definitions of $\widetilde{f}_i^k$ from the proof of Lemma 3, we have

$$\frac{\partial \widetilde{f}_i^A}{\partial v_i^A} = 1 - n_i^A,$$
$$\frac{\partial \widetilde{f}_i^B}{\partial v_i^A} = 0$$

By applying the implicit function theorem, it can easily be shown that, as an equilibrium property, $\partial h_i^A/\partial v_i^A > 0$ and $\partial h_i^B/\partial v_i^A \geq 0$, as shown in the proof of Lemma 2. We note that $\partial h_i^B/\partial v_i^A > 0$ if and only if $\beta^A$ or $\beta^B$ is strictly positive. Since $n_i^k = h_i^k/H^k$, the result follows. □

*Proof of Proposition 3.* Recall that the definitions of $(g_i^A, g_i^B)$ from the proof of Proposition 2 are given by

$$g_i^A =(1-n_i^A)[a_i^A - c_i^A - (1-\alpha^A)\log n_i^A + \beta^A \log n_i^B - CS^A] - 1 + \alpha^A + \beta^B \frac{n_i^B}{n_i^A} = 0, \quad \text{(A.7)}$$

$$g_i^B =(1-n_i^B)[a_i^B - c_i^B - (1-\alpha^B)\log n_i^B + \beta^B \log n_i^A - CS^B] - 1 + \alpha^B + \beta^A \frac{n_i^A}{n_i^B} = 0. \quad \text{(A.8)}$$

These equations imply that $v_i^A := a_i^A - c_i^A$ and $CS^A$ affects $n_i^A$ and $n_i^B$ only through $v_i^A - CS^A$. Hence, we have $\partial n_i^k/\partial v_i^k = -\partial n_i^k/\partial CS^k > 0$ and $\partial n_i^k/\partial v_i^l = -\partial n_i^k/\partial CS^l \geq 0$ for $k, l = A, B$, where $\partial n_i^k/\partial CS^k < 0$ $\partial n_i^k/\partial CS^l \leq 0$ are established in the proof of Proposition 2. □

*Proof of Proposition 4.* We consider $v_i^A > v_j^A$, $v_i^B = v_j^B$. Denote price-cost margin for group $k$ as $\mu_i^k = p_i^k - c_i^k$. Differences across platform depend only on cost-adjusted quality offered to group $A$, $v_i^A = a_i^A - c_i^A$. We can write
$\mu^k(v_i^A) = \mu^k(v_j^A) + \int_{v_j^A}^{v_i^A} \frac{d\mu^k(v^A)}{dv^A} dv^A$.

Hence, $\mu^k(v_i^A) > \mu^k(v_j^A)$ is implied by $\frac{d\mu^k(v^A)}{dv^A} > 0$. We express price-cost margins using the formula from Lemma 1.

$$p_i^B(h_i^A, h_i^B) - c_i^B = v_i^B - (1-\alpha^B)\log h_i^B + \beta^B \log h_i^A.$$

As in Proposition 3, we use the definitions of $\widetilde{f}_i^k$ from the proof of Lemma 3,Sign $\left(\frac{d\left[p_i^B(h_i^A, h_i^{B+}) - c_i^B\right]}{dv_i^A}\right) =$ Sign $\left(\frac{\partial h_i^A}{\partial v_i^A}\frac{\beta^B}{h_i^A} - \frac{\partial h_i^B}{\partial v_i^A}\frac{1-\alpha^B}{h_i^B}\right)$ which is positive if and only if

$$\beta^B \left\{ \frac{(1-n_i^B)^2 + n_i^B}{1-n_i^B}(1-\alpha^B) - \beta^A \frac{n_i^A}{1-n_i^B} + \beta^B \frac{n_i^A}{n_i^B} \right\} - (1-\alpha^B)\left[\beta^B(1-n_i^B) + \beta^A \frac{n_i^A}{n_i^B}\right]$$

is positive. This simplifies to

$$\frac{n_i^B}{1-n_i^B}\beta^B(1-\alpha^B) - \beta^A\beta^B \frac{n_i^A}{1-n_i^B} + (\beta^B)^2 \frac{n_i^A}{n_i^B} - (1-\alpha^B)\beta^A \frac{n_i^A}{n_i^B} > 0.$$

We now prove the statement of the proposition. (i) If $\beta^A > 0$ and $\beta^B = 0$, the above expression is negative, so $p_i^B - c_i^B$ is decreasing in $v_i^A$, implying that $p_i^B - c_i^B < p_j^B - c_j^B$ when $v_i^A > v_j^A$ and $v_i^B = v_j^B$. (ii) If $\beta^B > 0$ and $\beta^A = 0$, the above expression is positive, so $p_i^B - c_i^B$ is increasing in $v_i^A$, implying that $p_i^B - c_i^B > p_j^B - c_j^B$ when $v_i^A > v_j^A$ and $v_i^B = v_j^B$. □

*Proof of Remark 5.* Similar to the proof of Proposition 4, this proof relies on small variations of the cost-adjusted platform quality, to establish how the price-cost margin given in Lemma 1, $p_i^A(h_i^A, h_i^B) - c_i^A = v_i^A - (1-\alpha^A)\log h_i^A + \beta^A \log h_i^B$, reacts. We have

$$\begin{aligned} & \text{Sign}\left(\frac{d[p_i^A(h_i^{A+}, p_i^{B+}) - c_i^A]}{dv_i^A}\right) \\ = \; & \text{Sign}\left(1 - (1-n_i^A)\frac{-(1-\alpha^A)\frac{1}{h_i^A}\frac{\partial \tilde{f}_i^B}{\partial h_i^B} - \beta^A \frac{1}{h_i^B}\frac{\partial \tilde{f}_i^B}{\partial h_i^A}}{\frac{\partial \tilde{f}_i^A}{\partial h_i^A}\frac{\partial \tilde{f}_i^B}{\partial h_i^B} - \frac{\partial \tilde{f}_i^A}{\partial h_i^B}\frac{\partial \tilde{f}_i^B}{\partial h_i^A}}\right) \end{aligned}$$

After some calculations, it turns out that the above expression has the same sign as

$$\begin{aligned} & \left[n_i^A(p_i^A - c_i^A) + \beta^B \frac{n_i^B}{n_i^A}\right]\left[n_i^B(p_i^B - c_i^B) + \beta^A \frac{n_i^A}{n_i^B} + (1-\alpha^B)(1-n_i^B)\right] \\ & - \beta^B\left[\beta^A + \beta^B \frac{n_i^B}{n_i^A}(1-n_i^B)\right] \\ = & \left[\frac{n_i^A}{1-n_i^A}(1-\alpha^A) + \beta^B \frac{n_i^B}{n_i^A}\left(1 - \frac{n_i^A}{1-n_i^A}\right)\right]\left[\left(\frac{n_i^B}{1-n_i^B} + 1 - n_i^B\right)(1-\alpha^B) + \beta^A \frac{n_i^A}{n_i^B}\left(1 - \frac{n_i^B}{1-n_i^B}\right)\right] \\ & - \beta^B\left[\beta^A + \beta^B \frac{n_i^B}{n_i^A}(1-n_i^B)\right] \\ \geq & \left[\beta^B \frac{(1-n_i^B)n_i^B + (n_i^A - n_i^B)^2}{(1-n_i^A)n_i^A}\right]\left[\left(\frac{n_i^B}{1-n_i^B} + 1 - n_i^B\right)(1-\alpha^B) + \beta^A \frac{n_i^A}{n_i^B}\left(1 - \frac{n_i^B}{1-n_i^B}\right)\right] \\ & - \beta^B\left[\beta^A + \beta^B \frac{n_i^B}{n_i^A}(1-n_i^B)\right] \end{aligned}$$

where we obtain the last expression from the following calculation:

$$\begin{aligned} & \frac{n_i^A}{1-n_i^A}(1-\alpha^A) + \beta^B \frac{n_i^B}{n_i^A}\left(1 - \frac{n_i^A}{1-n_i^A}\right) \\ = \; & \frac{(n_i^A)^2(1-\alpha^A) + \beta^B n_i^B(1-2n_i^A)}{(1-n_i^A)n_i^A} \\ \geq \; & \beta^B \frac{(n_i^A)^2 + n_i^B - 2n_i^A n_i^B}{(1-n_i^A)n_i^A} \\ = \; & \beta^B \frac{(1-n_i^B)n_i^B + (n_i^A - n_i^B)^2}{(1-n_i^A)n_i^A}. \end{aligned}$$

In the following cases, $d(p_i^A - c_i^A)/dv_i^A$ positive:

1. When $\beta^B = 0$: this is straightforward.

2. When $\beta^A = 0$: to see this, for $\beta^A = 0$, the expression under consideration is

$$\begin{aligned} & \left[\frac{n_i^A}{1-n_i^A}(1-\alpha^A) + \beta^B \frac{n_i^B}{n_i^A}\left(1 - \frac{n_i^A}{1-n_i^A}\right)\right]\left(\frac{n_i^B}{1-n_i^B} + 1 - n_i^B\right)(1-\alpha^B) \\ & -(\beta^B)^2 \frac{n_i^B}{n_i^A}(1-n_i^B) \\ > \ & (\beta^B)^2 \frac{n_i^B}{n_i^A} \frac{n_i^B + (1-n_i^B)^2 - (1-n_i^B)}{(1-n_i^A)n_i^A} \\ > \ & 0. \end{aligned}$$

3. When $n_i^A \geq n_i^B$: to see this, the expression under consideration is greater than

$$\begin{aligned} & \beta^B \frac{(1-n_i^B)n_i^B}{(1-n_i^A)n_i^A}\left[\beta^A \frac{(1-n_i^A)n_i^A}{(1-n_i^B)n_i^B} + \beta^B(1-n_i^B)\right] \\ = \ & \beta^B\left[\beta^A + \frac{(1-n_i^B)}{(1-n_i^A)}\beta^B \frac{n_i^B}{n_i^A}(1-n_i^B)\right] \\ \geq \ & \beta^B\left[\beta^A + \beta^B \frac{n_i^B}{n_i^A}(1-n_i^B)\right] \end{aligned}$$

because $(1-n_i^B)/(1-n_i^A) \geq 1$ when $n_i^A \geq n_i^B$.

Therefore, if corresponding values of $(n^A, n^B)$ satisfy $n^A \geq n_i^B$ for all $v^A \in [v_j^A, v_i^A]$, we have the desired result. To establish this, it suffices to examine that $d(n^A/n^B)/dv^A > 0$ at $v^A$ such that $n^A = n^B$. To see this, note that

$$\begin{aligned} & \text{Sign}\left[\frac{d}{dv_i^A}\left(\frac{n_i^A}{n_i^B}\right)\right] \\ = \ & \text{Sign}\left[\frac{1}{h_i^A}\frac{\partial h_i^A}{\partial v_i^A} - \frac{1}{h_i^B}\frac{\partial h_i^B}{\partial v_i^A}\right] \\ = \ & \text{Sign}\left(-\frac{1}{h_i^A}\frac{\partial \widetilde{f}^B}{\partial h_i^B} - \frac{1}{h_i^B}\frac{\partial \widetilde{f}_i^B}{\partial h_i^A}\right) \\ = \ & \text{Sign}\left[n_i^A(p_i^A - c_i^A) + (1-n_i^B)(1-\alpha^B-\beta^B)\right], \end{aligned}$$

which is positive as long as $p_i^A - c_i^A \geq 0$, which always holds when $n_i^A \geq n_i^B$. Therefore, along the path from $v_j^A$ to $v_i^A$, if $n_j^A \geq n_j^B$, $n^A \geq n^B$ always holds for the intermediate values of $v^A$, holding other parameters fixed. Therefore, $p^A - c^A$ also monotonically increases on this path, establishing that $p_i^A - c_i^A > p_j^A - c_j^A$.

4. When $p_i^A - c_i^A \geq 0$ and $p_i^B - c_i^B \geq 0$: we consider the case where $n_i^B > n_i^A$ because we have established the desired result in the case where $n_i^A \geq n_i^B$. First, it is straightforward that a small change in $v_i^A$ increases $p_i^A - c_i^A$ in this case. What needs to be shown is that $p_i^A - c_i^A$

continuously increases as $v_i^A$ increases more. To see this, note that a small increase in $v_i^A$ increases $n_i^A/n_i^B$ if $p_i^A - c_i^A \geq 0$. Therefore, the sign of $p_i^A - c_i^A$, which is determined by the sign of $1 - \alpha^A - \beta^B n_i^B/n_i^A$, is positive for all $v \in [\underline{v}_i^A, \overline{v}_i^A]$ as long as $p_i^A - c_i^A \geq 0$ at $\underline{v}_i^A$. Furthermore, because $p_i^B - c_i^B \geq 0$ whenever $n_i^B > n_i^A$. Put together, at any point on the path $[\underline{v}_i^A, \overline{v}_i^A]$, the signs of $p_i^A - c_i^A$ and $p_i^B - c_i^B$ are positive, implying that the local increase in $p_i^A - c_i^A$ continues until the end. This establishes that $p_i^A - c_i^A > p_j^A - c_j^A$.

The remaining case is $\beta^A > 0$, $\beta^B > 0$, and $n_i^B > n_i^A$. It turns out that $d(p_i^A - c_i^A)/dv_i^A < 0$ may hold in this case. We show this by example. Suppose that $\beta^A = \beta^B = 0.995$, $\alpha^A = \alpha^B = 0$, $n_i^A = 0.2$, and $n_i^B = 0.25$. By Remark 4, we can obtain these participation levels with appropriate choices of the primitives of the model. Then,

$$\begin{aligned}
&\left[\frac{n_i^A}{1-n_i^A}(1-\alpha^A) + \beta^B \frac{n_i^B}{n_i^A}\left(1 - \frac{n_i^A}{1-n_i^A}\right)\right]\left[\left(\frac{n_i^B}{1-n_i^B} + 1 - n_i^B\right)(1-\alpha^B) + \beta^A \frac{n_i^A}{n_i^B}\left(1 - \frac{n_i^B}{1-n_i^B}\right)\right] \\
&- \beta^B\left[\beta^A + \beta^B \frac{n_i^B}{n_i^A}(1-n_i^B)\right] \\
&= -0.000911406.
\end{aligned}$$

□

*Proof of Proposition 5.* We note that

$$\begin{aligned}
\text{Sign}\left\{\frac{\partial}{\partial v_i^A}\left(\frac{\widetilde{n}_i^A}{\widetilde{n}_i^B}\right)\right\} &= \text{Sign}\left(\frac{\partial \widetilde{n}_i^A}{\partial v_i^A} n_i^B - \frac{\partial \widetilde{n}_i^B}{\partial v_i^A} n_i^A\right) \\
&= \text{Sign}\left(-\frac{\partial g_i^B}{\partial n_i^B} n_i^B - \frac{\partial g_i^B}{\partial n_i^A} n_i^A\right) \\
&= \text{Sign}\left(n_i^B(p_i^B - c_i^B) + (1-\alpha^B)(1-n_i^B) + \frac{n_i^A}{n_i^B}\beta^A - (1-n_i^B)\beta^B - \frac{n_i^A}{n_i^B}\beta^A\right) \\
&= \text{Sign}\left(\frac{n_i^B}{1-n_i^B}\left(1 - \alpha^B - \frac{n_i^A}{n_i^B}\beta^A\right) + (1-n_i^B)(1-\alpha^B-\beta^B)\right),
\end{aligned}$$

where

$$\frac{n_i^B}{1-n_i^B}\left(1 - \alpha^B - \frac{n_i^A}{n_i^B}\beta^A\right) + (1-n_i^B)(1-\alpha^B-\beta^B),$$

which is strictly positive whenever $p_i^B - c_i^B \geq 0$.

Since

$$\text{Sign}\left(p_i^k - c_i^k\right) = \text{Sign}\left(1 - \alpha^k - \beta^l \frac{n_i^l}{n_i^k}\right),$$

there exists a critical value $\overline{v}_i^A$ such that $p_i^B - c_i^B < 0$ if and only if $v_i^A > \overline{v}_i^A$. For $v_i^A \leq \overline{v}_i^A$,

$n_i^A/n_i^B$ is monotonically increasing in $v_i^A$. Hence, there exists another critical value $\underline{v}_i^A < \overline{v}_i^A$ such that $p_i^A - c_i^A < 0$ if and only if $v_i^A < \underline{v}_i^A$. The cross-section comparison then implies the statement of the proposition. □

*Proof of Proposition 6.* Using the pricing formula, we can write the profit of each platform as a function of its network sizes:

$$\Pi_i = \frac{n_i^A}{1-n_i^A}\left(1-\alpha^A-\beta^B\frac{n_i^B}{n_i^A}\right)+\frac{n_i^B}{1-n_i^B}\left(1-\alpha^B-\beta^A\frac{n_i^A}{n_i^B}\right).$$

Noting that

$$\begin{aligned}\frac{\partial g_i^B}{\partial n_i^B} &= -\frac{1}{n_i^B}\left[n_i^B(p_i^B-c_i^B)+(1-\alpha^B)(1-n_i^B)+\beta^A\frac{n_i^A}{n_i^B}\right]<0,\\ \frac{\partial g_i^B}{\partial n_i^A} &= \frac{1}{n_i^A}\left[\beta^B(1-n_i^B)+\beta^A\frac{n_i^A}{n_i^B}\right]\geq 0,\end{aligned}$$

where $(g_i^A, g_i^B)$ is define in the proof of Proposition 2, and using the implicit function theorem, we have that

$$\frac{\frac{\partial \tilde{n}_i^A}{\partial v_i^A}}{\frac{\partial \tilde{n}_i^B}{\partial v_i^A}} = -\frac{\frac{\partial g_i^B}{\partial n_i^B}}{\frac{\partial g_i^B}{\partial n_i^A}} = \frac{(p_i^B-c_i^B)+(1-\alpha^B)\frac{1-n_i^B}{n_i^B}+\frac{n_i^A\beta^A}{(n_i^B)^2}}{\frac{1-n_i^B}{n_i^A}\beta^B+\frac{\beta^A}{n_i^B}},$$

which can be rewritten as

$$\frac{\partial n_i^B}{\partial v_i^A} = \frac{\partial n_i^A}{\partial v_i^A}\frac{\frac{1-n_i^B}{n_i^A}\beta^B+\frac{\beta^A}{n_i^B}}{p_i^B-c_i^B+(1-\alpha^B)\frac{1-n_i^B}{n_i^B}+\frac{n_i^A\beta^A}{(n_i^B)^2}}.$$

Hence, $\partial\Pi_i/\partial v_i^A = (\partial n_i^A/\partial v_i^A)\Delta\Pi_i$, where

$$\begin{aligned}\Delta\Pi_i \;=\; & \frac{n_i^A}{(1-n_i^A)^2}\left(1-\alpha^A-\beta^B\frac{n_i^B}{n_i^A}\right)+\frac{1-\alpha^A}{1-n_i^A}-\frac{\beta^A}{1-n_i^B}\\ & +\frac{\frac{1-n_i^B}{n_i^A}\beta^B+\frac{\beta^A}{n_i^B}}{p_i^B-c_i^B+(1-\alpha^B)\frac{1-n_i^B}{n_i^B}+\frac{n_i^A\beta^A}{(n_i^B)^2}}\left[\frac{n_i^B}{(1-n_i^B)^2}\left(1-\alpha^B-\beta^A\frac{n_i^A}{n_i^B}\right)+\frac{1-\alpha^B}{1-n_i^B}-\frac{\beta^B}{1-n_i^A}\right].\end{aligned}$$

Therefore, $\partial\Pi_i/\partial v_i^A$ has the same sign as $\frac{\partial n_i^A}{\partial v_i^A}$ if $\Delta\Pi_i$ is positive.

The following calculations show that this is always the case. $\Delta\Pi_i$ has the same sign as

$$
\begin{aligned}
& \left(p_i^B - c_i^B + (1-\alpha^B)\frac{1-n_i^B}{n_i^B} + \frac{n_i^A\beta^A}{(n_i^B)^2}\right)\left[\frac{n_i^A}{(1-n_i^A)^2}\left(1-\alpha^A-\beta^B\frac{n_i^B}{n_i^A}\right)+\frac{1-\alpha^A}{1-n_i^A}-\frac{\beta^A}{1-n_i^B}\right] \\
& +\left(\frac{1-n_i^B}{n_i^A}\beta^B+\frac{\beta^A}{n_i^B}\right)\left[\frac{n_i^B}{(1-n_i^B)^2}\left(1-\alpha^B-\beta^A\frac{n_i^A}{n_i^B}\right)+\frac{1-\alpha^B}{1-n_i^B}-\frac{\beta^B}{1-n_i^A}\right] \\
= & \left(p_i^B - c_i^B + (1-\alpha^B)\frac{1-n_i^B}{n_i^B} + \frac{n_i^A\beta^A}{(n_i^B)^2}\right)\left[\frac{n_i^A}{(1-n_i^A)^2}\left(1-\alpha^A-\beta^B\frac{n_i^B}{n_i^A}\right)+\frac{1-\alpha^A}{1-n_i^A}\right] \\
& -\frac{\beta^A}{1-n_i^B}\left((1-\alpha^B)\frac{1-n_i^B}{n_i^B}+\frac{n_i^A\beta^A}{(n_i^B)^2}\right) \\
& +\frac{1-n_i^B}{n_i^A}\beta^B\left[\frac{n_i^B}{(1-n_i^B)^2}\left(1-\alpha^B-\beta^A\frac{n_i^A}{n_i^B}\right)+\frac{1-\alpha^B}{1-n_i^B}-\frac{\beta^B}{1-n_i^A}\right] \\
& +\frac{\beta^A}{n_i^B}\left(\frac{1-\alpha^B}{1-n_i^B}-\frac{\beta^B}{1-n_i^A}\right) \\
= & \left(p_i^B - c_i^B + (1-\alpha^B)\frac{1-n_i^B}{n_i^B} + \frac{n_i^A\beta^A}{(n_i^B)^2}\right)\left[\frac{n_i^A}{(1-n_i^A)^2}\left(1-\alpha^A-\beta^B\frac{n_i^B}{n_i^A}\right)+\frac{1-\alpha^A}{1-n_i^A}\right] \\
& +\frac{\beta^A}{(1-n_i^B)}(1-\alpha^B)-\frac{\beta^A}{1-n_i^B}\frac{n_i^A\beta^A}{(n_i^B)^2} \\
& +\frac{1-n_i^B}{n_i^A}\beta^B\left[\frac{n_i^B}{(1-n_i^B)^2}\left(1-\alpha^B-\beta^A\frac{n_i^A}{n_i^B}\right)+\frac{1-\alpha^B}{1-n_i^B}-\frac{\beta^B}{1-n_i^A}\right] \\
& -\frac{\beta^A}{n_i^B}\frac{\beta^B}{1-n_i^A} \\
> & \ \overline{\beta}^2 X_\Pi,
\end{aligned}
$$

where

$$
\begin{aligned}
X_\Pi = & \left(\frac{1-\frac{n_i^A}{n_i^B}}{1-n_i^B}+\frac{1-n_i^B}{n_i^B}+\frac{n_i^A}{(n_i^B)^2}\right)\left[\frac{n_i^A}{(1-n_i^A)^2}\left(1-\frac{n_i^B}{n_i^A}\right)+\frac{1}{1-n_i^A}\right]+\frac{1}{1-n_i^B}\left(1-\frac{n_i^A}{(n_i^B)^2}\right) \\
& +\frac{1-n_i^B}{n_i^B}\left[\frac{n_i^B}{(1-n_i^B)^2}\left(1-\frac{n_i^A}{n_i^B}\right)+\frac{1}{1-n_i^B}-\frac{1}{1-n_i^A}\right]-\frac{1}{n_i^B(1-n_i^A)} \\
= & \frac{1}{1-n_i^A}\left[\frac{1-\frac{n_i^A}{n_i^B}}{1-n_i^B}+\frac{1}{n_i^B}\left(\frac{n_i^A}{n_i^B}-1\right)\right]+\frac{n_i^A}{(1-n_i^A)^2}\left(1-\frac{n_i^B}{n_i^A}\right)\left(\frac{1-\frac{n_i^A}{n_i^B}}{1-n_i^B}+\frac{1-n_i^B}{n_i^B}+\frac{n_i^A}{(n_i^B)^2}\right) \\
& +\frac{1}{1-n_i^B}\left(1-\frac{n_i^A}{n_i^B}\right)+\frac{1}{n_i^B}.
\end{aligned}
$$

Thus, if $X_\Pi > 0$, we have that $\Delta\Pi_i > 0$.

We define $Y_\Pi := (1-n_i^A)^2(1-n_i^B)(n_i^B)^2 X_\Pi$, which has the same sign as $X_\Pi$.

$$\begin{aligned}
Y_\Pi &= (1-n_i^A)\left[n_i^B(n_i^B-n_i^A)+(1-n_i^B)(n_i^A-n_i^B)\right]\\
&\quad +(n_i^A-n_i^B)\left[n_i^B(n_i^B-n_i^A)+n_i^B(1-n_i^B)^2+(1-n_i^B)n_i^A\right]\\
&\quad +(1-n_i^B)^2n_i^B(n_i^B-n_i^A)+(1-n_i^A)^2(1-n_i^B)n_i^B\\
&= (1-n_i^A)\left[n_i^B(n_i^B-n_i^A)+(1-n_i^B)(n_i^A-n_i^B)\right]\\
&\quad +(n_i^A-n_i^B)\left[n_i^B(n_i^B-n_i^A)+(1-n_i^B)n_i^A\right]\\
&\quad +(1-n_i^A)^2(1-n_i^B)n_i^B\\
&= (1-n_i^A)\left[(1-n_i^A)(1-n_i^B)n_i^B+n_i^B(n_i^B-n_i^A)+(1-n_i^B)(n_i^A-n_i^B)\right]\\
&\quad +(n_i^A-n_i^B)\left[n_i^B(n_i^B-n_i^A)+(1-n_i^B)n_i^A\right]\\
&= (1-n_i^A)\left[n_i^A-3n_i^An_i^B+(1+n_i^A)(n_i^B)^2\right]\\
&\quad +(n_i^A-n_i^B)\left[n_i^B(n_i^B-n_i^A)+(1-n_i^B)n_i^A\right]\\
&= n_i^A-3n_i^An_i^B-(n_i^A)^2+3(n_i^A)^2n_i^B+(n_i^B)^2-(n_i^A)^2(n_i^B)^2\\
&\quad +(n_i^A-n_i^B)[n_i^A-2n_i^An_i^B+(n_i^B)^2]\\
&= n_i^A-4n_i^An_i^B+(n_i^A)^2n_i^B+3n_i^A(n_i^B)^2+(n_i^B)^2-(n_i^A)^2(n_i^B)^2-(n_i^B)^3\\
&= n_i^A[1-4n_i^B+3(n_i^B)^2]+(n_i^A)^2n_i^B+(n_i^B)^2-(n_i^A)^2(n_i^B)^2-(n_i^B)^3\\
&= n_i^A(1-n_i^B)(1-3n_i^B)+[(n_i^A)^2+n_i^B]n_i^B(1-n_i^B)\\
&= (1-n_i^B)\left[n_i^A(1-3n_i^B)+(n_i^B)^2+n_i^B(n_i^A)^2\right]\\
&= (1-n_i^B)\left[(n_i^B)^2-2n_i^An_i^B+(n_i^A)^2+n_i^A-(n_i^A)^2-n_i^An_i^B+n_i^B(n_i^A)^2\right]\\
&= (1-n_i^B)\left[(n_i^B-n_i^A)^2+n_i^A(1-n_i^A)(1-n_i^B)\right]\\
&> 0.
\end{aligned}$$

This establishes that $\Delta\Pi_i > 0$, which completes the proof.

Note that because the profit depends on $v_i^k$ only through $v_i^k - CS^k$ (see the proof of Proposition 3), the profit decreases in $CS^k$, which is used in the proof of Proposition 9. □

*Proof of Remark 6.* Suppose that $v_i > v_j$. The result $n_i > n_j$ follows from Proposition 3; $p_i - c_i > p_j - c_j$ follows from case (1) of Remark 5; and $\Pi_i > \Pi_j$ follows from Proposition 6. □

*Proof of Proposition 7.* We show each of the statements of the proposition.

1. Fix the characteristics of an entrant $(a_E^A, c_E^A, a_E^B, c_E^B)$. Then, we show that if $\max\{\beta^A, \beta^B\}$ is sufficiently small, the user surpluses for both groups increase with entry. To see this, consider the limit case of zero cross-group network effects (i.e., $\beta^A = \beta^B = 0$). When $\beta^A = \beta^B = 0$, $\widetilde{n}_i^k(CS^A, CS^B)$ depends only on $CS^k$. We have $\widetilde{n}_i^A(CS^A, CS^B) = \widetilde{n}_i^A(CS^A, 0)$

and $\widetilde{n}_i^B(0, CS^B)$. Given the pre-entry equilibrium user surpluses $(CS^{A*}, CS^{B*})$ and

$$\sum_{i=1}^{M} \widetilde{n}_i^A(CS^{A*}, 0) + \widetilde{n}_E^A(CS^{A*}, 0) > 1,$$
$$\sum_{i=1}^{M} \widetilde{n}_i^B(0, CS^{B*}) + \widetilde{n}_E^B(0, CS^{B*}) > 1,$$

the post-entry user surpluses $(CS^{A**}, CS^{B**})$ must satisfy $CS^{A**} > CS^{A*}$ and $CS^{B**} > CS^{B*}$. Because of the continuity of the model in parameters, we obtain the statement.

2. Take group-$A$ optimal equilibrium $(CS^{A*}, CA^{B*})$, which is also the equilibrium that minimizes the group-$B$ user surplus. Let $CS^{A**} > CS^{A*}$ and $CS^{B**} = \widehat{CS}^B(CS^{A**}) + \epsilon$, where $\epsilon > 0$ is a sufficiently small positive number such that $\widehat{CS}^B(CS^{A**}) + \epsilon < CS^{B*}$. Then, at $(CS^{A**}, CS^{B**})$,

$$\sum_i \widetilde{n}_i^A(CS^{A**}, CS^{B**}) < 1,$$
$$\sum_i \widetilde{n}_i^B(CS^{A**}, CS^{B**}) < 1.$$

This pair of $(CS^{A**}, CS^{B**})$ is consistent with entry of platform $E$ with post-entry market shares

$$n_E^A = 1 - \sum_i \widetilde{n}_i^A(CS^{A**}, CS^{B**}) > 0,$$
$$n_E^B = 1 - \sum_i \widetilde{n}_i^B(CS^{A**}, CS^{B**}) > 0.$$

Then, the proof of Remark 4 implies that there exists a type of platform that is consistent with $(n_E^A, n_E^B)$ and $(CS^{A**}, CS^{B**})$. Hence, there exists platform entry that induces $(CS^{A**}, CS^{B**})$ as an equilibrium outcome, and the lowest equilibrium group-$B$ user surplus is lower after the entry than the pre-entry level.

3. Consider the entry of new platform $E$ with characteristics $(a_E^A, c_E^A, a_E^B, c_E^B)$. At any pre-entry equilibrium user surpluses $(CS^A, CS^B)$, we have

$$\sum_{i=1}^{M} \widetilde{n}_i^k(CS^A, CS^B) + \widetilde{n}_E^k(CS^A, CS^B) = 1 + \widetilde{n}_E^k(CS^A, CS^B) > 1$$

for $k \in \{A, B\}$. Since $\widetilde{n}_j^k(CS^A, CS^B)$ is decreasing in $(CS^A, CS^B)$, $CS^A$ or $CS^B$ must be greater than the pre-entry level. Now, take the group-$A$ optimal equilibrium user surpluses $(CS^{A*}, CS^{B*})$. Then, there must be a post-entry equilibrium user surpluses $(CS^{A**}, CS^{B**})$ that satisfies $CS^{A**} > CS^{A*}$ or $CS^{B**} > CS^{B*}$, implying that maximal group-$A$ user surplus or minimal group-$B$ user surplus increases with entry.

□

*Proof of Example 1.* Consider a symmetric duopoly prior to entry. Then, in the pre-entry equilibrium, we have $n_i^{A*} = n_i^{B*} = 1/2$, $p_i^A - c_i^A = 2(1 - \alpha^A - \beta^B) > 0$, and $p_i^B - c_i^B = 2(1 - \alpha^B - \beta^A) > 0$ regardless of the type of the platforms $(v^A, v^B)$.

Now, suppose that the value of the post-entry aggregates is given by $(H^{A**}, H^{B**})$. Suppose that the types of the entrant and incumbent platforms are such that incumbents obtain the network sizes $(n_I^A, n_I^B)$ and, thus, the entrant obtains $(n_E^A, n_E^B) = (1 - 2n_I^A, 1 - 2n_I^B)$ – the existence of such types is guaranteed for any $(n_I^A, n_I^B) \in (0, 1/2)^2$ by Lemma 4.

Then, the post-entry equilibrium prices are given by

$$p_I^A - c_I^A = \frac{1}{1 - n_I^A}\left(1 - \alpha^A - \beta^B \frac{n_I^B}{n_I^A}\right),$$
$$p_E^B - c_E^B = \frac{1}{2n_I^B}\left(1 - \alpha^B - \beta^A \frac{1 - 2n_I^A}{1 - 2n_I^B}\right).$$

Hence, if

$$n_I^B > \frac{1 - \alpha^B}{\beta^A} n_I^A + \frac{1}{2}\left(\frac{1 - \alpha^B - \beta^A}{1 - \alpha^B}\right)$$

holds, we have both $p_I^A - c_I^A < 0$ and $p_E^B - c_E^B < 0$. Correspondingly, we have $p_I^B - c_I^B < 0$ and $p_E^A - c_E^A < 0$ by interchanging the labels of the two groups. □

*Proof of Proposition 8.*

1. We show by example within the setting in Example 1. The pre-entry equilibrium profit of the platform is given by

$$\Pi^D = n_i^A(p_i^A - c_i^A) + n_i^B(p_i^B - c_i^B)$$
$$= 2 - \alpha^A - \alpha^B - \beta^A - \beta^B.$$

Now consider entry that leads to the network sizes $(n_I^A, n_I^B)$ for the two incumbents and $(n_E^A, n_E^B) = (1 - 2n_I^A, 1 - 2n_I^B)$ for the entrant. When $n_I^A \simeq 1/2$ and $n_I^B \simeq 0$, the post-entry profit of each incumbent platform is given by

$$
\begin{aligned}
\Pi^T &= n_I^A(p_I^A - c_I^A) + n_I^B(p_I^B - c_I^B) \\
&= \frac{n_I^A}{1-n_I^A}(1-\alpha^A) - \frac{n_I^B}{1-n_I^A}\beta^B + \frac{n_I^B}{1-n_I^B}(1-\alpha^B) - \frac{n_I^A}{1-n_I^B}\beta^A \\
&\simeq 1 - \alpha^A - \frac{\beta^A}{2}.
\end{aligned}
$$

Hence, we have

$$\Pi^T - \Pi^D \simeq \alpha^B + \frac{\beta^A}{2} + \beta^B - 1,$$

which is positive if $\alpha^B + \beta^A/2 + \beta^B > 1$.

2. In the proof of Proposition 6, we showed that $\Pi_i$ is decreasing in $(CS^A, CS^B)$. Hence, for a change in the equilibrium value of $(CS^A, CS^B)$ leading to an increase of profit $\Pi_i$, either $CS^A$ or $CS^B$ must decrease. This means that for entry to increase an incumbent platform's profit, the user surplus for one group must decrease.

□

*Proof of Proposition 9.* Consider a local change in a parameter that leads to a local change of $(dCS^A, dCS^B)$ to $(CS^A, CS^B)$. Let $\Pi_E(CS^A, CS^B)$ be the post-entry profit of platform $E$ as a function of $(CS^A, CS^B)$. Then, the free-entry condition can be written as

$$\Pi_E(CS^A, CS^B) - K = 0. \tag{A.9}$$

In the last paragraph of the proof of Proposition 6, we showed that $\Pi_E(CS^A, CS^B)$ is strictly decreasing in $(CS^A, CS^B)$. Hence, applying the implicit function theorem to equation (A.9), we have that

$$\left.\frac{dCS^B}{dCS^A}\right|_{\Pi_E(CS^A,CS^B)=0} = -\frac{\frac{\partial \Pi_E}{\partial CS^A}}{\frac{\partial \Pi_E}{\partial CS^B}} < 0,$$

which completes the proof. □

## A.2. Platform compatibility

In this part of the appendix, we consider the effect of the degree of compatibility on market outcomes and focus on settings in which there are only within-group network effects. Thus the two groups operate independently and we can focus our attention on group $A$. Partial compatibility implies that a fraction $\lambda$ of network effects are industry-wide. Partial compatibility is gained if some of the functionalities are available to all users, not only those on the same platform, but also those on competing platforms. An example of a regulatory intervention with that goal is Article 7 in the Digital Markets Act (DMA) in the European Union. According to this regulation, a gatekeeper of a number-independent interpersonal communications service must "make the basic functionalities of its number-independent interpersonal communications services interoperable with the number-independent interpersonal communications services of another provider."[25]

We can easily adopt the analysis of the base model to analyze the price equilibrium. Using the first-order condition given by equation (7) adjusted by $\lambda$, we obtain

$$\frac{H^A}{H^A - h_i^A} = \frac{a_i^A - c_i^A}{1 - (1-\lambda)\alpha^A} - \log h_i^A, \tag{A.10}$$

which implicitly defines a solution $h_i(H^A;\lambda)$. We note that, as $\lambda$ increases, the right-hand side decreases. This implies that an increase in compatibility pushes the function $h_i(\cdot;\lambda)$ downward. Since this holds for all functions $h_i$, $i \in \{1,\ldots,M\}$, it must be that the equilibrium aggregate $H^A$ decreases in $\lambda$.

We know that if $a_i^A - c_i^A > a_j^A - c_j^A$ platform $i$ has a larger market share than platform $j$. How does the relative market size $n_i^A/n_j^A$ change as compatibility increases? From equation (A.10) we see that $h_i$ receives a stronger downward push than $h_j$ as compatibility increases. This tends to reduce the market size asymmetry between firms. Also the equilibrium value of the aggregate changes in compatibility: because of the downward shift, the equilibrium value of $H^A$ must decrease.

We now take a closer look at the model to answer the question of how a change in the degree of compatibility affects market shares. Denoting $\tilde{\alpha}^A := \alpha^A(1-\lambda)$, the first-order condition (A.10) can be rewritten as

$$\frac{H^A - h_i^A}{H^A}\left(a_i^A - c_i^A + (1-\tilde{\alpha}^A)\log h_i^A\right) - (1-\tilde{\alpha}^A) = 0$$

or, equivalently,

$$(1-n_i^A)\left(\frac{v_i^A}{1-\tilde{\alpha}^A} - \log n_i^A - \log H^A\right) - 1 = 0$$

This defines platform $i$'s market share as a function of the aggregate $\widetilde{n}_i^A(H^A)$, which has slope

$$\frac{d\widetilde{n}_i^A}{dH^A} = \frac{-\frac{1-n_i^A}{H^A}}{\frac{1}{1-n_i^A} + \frac{1-n_i^A}{n_i^A}} < 0.$$

[25] The provision applies only to gatekeeper platforms and interoperability has to be offered upon the request of another provider. As a caveat, our model does not accommodate the situation that some but not all of the competing providers ask for interoperability.

The equilibrium condition for $H^A$ is $\sum_{i=1}^M \widetilde{n}_i^A(H^A) = 1$. We obtain the result that lower-quality platforms gain market share when the degree of compatibility is increased, while higher-quality platforms lose. In other words, industry concentration (e.g., measured by the HHI) goes down. This is formally stated in the following proposition.

**Proposition A.1.** *Supppose that $\beta^A = \beta^B = 0$ and order platforms such that $v_j^A \leq v_{j+1}^A$ for all $j \in \{1, \ldots, M-1\}$. Then an increase in the degree of compatibility $\lambda$ affects market shares as follows: there exists a critical platform $\hat{j} \in \{1, \ldots, M-1\}$ such that for all $j > \hat{j}$ market share decreases ($dn_i^{A*}/d\lambda < 0$), and for all $j \leq \hat{j}$ market share (weakly) increases ($dn_i^{A*}/d\lambda \geq 0$, where the inequality must be strict for $j = 1$ and for all $j < \hat{j}$ with $v_j^A < v_{\hat{j}}^A$).*

*Proof.* The market share of platform $i$ changes with $\tilde{\alpha}^A$

$$\begin{aligned}
\frac{d\widetilde{n}_i^A}{d\tilde{\alpha}^A} &= \frac{v_i^A}{(1-\tilde{\alpha}^A)^2} \frac{1-n_i^A}{\frac{1}{1-n_i^A} + \frac{1-n_i^A}{n_i^A}} \\
&= -\frac{H^A v_i^A}{(1-\tilde{\alpha}^A)^2} \frac{\partial \widetilde{n}_i^A}{\partial H^A}.
\end{aligned}$$

The aggregate $H^{A*}$ changes with $\tilde{\alpha}^A$ as follows:

$$\begin{aligned}
\frac{dH^{A*}}{d\tilde{\alpha}^A} &= \frac{-\sum_{i=1}^M \frac{\partial \widetilde{n}_i^A}{\partial \tilde{\alpha}^A}}{\sum_{i=1}^M \frac{\partial \widetilde{n}_i^A}{\partial H^A}} \\
&= \frac{\sum_{i=1}^M \frac{H^A v_i^A}{(1-\tilde{\alpha}^A)^2} \frac{\partial \widetilde{n}_i^A}{\partial H^A}}{\sum_{i=1}^M \frac{\partial \widetilde{n}_i^A}{\partial H^A}}.
\end{aligned}$$

We have

$$\begin{aligned}
\frac{dn_i^{A*}}{d\tilde{\alpha}^A} &= \frac{\partial \widetilde{n}_i^A}{\partial \tilde{\alpha}^A} + \frac{dH^{A*}}{d\tilde{\alpha}^A} \frac{\partial \widetilde{n}_i^A}{\partial H^A} \\
&= \left[-\frac{\partial \widetilde{n}_i^A}{\partial H^A}\right] \left(\frac{H^A v_i^A}{(1-\tilde{\alpha}^A)^2} - \frac{dH^{A*}}{d\tilde{\alpha}^A}\right) \\
&= \left[-\frac{\partial \widetilde{n}_i^A}{\partial H^A}\right] \frac{H^A}{(1-\tilde{\alpha}^A)^2} \left(v_i^A - \frac{\sum_{j=1}^M v_j^A \left[-\frac{\partial \widetilde{n}_j^A}{\partial H^A}\right]}{\sum_{j=1}^M \left[-\frac{\partial \widetilde{n}_j^A}{\partial H^A}\right]}\right) \\
&= \underbrace{\left[-\frac{\partial \widetilde{n}_i^A}{\partial H^A}\right]}_{>0} \frac{H^A}{(1-\tilde{\alpha}^A)^2} \left(\frac{\sum_{j=1}^M (v_i^A - v_j^A) \left[-\frac{\partial \widetilde{n}_j^A}{\partial H^A}\right]}{\sum_{j=1}^M \left[-\frac{\partial \widetilde{n}_j^A}{\partial H^A}\right]}\right).
\end{aligned}$$

Therefore, there exists a critical platform $\hat{j} \leq M-1$ such that for all $j > \hat{j}$, $dn_j^{A*}/d\tilde{\alpha}^A > 0$, and for all $j \leq \hat{j}$ with $dn_j^{A*}/d\tilde{\alpha}^A \leq 0$. This last inequality must be strict for $j = 1$. It must also be strict for all $j < \hat{j}$ in platform oligopoly provided that $v_j^A < v_{\hat{j}}^A$. Since an increase in the degree of compatibility implies a decrease in $\tilde{\alpha}^A$, the result follows. □

Prices are determined according to Lemma 1 through $p_i^A = a_i^A - [1-(1-\lambda)\alpha^A]\log h_i^A$. Increased compatibility leads to a downward shift of $h_i^A$ and the equilibrium value of the aggregate decreases. More compatibility reduces the equilibrium value of $h_i^A$. However, a larger $\lambda$ leads to an increase of $[1-(1-\lambda)\alpha^A]$, which points in the opposite direction to $h_i^A$.

User surplus is $[1-(1-\lambda)\alpha^A]\log H^A$. The term in square brackets increases in the degree of compatibility $\lambda$, which captures the direct effect of increased compatibility on user surplus. By contrast, as just shown, $H^A$ decreases. The decrease in $H^A$ captures the strategic effect that an increase in partial compatibility causes the platforms to compete less intensely for users.

Consider the special case of symmetric platforms, implying that $h_i^A/H^A = 1/M$. The first-order condition can then be rewritten as

$$\log H^A = \frac{a^A - c^A}{1-(1-\lambda)\alpha^A} + \log M - \frac{M}{M-1}.$$

Thus, user surplus can be expressed as

$$a^A - c^A + [1-(1-\lambda)\alpha^A]\left(\log M - \frac{M}{M-1}\right).$$

Under symmetry, welfare increases in the degree of compatibility if and only if $\log M > \frac{M}{M-1}$. This implies that welfare decreases with compatibility if and only if $M = 2$ or $M = 3$, while it increases for $M \geq 4$. With a sufficiently large number of platforms, the strategic effect is less pronounced, and thus the direct effect dominates.[26]

When platforms are asymmetric, compatibility mitigates the asymmetry of market outcomes, as observed in Proposition A.1. This gives rise to an additional effect pushing down the price of large platforms. Naturally, this effect is strong when the asymmetry is large. To illustrate the role of asymmetry, consider a duopoly. As shown above, in this case, the strategic effect dominates under symmetry. In the following proposition, we establish that even under duopoly an increase in the degree of compatibility lowers the price set by a larger platform and increases user surplus if the asymmetry between platforms is sufficiently large.

[26]Starting with Katz and Shapiro (1985), earlier literature has looked at the welfare effect of (no versus full) compatibility under Cournot competition; Amir, Evstigneev and Gama (2021) provides conditions under which full compatibility leads to a larger consumer surplus than no compatibility. For an extension to two-sided platforms, see Shekhar, Petropoulos, Van Alstyne and Parker (2022). Grilo, Shy and Thisse (2001) provide an early analysis of price competition with direct network effects and product differentiation but with a different focus.

**Proposition A.2.** *Suppose that $\beta^A = \beta^B = 0$, $M = 2$, and $v_1^A \le v_2^A$. (1) There exists a critical value $\Delta_p v^A \in (0, \infty)$ such that the equilibrium price set by platform 1, $p_1^{A*}$ decreases with $\lambda$ if and only if $v_1^A - v_2^A \ge \Delta_p v^A$. (2) There exists a critical value $\Delta_{CS} v^A \in (0, \infty)$ such that the equilibrium user surplus $CS^{A*}$ increases with $\lambda$ if and only if $v_1^A - v_2^A \ge \Delta_{CS} v^A$.*

*Proof.* We first show Proposition A.2-2 and then show Proposition A.2-1. Here we make use of a derivation in the proof of Proposition A.3 below, which does not rely on any of the results obtained in the current proof.

Supposing that $\beta^A = \beta^B = 0$, we can simplify $\Omega^A$, which appears in equation (A.11) in the proof of Proposition A.3, to

$$\begin{aligned}\tilde{\Omega}^A(n_1^A, \Delta v^A) &= \frac{\Delta v^A}{1-\tilde{\alpha}^A} - [\log n_1^A - \log(1-n_1^A)] - \left(\frac{1}{1-n_1^A} - \frac{1}{n_1^A}\right) \\ &= \Delta \tilde{v}^A - [\log n_1^A - \log(1-n_1^A)] - \left(\frac{1}{1-n_1^A} - \frac{1}{n_1^A}\right),\end{aligned}$$

where $\Delta v^A := v_1^A - v_2^A$ and $\Delta \tilde{v}^A := \Delta v^A/(1-\tilde{\alpha}^A)$. The proof of Proposition A.3 shows that the equilibrium market share of platform 1 is given by $\Omega^A(n_1^A, \Delta v^A)$. Since $\frac{\partial \tilde{\Omega}^A}{\partial \Delta \tilde{v}^A} = 1$ and

$$\begin{aligned}\frac{\partial \tilde{\Omega}^A}{\partial n_1^A} &= -\frac{1}{n_1^A} - \frac{1}{1-n_1^A} - \frac{1}{(1-n_1^A)^2} - \frac{1}{(n_1^A)^2} \\ &= -\frac{1}{n_1^A(1-n_1^A)} - \frac{2(n_1^A)^2 - 2n_1^A + 1}{(n_1^A)^2(1-n_1^A)^2} \\ &= -\frac{(n_1^A)^2 - n_1^A + 1}{(n_1^A)^2(1-n_1^A)^2},\end{aligned}$$

we can write

$$\frac{dn_1^{A*}}{d\Delta \tilde{v}^A} = \frac{(n_1^A)^2(1-n_1^A)^2}{(n_1^A)^2 - n_1^A + 1}.$$

Next, noting that

$$\frac{\partial \Delta \tilde{v}^A}{\partial \tilde{\alpha}^A} = \frac{\Delta v^A}{(1-\tilde{\alpha}^A)^2} = \frac{1}{1-\tilde{\alpha}^A}\Delta \tilde{v}^A$$

and

$$CS^A = v_1^A - (1-\tilde{\alpha}^A)\log n_1^A - \frac{1-\tilde{\alpha}^A}{1-n_1^A},$$

we have

$$\begin{aligned}\frac{\partial CS^{A*}}{\partial \tilde{\alpha}^A} &= \log n_1^A + \frac{1}{1-n_1^A} - \frac{dn^{A*}}{d\Delta\tilde{v}^A}\Delta\tilde{v}^A\left[\frac{1}{n_1^A} + \frac{1}{(1-n_1^A)^2}\right] \\ &= \log n_1^A + \frac{1}{1-n_1^A} - \frac{n_1^A}{(n_1^A)^2 - n_1^A + 1}\left((1-n_1^A)^2 + n_1^A\right)\Delta\tilde{v}^A \\ &= \log n_1^A + \frac{1}{1-n_1^A} - n_1^A\Delta\tilde{v}^A.\end{aligned}$$

Since

$$\Delta\tilde{v}^A = \log n_1^A - \log(1-n_1^A) + \frac{1}{1-n_1^A} - \frac{1}{n_1^A},$$

we can write

$$\frac{\partial CS^{A*}}{\partial \tilde{\alpha}^A} = (1-n_1^A)\log n_1^A + 2 + n_1^A\log(1-n_1^A).$$

When $n_1^A = 1/2$, the inequality

$$\frac{\partial CS^{A*}}{\partial \tilde{\alpha}^A} = -\log 2 + 2 > 0$$

holds. As $n_1^A \to 1$, we have the limit result $\frac{\partial CS^{A*}}{\partial \tilde{\alpha}^A} \to -\infty$. Finally, since

$$\frac{\partial^2 CS^{A*}}{\partial \tilde{\alpha}^A \partial n_1^A} = -\left[\log n_1^A - \log(1-n_1^A)\right] + \frac{1-n_1^A}{n_1^A} - \frac{n_1^A}{1-n_1^A} < 0,$$

there exists a critical value $\hat{n}_{1,CS}^A \in (1/2, 1)$ of the market share of platform 1 such that user surplus is decreasing in $\tilde{\alpha}^A$ if and only if $n_1^{A*} > \hat{n}_{1,CS}^A$. Because $n_1^{A*}$ is increasing in $\Delta\tilde{v}^A$, which is increasing in $\Delta v^A$, there exists $\Delta_{CS}v^A > 0$ such that $dp_1^{A*}/d\tilde{\alpha}^A > 0$ if and only if $v_1^A - v_2^A > \Delta_{CS}v^A$. Because $\tilde{\alpha}^A$ is decreasing in $\lambda$, we obtain Proposition A.2-2

Next, consider the impact of $\tilde{\alpha}^A$ on prices. Since $p_1^A = \frac{1-\tilde{\alpha}^A}{1-n_1^A}$, the effect of $\tilde{\alpha}^A$ on the equilibrium prices is given by

$$\begin{aligned}\frac{dp_1^{A*}}{d\tilde{\alpha}^A} &= -\frac{1}{1-n_1^A} + \Delta\tilde{v}^A\frac{(n_1^A)^2}{(n_1^A)^2 - n_1^A + 1} \\ &= \frac{1}{(n_1^A)^2 - n_1^A + 1}\left[(n_1^A)^2\left(\log n_1^A - \log(1-n_1^A) + \frac{1}{1-n_1^A} - \frac{1}{n_1^A}\right) - \frac{(n_1^A)^2 - n_1^A + 1}{1-n_1^A}\right] \\ &= \frac{1}{(n_1^A)^2 - n_1^A + 1}\left[(n_1^A)^2\left(\log n_1^A - \log(1-n_1^A) - \frac{1}{n_1^A}\right) - 1\right]\end{aligned}$$

The function

$$(n_1^A)^2\left(\log n_1^A - \log(1-n_1^A)\right) - n_1^A$$

has the derivative

$$\begin{aligned} & 2n_1^A\left(\log n_1^A - \log(1-n_1^A)\right) + n_1^A + \frac{(n_1^A)^2}{1-n_1^A} - 1 \\ = \;& 2n_1^A\left(\log n_1^A - \log(1-n_1^A)\right) + \frac{(2n_1^A-1)}{1-n_1^A} > 0 \end{aligned}$$

for all $n_1^A > 1/2$. Thus, there exists a critical value $\hat{n}_{1,p}^A \in (1/2, 1)$ of the market share of platform 1 such that $p_1^{A*}$ is increasing in $\tilde{\alpha}^A$ if and only if $n_1^{A*} > \hat{n}_{1,p}^A$. Therefore, there exists $\Delta_p v^A > 0$ such that $dp_1^{A*}/d\tilde{\alpha}^A > 0$ if and only if $v_1^A - v_2^A > \Delta_p v^A$. Because $\tilde{\alpha}^A$ is decreasing in $\lambda$, we obtain Proposition A.2-1.

□

Our analysis has focused on the case with zero cross-group network effects. We take a quick look at cross-group network effects when platforms are symmetric. When cross-group network effects are positive, a group-$k$ consumer's utility from joining platform $i$ is given by $a_i^k - p_i^k + (1-\lambda)\alpha^k \log n_i^k + (1-\lambda)\beta^k \log n_i^l + \varepsilon_i^k$. Let $\tilde{\alpha}^k := (1-\lambda)\alpha^k$ and $\tilde{\beta}^k := (1-\lambda)\beta^k$. Then, from the first-order condition and the fact that $h_i^k/H^k = 1/M$, the symmetric equilibrium price-cost margin for group-$k$ users is given by $p^k - c^k = (1-\tilde{\alpha}^k - \tilde{\beta}^l)\frac{M}{M-1}$, which is increasing in $\lambda$, as increased compatibility relaxes price competition between platforms.

The symmetric model allows us to obtain insights into which user group benefits from increased compatibility. With partial compatibility, the expression for user surplus can be written as

$$CS^{k*} = v^k + (1-\tilde{\alpha}^k - \tilde{\beta}^k)\log M - (1-\tilde{\alpha}^k - \tilde{\beta}^l)\frac{M}{M-1},$$

for $k, l \in \{A, B\}$, $l \neq k$. The derivative with respect to the degree of partial compatibility is

$$\frac{\partial CS^{k*}}{\partial \lambda} = (\alpha^k + \beta^k)\log M - (\alpha^k + \beta^l)\frac{M}{M-1}.$$

Given a large number of platforms, $M$, partial compatibility tends to be beneficial for users in either group because an increase in compatibility has a strong direct effect on users by expanding interaction possibilities within and across groups for the service features that become compatible. The associated consumer benefit then dominates the loss from reduced price competition. Considering group-$k$ user surplus, we observe that increased compatibility tends to be beneficial if $\beta^k$ is large relative to $\beta^l$, $l \neq k$. The group that experiences rather small benefits from cross-group network effects tends to be harmed by increased compatibility.

To address the effect of compatibility on industry concentration under cross-group network effects, we restrict attention to the duopoly case. In line with Proposition A.1, we establish that compatibility mitigates industry concentration even in the presence of cross-group network effects.

**Proposition A.3.** *Suppose that $M = 2$ and that $v_1^k := a_1^k - c_1^k > a_2^k - c_2^k =: v_2^k$ for $k \in \{A, B\}$, that is, platform 1 is more efficient than platform 2. Then, the equilibrium market share of platform 1, $n_1^{k*}$ decreases with the degree of compatibility $\lambda$.*

*Proof.* Suppose that $M = 2$; that is, platforms are duopolists. Also suppose that $v_1^k := a_1^k - c_1^k > a_2^k - c_2^k =: v_2^k$ for $k = A, B$; that is, platform 1 is more efficient than platform 2 is. In this setting, the equilibrium object can be summarized by $(n_1^A, n_1^B)$, $n_2^k = 1 - n_1^k$ for $k \in \{A, B\}$.

Noting that, from the first-order conditions (A.5) and (A.6), we have

$$CS^A = v_2^A - (1 - \tilde{\alpha}^A)\log n_2^A + \tilde{\beta}^A \log n_2^B - \frac{1}{1 - n_2^A}\left(1 - \tilde{\alpha}^A - \tilde{\beta}^B \frac{n_2^B}{n_2^A}\right),$$
$$CS^B = v_2^B - (1 - \tilde{\alpha}^B)\log n_2^B + \tilde{\beta}^B \log n_2^A - \frac{1}{1 - n_2^B}\left(1 - \tilde{\alpha}^B - \tilde{\beta}^A \frac{n_2^A}{n_2^B}\right),$$

the equilibrium condition for the market share of platform 1, $(n_1^A, n_1^B)$, is given by the system of equations:

$$\Omega^A(n_1^A, n_1^B, \Delta v^A) = 0$$
$$\Omega^B(n_1^A, n_1^B, \Delta v^A) = 0,$$

where

$$\begin{aligned}
\Omega^A(n_1^A, n_1^B, \Delta v^A) &= v_1^A - (1 - \tilde{\alpha}^A)\log n_1^A + \tilde{\beta}^A \log n_1^B - CS^A - \frac{1}{1 - n_1^A}\left(1 - \tilde{\alpha}^A - \tilde{\beta}^B \frac{n_1^B}{n_1^A}\right) \\
&= v_1^A - v_2^A - (1 - \tilde{\alpha}^A)\log \frac{n_1^A}{n_2^A} + \tilde{\beta}^A \log \frac{n_1^B}{n_2^B} \\
&\quad -\frac{1}{1 - n_1^A}\left(1 - \tilde{\alpha}^A - \tilde{\beta}^B \frac{n_1^B}{n_1^A}\right) + \frac{1}{1 - n_2^A}\left(1 - \tilde{\alpha}^A - \tilde{\beta}^B \frac{n_2^B}{n_2^A}\right) \\
&= \Delta v^A - (1 - \tilde{\alpha}^A)\left(\log n_1^A - \log(1 - n_1^A)\right) + \tilde{\beta}^A\left(\log n_1^B - \log(1 - n_1^B)\right) \\
&\quad -\frac{1}{1 - n_1^A}\left(1 - \tilde{\alpha}^A - \tilde{\beta}^B \frac{n_1^B}{n_1^A}\right) + \frac{1}{n_1^A}\left(1 - \tilde{\alpha}^A - \tilde{\beta}^B \frac{1 - n_1^B}{1 - n_1^A}\right) \qquad \text{(A.11)}
\end{aligned}$$

and $\Omega^B(n_1^A, n_1^B, \Delta v^B)$ is analogously defined.

From Proposition 4, we know that any solution to this system of equations lies in $(1/2, 1)^2$.

We can rewrite $\Omega^A$ as

$$\Omega^A(n_1^A, n_1^B, \Delta v^A) = \Delta v^A - (1-\tilde{\alpha}^A)[\log n_1^A - \log(1-n_1^A)] + \tilde{\beta}^A[\log n_1^B - \log(1-n_1^B)] \\ - \frac{1}{n_1^A(1-n_1^A)}\left[(1-\tilde{\alpha}^A)(2n_1^A-1) - \tilde{\beta}^B(2n_1^B-1)\right],$$

Thus, we have the partial derivatives with respect to $n_1^A$ and $n_1^B$:

$$\begin{aligned}
\frac{\partial\Omega^A}{\partial n_1^A} &= -(1-\tilde{\alpha}^A)\frac{1}{n_1^A(1-n_1^A)} - \frac{(2n_1^A-1)}{[n_1^A(1-n_1^A)]^2}\left[(1-\tilde{\alpha}^A)(2n_1^A-1) - \tilde{\beta}^B(2n_1^B-1)\right] \\
&\quad -\frac{2(1-\tilde{\alpha}^A)}{n_1^A(1-n_1^A)} \\
&= -\frac{1}{(n_1^A)^2(1-n_1^A)^2}\left[(1-\tilde{\alpha}^A)\left[3n_1^A(1-n_1^A) + (2n_1^A-1)^2\right] + (2n^A-1)(2n_1^B-1)\tilde{\beta}^B\right] \\
&< -\frac{1}{(n_1^A)^2(1-n_1^A)}(1-\tilde{\alpha}^A)[2-n_1^A] < 0. \\
\frac{\partial\Omega^A}{\partial n_1^B} &= \tilde{\beta}^A\frac{1}{n_1^B(1-n_1^B)} + \tilde{\beta}^B\frac{2}{n_1^A(1-n_1^A)} > 0.
\end{aligned}$$

The partial derivative with respect to $\lambda$ is:

$$\frac{\partial\Omega^A}{\partial\lambda} = -\alpha^A\log\left(\frac{n_1^A}{1-n_1^A}\right) - \beta^A\log\left(\frac{n_1^B}{1-n_1^B}\right) - \frac{\alpha^A(2n_1^A-1)+\beta^B(2n_1^B-1)}{n_1^A(1-n_1^A)}$$

To conduct comparative statics with respect to $\lambda$, we can write

$$\frac{dn_1^A}{d\lambda} = \frac{-\overbrace{\frac{\partial\Omega^A}{\partial\lambda}}^{(-)}\overbrace{\frac{\partial\Omega^B}{\partial n_1^B}}^{(-)} + \overbrace{\frac{\partial\Omega^B}{\partial\lambda}}^{(-)}\overbrace{\frac{\partial\Omega^A}{\partial n_1^B}}^{(+)}}{\frac{\partial\Omega^A}{\partial n_1^A}\frac{\partial\Omega^B}{\partial n_1^B} - \frac{\partial\Omega^A}{\partial n_1^B}\frac{\partial\Omega^B}{\partial n_1^A}}$$

Therefore, once we establish that

$$\frac{\partial\Omega^A}{\partial n_1^A}\frac{\partial\Omega^B}{\partial n_1^B} - \frac{\partial\Omega^A}{\partial n_1^B}\frac{\partial\Omega^B}{\partial n_1^A} > 0, \tag{A.12}$$

we know that $dn_1^A/d\lambda < 0$ and analogously $dn_2^A/d\lambda < 0$.

To show this, we write the left-hand side of inequality (A.12) as

$$\frac{\partial\Omega^A}{\partial n_1^A}\frac{\partial\Omega^B}{\partial n_1^B} - \frac{\partial\Omega^A}{\partial n_1^B}\frac{\partial\Omega^B}{\partial n_1^A} = \frac{Z_1}{(n_1^A)^2(1-n_1^A)^2(n_1^B)^2(1-n_1^B)^2},$$

where

$$
\begin{aligned}
Z_1 &= \Big[(1-\tilde{\alpha}^A)\left[3n_1^A(1-n_1^A)+(2n_1^A-1)^2\right]+(2n_1^A-1)(2n_1^B-1)\tilde{\beta}^B\Big] \\
&\quad \times \Big[(1-\tilde{\alpha}^B)\left[3n_1^B(1-n_1^B)+(2n_1^B-1)^2\right]+(2n_1^B-1)(2n_1^A-1)\tilde{\beta}^A\Big] \\
&\quad - \Big[\tilde{\beta}^A n_1^A(1-n_1^A)+2\tilde{\beta}^B n_1^B(1-n_1^B)\Big]\Big[\tilde{\beta}^B n_1^B(1-n_1^B)+2\tilde{\beta}^A n_1^A(1-n_1^A)\Big] \\
&> \Big[\max\{\tilde{\beta}^A,\tilde{\beta}^B\}\Big]^2 \times Z_2,
\end{aligned}
$$

with

$$
\begin{aligned}
Z_2 &= 3n_1^A(1-n_1^A)(2n_1^B-1)^2+3n_1^B(1-n_1^B)(2n_1^A-1)^2+(2n_1^A-1)^2(2n_1^B-1)^2 \\
&\quad -2\left[n_1^A(1-n_1^A)-n_1^B(1-n_1^B)\right]^2.
\end{aligned}
$$

Inequality (A.12) is satisfied if and only if $Z_1>0$.

Without loss of generality, suppose that, on the larger platform, there are weakly more group-$A$ users than group-$B$ users, $n_1^A \geq n_1^B > 1/2$. Then, we have

$$
\begin{aligned}
\frac{\partial Z_2}{\partial n_1^B} &= 4(2n_1^B-1)[n_1^B(1-n_1^B)-n_1^A(1-n_1^A)] \\
&\quad +12n_1^A(1-n_1^A)(2n_1^B-1)+(2n_1^B-1)(2n_1^A-1)^2>0.
\end{aligned}
$$

Hence, if $Z_2 \geq 0$ at $n_1^B=1/2$, $Z_2>0$ for all $n_1^B \in (1/2, n_1^A]$. At $n_1^B=1/2$, we have

$$
\begin{aligned}
Z_2\big|_{n_1^B=1/2} &= \frac{3}{4}(2n_1^A-1)^2-2\left[n_1^A(1-n_1^A)-\frac{1}{4}\right]^2 \\
&= \frac{5-16Z_3}{8},
\end{aligned}
$$

where $Z_3$ is defined as

$$
Z_3 = n_1^A[1-(2-n_1^A)(n_1^A)^2].
$$

Function $Z_3$ has the first-order and second-order derivatives with respect to $n_1^A$:

$$
\begin{aligned}
\frac{\partial Z_3}{\partial n_1^A} &= 1-(n_1^A)^2(6-4n_1^A), \\
\frac{\partial^2 Z_3}{\partial (n_1^A)^2} &= -12n_1^A(1-n_1^A)<0.
\end{aligned}
$$

Noting that $\partial Z_3/\partial n_1^A=0$ at $n_1^A=1/2$, $Z_3$ is maximized at $n_1^A=1/2$ with maximum value

$Z_3|_{n_1^A=1/2} = 5/16$. Hence, $5 - 16Z_3$ is minimized at $n_1^A = 1/2$, with the minimum

$$(5 - 16Z_3)\big|_{n_1^A=1/2} = 0.$$

This establishes that $Z_1 > 0$ for all $n_1^A$ and $n_1^B \in (1/2, n_1^A]$ and, thus, inequality (A.12) is satisfied. □

## A.3. Supplementary material on partial market coverage

The main analysis also assumes that there is no outside option. We relax this assumption in three different extensions.

### A.3.1. Outside options subject to network effects

A straightforward way to introduce partial coverage is to assume that the outside option is also subject to the same network effects and idiosyncratic taste shocks as the for-profit platforms. This applies if choosing the outside option does not mean abstaining from the market but choosing a non-commercial offer. In the case of software, this could be open-source software that is provided free of charge. In the case of content platforms, this could a public platform that is free of charge on both sides. Our model in Section 2 can easily accommodate such a free platform by adding platform 0 that offers quality $a_0^k$ to side $k \in \{A, B\}$ at zero price, $p_0^k = 0$. The free platform 0 provides the utility

$$u_0^k = a_0^k + \alpha^k \log n_0^k + \beta^k \log n_0^l + \varepsilon_0^k$$

for $k \in \{A, B\}$. If instead the platform offers its services at fixed fees, $a_0^k$ stands for the quality net of the respective fee. With the same change of variables as for the strategic platforms $i \in \{1, \ldots, M\}$, platform 0 then offers $(h_0^A, h_0^B)$, which is independent of the choices offered by the for-profit platforms, and we write $H^k = \sum_{i=0}^{M} h_i^k$. Our equilibrium characterization of the participation game (Remark 1) and the existence of an ordered set of price equilibria (Proposition 2) generalize to the introduction of such an outside option.

**Equilibrium characterization and comparative statics** We recall that group-$k$ user surplus depends linearly on $\log H^k$ and $\log H^l$,

$$CS^A = (1 - \alpha^A) \log H^A - \beta^A \log H^B,$$
$$CS^B = (1 - \alpha^B) \log H^B - \beta^B \log H^A.$$

Inverting this relation, we can write $(H^A, H^B)$ as functions of $(CS^A, CS^B)$:

$$\log H^A = \Gamma^{AA} CS^A + \Gamma^{AB} CS^B,$$
$$\log H^B = \Gamma^{BB} CS^B + \Gamma^{BA} CS^A.$$

Hence, we can write the market share of the outside platform as a function of $(CS^A, CS^B)$:

$$n_0^A(h_0^A, CS^A, CS^B) = \frac{h_0^A}{\exp\left(\Gamma^{AA} CS^A + \Gamma^{AB} CS^B\right)},$$
$$n_0^B(h_0^B, CS^A, CS^B) = \frac{h_0^B}{\exp\left(\Gamma^{BB} CS^B + \Gamma^{BA} CS^A\right)}.$$

This implies that the market shares of the outside option $n_0^k(h_0^k, CS^A, CS^B)$ is strictly decreasing in $CS^A$ and $CS^B$ if $\beta^k > 0$; it is strictly decreasing in $CS^A$ and independent of $CS^B$ if $\beta^k = 0$.

The equilibrium condition for the user surpluses $(CS^A, CS^B)$ is now written as

$$n_0^A(h_0^A, CS^A, CS^B) + \sum_{i=1}^{M} \widetilde{n}_i^A(CS^A, CS^B) = 1,$$
$$n_0^B(h_0^B, CS^A, CS^B) + \sum_{i=1}^{M} \widetilde{n}_i^B(CS^A, CS^B) = 1$$

The characterization results presented in Section 4 continue to hold.

Given the presence of an outside platform, it is possible examine the impact of the value of the outside platform on the surplus of each user group. By applying the implicit function theorem, we have that

$$\begin{pmatrix} \sum_{i=0}^{M} \left( \frac{\partial \widetilde{n}_i^A}{\partial CS^A} \right) & \sum_{i=0}^{M} \left( \frac{\partial \widetilde{n}_i^A}{\partial CS^B} \right) \\ \sum_{i=0}^{M} \left( \frac{\partial \widetilde{n}_i^B}{\partial CS^A} \right) & \sum_{i=0}^{M} \left( \frac{\partial \widetilde{n}_i^B}{\partial CS^B} \right) \end{pmatrix} \begin{pmatrix} \frac{dCS^A}{dh_0^A} \\ \frac{dCS^B}{dh_0^A} \end{pmatrix} = - \begin{pmatrix} \frac{\partial n_0^A}{\partial h_0^A} \\ 0 \end{pmatrix}$$

Applying this comparative statics result to extremal equilibria (i.e., the equilibrium user surpluses ($CS^{A*}, CS^{B*}$) that maximize group-$A$ or group-$B$ user surplus), we obtain

$$\frac{dCS^{A*}}{dh_0^A} = \frac{-\frac{\partial n_0^A}{\partial h_0^A}\left[\sum_{i=0}^M \left(\frac{\partial \tilde{n}_i^B}{\partial CS^B}\right)\right]}{\det\begin{pmatrix} \sum_{i=0}^M \left(\frac{\partial \tilde{n}_i^A}{\partial CS^A}\right) & \sum_{i=0}^M \left(\frac{\partial \tilde{n}_i^A}{\partial CS^B}\right) \\ \sum_{i=0}^M \left(\frac{\partial \tilde{n}_i^B}{\partial CS^A}\right) & \sum_{i=0}^M \left(\frac{\partial \tilde{n}_i^B}{\partial CS^B}\right)\end{pmatrix}} > 0,$$

$$\frac{dCS^{B*}}{dh_0^A} = \frac{\frac{\partial n_0^A}{\partial h_0^A}\left[\sum_{i=0}^M \left(\frac{\partial \tilde{n}_i^B}{\partial CS^A}\right)\right]}{\det\begin{pmatrix} \sum_{i=0}^M \left(\frac{\partial \tilde{n}_i^A}{\partial CS^A}\right) & \sum_{i=0}^M \left(\frac{\partial \tilde{n}_i^A}{\partial CS^B}\right) \\ \sum_{i=0}^M \left(\frac{\partial \tilde{n}_i^B}{\partial CS^A}\right) & \sum_{i=0}^M \left(\frac{\partial \tilde{n}_i^B}{\partial CS^B}\right)\end{pmatrix}} \leq 0.$$

This shows that if the outside option becomes more attractive for group-$k$ users, user surplus of this group will increase, whereas user surplus of the other group will (weakly) decrease.

### A.3.2. User opt-in

Another approach to introducing partial coverage is to postulate that users have heterogeneous opportunity costs to become active and make the following sequential decisions: first, after learning their opportunity cost of joining but before learning their idiosyncratic taste realization for the different platforms, $(\varepsilon_1^k, \ldots, \varepsilon_M^k)$, they decide whether to become active (e.g. by buying the necessary hardware that enables them to install a software package) and, second, after learning their taste realization they decide which platform to join (e.g. by buying one of the competing software packages). We analyze the models in which users do not observe prices at the first stage, but correctly predict equilibrium prices, given the parameters of the model); they observe them at the second stage. Thus, the timing is as follows: At the first stage, users decide whether to opt in and platforms set prices for each user group; at the second stage, participating users, after observing prices and individual tastes regarding the different platforms, decide which platform to join.[27]

Given network sizes $(n_i^A, n_i^B)_{i=1}^M$ and prices $(p_i^A, p_i^B)_{i=1}^M$, the expected indirect utility of a group-$k$ user from opting in the market is given by $CS^k$, where

$$\begin{aligned} CS^k(\mathbf{n}, \mathbf{p}) &= \mathbb{E}\left[\max_{i=1,\ldots,M}\left\{a_i^k - p_i^k + \alpha^k \log n_i^k + \beta^k \log n_i^l + \varepsilon_i^k\right\}\right] \\ &= \log\left[\sum_{j=1}^M y_j^k \left(n_j^k\right)^{\alpha^k}\left(n_j^l\right)^{\beta^k}\right] \end{aligned}$$

[27]For the analysis, it does not matter whether platforms set prices at the first stage or at an intermediate stage. The analysis would need to be modified if platforms set prices first and users observed prices before deciding whether to opt in. In this alternative, less tractable version, a platform's price decision would affect overall participation levels.

for $k, l \in \{A, B\}, l \neq k$.

The numbers of opting-in users are given by $N^k := \sum_{j=1}^{M} n_j^k$. The idiosyncratic taste shocks for staying out of the market is given by $a_0^k + \theta\varepsilon_0^k$, where $\varepsilon_0^k$ is drawn from a distribution function $\Psi^k$ that has a density function $\psi^k$, leading to the mass of opting-in group-$k$ users $N^k = \Psi^k\left(\frac{CS^k - a_0^k}{\theta}\right)$. We assume that $\Psi^k(x) \in [\underline{N}^k, 1]$ for $\underline{N}^k \in (0, 1)$ and that it is strictly increasing in $x$ for values of $x$ such that $\Psi^k(x) \in (\underline{N}^k, 1)$.[28]

We make the following assumption on the shapes of $\Psi^A(\cdot)$ and $\Psi^B(\cdot)$:

**Assumption A.1.** *For $k \in \{A, B\}$, the following inequality holds:*

$$\theta^k \inf_{x \in \text{Supp}\Psi^k} \left\{ \frac{\Psi^k(x)}{\psi^k(x)} \right\} > \alpha^k + \beta^k.$$

**Assumption A.2.** *For $k, l \in \{A, B\}$ and $l \neq k$,*

$$\alpha^k + \frac{\beta^l}{\underline{N}^k} < 1.$$

Assumption A.1 ensures that the user participation equilibrium in the first stage is unique, and Assumption A.2 is made to guarantee the existence of price equilibrium in any subgame after users make their opt-in decisions.

**Participation equilibrium** First, consider the network size of platform $i$, $(n_i^A, n_i^B)$, which is given by the user mass $(N^A, N^B)$ and the conditional choice probability $(s_i^A, s_i^B)$, where $s_i^A := n_i^A / N^A$ and $s_i^B := n_i^B / N^B$, respectively. Then, in the fulfilled expectation equilibrium, we have

$$\begin{aligned} s_i^k &= \frac{\exp(a_i^k - p_i^k)\left(n_i^k\right)^{\alpha^k}\left(n_i^l\right)^{\beta^k}}{\sum_{j=1}^{M} \exp(a_j^k - p_j^k)\left(n_j^k\right)^{\alpha^k}\left(n_j^l\right)^{\beta^k}} \\ &= \frac{\exp(a_i^k - p_i^k)\left(s_i^k\right)^{\alpha^k}\left(s_i^l\right)^{\beta^k}}{\sum_{j=1}^{M} \exp(a_j^k - p_j^k)\left(s_j^k\right)^{\alpha^k}\left(s_j^l\right)^{\beta^k}}, \end{aligned}$$

leading to the formula

$$s_i^k = \frac{h_i^k}{H^k}. \tag{A.13}$$

[28] The lower bound $\underline{N}^k$ is imposed in order to simplify the proof of equilibrium existence.

Hence, we have

$$CS^A = \alpha^A \log N^A + \beta^A \log N^B + V^A,$$
$$CS^B = \alpha^B \log N^B + \beta^B \log N^A + V^B,$$

where

$$V^A := (1-\alpha^A)\log H^A - \beta^A \log H^B,$$
$$V^B := (1-\alpha^B)\log H^B - \beta^B \log H^A.$$

Note that in the baseline model of the main text, we always have $N^A = N^B = 1$, which gives the identity $CS^k = V^k$.

The value of $(N^A, N^B)$ in the fulfilled-expectation equilibrium are characterized by the system of equations

$$N^A - \Psi^A\left(\frac{\alpha^A \log N^A + \beta^A \log N^B + V^A - a_0^A}{\theta^A}\right) = 0, \quad \text{(A.14)}$$
$$N^B - \Psi^B\left(\frac{\alpha^B \log N^B + \beta^B \log N^A + V^B - a_0^B}{\theta^B}\right) = 0. \quad \text{(A.15)}$$

We obtain the following characterization of participation equilibrium.

**Proposition A.4.** *Under Assumption A.1, there exists a unique interior participation equilibrium with the following properties:*

1. *Network size* $(n_i^A, n_i^B)$ *is given by* $n_i^k = N^k s_i^k$*, where* $N^A(V^A, V^B)$ *and* $N^B(V^A, V^B)$ *are given by the solution to the system of equations* (A.14) *and* (A.15)*;*
2. $N^k$ *is increasing in* $V^k$ *and* $V^l$*;*
3. $N^k/N^l$ *is increasing in* $V^k$ *and decreasing in* $V^l$*.*

*Proof.* Applying the implicit function theorem to the system of equations (A.14)-(A.15), we obtain

$$\frac{dN^A}{dV^A} = \frac{\frac{\psi^A}{\theta^A}\left(1 - \frac{\alpha^B}{\theta^B}\frac{\psi^B}{\Psi^B}\right)}{\left(1 - \frac{\alpha^A}{\theta^A}\frac{\psi^A}{\Psi^A}\right)\left(1 - \frac{\alpha^B}{\theta^B}\frac{\psi^B}{\Psi^B}\right) - \frac{\beta^A\beta^B}{\theta^A\theta^B}\frac{\psi^A\psi^B}{\Psi^A\Psi^B}},$$
$$\frac{dN^B}{dV^A} = \frac{\beta^B \frac{\psi^B}{\theta^B}\frac{\psi^A}{\Psi^A\theta^A}}{\left(1 - \frac{\alpha^A}{\theta^A}\frac{\psi^A}{\Psi^A}\right)\left(1 - \frac{\alpha^B}{\theta^B}\frac{\psi^B}{\Psi^B}\right) - \frac{\beta^A\beta^B}{\theta^A\theta^B}\frac{\psi^A\psi^B}{\Psi^A\Psi^B}}$$

and

$$
\begin{aligned}
\frac{d}{dV^A}\left(\frac{N^A}{N^B}\right) &= \frac{N^A}{N^B}\left(\frac{dN^A}{dV^A}\frac{1}{N^A} - \frac{dN^B}{dV^A}\frac{1}{N^B}\right) \\
&= \frac{\frac{N^A}{N^B}\frac{1}{\theta^A}\frac{\psi^A}{\Psi^A}\left(1 - \frac{\psi^B}{\Psi^B}\frac{\alpha^B+\beta^B}{\theta^B}\right)}{\left(1 - \frac{\alpha^A}{\theta^A}\frac{\psi^A}{\Psi^A}\right)\left(1 - \frac{\alpha^B}{\theta^B}\frac{\psi^B}{\Psi^B}\right) - \frac{\beta^A\beta^B}{\theta^A\theta^B}\frac{\psi^A\psi^B}{\Psi^A\Psi^B}}
\end{aligned}
$$

Under Assumption A.1, all the terms are nonnegative.

□

For illustration, we consider a particular functional form of the distribution of $\varepsilon_0^k$.

**Example A.1.** *If outside options are given by $a_0^k + \theta^k \varepsilon_0^k$, $k \in \{A, B\}$, where*

$$
\varepsilon_0^k = \begin{cases} -\infty & \textit{with probability } \underline{N}^k, \\ -\epsilon_0^k, \ (\epsilon_0^k \sim \mathrm{Exp}(1)) & \textit{with probability } 1 - \underline{N}^k, \end{cases}
$$

*we have*

$$
\begin{aligned}
\Psi^k\left(\frac{CS^k - a_0^k}{\theta^k}\right) &= \Pr\left(CS^k \geq a_0^k + \theta^k \varepsilon_0^k\right) \\
&= \underline{N}^k + (1 - \underline{N}^k)\min\left\{1, \exp\left(\frac{CS^k - a_0^k}{\theta^k}\right)\right\}.
\end{aligned}
$$

*and*

$$
\begin{aligned}
\frac{\Psi^k(x)}{\psi^k(x)} &= \frac{\underline{N}^k + (1 - \underline{N}^k)\exp(x)}{(1 - \underline{N}^k)\exp(x)} \\
&> 1.
\end{aligned}
$$

**Price equilibrium** In the pricing game played by platforms, they take user masses $(N^A, N^B)$ as given, and maximize the profit

$$
\begin{aligned}
\Pi_i &= N^A s_i^A (p_i^A - c_i^A) + N^B s_i^B (p_i^B - c_i^B) \\
&= N^A \frac{h_i^A}{H^A}[a_i^A - c_i^A - (1 - \alpha^A)\log h_i^A + \beta^A \log h_i^B] \\
&\quad + N^B \frac{h_i^B}{H^B}[a_i^B - c_i^B - (1 - \alpha^B)\log h_i^B + \beta^B \log h_i^A].
\end{aligned}
$$

The partial derivative is given by

$$\begin{aligned}\frac{\partial \Pi_i}{\partial h_i^A} &= N^A\left[\left(\frac{1}{H^A}-\frac{h_i^A}{(H^A)^2}\right)(p_i^A-c_i^A)-\frac{1-\alpha^A}{H^A}\right]+N^B\frac{h_i^B}{H^B}\beta^B\frac{1}{h_i^A}\\ &= \frac{N^A}{H^A}\left[\left(1-\frac{h_i^A}{H^A}\right)(p_i^A-c_i^A)-1+\alpha^A+\frac{N^B}{N^A}\beta^B\frac{h_i^B}{H^B}\frac{H^A}{h_i^A}\right].\end{aligned}$$

Define

$$\iota^A(s_i^A,s_i^B,N^A/N^B):=\left(1-s_i^A\right)\left(a_i^A-c_i^A-(1-\alpha^A)\log s_i^A+\beta^A\log s_i^B-V^A\right)-1+\alpha^A+\frac{N^B}{N^A}\beta^B\frac{s_i^B}{s_i^A},$$
$$\iota^B(s_i^A,s_i^B,N^A/N^B):=\left(1-s_i^B\right)\left(a_i^B-c_i^B-(1-\alpha^B)\log s_i^B+\beta^B\log s_i^A-V^B\right)-1+\alpha^B+\frac{N^A}{N^B}\beta^A\frac{s_i^A}{s_i^B}.$$

Lemma 3 and Assumption A.2 imply that $\partial\iota^k/\partial s_i^k<0$, $\partial\iota^k/\partial s_i^l>0$, and $(\partial\iota^A/\partial s_i^A)(\partial\iota^B/\partial s_i^B)-(\partial\iota^A/\partial s_i^B)(\partial\iota^B/\partial s_i^A)>0$. Furthermore, we have $\partial\iota^k/\partial(N^l/N^k)>0$. Hence, this gives the implicit share functions $s_i^A(V^A,V^B,N^A/N^B)$ and $s_i^B(V^A,V^B,N^A/N^B)$, which are the same as network size functions $(\widetilde{n}_i^A,\widetilde{n}_i^B)$ in the proof of Proposition 2, where $\beta^A$ and $\beta^B$ are replaced by $N^A\beta^A/N^B$ and $N^B\beta^B/N^A$. We have $\partial s_i^k/\partial(N^k/N^l)<0$ and $\partial s_i^k/\partial(N^k/N^k)>0$.

Accordingly, the equilibrium condition for $V^A$, $V^B$, given $(N^A,N^B)$, is given by

$$\sum_{j=1}^M s_j^A(V^A,V^B,N^A/N^B)=1,$$
$$\sum_{j=1}^M s_j^B(V^A,V^B,N^A/N^B)=1.$$

Proposition 2 implies that there exists a solution to the system of equations, and they are ordered in the sense that for any two solutions $(V^{A\circ},V^{B\circ})$ and $(V^{A\circ\circ},V^{B\circ\circ})$, we have

$$(V^{A\circ}-V^{A\circ\circ})(V^{B\circ}-V^{B\circ\circ})<0.$$

Let $\mathcal{V}_{N^A/N^B}$ be the set of $(V^A,V^B)$ that satisfies the above equilibrium condition.

We turn to the condition under which $(V^A,V^B)$ constitutes a fulfilled-expectation equilibrium. The pair $(V^A,V^B)$ constitutes a fulfilled-expectation equilibrium if $(V^A,V^B)\in\mathcal{V}(N^A/N^B)$, and

$$\frac{N^A}{N^B}=\frac{N^A(V^A,V^B)}{N^B(V^A,V^B)}. \tag{A.16}$$

Let $\mathcal{V}_\lambda$ be the set of equilibrium values of $(V^A,V^B)$ for a given value of $N^A/N^B=\lambda$, and define

$$\overline{V}^A(N^A/N^B) := \max\{V^A : (V^A, V^B) \in \mathcal{V}_{N^A/N^B}\}$$
$$\underline{V}^A(N^A/N^B) := \min\{V^A : (V^A, V^B) \in \mathcal{V}_{N^A/N^B}\}.$$

Analogously, define $\overline{V}^B(N^A/N^B)$ and $\underline{V}^B(N^A/N^B)$ accordingly.

By invoking the implicit function theorem, we have

$$\begin{aligned}
\text{Sign}\left(\frac{\partial \overline{V}^A}{\partial(N^A/N^B)}\right) &= \text{Sign}\left\{-\left(\sum \frac{\partial s^A}{\partial(N^A/N^B)}\right)\left(\sum \frac{\partial s^B}{\partial CS^B}\right) + \left(\sum \frac{\partial s^B}{\partial(N^A/N^B)}\right)\left(\sum \frac{\partial s^A}{\partial CS^B}\right)\right\} \\
&= -, \\
\text{Sign}\left(\frac{\partial \underline{V}^B}{\partial(N^A/N^B)}\right) &= \text{Sign}\left\{-\left(\sum \frac{\partial s^B}{\partial(N^A/N^B)}\right)\left(\sum \frac{\partial s^A}{\partial CS^A}\right) + \left(\sum \frac{\partial s^A}{\partial(N^A/N^B)}\right)\left(\sum \frac{\partial s^B}{\partial CS^A}\right)\right\} \\
&= +
\end{aligned}$$

Hence, $\overline{V}^A(N^A/N^B)$ and $\underline{V}^A(N^A/N^B)$ are decreasing functions, while $\overline{V}^B(N^A/N^B)$ and $\underline{V}^B(N^A/N^B)$ are increasing functions.

Define

$$\begin{aligned}
\mathcal{V} &:= \left\{(V^A, V^B) : \exists\lambda \text{ such that } \lambda = \frac{N^A(V^A, V^B)}{N^B(V^A, V^B)} \text{ and } (V^A, V^B) \in \mathcal{V}_\lambda\right\}, \\
\mathcal{V}^A &:= \left\{V^A : \exists V^B \text{ such that } (V^A, V^B) \in \mathcal{V}\right\} \\
\mathcal{V}^B &:= \left\{V^B : \exists V^A \text{ such that } (V^A, V^B) \in \mathcal{V}\right\}
\end{aligned}$$

We obtain the following characterization of the set of equilibrium values of $(V^A, V^B)$.

**Lemma A.1.** *The $V^A$-maximizing equilibrium is the $V^B$-minimizing equilibrium. Formally, if $(V^{A*}, V^{B*}) \in \mathcal{V}$ satisfies*

$$V^{A*} \geq V^A$$

*for all $V^A \in \mathcal{V}^A$, then*

$$V^{B*} \leq V^B$$

*for all $V^B \in \mathcal{V}^B$. Similarly, the $V^B$-maximizing equilibrium is the $V^A$-minimizing equilibrium.*

*Proof.* Define $(E^A, E^B) := (V^A, -V^B)$, and define

$$\lambda(E^A, E^B) := \frac{N^A(E^A, -E^B)}{N^B(E^A, -E^B)}$$

Then, $\lambda(\cdot,\cdot)$ is an increasing function. Next, define $\Phi$ as

$$\Phi(E^A, E^B) := \{(e^A, e^B) : (e^A, -e^B) \in \mathcal{V}(\lambda(E^A, E^B))\}.$$

Define $\overline{\Phi}$ and $\underline{\Phi}$ by

$$\begin{aligned}\overline{\Phi}(E^A, E^B) &:= \sup \Phi(E^A, E^B) = (\overline{V}^A(\lambda(E^A, E^B)), -\underline{V}^B(\lambda(E^A, E^B))), \text{ and} \\ \underline{\Phi}(E^A, E^B) &:= \inf \Phi(E^A, E^B) = (\underline{V}^A(\lambda(E^A, E^B)), -\overline{V}^B(\lambda(E^A, E^B))).\end{aligned}$$

Because $\lambda$ is increasing in $(E^A, E^B)$ and $(\overline{V}^A(\lambda), \underline{V}^A(\lambda))$ and $(-\overline{V}^B(\lambda), -\underline{V}^B(\lambda))$ are decreasing in $\lambda$, both $\overline{\Phi}$ and $\underline{\Phi}$ are decreasing in $(E^A, E^B)$.

Let $\overline{E} = (\overline{E}^A, \overline{E}^B)$ be the solution to

$$(E^A, E^B) = \overline{\Phi}(E^A, E^B).$$

Because $\overline{\Phi}(E)$ is a decreasing function, we have

$$\overline{E} = \sup \left\{E : \overline{\Phi}(E) \geq E\right\}.$$

Hence, for any $E = (E^A, E^B)$ such that $E \in \Phi(E)$,

$$E \leq \overline{\Phi}(E) \leq \overline{E},$$

which implies that for any equilibrium $(V^A, V^B) \in \mathcal{V}$, we have

$$\begin{aligned}V^A &\leq \overline{E}^A =: \overline{V}^{A*}, \\ V^B &\geq -\overline{E}^B =: \underline{V}^{B*}.\end{aligned}$$

Hence, the $V^A$-maximizing equilibrium is the $V^B$-minimizing equilibrium. Similarly, we can show that the $V^A$-minimizing equilibrium is the $V^B$-maximizing equilibrium. □

However, for the ranking of equilibria, the variable of interest is user surplus $CS^k$ of each group $k \in \{A, B\}$ and not $V^k$.

**Proposition A.5.** *Let* $\overline{\alpha} := \max\left\{\alpha^A, \alpha^B\right\}$, $\overline{\beta} := \max\left\{\beta^A, \beta^B\right\}$, *and* $\underline{\theta} := \min\left\{\theta^A, \theta^B\right\}$. *There exists* $\underline{\eta} > 0$ *such that if*

$$\max\left\{\sup_x \frac{\psi^A(x)}{\Psi^A(x)}, \sup_x \frac{\psi^B(x)}{\Psi^B(x)}\right\} \frac{\max\{\overline{\alpha}, \overline{\beta}\}}{\underline{\theta}} \leq \underline{\eta}, \tag{PC}$$

*then the $CS^A$-maximal equilibrium is the $CS^B$-minimal equilibrium and the $CS^A$-minimal equilibrium is the $CS^B$-maximal equilibrium.*

*Proof.* Suppose that condition (PC) holds. The result follows from Lemma A.1 if we can show that the $V^k$-maximizing (resp. $V^k$-minimizing) equilibrium is the $CS^k$-maximal (resp. $CS^k$-minimal) equilibrium. This is done in the following. Recalling that

$$\frac{dN^A}{dV^A} = \frac{\frac{\psi^A}{\theta^A}\left(1-\frac{\alpha^B}{\theta^B}\frac{\psi^B}{\Psi^B}\right)}{\left(1-\frac{\alpha^A}{\theta^A}\frac{\psi^A}{\Psi^A}\right)\left(1-\frac{\alpha^B}{\theta^B}\frac{\psi^B}{\Psi^B}\right)-\frac{\beta^A\beta^B}{\theta^A\theta^B}\frac{\psi^A\psi^B}{\Psi^A\Psi^B}},$$
$$\frac{dN^B}{dV^A} = \frac{\beta^B\frac{\psi^B}{\theta^B}\frac{\psi^A}{\Psi^A\theta^A}}{\left(1-\frac{\alpha^A}{\theta^A}\frac{\psi^A}{\Psi^A}\right)\left(1-\frac{\alpha^B}{\theta^B}\frac{\psi^B}{\Psi^B}\right)-\frac{\beta^A\beta^B}{\theta^A\theta^B}\frac{\psi^A\psi^B}{\Psi^A\Psi^B}},$$

we have

$$\begin{aligned}
\frac{dCS^A}{dV^A} &= 1+\underbrace{\alpha^A\frac{dN^A}{dV^A}\frac{1}{N^A}}_{\geq 0}+\underbrace{\beta^A\frac{dN^B}{dV^A}\frac{1}{N^B}}_{\geq 0}\geq 1,\\
\frac{dCS^A}{dV^B} &= \alpha^A\frac{dN^A}{dV^B}\frac{1}{N^A}+\beta^A\frac{dN^B}{dV^B}\frac{1}{N^B}\\
&= \frac{\beta^A\frac{\psi^B}{\Psi^B\theta^B}}{1-\frac{\alpha^A}{\theta^A}\frac{\psi^A}{\Psi^A}-\frac{\alpha^B}{\theta^B}\frac{\psi^B}{\Psi^B}-\frac{\beta^A\beta^B}{\theta^A\theta^B}\frac{\psi^A\psi^B}{\Psi^A\Psi^B}+\frac{\alpha^A}{\theta^A}\frac{\psi^A}{\Psi^A}\frac{\alpha^B}{\theta^B}\frac{\psi^B}{\Psi^B}}\\
&\leq \frac{\beta^A\frac{\psi^B}{\Psi^B\theta^B}}{1-\frac{\alpha^A}{\theta^A}\frac{\psi^A}{\Psi^A}-\frac{\alpha^B}{\theta^B}\frac{\psi^B}{\Psi^B}-\frac{\beta^A\beta^B}{\theta^A\theta^B}\frac{\psi^A\psi^B}{\Psi^A\Psi^B}}\\
&\leq \frac{\underline{\eta}}{1-2\underline{\eta}-\underline{\eta}^2}.
\end{aligned}$$

Therefore, for any $(V^{A1}, V^{B1})$ and $(V^{A2}, V^{B2})$ such that $V^{A1} > V^{A2}$ and $V^{B1} < V^{B2}$, the associated user surpluses $(CS^{A1}, CS^{B1})$ and $(CS^{A2}, CS^{B2})$ satisfy

$$CS^{A1}-CS^{A2}\geq (V^{A1}-V^{A2})-(V^{B2}-V^{B1})\frac{\underline{\eta}}{1-2\underline{\eta}-\underline{\eta}^2}>0,$$
$$CS^{B2}-CS^{B1}\geq (V^{B2}-V^{B1})-(V^{A1}-V^{A2})\frac{\underline{\eta}}{1-2\underline{\eta}-\underline{\eta}^2}>0,$$

when $\underline{\eta}$ is sufficiently small. □

In the following remark we show that the condition stated in the previous proposition can indeed be satisfied.

**Remark A.1.** *In the setting of Example A.1, if $CS^k \geq a_0^k$, then $\Psi^k\left(\frac{CS^k-a_0^k}{\theta^k}\right) = 1$ and $\psi^k\left(\frac{CS^k-a_0^k}{\theta^k}\right) = 0$. If $CS^k < a_0^k$, then $\Psi^k\left(\frac{CS^k-a_0^k}{\theta^k}\right) = \underline{N}^k + (1-\underline{N}^k)\exp[(CS^k-a_0^k)/\theta^k]$*

*and*

$$\frac{\psi^k(x)}{\Psi^k(x)} = \frac{1}{1 + \frac{N^k}{1-\underline{N}^k}\exp(-x)} < 1.$$

*Hence, the condition stated in Proposition A.5 holds if* $\max\left\{\overline{\alpha}, \overline{\beta}\right\}/\underline{\theta} < \underline{\eta}$*, which is satisfied for* $\underline{\theta}$ *sufficiently large.*

**Decentralization result and equilibrium characterization** We state a generalized version of Remark 4.

**Remark A.2.** *For any pair of* $(V^A, V^B)$ *and network sizes* $(n_i^A, n_i^B)_{i=1}^M$ *such that* $\sum_{i=1}^M n_i^A \in [\underline{N}^A, 1]$ *and* $\sum_{i=1}^M n_i^B \in [\underline{N}^B, 1]$*, there exist a vector of types* $(v_i^A, v_i^B)_{i=1}^M$ *and a pair of* $(a_0^A, a_0^B)$ *such that* $(n_i^A, n_i^B)_i^M$ *is supported as a vector of equilibrium network sizes.*

Because the shape of the implicit best reply functions $\tilde{h}_i^k(H^A, H^B)$ remains unchanged, all the results obtained in Section 4 remain unchanged.

**Comparative statics: case of network goods** Consider the welfare effect of an increase in the value of outside options $a_0^A$ on user surplus. When $\beta^A = \beta^B = 0$, we can show that the equilibrium aggregate $H^{k**}$ is independent of $N^k$. Hence, the equilibrium condition for $N^k$ is

$$N^k = \Psi^k\left(\frac{\alpha^k \log N^k + (1-\alpha^k)\log H^{k*} - a_0^k}{\theta^k}\right),$$

which implicitly defines a unique solution $N^{k**}$ if

$$1 - \frac{\alpha^A}{\theta^A}\frac{\psi^A}{\Psi^A} > 0. \tag{A.17}$$

Overall group-$k$ user surplus is given by

$$TCS^k(CS^k, a_0^k) = a_0^k + \theta\mathbb{E}[\varepsilon_0^k] + \int_{-\infty}^{\frac{CS^k - a_0}{\theta}} \theta\Psi^k(x)dx$$

Hence, as comparative statics, we have

$$\frac{dTCS^k}{da_0^k} = 1 - \Psi^k\left(\frac{CS^k - a_0^k}{\theta^k}\right) + \Psi^k\left(\frac{CS^k - a_0^k}{\theta^k}\right)\frac{dCS^k}{da_0^k},$$

where

$$\frac{dCS^k}{da_0^k} = -\frac{\alpha^k}{N^k}\frac{\frac{\psi^k}{\theta^k}}{1-\frac{\psi^k}{\theta^k}\frac{\alpha^k}{N^k}}.$$

**Remark A.3.** *In the setting of Example A.1, there exist parameter constellations such that an increase in $a_0^k$ reduces the overall user surplus.*

*Proof.* When $\Psi^k(x) = \exp(x)$, we have that $\psi^k/\Psi^k = 1$ and, thus, using inequality (A.17), equilibrium uniqueness is guaranteed if $1 - \alpha^k/\theta^k > 0$, which is equivalent to $\theta^k > \alpha^k$.

As for the comparative statics, we have

$$\frac{dCS^k}{da_0^k} = \frac{\alpha^k}{\theta^k - \alpha^k}. \tag{A.18}$$

In our example,

$$\frac{dTCS^k}{da_0^k} = 1 - \exp\left(\frac{CS^k - a_0^k}{\theta^k}\right)\frac{\theta^k}{\theta^k - \alpha^k}.$$

Hence, $dTCS^k/da_0^k < 0$ holds for given $CS^k$ when $\alpha^k$ is sufficiently close to $\theta^k$. □

### A.3.3. Outside options with fixed values

We return to the setting in which users first observe prices and simultaneously decide whether and, if so, which platform to join. However, the outside option does not feature network effects. More specifically, suppose that each group-$k$ consumer has an outside option with value $a_0^k + \varepsilon_0^k$, where $\varepsilon_0^k$ is drawn from an i.i.d. type-I extreme-value distribution. In the following, we use $y_0^k := \exp(a_0^k)$ as a primitive.

**Participation equilibrium** Let $\phi^A(H^A, H^B)$ and $\phi^B(H^A, H^B)$ be the (unique) solution to the system of equations

$$\begin{aligned}
\log\phi^A - \alpha^A \log(y_0^A\phi^A + H^A) - \beta^A \log(y_0^B\phi^B + H^B) &= 0,\\
\log\phi^B - \alpha^B \log(y_0^B\phi^B + H^B) - \beta^B \log(y_0^A\phi^A + H^A) &= 0.
\end{aligned}$$

Then, the demand for platforms in a partially covered market are characterized as follows.

**Proposition A.6.** *In the unique interior participation equilibrium, the demand for platform i* *is given by the functions*

$$n_i^A(h_i^A, H^A, H^B) = \frac{h_i^A}{H_+^A(H^A, H^B)}, \tag{A.19}$$

$$n_i^B(h_i^B, H^A, H^B) = \frac{h_i^B}{H_+^B(H^A, H^B)}. \tag{A.20}$$

*where*

$$H_+^A(H^A, H^B) := y_0^A \phi^A(H^A, H^B) + H^A,$$
$$H_+^B(H^A, H^B) := y_0^B \phi^B(H^A, H^B) + H^B,$$

*are the augmented aggregates.*

*Proof.* Group-$k$ demand of platform $i$ is implicitly defined by

$$n_i^k = \frac{h_i^k(p_i^k, p_i^l)}{H_{0i}^k(y_0^k, n_i^k, n_i^l, p_i^k, p_i^l) + H^k(p)}, \tag{A.21}$$

where

$$\begin{aligned} H_{0i}^k(y_0^k, n_i^k, n_i^l, p_i^k, p_i^l) \quad &:= \quad y_0^k \frac{\exp[(\Gamma^{kk} - 1)(a_i^k - p_i^k) + \Gamma^{kl}(a_i^l - p_i^l)]}{(n_i^k)^{\alpha^k}(n_i^l)^{\beta^k}} \\ &= \quad y_0^k \frac{h_i^k}{\exp(a_i^k - p_i^k)(n_i^k)^{\alpha^k}(n_i^l)^{\beta^k}}. \end{aligned} \tag{A.22}$$

Equations (A.21) and (A.22), along with the fact that $n_i^k/n_j^k = h_i^k/h_j^k$ implies that $H_{0i}^k = H_{0j}^k$ for all $i, j \in \{1, \ldots, M\}$. This implies that there exist $\phi^k$, $k \in \{A, B\}$, such that $H_{0i}^k = y_0^k \phi^k$ and

$$\frac{\exp[(\Gamma^{kk} - 1)(a_i^k - p_i^k) + \Gamma^{kl}(a_i^l - p_i^l)]}{(n_i^k)^{\alpha^k}(n_i^l)^{\beta^k}} = \phi^k,$$

for all $k \in \{A, B\}$ and $i \in \{1 \ldots, M\}$. These equations can be rewritten as

$$\alpha^A \log n_i^A + \beta^A \log n_i^B + \log \phi^A - (\Gamma^{AA} - 1)(a_i^A - p_i^A) - \Gamma^{AB}(a_i^B - p_i^B) = 0,$$
$$\alpha^B \log n_i^B + \beta^B \log n_i^A + \log \phi^B - (\Gamma^{BB} - 1)(a_i^B - p_i^B) - \Gamma^{BA}(a_i^A - p_i^A) = 0.$$

Solving for this system of equations, we obtain

$$\begin{aligned}\log n_i^A &= \frac{\beta^A \log \phi^B - \alpha^B \log \phi^A}{\alpha^A\alpha^B - \beta^A\beta^B} \\ &\quad + \frac{[\alpha^B(\Gamma^{AA}-1) - \beta^A\Gamma^{BA}](a_i^A - p_i^A)}{\alpha^A\alpha^B - \beta^A\beta^B} + \frac{[\alpha^B\Gamma^{AB} - \beta^A(\Gamma^{BB}-1)]\exp(a_i^B - p_i^B)}{\alpha^A\alpha^B - \beta^A\beta^B} \\ &= \log h_i^A + \frac{\beta^A \log \phi^B - \alpha^B \log \phi^A}{\alpha^A\alpha^B - \beta^A\beta^B}, \end{aligned} \tag{A.23}$$

where we used the relations

$$\begin{aligned}\alpha^B(\Gamma^{AA}-1) - \beta^A\Gamma^{BA} &= \frac{\alpha^B[(1-\alpha^B)\alpha^A + \beta^A\beta^B] - \beta^A\beta^B}{(1-\alpha^A)(1-\alpha^B) - \beta^A\beta^B} = (\alpha^A\alpha^B - \beta^A\beta^B)\Gamma^{AA}, \\ \alpha^B\Gamma^{AB} - (\beta^A\Gamma^{BB}-1) &= \frac{\alpha^B\beta^A - \beta^A[(1-\alpha^A)\alpha^B + \beta^A\beta^B]}{(1-\alpha^A)(1-\alpha^B) - \beta^A\beta^B} = (\alpha^A\alpha^B - \beta^A\beta^B)\Gamma^{AB},\end{aligned}$$

to obtain equation (A.23). Similarly, we have

$$\log n_i^B = \log h_i^B + \frac{\beta^B \log \phi^A - \alpha^A \log \phi^B}{\alpha^A\alpha^B - \beta^A\beta^B}. \tag{A.24}$$

From the equation $H_{0i}^k = y_0^k \phi^k$, equation (A.21) can be rewritten as

$$n_i^k = \frac{h_i^k}{y_0^k \phi^k + H^k}.$$

Hence, we can combine the equations

$$\begin{aligned}\log n_i^A &= \log h_i^A - \log\left(y_0^A\phi^A + H^A\right), \\ \log n_i^B &= \log h_i^B - \log\left(y_0^B\phi^B + H^B\right),\end{aligned}$$

with equations (A.23) and (A.24) to write the equations that determine the values of $(\phi^A, \phi^B)$ as a function of $(H^A, H^B)$:

$$\begin{aligned}f^A(\phi^A, \phi^B, H^A, H^B) &= 0, \\ f^B(\phi^A, \phi^B, H^A, H^B) &= 0,\end{aligned}$$

where

$$f^A(\phi^A, \phi^B, H^A, H^B) := \log \phi^A - \alpha^A \log(y_0^A\phi^A + H^A) - \beta^A \log(y_0^B\phi^B + H^B), \tag{A.25}$$

$$f^B(\phi^A, \phi^B, H^A, H^B) := \log \phi^B - \alpha^B \log(y_0^B\phi^B + H^B) - \beta^B \log(y_0^A\phi^A + H^A). \tag{A.26}$$

Next we show that the system of equations (A.25) and (A.26) has a unique solution. To do so, we first show that, for any given $\phi^B$, there exists a unique value $\tilde{\phi}^A(\phi^B)$ that solves equation (A.25). Then, we show that there exists a unique $\phi^B$ that solves $f^B(\tilde{\phi}^A(\phi^B), \phi^B) = 0$. To show the first statement, note that

$$\lim_{\phi^A \to 0} f^A(\phi^A, \phi^B, H^A, H^B) = -\infty,$$

$$\lim_{\phi^A \to \infty} f^A(\phi^A, \phi^B, H^A, H^B) = \lim_{\phi^A \to \infty} \log\left[\frac{\phi^A}{(y_0^A \phi^A + H^A)^{\alpha^A}}\right] - \beta^A \log(y_0^B \phi^B + H^B) = \infty,$$

$$\frac{\partial f^A(\phi^A, \phi^B, H^A, H^B)}{\partial \phi^A} = \frac{1}{\phi^A} \frac{(1-\alpha^A) y_0^A \phi^A + H^A}{y_0^A \phi^A + H^A} > 0,$$

which establishes the existence and uniqueness of $\tilde{\phi}^A(\phi^B) \in (0, \infty)$. Note that

$$\frac{\partial \tilde{\phi}^A}{\partial \phi^B} = -\frac{\frac{\partial f^A}{\partial \phi^B}}{\frac{\partial f^A}{\partial \phi^A}} = \frac{\phi^A}{\phi^B} \frac{\beta^A \frac{y_0^B \phi^B}{y_0^B \phi^B + H^B}}{1 - \alpha^A \frac{y_0^A \phi^A}{y_0^A \phi^A + H^A}} \in \left(0, \frac{\phi^A}{\phi^B}\right)$$

and $\lim_{\phi^B \to \infty}\left[\tilde{\phi}^A(\phi^B)\right] = \infty$. Furthermore, by using l'Hôpital's rule, we obtain that

$$\lim_{\phi^B \to \infty}\left(\frac{\tilde{\phi}^A(\phi^B)}{\phi^B}\right) = \lim_{\phi^B \to \infty}\left(\frac{\partial \tilde{\phi}^A(\phi^B)}{\partial \phi^B}\right) = \lim_{\phi^B \to \infty}\left(\frac{\partial \tilde{\phi}^A(\phi^B)}{\partial \phi^B}\right) \frac{\beta^A \frac{y_0^B \phi^B}{y_0^B \phi^B + H^B}}{1 - \alpha^A \frac{y_0^A \phi^A}{y_0^A \phi^A + H^A}}.$$

This implies that $\lim_{\phi^B \to \infty}[\tilde{\phi}^A(\phi^B)/\phi^B] = 0$.

Next, we show the existence and the uniqueness of the value $\phi^B$ that solves $f^B(\tilde{\phi}^A(\phi^B), \phi^B) = 0$. First, we have

$$\frac{df^B(\tilde{\phi}^A(\phi^B), \phi^B)}{d\phi^B} = \frac{1}{\phi^B} - \alpha^B \frac{y_0^B}{y_0^B \phi^B + H^B} - \frac{\partial \tilde{\phi}^A}{\partial \phi^B} \beta^B \frac{y_0^A}{y_0^A \phi^A + H^A} \tag{A.27}$$

$$= \frac{1}{\phi^B}\left[1 - \alpha^B \frac{y_0^B \phi^B}{y_0^B \phi^B + H^B} - \beta^B \beta^A \frac{\frac{y_0^A \phi^A}{y_0^A \phi^A + H^A} \frac{y_0^B \phi^B}{y_0^B \phi^B + H^B}}{1 - \alpha^A \frac{y_0^A \phi^A}{y_0^A \phi^A + H^A}}\right] \tag{A.28}$$

$$> 0, \tag{A.29}$$

from the assumption that $1 - \alpha_k > \beta^l$ for $k, l = \{A, B\}$, $l \neq k$. We also have

$$\lim_{\phi^B \to 0} f^A(\tilde{\phi}^A(\phi^B), \phi^B) = -\infty,$$

$$\begin{aligned}
\lim_{\phi^B \to \infty} f^A(\tilde{\phi}^A(\phi^B), \phi^B) &= \lim_{\phi^B \to \infty} \log \left[ \frac{\phi^B}{(y_0^B \phi^B + H^B)^{\alpha^B} \left( y_0^A \tilde{\phi}^A(\phi^B) + H^B \right)^{\beta^B}} \right] \\
&= \lim_{\phi^B \to \infty} \log \left[ \frac{\left(\phi^B\right)^{1-\alpha^B-\beta^B} \left(\frac{\phi^B}{\phi^A}\right)^{\beta^B}}{\left(y_0^B\right)^{\alpha^B} \left(y_0^A\right)^{\beta^B}} \left( \frac{y_0^B \phi_0^B}{y_0^B \phi^B + H^B} \right)^{\alpha^B} \left( \frac{y_0^A \tilde{\phi}^A(\phi^B)}{y_0^A \tilde{\phi}^A(\phi^B) + H^B} \right)^{\beta^B} \right] \\
&= \infty.
\end{aligned}$$

Hence, we have the unique solution to the equation $f^B(\tilde{\phi}^A(\phi^B), \phi^B) = 0$. Let $\phi^A(H^A, H^B)$ and $\phi^B(H^B, H^A)$ be the solution to this system of equations.

Group-$k$ demand of platform $i$ is now written as

$$n_i^k(h_i^A, H^A, H^B) = \frac{h_i^k}{H_+^k(H^A, H^B)},$$

which completes the proof. □

Note that this extension nests as special cases (i) the standard logit demand oligopoly model with an outside option (i.e., no network effects) and (ii) our main model (i.e., no outside option).[29]

User surplus in the interior participation equilibrium is given by

$$\begin{aligned}
CS^A &= \log \left( y_0^A + \sum_j \exp(a_j^A - p_j^A)(n_j^A)^{\alpha^k} (n_j^B)^{\beta^k} \right) \\
&= \log(y_0^A \phi^A + H^A) - \log \phi^A \\
&= (1 - \alpha^A) \log(y_0^A \phi^A + H^A) - \beta^A \log(y_0^B \phi^B + H^B).
\end{aligned}$$

Therefore, group-$k$ user surplus depend on $(H^A, H^B)$ and $(y_0^A, y_0^B)$.

[29] Absent network effects (i.e., $\alpha^A = \alpha^B = \beta^A = \beta^B = 0$), we must have $\phi^A = \phi^B = 1$. Then, the demand is given by $n_i^A = \frac{\exp(a_i^A - p_i^A)}{y_i^A + \sum_{j=1}^M \exp(a_j^A - p_j^A)}$. Absent the outside option (i.e., $y_0^A = y_0^B = 0$), the demand for platform $i$ is given by equation (3).

Group-$A$ demand for the outside option can be written as a function of $CS^A$:

$$\begin{aligned} n_0^A &= \frac{y_0^A \phi^A(H^A, H^B)}{y_0^A \phi^A(H^A, H^B) + H^A} \\ &= \frac{y_0^A}{\exp(CS^A)}, \end{aligned}$$

where we used the relation that $CS^A = \log(y_0^A \phi^A + H^A) - \log \phi^A$. Similarly, we have

$$n_0^B(CS^B) := \frac{y_0^B}{\exp(CS^B)}.$$

To see how augmented aggregates $(H_+^A, H_+^B)$ depend on the original aggregates $(H^A, H^B)$, recall that $f^A$ and $f^B$ are defined in equations (A.25) and (A.26). Since

$$\begin{array}{ll} \frac{\partial f^A}{\partial \phi^A} = \frac{1}{\phi^A}\left(1 - \alpha^A n_0^A\right), & \frac{\partial f^A}{\partial \phi^B} = -\beta^A \frac{1}{\phi^B} n_0^B, \\ \frac{\partial f^B}{\partial \phi^A} = -\beta^B \frac{1}{\phi^A} n_0^A, & \frac{\partial f^B}{\partial \phi^B} = \frac{1}{\phi^B}\left(1 - \alpha^B n_0^B\right), \\ \frac{\partial f^A}{\partial H^A} = -\alpha^A \frac{n_0^A}{y_0^A \phi^A}, & \frac{\partial f^B}{\partial H^A} = -\beta^B \frac{n_0^A}{y_0^A \phi^A}, \end{array}$$

using the implicit function theorem, we obtain the derivatives

$$\begin{aligned} \frac{\partial \phi^A}{\partial H^A} &= \frac{-\frac{\partial f^A}{\partial H^A}\frac{\partial f^B}{\partial \phi^B} + \frac{\partial f^B}{\partial H^A}\frac{\partial f^A}{\partial \phi^B}}{\frac{\partial f^A}{\partial \phi^A}\frac{\partial f^B}{\partial \phi^B} - \frac{\partial f^A}{\partial \phi^B}\frac{\partial f^B}{\partial \phi^A}} \\ &= \frac{1}{y_0^A}\left(\frac{1 - \alpha^B n_0^B}{(1 - \alpha^A n_0^A)(1 - \alpha^B n_0^B) - \beta^A \beta^B n_0^A n_0^B} - 1\right), \\ \frac{\partial \phi^B}{\partial H^A} &= \frac{-\frac{\partial f^A}{\partial \phi^A}\frac{\partial f^B}{\partial H^A} + \frac{\partial f^B}{\partial \phi^A}\frac{\partial f^A}{\partial H^A}}{\frac{\partial f^A}{\partial \phi^A}\frac{\partial f^B}{\partial \phi^B} - \frac{\partial f^A}{\partial \phi^B}\frac{\partial f^B}{\partial \phi^A}} \\ &= \frac{\phi^B}{y_0^A \phi^A}\frac{\beta^B n_0^A}{(1 - \alpha^A n_0^A)(1 - \alpha^B n_0^B) - \beta^A \beta^B n_0^A n_0^B}. \end{aligned}$$

Hence, we obtain the derivative of the augmented aggregates $(H_+^A, H_+^B)$ with respect to the original aggregates $(H^A, H^B)$:

$$\begin{aligned} \frac{\partial H_+^A(H^A, H^B)}{\partial H^A} &= \Gamma_0^{AA}(n_0^A, n_0^B), \\ \frac{\partial H_+^A(H^A, H^B)}{\partial H^B} &= \frac{y_0^A \phi^A}{y_0^B \phi^B}\Gamma_0^{AB}(n_0^A, n_0^B), \end{aligned}$$

where

$$\Gamma_0^{kk}(n_0^A, n_0^B) := \frac{1-\alpha^l n_0^l}{(1-\alpha^A n_0^A)(1-\alpha^B n_0^B) - \beta^A\beta^B n_0^A n_0^B},$$
$$\Gamma_0^{kl}(n_0^A, n_0^B) = \frac{\beta^k n_0^l}{(1-\alpha^A n_0^A)(1-\alpha^B n_0^B) - \beta^A\beta^B n_0^A n_0^B}$$

for $k \in \{A, B\}$.

**Price equilibrium** The profit of platform $i$ is given by

$$\Pi_i = \frac{h_i^A}{H_+^A(H^A, H^B)}[p_i^A(h_i^A, h_i^B) - c_i^A] + \frac{h_i^B}{H_+^B(H^A, H^B)}[p_i^B(h_i^A, h_i^B) - c_i^B]$$

The partial derivative with respect to $h_i^A$ is given by

$$\begin{aligned}
\frac{\partial \Pi_i}{\partial h_i^A} &= \left[\frac{1}{H_+^A} - \frac{\partial H_+^A}{\partial H^A}\frac{h_i^A}{(H_+^A)^2}\right][p_i^A(h_i^A, h_i^B) - c_i^A] - \frac{h_i^A}{H_+^A}\frac{1-\alpha^A}{h_i^A} \\
&\quad + \frac{h_i^B}{H_+^B}\frac{\beta^B}{h_i^A} - \frac{\partial H_+^B}{\partial H^A}\frac{h_i^B}{(H_+^B)^2}[p_i^B(h_i^A, h_i^B) - c_i^B] \\
&= \frac{1}{H_+^A}\Bigg\{\left(1 - \Gamma_0^{AA}(n_0^A, n_0^B)n_i^A\right)[p_i^A(h_i^A, h_i^B) - c_i^A] - 1 + \alpha^A \\
&\quad + \frac{h_i^B/H_+^B}{h_i^A/H_+^A}\beta^B - \Gamma_0^{BA}(n_0^A, n_0^B)\frac{y_0^B\phi^B/H_+^B}{y_0^A\phi^A/H_+^A}n_i^B[p_i^B(h_i^A, h_i^B) - c_i^B]\Bigg\}
\end{aligned}$$

The best response is characterized by the first-order conditions $\partial\Pi_i/\partial h_i^A = 0$ and $\partial\Pi_i/\partial h_i^B = 0$.

In the baseline model without outside options, implicit best-response functions exist and are uniquely determined by the first-order conditions. In this extension, we do not have the analytical proof of the existence of the implicit best reply. Instead, we have the following characterization of the best-response behaviors of the platforms.

**Remark A.4.** *The market shares of the profit-maximizing platform i, $(\tilde{n}_i^A, \tilde{n}_i^B)$ (presuming that best responses exist) are given as functions of user surpluses $(CS^A, CS^B)$, implicitly defined by the system of equations*

$$\begin{aligned}
a_i^k - c_i^k - CS^k - (1-\alpha^k)\log n_i^k + \beta^k \log n_i^l &= \Upsilon_i^{kk}(n_i^A, n_i^B, n_0^A(CS^A), n_0^B(CS^B))\left(1 - \alpha^k - \beta^l\frac{n_i^l}{n_i^k}\right) \\
&\quad + \Upsilon_i^{lk}(n_i^A, n_i^B, n_0^A(CS^A), n_0^B(CS^B))\left(1 - \alpha^l - \beta^k\frac{n_i^k}{n_i^l}\right),
\end{aligned}$$

*where*

$$\Upsilon_i^{kk}(n_i^A, n_i^B, n_0^A, n_0^B) := \frac{1 - \Gamma_0^{ll}(n_0^A, n_0^B)n_i^l}{[1 - \Gamma_0^{AA}(n_0^A, n_0^B)n_i^A]1 - \Gamma_0^{BB}(n_0^A, n_0^B)n_i^B] - \Gamma_0^{BA}(n_0^A, n_0^B)\Gamma_0^{AB}(n_0^A, n_0^B)n_i^A n_i^B},$$

$$\Upsilon_i^{kl}(n_i^A, n_i^B, n_0^A, n_0^B) := \frac{\Gamma_0^{lk}(n_0^A, n_0^B)n_i^l}{[1 - \Gamma_0^{AA}(n_0^A, n_0^B)n_i^A]1 - \Gamma_0^{BB}(n_0^A, n_0^B)n_i^B] - \Gamma_0^{BA}(n_0^A, n_0^B)\Gamma_0^{AB}(n_0^A, n_0^B)n_i^A n_i^B},$$

*for* $k, l \in \{A, B\}$, $l \neq k$.

*Proof.* The first-order condition $\partial \Pi_i / \partial h_i^A = 0$ implies that

$$\begin{aligned} p_i^A - c_i^A &= \frac{1}{1 - \Gamma_0^{AA} n_i^A}\left[1 - \alpha^A - \beta^B \frac{n_i^B}{n_i^A} + \Gamma_0^{BA}\frac{n_0^B}{n_0^A} n_i^B (p_i^B - c_i^B)\right] \\ &= \frac{1}{1 - \Gamma_0^{AA} n_i^A}\left(1 - \alpha^A - \beta^B \frac{n_i^B}{n_i^A}\right) \\ &\quad + \frac{\Gamma_0^{BA}}{1 - \Gamma_0^{AA} n_i^A}\frac{n_0^B}{n_0^A} n_i^B \frac{1}{1 - \Gamma_0^{BB} n_i^B}\left[1 - \alpha^B - \beta^A \frac{n_i^A}{n_i^B} + \Gamma_0^{AB}\frac{n_0^A}{n_0^B} n_i^A (p_i^A - c_i^A)\right], \end{aligned}$$

leading to the equation

$$\begin{aligned} &\frac{(1 - \Gamma_0^{AA} n_i^A)(1 - \Gamma_0^{BB} n_i^B) - \Gamma_0^{BA}\Gamma_0^{AB} n_i^A n_i^B}{(1 - \Gamma_0^{AA} n_i^A)(1 - \Gamma_0^{BB} n_i^B)}[p_i^A(h_i^A, h_i^B) - c_i^A] \qquad \text{(A.30)} \\ = &\frac{1}{1 - \Gamma_0^{AA} n_i^A}\left(1 - \alpha^A - \beta^B \frac{n_i^B}{n_i^A}\right) + \frac{\Gamma_0^{BA} n_i^B}{(1 - \Gamma_0^{AA} n_i^A)(1 - \Gamma_0^{BB} n_i^B)}\frac{n_0^B}{n_0^A}\left(1 - \alpha^B - \beta^A \frac{n_i^A}{n_i^B}\right). \end{aligned}$$

Similarly, the first-order condition $\partial \Pi_i / \partial h_i^B = 0$ leads to the equation

$$\begin{aligned} &\frac{(1 - \Gamma_0^{AA} n_i^A)(1 - \Gamma_0^{BB} n_i^B) - \Gamma_0^{BA}\Gamma_0^{AB} n_i^A n_i^B}{(1 - \Gamma_0^{AA} n_i^A)(1 - \Gamma_0^{BB} n_i^B)}[p_i^B(h_i^A, h_i^B) - c_i^B] \qquad \text{(A.31)} \\ = &\frac{1}{1 - \Gamma_0^{BB} n_i^B}\left(1 - \alpha^B - \beta^A \frac{n_i^A}{n_i^B}\right) + \frac{\Gamma_0^{AB} n_i^A}{(1 - \Gamma_0^{BB} n_i^B)(1 - \Gamma_0^{AA} n_i^A)}\frac{n_0^A}{n_0^B}\left(1 - \alpha^A - \beta^B \frac{n_i^B}{n_i^A}\right) \end{aligned}$$

Equations (A.30) and (A.31) define the implicit best reply function, if it exists.

Because of the relation

$$\begin{aligned} (1 - \alpha^k)\log h_i^k - \beta^k \log h_i^l &= (1 - \alpha^k)\log H_+^k - \beta^k \log H_+^l + (1 - \alpha^k)\log n_i^k - \beta^k \log n_i^l \\ &= CS^k + (1 - \alpha^k)\log n_i^k - \beta^k \log n_i^l \end{aligned}$$

for $k \in \{A, B\}$, the group-$k$ price-cost margin $p_i^k(h_i^A, h_i^B) - c_i^A$ can be expressed as

$$p_i^k(h_i^A, h_i^B) - c_i^A = a_i^k - c_i^k - CS^k - (1 - \alpha^k)\log n_i^k + \beta^k n_i^l.$$

Using this relation and noting that $n_0^k = n_0^k(CS^k)$ for $k \in \{A, B\}$, we can rearrange the first-order conditions (A.30) and (A.31) to obtain the expression in Remark A.4.

$\square$

The equilibrium conditions are

$$\sum_{i=1}^{M} \widetilde{n}_i^A(CS^A, CS^B) + \frac{y_0^A}{\exp(CS^A)} = 1,$$
$$\sum_{i=1}^{M} \widetilde{n}_i^B(CS^A, CS^B) + \frac{y_0^B}{\exp(CS^B)} = 1$$

When $y_0^A = y_0^B = 0$, the setting corresponds to the main model (i.e., the model without outside options). Hence, a price equilibrium exists (Proposition 2), and the characterization results of Section 4 hold. Because of continuity, these results carry over to this extension with sufficiently unattractive outside options (i.e., $(y_0^A, y_0^B)$ sufficiently small).

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