Discussion Paper Series – CRC TR 224

Discussion Paper No. 431
Project C 02

Evaluating the Financial Instability Hypothesis: A Positive and Normative Analysis of Leveraged Risk-Taking and Extrapolative Expectations

Antoine Camous[1]
Alejandro Van der Ghote[2]

May 2024
(First version: May 2023)

[1]University of Mannheim, Email: camous@uni-mannheim.de
[2]European Central Bank, Email: Alejandro.Van_der_Ghote@ecb.europa.eu

Support by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through CRC TR 224 is gratefully acknowledged.

Collaborative Research Center Transregio 224 - www.crctr224.de
Rheinische Friedrich-Wilhelms-Universität Bonn - Universität Mannheim

Evaluating the Financial Instability Hypothesis:
A Positive and Normative Analysis of Leveraged Risk-Taking and Extrapolative Expectations*

Antoine Camous
University of Mannheim

Alejandro Van der Ghote
European Central Bank

May 10, 2024

Abstract

Classical accounts of financial crises emphasize the joint contribution of extrapolative beliefs and leveraged risk-taking to financial instability. This paper proposes a simple macro-finance framework to evaluate these views. We find a novel interplay between non-rational extrapolation and investment risk-taking that amplifies financial instability relative to a rational expectation benchmark. Furthermore, the analysis provides guidance on the design of cyclical policy intervention. Specifically, extrapolative expectations command tighter financial regulation, irrespective of the regulator's degree of non-rational extrapolation.

Keywords: Non-rational Extrapolation in Expectations, Financial Stability, Financial Regulation

JEL codes: E44, E71, G01

*We thank Klaus Adam, Russell Cooper, Wouter den Haan, Mark Gertler, Urban Jermann, Paymon Khorrami, Wenhao Li, Federico Maldelman, Andreas Schaab, and Dejanir Silva for highly valuable comments. We also thank seminar participants at various institutions and conferences. Funding by the German Research Foundation (DFG) through CRC TR 224 (Project C02) is gratefully acknowledged. The views expressed in this article are those of the authors and do not necessarily reflect the views of the European Central Bank or the Eurosystem. Camous: University of Mannheim, D-68131, Mannheim, Germany. Email: camous@uni-mannheim.de. Van der Ghote: European Central Bank, Monetary Policy Research Division, Sonnemannstrasse 20, D-60314, Frankfurt, Germany. Email: Alejandro.Van_der_Ghote@ecb.europa.eu.

1 Introduction

Hyman Minsky (1977, 1986) and Charles P. Kindleberger (1978) have popularized the view that financial markets are prone to robust expansion leading to asset price collapses and economic contractions. Kindleberger's "Anatomy of a Typical Crises" (1978, Chapter 2) identifies a sequence starting with extended periods of prosperity and investment gains, propelling a surge in leveraged risk-taking fueled by investors' optimism:

> *"During the expansion phase, investors become more optimistic about the future, revising upward their estimates of the profitability of a wide range of investments, and thus, they become more eager to borrow."*

The seeds of financial instability lie with highly leveraged investors exposed at the cycle's peak to minor declines in asset prices. Dominos unfold as losses on asset values trigger credit restrictions and force investors to sell assets at discount prices – a phenomenon called "fire-sales" – rapidly transforming a minor downturn into a collapse of asset prices, intensified by investors pessimism:

> *"Soon, some of the investors who had financed most of their purchases with borrowed money become distress sellers (. . .) lead to sharp declines in the prices of the assets, and a crash and panic may follow."*

The objective of this study is to assess these narratives and their policy implications. We propose a model featuring the three critical components of Minsky and Kindleberger's Financial Instability Hypothesis (FIH): investment decisions are undertaken by leveraged investors, even when investment projects are inherently risky; the reallocation of assets across investment projects is difficult, even when these projects present similar characteristics; and investors' appreciation of future economic developments are tied to recent economic events, even in the presence of incompatible statistical evidence.

Our analysis considers a macroeconomic framework that emphasizes the role of credit market frictions in business fluctuations and financial stability (as initiated by Bernanke and Gertler 1989; Kiyotaki and Moore 1997; Bernanke, Gertler and Gilchrist 1999). At the crux of that framework is a financial sector whose capitalization directly influences the supply of credit and the resilience of economic activity to different economic disturbances. One prominent approach—first espoused by Shleifer and Vishny (1992) and Brunnermeier and Sannikov (2014)—highlights the role of fire sales as the outcome of a two-way loop between asset prices and financial capitalization that amplifies the economic effects of the disturbances. Another important approach—promoted by Bordalo, Gennaioli and Shleifer (2018) and Maxted (2023)—underscores the role of non-rational extrapolative (or diagnostic) expectations as a source of volatility in asset prices and financial instability.[1] Our model brings these two approaches together, and to evaluate the FIH appropriately, it considers in addition the possibility of risk-taking in real investments. Procyclical, real risk-taking is widely documented in formal empirical studies, notably during the boom-bust cycle that culminated in the Global Financial Crisis (Mian and Sufi 2011; Dell'Ariccia, Igan and Laeven 2012). We find that a real risk-taking channel creates a novel mechanism that ties non rational extrapolation and endogenous aggregate risk, leading to an amplification of financial and economic instability. In addition, the analysis prescribes the design of tighter financial regulations, regardless of the regulator's degree of diagnostic expectation, relative to the optimal design derived assuming agents form rational expectations.

Our modeling environment features a single productive asset and two production technologies. One of the technologies is more productive on average, but it is also riskier. Because agents are risk neutral, if expectations were rational and there were no financial frictions, real risk-taking would always be perceived as privately optimal and would never be financially constrained. However, because agents form diagnostic expectations (i.e., they assign excessive likelihood to possible events in the future that are reminiscent of those realized in the recent past), real risk-taking can temporarily be perceived as privately suboptimal when productive technologies have been impacted by a sequence of adverse disturbances. Moreover, because only some expert investors subject to leverage restrictions can channel the asset to the productive technology, real risk-taking may occasionally be financially constrained when the experts' aggregate capitalization is sufficiently low. The interplay between the agents' perceptions and the experts' financial capacity naturally determines the allocation of the asset between the technologies (i.e., the degree of real risk-taking) and the asset price in equilibrium. Real risk-taking falls and a fire sales episode occurs when adverse disturbances erode the experts' wealth share sufficiently to force them to sell assets to non-experts. As in Brunnermeier and Sannikov (2014), the resilience of the share of experts' wealth to disturbances provides a natural measure of the

[1] The concept of diagnostic expectations is grounded of the psychological theory of "the representativeness heuristic" (Tversky and Kahneman 1983). The systematic forecast errors on asset returns generated by the expectations are consistent with those documented empirically (Bordalo, Gennaioli and Shleifer 2018; Bordalo et al. 2019).

stability of the financial system.

Our first main finding is that diagnostic expectations intensify financial instability relative to a benchmark economy in which agents form rational expectations. The mechanism behind this result is as follows. As noted previously, a favorable sequence of technological disturbances boosts the wealth share of experts (and thus improves financial stability) whereas an unfavorable sequence of the disturbances deteriorates it. Moreover, as in Bordalo, Gennaioli and Shleifer (2018) and Maxted (2023), a favorable sequence renders the agents over optimistic about the prospects of the technologies, whereas an unfavorable sequence makes them over pessimistic about future returns. Together, these two elements generate a negative co-movement between the experts' wealth share and the systematic forecasts errors resulting from extrapolative expectations. In addition, in our environment, excessive extrapolation encourages real risk-taking when the wealth share is high (and the forecast errors are negative), whereas it discourages it when the wealth share is low. This in turn strengthens the impact of forecast errors during upturn while it weakens it during downturns. This asymmetry, in general, is sufficiently strong to tilt leftward the ergodic (i.e., long-run) distribution of the expert's wealth share relative to the rational expectations benchmark, thus deteriorating stability in both the financial system and the real economy, in line with the FIH.

In an extension to our baseline environment, we consider an alternative process for extrapolative expectations (arguably a more realistic one), in which agents form diagnostic expectations over a risk-adjusted measure of the return on the asset (as in Barberis et al. 2015), rather than over the disturbance (as in Bordalo, Gennaioli and Shleifer 2018 and Maxted 2023). Under this approach, we find a novel two-way loop between non-rational extrapolation and real risk-taking, which further exacerbates instability in the financial system and the real economy. The increased instability stems from the interplay between asset price volatility and agents' expectations, which in turn render systematic forecast errors more persistent. In another extension, we consider an alternative market-based, restriction on leverage tied to the present discounted value of future investment profits (as in Gertler and Kiyotaki 2010 and Gertler and Karadi 2011). Under this specification, non-rational extrapolation directly influences the perception of future profits—and thus the leverage limit as well—but these effects are generally of second-order importance, because the financing constraint becomes slack when the experts are well capitalized and optimism is strong.

Kindleberger and Minsky devote discussions to cyclical policy interventions designed to temper financial instability and associated undesirable economic volatility:

> *"The appearance of a mania or a bubble raises the policy issue of whether governments should seek to moderate the surge in asset prices to reduce the likelihood or the severity of the ensuing financial crisis or to ease the economic hardship that occurs when asset prices begin to decline."* (Ibid.)

Our analysis allows us to determine whether policy intervention is required at each particular stage of the financial cycle. Specifically, we analytically characterize a socially optimal allocation and compare its properties with the competitive equilibrium.[2] Two contrasting elements stand out. First, the socially optimal allocation internalizes the collective effects of individual decisions on aggregate prices and fluctuations, whereas individual decisions in the competitive equilibrium do not. Second, the socially optimal allocation is fundamentally related to the social value of the asset, whereas individual decisions in a competitive equilibrium are inherently tied to the asset price.

These differences motivate an active role for financial regulation, even under rational expectations. However, under diagnostic expectations, we find that restrictions on financial leverage and risk-taking are tighter, regardless of the regulator's degree of diagnostic expectations. Importantly, the cyclical nature of the restrictions depends on the degree of diagnosticity of the planner. Specifically, a benevolent planner (i.e., a planner whose expectations are equally diagnostic to those of private agents) restricts the allocation of the asset to the productive technology during recoveries from busts. By contrast, a paternalistic planner (i.e., a planner with rational expectations in an environment in which private agents have diagnostic expectations) restricts it during booms. Overall, our analysis identifies how macro-prudential regulations are required at different phases of the financial cycle depending on the expectation processes of regulators.

Related literature. A wealth of literature has studied the implications of financial and behavioral frictions on financial markets and macroeconomic outcomes.[3] Among those studies, ours characterizes the nonlinear stochastic global dynamics in a continuous time

[2]Using constrained efficiency rather than parametrized instruments means that there is no need to commit to an arbitrary set of policy instruments, but rather let the model guide the choice of instruments, as in Di Tella (2019).

[3]Recent studies that incorporate diagnostic expectations into dynamic general equilibrium models include Krishnamurthy and Li (2020); Bianchi, Ilut and Saijo (2021); L'Huillier, Singh and Yoo (2021); L'Huillier, Phelan and Wieman (2022).

general equilibrium framework, and hence is closest to Brunnermeier and Sannikov (2014) and Maxted (2023). As previously mentioned, Brunnermeier and Sannikov (2014) emphasize the role of fire sales to financial instability in an economy with rational expectations, whereas Maxted (2023) highlights contribution of diagnostic expectations to financial instability in an economy without fire sales or real risk-taking. To the best of our knowledge, our study is the first to incorporate extrapolative expectations with financial frictions and risk-taking in real investments to appropriately evaluate the FIH.

Importantly, our analysis establishes that extrapolative expectations amplify financial instability when one accounts for the real risk taking channel. This contrasts with results presented in Maxted (2023), where the effects of extrapolative beliefs on asset prices and the dynamics of net worth tends to be symmetric along the financial cycle, hence positive forecasts errors during busts compensate negative forecasts errors during booms. Instead, our environment with endogenous aggregate risk unambiguously highlights an amplification of financial instability under diagnostic expectations relative to rational expectations. Indeed, once we account for endogenous risk taking, the effects of extrapolative beliefs are asymmetric along the financial cycle. This feature contributes to enhance financial instability, a view that aligns closely with the classical analyses of financial crises.

A central piece of analysis derives the implications of financial instability and extrapolative beliefs for macro-prudential regulation. As noted in Phelan (2016), Brunnermeier and Sannikov (2014) is a critical contribution, but the model is not designed to study leverage regulation: "limiting leverage can improve stability, but it is very difficult to improve the welfare of either households or banks." In contrast, our analysis points to beneficial leverage restrictions when agents form extrapolative expectations. Overall, our study completes the review of the FIH and identifies conditions that motivate beneficial policy interventions and leverage restrictions at each phase of the financial cycle.

Finally, our study differs from other analyses that focuses on the normative implications of non-rational expectations for financial regulation. Fontanier (2022) presents a stylized environment, in which externalities may arise when the non-rational component of expectations is tied to asset prices. To conduct the welfare analysis, Fontanier (2022) restricts attention to the socially optimal allocation derived by a paternalistic planner. Instead, our study considers socially optimal allocations for both paternalistic and benevolent planners in a nonlinear stochastic environment. Dávila and Walther (2021) also study the optimal design of financial regulation when private agents have distorted beliefs relative to a planner, but they focus on implications for the optimal regulation of differences in beliefs between investors and creditors. Finally, Farhi and Werning (2020) study an economy with diagnostic expectations in which social inefficiencies may arise from aggregate demand externalities. Their study focuses on the implications of extrapolative expectations for the coordination of monetary policy and macro-prudential policy.

Layout. The paper is structured as follows. Section 2 describes the model and characterizes its equilibrium. Section 3 conducts the positive analysis, and section 4 explores additional properties under alternative model specifications. Section 5 presents the normative analysis. Section 6 concludes. An Appendix to the paper provides the proofs of all lemmas, propositions and corollaries stated in the sections.

2 The Model

Consider a production economy with financial frictions in which agents form extrapolative (or diagnostic) expectations over future events. Of particular interest are the joint implications of diagnostic expectations and financial frictions on the equilibrium outcome (section 3 and section 4) and allocative efficiency (section 5).

2.1 Environment

Time $t \in \mathbb{R}_+$ is continuous and unbounded. There is a single real asset $k_t \geq 0$ and a single output good $y_t \geq 0$. The asset can be allocated between two production technologies.

Technologies. The technologies produce the output good using the asset according to

$$y_{j,t} = A_j k_{j,t} \geq 0 \ , \tag{1}$$

where $A_j > 0$ is the productivity of technology $j \in \{1, 2\}$ and $k_{j,t} \geq 0$ are the units of the asset allocated to that technology. In addition, the technologies allow for internal reinvestment of the asset, at a standard rate of return, $\mathcal{I}_j(\iota_{j,t})k_{j,t}dt$, which satisfies $\mathcal{I}_j(0) = 0$, $\mathcal{I}_j'(\cdot) > 0$, and $\mathcal{I}_j''(\cdot) < 0$, where $\iota_{j,t} \in [0, A_j]$ is the reinvestment rate per unit of the asset. One of the technologies (i.e., $j = 1$) is more productive but it is also riskier. Formally, $A_1 \geq A_2$, $\mathcal{I}_1(\cdot) \geq \mathcal{I}_2(\cdot)$, and $\sigma_1 > \sigma_2 \geq 0$, with

$$\frac{dk_{j,t}}{k_{j,t}} = \mathcal{I}_j(\iota_{j,t})dt + \sigma_j dZ_t \ , \tag{2}$$

where $dZ_t \sim_{i.i.d.} \mathcal{N}(0, dt)$ is a Brownian disturbance common to the technologies. The disturbance can be interpreted as an aggregate shock to the productive quality of the asset, or in other words, as a *quality shock*.

Preferences. The economy is populated by a continuum of agents with linear preferences over the output good. Thus, agents only derive utility from the present value of consumption flows, discounted at subjective time discount rate, $r > 0$. All agents share common expectations but these feature non-rational (or diagnostic) extrapolation.

Non-rational Extrapolation. If expectations were rational, agents would not extrapolate from past disturbances the likelihood of future disturbances, because the process for the disturbance is serially uncorrelated. Thus, agents would correctly forecast $\hat{E}_t[dZ_t] = E_t[dZ_t] = 0$, where hat variables indicate perceptions. By contrast, under extrapolative expectations, agents rely on recently realized disturbances to estimate the future average disturbance, and hence they make systematic forecast errors. Notably, when recent disturbances are sufficiently negative, agents incorrectly forecast $\hat{E}_t[dZ_t] << 0$ and thus perceive the *unproductive* technology (i.e., $j = 2$) as the most profitable.

Formally—following Maxted (2023)—we assume agents synthesize information about past disturbances as

$$\omega_t \equiv \int_0^t e^{-\delta(t-s)} dZ_s \ , \tag{3}$$

where $\delta > 0$ indicates the memory rate of decay at which past realizations are discounted. Agents then use recent information $\omega_t \in \mathbb{R}$ to forecast future disturbances according to

$$\hat{E}_t[dZ_t] \equiv E_t[d\hat{Z}_t], \text{ with } d\hat{Z}_t \equiv \hat{\mu}\omega_t dt + dZ_t, \tag{4}$$

where parameter $\hat{\mu} > 0$ is the extrapolative (or diagnostic) weight of information on expectation formation. Thus, forecast $\hat{E}_t[dZ_t] = \hat{\mu}\omega_t dt$ positively depends on information ω_t, and forecast errors $\hat{E}_t[dZ_t] \neq 0$ are possible. Moreover, if information $\omega_t < 0$ is sufficiently negative, estimates about the growth rate of the asset under the technologies satisfy $\hat{E}_t[dk_2/k_2] > \hat{E}_t[dk_1/k_1]$. Based on these results—and because past disturbances do not have predictive power over future disturbances—in what follows, we interpret ω_t as *sentiment*.[4]

[4]Note that with $\hat{\mu} = 0$ expectations are rational.

Perceived Returns on Technologies. The asset can be traded in spot markets at a price $q_t > 0$. We postulate that the price evolves according to an Ito process

$$\frac{dq_t}{q_t} = \mu_{q,t}dt + \sigma_{q,t}dZ_t \ , \ \text{with} \ \ \sigma_{q,t} \geq 0 \ , \tag{5}$$

where $\mu_{q,t} \in \mathbb{R}$ and $\sigma_{q,t} \geq 0$ are endogenous drift and diffusion processes to be determined in equilibrium and dZ_t is the aggregate disturbance introduced in (2). Let $dR_{j,t} \in \mathbb{R}$ denote the rate of return to allocate the asset to technology j, defined as

$$dR_{j,t} \equiv \frac{A_j - \iota_{j,t}}{q_t}dt + \frac{d\left(q_t k_{j,t}\right)}{q_t k_{j,t}} \ , \tag{6}$$

where the first term on the RHS is the dividend yield from operating the technology and the second term is the percentage change of the market value of asset holdings. Applying Ito's product rule, one gets

$$dR_{j,t} = \Big[\frac{A_j - \iota_{j,t}}{q_t} + \mu_{q,t} + \mathcal{I}_j(\iota_{j,t}) + \sigma_{q,t}\sigma_j\Big]dt + (\sigma_{q,t} + \sigma_j)dZ_t \ , \tag{7}$$

which implies that agents' forecasts are

$$\hat{E}_t[dR_{j,t}] = \left[\frac{A_j - \iota_{j,t}}{q_t} + \mu_{q,t} + \mathcal{I}_j(\iota_{j,t}) + \sigma_{q,t}\sigma_j + \left(\sigma_{q,t} + \sigma_j\right)\hat{\mu}\omega_t\right] dt \ , \tag{8}$$

The last term on the RHS of (7) reflects the influence of diagnostic expectations over the perceptions of price risk $\sigma_{q,t}dZ_t$ and quality risk $\sigma_j dZ_t$. Notably, if sentiment ω_t is sufficiently low, diagnostic perceptions towards the risks are sufficiently strong to render $\hat{E}_t[dR_{2,t}] > \hat{E}_t[dR_{1,t}]$. Consequently, when sentiment is relatively high, agents correctly perceive the *productive* technology (i.e., $j = 1$) as the most profitable investment, but when sentiment is sufficiently low, they incorrectly perceive the *unproductive* technology (i.e., $j = 2$) as the most profitable.

Frictions. There are two types of agents: households and financiers. Households can only operate the unproductive technology whereas financiers can only operate the productive technology. Both types of agents can issue debt, but only financiers are subject to a financing constraint. This constraint is motivated by a standard agency problem in credit markets that allows financiers to walk away with a fraction of their assets immediately after issuing debt. The constraint restricts asset holdings of financiers to satisfy $q_t k_{1,t} \leq \lambda n_t$, where $n_t \geq 0$ is their net worth and parameter $\lambda - 1 > 0$ is the upper limit on their debt-to-net-worth ratio. For simplicity, we assume debt is short term and non-contingent, meaning that debt issued at time t matures at time $t + dt$ and promises a fixed rate of return regardless of realization dZ_t. Given linearity in preferences over consumption of the output good, the interest rate on debt is given by the agents' subjective time discount rate, rdt.

Portfolio Problems. Agents are competitive. Households maximize the present discounted value of consumption subject to the law of motion of their wealth. Formally, they solve

$$\max_{c_t, \iota_{2,t}, k_{2,t} \geq 0} \hat{E}_t \int_t^{+\infty} e^{-r(s-t)} c_s ds \ , \tag{9}$$

subject to

$$dw_s = dR_{2,s} q_s k_{2,s} + r(w_s - q_s k_{2,s})ds - c_s ds + \tau_s ds \ , \tag{10}$$

where $c_t \geq 0$ is consumption, $w_t \in \mathbb{R}$ is wealth, and $\tau_t \in \mathbb{R}$ are net transfers from financiers. As in Gertler and Kiyotaki (2010), Gertler and Karadi (2011), and Maggiori (2017), financiers do not consume, but rather each of them pays out dividends to a unique household. They do so according to an exogenous Poisson process with arrival rate $\theta > 0$. When they pay out, financiers transfer their entire net worth to their associated household, and immediately afterwards, they are replaced by an identical newcomer whose starting net worth is specified below. Financiers maximize the present discounted value of dividend payouts

$$\max_{\iota_{1,s}, k_{1,s} \geq 0} \hat{E}_t \int_t^{+\infty} \theta e^{-(r+\theta)(s-t)} n_s ds \ , \tag{11}$$

subject to the law of motion of net worth

$$dn_s = dR_{1,s} q_s k_{1,s} - r(q_s k_{1,s} - n_s)ds \ , \tag{12}$$

and the collateral constraint

$$q_s k_{1,s} \leq \lambda n_s \ . \tag{13}$$

Equilibrium. A competitive equilibrium is an allocation $\{\iota_{1,t}, k_{1,t}, \iota_{2,t}, k_{2,t}, c_t\}$ and an asset price process $\{q_t, \mu_{q,t}, \sigma_{q,t}\}$ such that (i) the allocation solves portfolio problems (9)-

(10) and (11)-(13) given the price process; (*ii*) the markets for the good, the asset, and debt clear.

2.2 Solving the Equilibrium

To solve the equilibrium we proceed as follow. First, we derive the optimal choices of households and financiers, which combined with market clearing deliver analytical equilibrium conditions. Then, we restrict attention to a Markov equilibrium, which allows to characterize equilibrium as a tractable system of second-order partial differential equations (PDEs).

2.2.1 Households' Problem

The lemma below characterizes the optimal choices of households.

Lemma 1. *At any given time t, households are indifferent among any consumption rate c_t. Moreover, they choose reinvestment rate $\iota_{2,t}$ and asset holding $k_{2,t}$ as follows:*

$$\mathcal{I}_2'(\iota_{2,t}) = \frac{1}{q_t} \ , \tag{14}$$

and

$$q_t k_{2,t} \left[\begin{array}{ll} = 0 & \text{if } \alpha_{2,t} < 0 \\ \in [0, +\infty) & \text{if } \alpha_{2,t} = 0 \end{array} \right. , \tag{15}$$

where the estimated risk-adjusted excess return to allocate the asset to the unproductive technology over holding debt, that is, $\alpha_{2,t} \leq 0$, is given by

$$\alpha_{2,t} \equiv \frac{1}{dt} \hat{E}_t \left[dR_{2,t} \right] - r \leq 0 \ . \tag{16}$$

The intuition behind this proposition is as follows. Households are indifferent among any consumption rate because the interest rate on debt equals their subjective time discount rate. When $\alpha_{2,t} < 0$, households strictly prefer holding debt securities to allocating the asset to the unproductive technology. Thus, $k_{2,t} = 0$ is optimal. By contrast, when $\alpha_{2,t} = 0$, households are indifferent between the two investment opportunities, and therefore, any $k_{2,t} \geq 0$ is optimal. Excess return $\alpha_{2,t} \leq 0$ cannot be positive in equilibrium.

Otherwise, households would take unbounded leveraged positions on the asset, since they are not subject to financing constraints. Reinvestment rule (14) indicates that reinvestment positively depends on asset price q_t.

2.2.2 Financiers' Problem

Let $V_t \geq 0$ be the value function associated with problem (11)-(13). We postulate that the value is linear in net worth. Formally, $V_t \equiv v_t n_t$, where marginal value of net worth $v_t \geq 1$ is endogenous but independent of individual choices. In addition, we postulate that marginal value v_t evolves stochastically over time, according to an Ito process with disturbance dZ_t and endogenous drift and diffusion $\mu_{v,t} \in \mathbb{R}$ and $\sigma_{v,t} \leq 0$, respectively. The following lemma characterizes the optimal choices of financiers.

Lemma 2. *At any given time t, financiers choose reinvestment rate $\iota_{1,t}$ and asset holding $k_{1,t}$ as follows:*

$$\mathcal{I}_1'(\iota_{1,t}) = \frac{1}{q_t} \ , \tag{17}$$

and

$$\frac{q_t k_{1,t}}{n_t} \begin{bmatrix} = 0 & \text{if } \alpha_{1,t} < 0 \\ \in [0, \lambda] & \text{if } \alpha_{1,t} = 0 \\ = \lambda & \text{if } \alpha_{1,t} > 0 \end{bmatrix} , \tag{18}$$

where the estimated risk-adjusted excess return to allocate the asset to the productive technology over holding debt, namely, $\alpha_{1,t} \in \mathbb{R}$, is given by

$$\alpha_{1,t} \equiv \frac{1}{dt}\hat{E}_t\left[dR_{1,t}\right] - r + \left(\sigma_{q,t} + \sigma_1\right)\sigma_{v,t} \ . \tag{19}$$

The marginal value of net worth, v_t, satisfies

$$0 = \alpha_{1,t}\frac{q_t k_{1,t}}{n_t} + \mu_{v,t} + \hat{\mu}\omega_t\sigma_{v,t} + \frac{\theta}{v_t} - \theta \ . \tag{20}$$

When $\alpha_{1,t} > 0$, financiers expect a positive excess return to allocating the asset to the productive technology. Thus, they take leveraged positions on the asset until they hit their limit on debt. When $\alpha_{1,t} = 0$, financiers are willing to take any position on the asset, because they are indifferent between the two investment alternatives. Lastly, when

$\alpha_{1,t} < 0$, financiers do not acquire the asset, because they expect a higher return from holding debt. The last term in (19) is a compensation for holding quality risk $\sigma_1 dZ_t$ and price risk $\sigma_{q,t} dZ_t$. This term results from the collateral constraint—which makes financiers concerned with the co-movement between the return on investments and the rate of change in the marginal value of net worth. Reinvestment rule (17) is analogous to reinvestment rule (14).

Condition (20) expresses marginal value v_t as a present discounted value of expected rents $\alpha_{1,t} q_t k_{1,t}/n_t \geq 0$. These rents are the profits earned by financiers from operating the productive technology. If financiers never earn any rent—and thus the collateral constraint is always slack—then $v_t = 1$. By contrast, if $\alpha_{1,t} > 0$ at least occasionally, then $v_t \geq 1$.

2.2.3 Equilibrium Characterization

We postulate that in equilibrium households and financiers cannot simultaneously be marginal buyers of the asset. Put formally, excess returns $\alpha_{1,t}$ and $\alpha_{2,t}$ cannot simultaneously be null "almost surely." The following proposition characterizes the equilibrium.

Proposition 1. *Let $\eta_t \equiv n_t/q_t k_t \in [0,1]$ be the aggregate net worth of financiers as a share of total wealth and let $\kappa_t \equiv k_{1,t}/k_t \in [0,1]$ be the aggregate share of the asset allocated to the productive technology. Then, the equilibrium outcome is partitioned into the following three regimes,*

1. *Financially unconstrained regime:* $\kappa_t = 1 \leq \lambda\eta_t$, $\alpha_{1,t} = 0$, $\alpha_{2,t} < 0$;
2. *Financially constrained regime:* $\kappa_t = \lambda\eta_t \in [0,1]$, $\alpha_{1,t} > 0$, $\alpha_{2,t} = 0$; (21)
3. *Precautionary regime:* $\kappa_t = 0$, $\alpha_{1,t} < 0$, $\alpha_{2,t} = 0$;

The equilibrium allocation is summarized as $\{\iota_{1,t}, \iota_{2,t}, \kappa_t\}$, and the equilibrium is characterized by $\{(14), (16), (17), (19), (20), (21)\}$. The equilibrium utility of households per unit of the asset, namely, $u_t > 0$, satisfies

$$
\begin{aligned}
0 \; = \; & \kappa_t \left\{A_1 - \iota_{1,t} + \left[\mathcal{I}_1(\iota_{1,t}) + \sigma_1 \hat{\mu}\omega_t\right] u_t\right\} + \\
& + (1-\kappa_t)\left\{A_2 - \iota_{2,t} + \left[\mathcal{I}_2(\iota_{2,t}) + \sigma_2 \hat{\mu}\omega_t\right] u_t\right\} + \hat{E}_t\left[du_t\right] - r u_t.
\end{aligned}
\tag{22}
$$

Notations do not distinguish between individual and aggregate variables because in equilibrium a representative household and a representative financiers exist.[5] The equilibrium regimes directly follow from combining the optimality conditions from lemmas 1 and 2 together with market clearing for the asset. In equilibrium, consumption per unit of the asset c_t/k_t is given by net output flows $y_t/k_t = (A_1 - \iota_{1,t})\,\kappa_t + (A_2 - \iota_{2,t})\,(1-\kappa_t)$. Utility u_t is the present discounted value of consumption per unit of the asset. Thus, it is interpreted as the *social value* of the asset.

2.2.4 Markov Equilibrium

For tractability, we restrict attention to a Markov equilibrium, which allows to reduce the equilibrium conditions to a system of second-order PDEs. As is common practice, we no longer report the time subscript.

Definition 1. *A Markov equilibrium is a set of state variables* $\{\eta, \omega, k\}$ *and a set of mappings* $\{q, v\}$ *defined over states* $\{\eta, \omega\}$*, such that (i) the mappings satisfy conditions* $\{(14), (16), (17), (19), (20), (21)\}$ *and (ii) the states evolve according to laws of motion consistent with the conditions.*

Wealth share η relates to the tightness of the collateral constraint while sentiment ω indicates the degree of extrapolation relative to the rational expectations benchmark. The aggregate quantity of the asset is also a state variable, but it is not relevant for the derivations, because the equilibrium is scale invariant with respect to k. Mappings $\{q, v\}$ alone are sufficient to characterize the equilibrium because any other endogenous variable can be expressed as a function of those mappings or of their partial derivatives with respect to the states.

Regarding the laws of motion, the aggregate quantity of the asset evolves endogenously, according to

$$\frac{dk}{k} = \mu_k dt + \sigma_k dZ \ , \tag{23}$$

with

$$\mu_k = \kappa \mathcal{I}_1(\iota_1) + (1-\kappa)\,\mathcal{I}_2(\iota_2) \ , \tag{24}$$

$$\sigma_k = \kappa\sigma_1 + (1-\kappa)\,\sigma_2 \ . \tag{25}$$

[5] There is a representative household because individual households are identical. A representative financier exists because the behavior of individual financiers is linear in net worth.

The wealth share of financiers evolves endogenously as well, according to

$$\frac{d\eta}{\eta} = \mu_\eta dt + \sigma_\eta dZ \ , \tag{26}$$

with

$$\begin{aligned} \mu_\eta &= \left[\frac{A_1 - \iota_1}{q} + \mathcal{I}_1(\iota_1) + \sigma_q \sigma_1\right]\phi + (\mu_q - r)(\phi - 1) - \mu_k \quad (27) \\ &\quad -\sigma_q\sigma_k + (\sigma_q + \sigma_k)\left[(\sigma_q + \sigma_k) - \phi(\sigma_q + \sigma_1)\right] - \left(\theta - \frac{\gamma}{\eta}\right) , \\ \sigma_\eta &= \phi(\sigma_q + \sigma_1) - (\sigma_q + \sigma_k) \ , \quad (28) \end{aligned}$$

where $\phi \equiv qk_1/n \geq 0$ is the leverage multiple of financiers and the last term in μ_η is the net transfers from financiers to households.[6] The first term of the transfers is the aggregate dividend payout and the second term is the starting endowment of newcomers. Each newcomer receives $\gamma/\theta > 0$ units of the asset from a unique household (as in Gertler and Kiyotaki 2010 and Gertler and Karadi 2011).

Lastly, sentiment evolves exogenously, according to

$$d\omega = -\delta\omega dt + dZ \ . \tag{29}$$

Proposition 2. *The Markov equilibrium can be analytically characterized as the solution to a system of second-order PDEs for* $\{q, v\}$ *in* $\{\eta, \omega\}$.

To conduct the positive and the normative analysis, when necessary, we solve the PDEs numerically, using spectral methods. To do so, we parametrize return functions $\mathcal{I}_j(\cdot)$ and assign numerical values to the parameters, as detailed in the next subsection.

2.3 Parametrization and Parameter Values

In our main exposition, we consider a riskless unproductive technology without reinvestment opportunities. Formally, $\mathcal{I}_2(\cdot) = 0$ and $\sigma_2 = 0$. In section 4, we investigate the robustness of our results to alternative technological specifications. Throughout the analysis, as is common in the literature (e.g., Brunnermeier and Sannikov 2014; Phelan 2016; He and Krishnamurthy 2019), we consider quadratic costs for reinvestment. That is, $\mathcal{I}_1(\iota_1) = \chi_1\sqrt{\iota_1}$,

[6] This law of motion follows from applying Ito's quotient rule to $\eta = n/qk$ and then subtracting from the resulting expression the net transfers from financiers to households, $\theta - \gamma/\eta$. Note then that $\tau = (\theta\eta - \gamma)qk$.

where $\chi_1 > 0$ is a parameter.

Table 1 reports parameter values for the baseline case. The time frequency is annual. Parameters for technologies and agents are either taken from other studies or set to match unconditional averages in an economy with rational expectations and financial frictions.[7] The targeted moments are standard and in particular are consistent with Gertler and Kiyotaki (2010) and Gertler and Karadi (2011). To compute the averages, we use the limiting probability density function of the state, which measures the share of time the economy spends on average at each state point over a sufficiently long (i.e., infinite) time horizon.

Table 1: Parameter Values

Description	Parameter	Value	Target / Source
Panel A. Technologies			
Productivity gap	$A_1 - A_2$	0.35	Av. credit spread (1%)
Quality risk	σ_1	3%	Av. volatility of output (4%)
Return on reinvestment	χ_1	1.9%	Av. investment-output ratio (20%)
Panel B. Agents			
Subjective time discount rate	r	2%	Interest rate
Limit on debt	$\lambda - 1$	3	Av. leverage multiple (3.7)
Frequency of dividend payouts	θ	10%	Av. life span of financiers (10 years)
Starting endowment of financiers	γ/θ	15%	Av. wealth share of financiers (25%)
Panel C. Expectation formation			
Memory decay rate	δ	0.85	Corr. sentiment-wealth share (0.71)
Extrapolation weight	$\hat{\mu}$	0.2	Output bias (0.75%)

Notes: The table reports the parameter values in the baseline specification of the model. The time frequency is annual.

The productivity of the productive technology, $A_1 = 1$, is normalized to 1. Productivity gap $A_1 - A_2 = 0.35$ targets an excess return to operating the productive technology of $E[dR_1] - r = 1\%$. The investment return $\chi_1 = 1.9\%$ targets an average ratio of reinvestment to output of $E[\iota/[A_1\kappa + A_2(1-\kappa)]] = 20\%$, whereas volatility $\sigma_1 = 3\%$ targets an average volatility of detrended output of $Var[A_1\kappa + A_2(1-\kappa)] = 4\%$. The subjective time discount

[7] This approach allows direct comparison with relevant literature. We provide sensitivity analysis of our results at different steps of the exposition.

rate is consistent with a standard value for the real interest rate of $r = 2\%$.[8] The debt limit of financiers, $\lambda - 1 = 3$, targets an average leverage multiple of financiers of $E[\phi] = 3.7$. The average frequency of dividend payouts is $\theta = 10\%$, and the endowment of financiers satisfies $\gamma/\theta = 15\%$. The former value implies an average lifespan of financial firms of 10 years, whereas the latter value targets an average share of wealth of financiers of $E[\eta] = 25\%$.

Finally, the extrapolation weight $\hat{\mu} = 0.2$ targets an output bias of 0.75% for a standard deviation in sentiment, as reported by Bordalo et al. (2020). The persistence parameter $\delta = 0.85$ targets a correlation between sentiment and wealth share of $Corr[\omega, \eta] = 71\%$, as in Maxted (2023).[9]

We perform robustness analyses in section 4, where we consider alternative specifications for the formation of diagnostic expectations, the collateral constraint, and differences across the technologies.

3 Positive Analysis

We can now derive the equilibrium outcome and investigate its positive properties. To clarify exposition, first, we examine three simplified versions of the model, and then we consider the original model economy.

3.1 Rational Expectations and No Financial Frictions

Consider first an economy with rational expectations and without financial frictions. Formally, extrapolation weight $\hat{\mu} = 0$ is null, financiers' net worth $n \in \mathbb{R}$ can be negative, and leverage limit $\lambda = +\infty$ is unbounded.[10] The following corollary derives from proposition 1 and describes the equilibrium outcome in this economy.

Corollary 1. *In the economy with rational expectations and without financial frictions,*

[8] In the economy with rational expectations and without financial frictions (presented in subsection 3.1), the growth rate of the economy is constant over time and under the parameter values presented in Table 1, it takes value $E[\kappa \mathcal{I}_1(\iota_1)] = 1.38\% < r = 2\%$. This guarantees that relevant present discounted values, such as those for consumption or output, are bounded and dynamics are not explosive.

[9] Table 2 reports the unconditional frequencies of the three equilibrium regimes under rational and diagnostic expectations. Under rational expectations, the precautionary regime does not occur, while under diagnostic expectations, it occurs approximately once every 500 years.

[10] If the net worth of financiers could not be negative, boundary condition $(\sigma_q + \sigma_1) \to 0$ as $\eta \to 0$ would be required to ensure $n \geq 0$—as in Maggiori (2017). The reason is that external financing is limited to non-contingent debt and the asset is risky. The possibility of $n < 0$ can then be interpreted as the possibility of issuing risky debt (i.e., debt whose rate of return depends on shock dZ) or of issuing equity.

neither sentiment ω nor wealth share η influence the equilibrium outcome. The asset price is a constant that satisfies

$$\alpha_1 = 0 \Leftrightarrow \frac{A_1 - \iota_1}{q} + \mathcal{I}_1(\iota_1) - r = 0 \ , \ \textit{with} \ \ \mathcal{I}_1'(\iota_1) = \frac{1}{q} \ . \tag{30}$$

Value $v = 1$ is also a constant. The aggregate quantity of the asset is allocated to the productive technology, that is, $\kappa = 1$. The social value of the asset equals the asset price, that is, $u = q$.

In this economy, out of the three regimes presented in proposition 1, only the *financially unconstrained* occurs. The economy fluctuates—because of variations in the aggregate quantity of the asset k—but no deviation from linear trend k occurs. Thus, there is no notion of an economic or of a financial cycle.

3.2 Diagnostic Expectations but No Financial Frictions

Consider next an economy with diagnostic expectations and without financial frictions. That is, the extrapolation weight $\hat{\mu} > 0$ is positive, net worth $n \in \mathbb{R}$ can be negative, and leverage limit $\lambda = +\infty$ is unbounded.

Corollary 2. *In the economy with diagnostic expectations and without financial frictions, sentiment ω is the only relevant state that affects the equilibrium outcome. A threshold state $\bar{\omega} < 0$ exists such that*

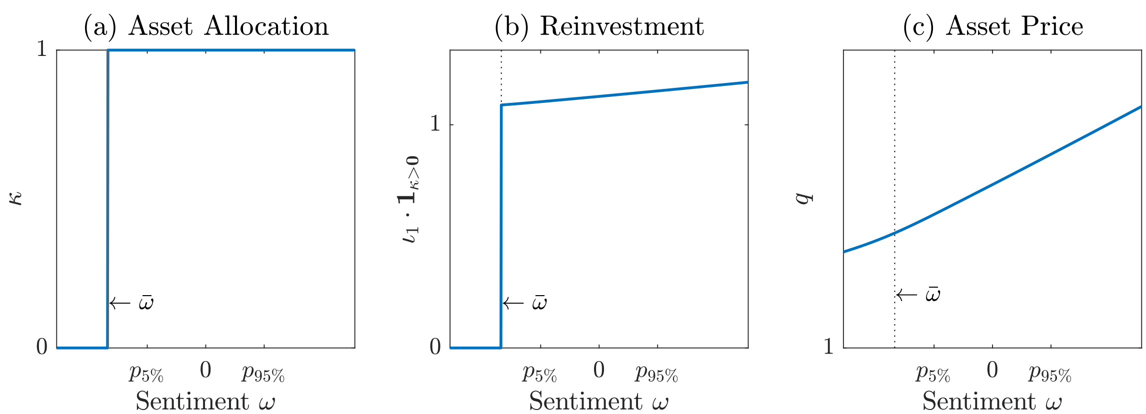
$$\begin{array}{ll} \textit{if}\ \omega < \bar{\omega} \Rightarrow & \kappa = 0 \ , \ \ \alpha_1 < 0 \ , \ \ \alpha_2 = 0 \ ; \\ \textit{if}\ \omega > \bar{\omega} \Rightarrow & \kappa = 1 \ , \ \ \alpha_1 = 0 \ , \ \ \alpha_2 < 0 \ ; \end{array} \ . \tag{31}$$

The threshold state $\bar{\omega} < 0$ is the solution to

$$\alpha_1 = \alpha_2 = 0 \Rightarrow \frac{A_1 - \iota_1 - A_2}{q} + \mathcal{I}_1(\iota_1) + (\sigma_q + \hat{\mu}\omega)\,\sigma_1 = 0 \ . \tag{32}$$

The equilibrium outcome features two well-demarcated regimes. When $\omega < \bar{\omega}$, the economy operates in a *precautionary* regime, in which the aggregate quantity of the asset is allocated to the unproductive technology and the asset is priced according to $\alpha_2 = 0$. In this regime, households are the marginal buyers of the asset, whereas financiers strictly prefer to acquire debt rather than to operate the productive technology. Output flows $y/k = A_2 < A_1$ are low, as are asset price q, aggregate growth rate $\mu_k = 0$, and aggregate risk $\sigma_k = 0$. By contrast, when $\omega > \bar{\omega}$, the economy operates in a *non-precautionary* and *financially unconstrained* regime, in which the aggregate quantity of the asset is allocated to the productive technology, financiers are the marginal buyers of the asset, and households strictly prefer to hold debt rather than to operate the unproductive technology. Formally, $\kappa = 1$ and $\alpha_1 = 0$. In this case, aggregate output flows $y/k = A_1$, asset price q, aggregate growth rate $\mu_k = \mathcal{I}_1(\iota_1)$, and aggregate risk $\sigma_k = \sigma_1$ are high. The outcome repeatedly alternates between these two regimes according to the law of motion (29). This law of motion generates a stationary distribution of sentiment of $\omega \sim \mathcal{N}[0, 1/(2\delta)]$.[11]

FIGURE 1: DIAGNOSTIC EXPECTATIONS AND NO FINANCIAL FRICTIONS



Notes: The figure plots the allocation of the asset between the technologies (panel a), the reinvestment rate conditional on a positive allocation of the asset to the productive technology (panel b), and the asset price (panel c) as a function of the relevant state of the economy, i.e., sentiment ω. Variables are normalized by their respective value in the economy in section 3.1. Threshold $\bar{\omega} < 0$ separates the precautionary and the non-precautionary regimes. Point $p_{x\%}$ in the x-axis, with $x \in \{5; 95\}$, indicates the $x\%$-percentile of the limiting distribution of sentiment.

This economy exhibits recurrent boom-bust cycles in aggregate output, asset prices, and economic growth rates. The driver of the cycles is sentiment. Risk-taking in real investments κ, price q, growth rate $\mu_k = \kappa\mathcal{I}_1(\iota_1)$, and aggregate risk $\sigma_k = \kappa\sigma_1$ are pro-cyclical.

[11]The regimes correspond to the first and third, respectively, in expression (21). The threshold state that separates the two regimes is negative because (i) the asset price is positively related to sentiment—that is, $\sigma_q > 0$—and (ii) the productive technology is riskier but yields higher dividend returns than the unproductive technology.

Forecast errors $-\hat{E}[dZ] = -\hat{\mu}\omega \neq 0$ are instead counter-cyclical. Because of these errors, the persistence of sentiment and the tails of its stationary distribution are perceived to be larger than what they actually are. Put formally, under diagnostic expectations, sentiment is perceived to fluctuate according to $d\omega = -(\delta - \hat{\mu})\,\omega dt + dZ$ and $\omega \sim \mathcal{N}[0, 1/(2(\delta - \hat{\mu}))]$. Lastly, the pro-cyclicality of the asset price strengthens a positive interaction between the price and reinvestment when sentiment is sufficiently high. This effect contributes to generate higher asset prices and reinvestment rates relative to their corresponding levels in the first economy (Figure 1, panel c).[12]

3.3 Financial Frictions but Rational Expectations

Now consider an economy with rational expectations and financial frictions. That is, expectation weight $\hat{\mu} = 0$ is null, net worth $n \geq 0$ cannot be negative, and leverage limit $\lambda < +\infty$ is bounded.

Corollary 3. *In the economy with rational expectations and financial frictions, wealth share η is the only relevant state that affects the equilibrium outcome. A threshold state $\bar{\eta} \in (0, 1)$ exists such that*

$$\begin{array}{llll} \text{if } \eta < \bar{\eta} \Rightarrow & \kappa = \lambda\eta < 1\,, & \alpha_1 > 0\,, & \alpha_2 = 0\,; \\ \text{if } \eta > \bar{\eta} \Rightarrow & \kappa = 1\,, & \alpha_1 = 0\,, & \alpha_2 < 0\,; \end{array} . \tag{33}$$

The threshold state $\bar{\eta} \in (0, 1)$ is the solution to

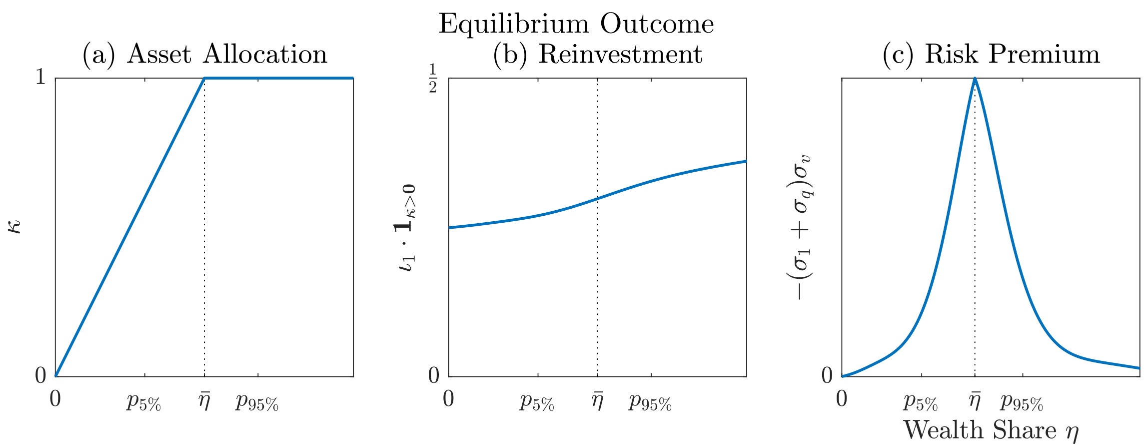
$$\lambda\eta = 1 \Leftrightarrow \eta = \frac{1}{\lambda}\,. \tag{34}$$

As in the economy in subsection 3.2, the equilibrium outcome features two well-delimited regimes. In contrast to that economy, however, the regimes are inherently determined by

[12] In Figure 1, the price and reinvestment exceed their levels in the first economy when sentiment is low. This effect is a consequence of an asymmetric effect of sentiment on the price. High sentiment exerts upward pressure on the price, whereas low sentiment exerts downward pressure. These pressures are not only exerted on impact, but also effective throughout the state space. The reason is that the price is forward-looking. The upward pressure is relatively stronger, nonetheless, because when sentiment is low, the aggregate quantity of the asset is allocated to the unproductive technology, which eliminates the exposure of the asset to quality risk as well as the direct negative effect of low sentiment on the price. Under the baseline parameter values, the upward pressure is sufficiently strong to support a price above its level in the first economy throughout the grid of sentiment used in the numerical solution.

the financial capacity of financiers to acquire assets and operate the productive technology. Specifically, when $\lambda\eta < 1$, the economy operates in a *financially constrained* regime in which financiers are constrained by their leverage limit to acquire assets. Accordingly, households hold the remnant share of the asset and are the marginal buyers. That is, $\kappa = \lambda\eta < 1$ and $\alpha_2 = 0 < \alpha_1$ (Figure 2, panel (a)). By contrast, when $\lambda\eta \geq 1$, the economy operates in a *financially unconstrained* regime in which financiers are marginal buyers, hold the aggregate quantity of the asset, and households only hold debt issued by financiers.

Figure 2: Rational Expectations and Financial Frictions

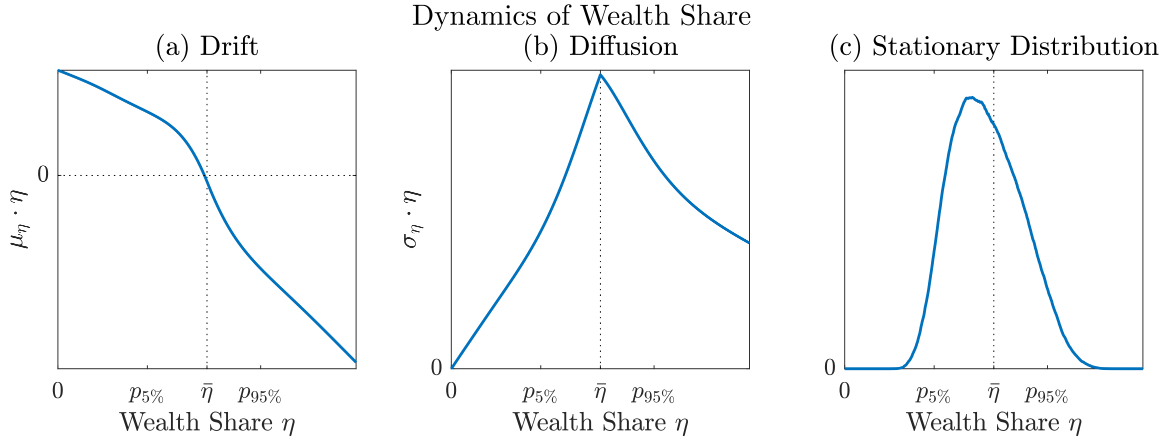


Notes: The figure plots the allocation of the asset between the technologies (panel a), the reinvestment rate conditional on a positive allocation of the asset to the productive technology (panel b), and the price of risk (panel c) as a function of the relevant state of the economy, i.e., financiers' wealth share η. The variables in the figure (except the risk premium) are normalized by their values in the economy in section (3.1). Threshold $\bar{\eta}$ separates the financially constrained and the financially unconstrained regimes. Point $p_{x\%}$ indicates the x%-percentile of the limiting distribution of the wealth share of financiers.

Aggregate output and the asset price are increasing in the wealth share—as is investment rate ι_1—because financiers operate the productive technology. By contrast, value v is decreasing in the wealth share, because the rents from operating the productive technology are positive $\alpha_1\lambda > 0$ when the wealth share is low $\eta < 1/\lambda$, and null otherwise. A counter-cyclical marginal net worth v creates a negative risk-premium term in (19), $(\sigma_q + \sigma_1)\,\sigma_v \leq 0$, which reflects financiers' effective risk aversion in the presence of financial constraints (Figure 2, panel (c)).

The equilibrium outcome repeatedly alternates between the two financial regimes (Fig-

Figure 3: Rational Expectations and Financial Frictions



Notes: The figure plots the drift (panel a), the diffusion (panel b), and the limiting distribution (panel c) of the wealth share that satisfies (26) $d\eta = \mu_\eta \eta dt + \sigma_\eta \eta dZ$. Threshold $\bar{\eta}$ separates the financially constrained and the financially unconstrained regimes. Point $p_{x\%}$ indicates the x%-percentile of the limiting distribution of the wealth share of financiers.

ure 3) according to the law of motion (26). Fluctuations display two properties. First, fluctuations are mean reverting around a stochastic steady state (i.e., η such that $\mu_\eta \eta = 0$—panel a). This is a consequence of the counter-cyclicality of rents $\alpha_1 \lambda$ and the a-cyclicality of dividend payouts. Second, fluctuations are stochastic (panel b), which is a consequence of a positive interaction between net-worth risk $\sigma_\eta = (\phi - 1)\,\sigma_q + (1-\eta)\,\phi\sigma_1$ and price risk $\sigma_q = \varepsilon_q \sigma_\eta$, where $\varepsilon_q \equiv (\partial q / \partial \eta)\,(\eta/q) \geq 0$ is the elasticity of the asset price with respect to the wealth share.[13] This interaction generates endogenous financial amplification of disturbances to the wealth share and the asset price according to

$$\frac{\sigma_\eta}{\sigma_1} = \frac{\phi - \phi\eta}{1 - (\phi - 1)\,\varepsilon_q} \geq 0 \quad \text{and} \quad \frac{\sigma_q}{\sigma_1} = \frac{(\phi - \phi\eta)\,\varepsilon_q}{1 - (\phi - 1)\,\varepsilon_q} \geq 0, \text{ with } \phi = \min\left\{\frac{1}{\eta}, \lambda\right\}. \tag{35}$$

In the financially constrained regime, notably, this amplification is characterized by fire sales and a reallocation of the asset from the productive to the unproductive technology. Indeed, when negative disturbances erode the wealth share, i.e., $d\eta < 0$, financiers are forced to sell the asset to households at discount price to meet a tighter collateral constraint.

All in all, like its counterpart in subsection 3.2, this economy exhibits recurrent boom-bust cycles in aggregate output, asset prices, and economic growth rates. Risk-taking in

[13]Formula $\sigma_q = \varepsilon_q \sigma_\eta$ follows from Ito's Lemma.

real investments κ, price q, growth rate $\mu_k = \kappa \mathcal{I}_1(\iota_1)$, and aggregate risk $\sigma_k = \kappa\sigma_1$ are also pro-cyclical. By contrast, sentiment does not influence economic cycles, and forecasts are not subject to systematic errors. Rather, the wealth share of financiers is the driver of the cycles, and conditional forecast errors on average are null. Finally, these cycles feature fire sales and asset reallocation with recessionary implications when the collateral constraint is binding and negative disturbances hit the technologies. Endogenous risk is time-varying and peaks when the collateral constraint is locally occasionally binding.

3.4 Diagnostic Expectations and Financial Frictions

Finally, consider the whole economy presented in section 2, with diagnostic expectations and financial frictions. The following corollary from proposition 1 describes the equilibrium outcome.

Corollary 4. *In the economy with diagnostic expectations and financial frictions, both sentiment ω and wealth share η affect the equilibrium outcome. Thresholds $\bar{\omega} < 0$ and $\bar{\eta} \in (0,1)$ partition the state space as follows:*

$$\begin{aligned} &\text{if } \omega < \bar{\omega} \Rightarrow && \kappa = 0\,, && \alpha_1 < 0\,,\ \alpha_2 = 0\,; \\ &\text{if } \omega > \bar{\omega} \text{ and } \eta < \bar{\eta} \Rightarrow && \kappa = \lambda\eta\,, && \alpha_1 > 0\,,\ \alpha_2 = 0\,; \\ &\text{if } \omega > \bar{\omega} \text{ and } \eta > \bar{\eta} \Rightarrow && \kappa = 1\,, && \alpha_1 = 0\,,\ \alpha_2 < 0\,; \end{aligned} \tag{36}$$

Threshold process $\bar{\omega}$ is the solution to

$$\alpha_1 = \alpha_2 = 0 \Rightarrow \frac{A_1 - \iota_1 - A_2}{q} + \mathcal{I}_1(\iota_1) + (\sigma_q + \hat{\mu}\omega)\,\sigma_1 + (\sigma_q + \sigma_1)\,\sigma_v = 0\,. \tag{37}$$

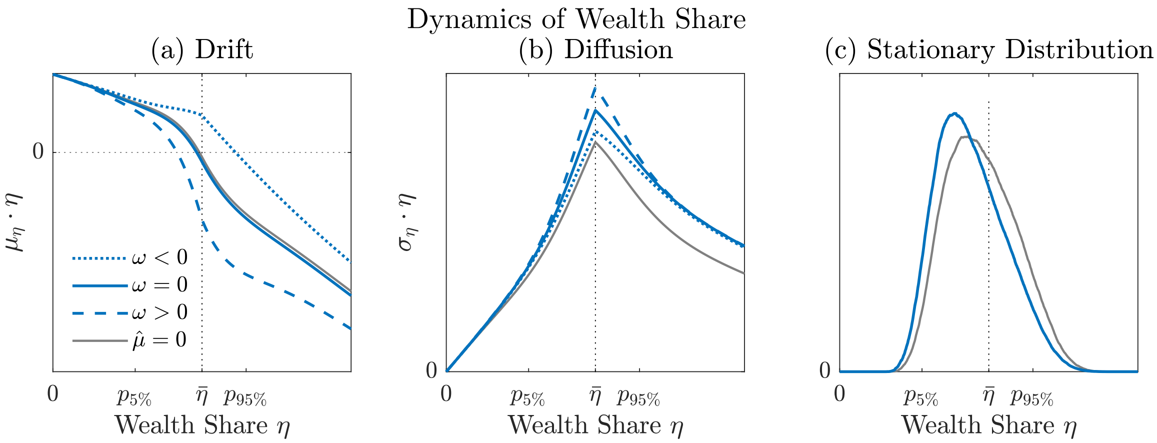
and threshold $\bar{\eta}$ is the solution to

$$\lambda\eta = 1 \Leftrightarrow \eta = \frac{1}{\lambda}\,. \tag{38}$$

With financial frictions, the characterization of sentiment threshold $\bar{\omega}$ includes a risk-premium term $(\sigma_q + \sigma_1)\,\sigma_v \leq 0$. Everything else being the same, this term reduces the perceived relative value of the productive technology, because operating that technology exposes de facto risk-averse financiers to aggregate risk. The equilibrium outcome repeatedly alternates among the aforementioned three regimes—*precautionary*, *financially constrained*, and *financially unconstrained*—according to the laws of motion (26) and (29).

In these cycles, sentiment and the wealth share co-move positively, because current disturbances positively affect both the perceived likelihood of future disturbances and the excess returns earned by financiers—as respectively shown by equations (29) and (35).

FIGURE 4: DIAGNOSTIC EXPECTATIONS AND FINANCIAL FRICTIONS



Notes: The figure plots the drift (panel a), the diffusion (panel b), and the limiting distribution (panel c) of the wealth share that satisfies (26) $d\eta = \mu_\eta \eta dt + \sigma_\eta \eta dZ$. Grey lines refer to the economy with rational expectations (i.e., section 3.3), whereas blue lines refer to the economy with diagnostic expectations (i.e., section 3.4). For the latter economy, variables are plotted for three different values of sentiment.

Figure (4) highlights the key takeaways from this economy and our main positive result: diagnostic expectations intensify financial instability relative to the rational expectation benchmark—as measured by a leftward shift in the stationary marginal distribution of the wealth share of financiers (Figure 4, panel c).[14] This additional instability results from the following two interactions between diagnostic expectations and financial frictions.

First, relative to an environment with rational expectations, the positive co-movement between sentiment and the wealth share strengthens the positive interaction between risks σ_η and σ_q. Following adverse disturbances, for instance, the fall in sentiment further depresses the asset price, which further deteriorates the wealth share, thus intensifying the interaction (panel b).[15] In the financially constrained regime, this implies stronger fire

[14]We also verify that the stationary distribution under rational expectations first-order stochastic dominates the stationary distribution under diagnostic expectations, namely, that the probability that financial conditions are lower than any $\eta \in (0, 1)$ is higher under the latter expectations.

[15]Put more formally, $dZ < 0$ exerts downward pressure on ω on impact, which reduces the first term in $\sigma_q = \varepsilon_{q,\omega}/\omega + \varepsilon_{q,\eta}\sigma_\eta$, with $\varepsilon_{q,\omega} \equiv (\partial q/\partial \omega)(\omega/q) \geq 0$ and $\varepsilon_{q,\eta} \equiv (\partial q/\partial \eta)(\eta/q) \geq 0$. This reduction, in turn, intensifies the interaction between the first term in $\sigma_\eta = (\phi - 1)\sigma_q + \phi\sigma_1 - \sigma_k$ and σ_q.

sales and more recessionary asset reallocation when sufficiently large adverse disturbances occur.

Second, everything else being the same, when both sentiment and financiers' wealth share are high, negative forecast errors $-\hat{E}[dZ|\cdot] = -\hat{\mu}\omega$ exert upward pressure on the asset price. The higher price then deteriorates excess return $E[dR_1 - r|\cdot]$, which eventually hurts the profitability of financiers—as measured by conditional average growth rate $\mu_\eta = E[d\eta/\eta|\cdot]$. The opposite naturally happens when sentiment and the wealth share are instead low. In contrast to Maxted (2023), however, because forecast errors and aggregate risk $\sigma_k = \kappa\sigma_1$ are negatively related, the former effects dominate (panel a). This asymmetry then exerts leftward pressure on the stationary marginal distribution of the wealth share relative to the rational expectation benchmark.

Overall, this economy features both systematic forecast errors and financial amplification effects. Relative to the economy with rational expectations, fire sales events are stronger and reallocation of assets across technologies is swifter. Moreover, because of counter-cyclical forecast errors and pro-cyclical risk-taking, financial markets are more unstable and economic cycles are more volatile. These results are closely in line with the view espoused by the FIH.

4 Alternative Specifications

The key takeaway from the positive analysis is that diagnostic expectations intensify financial instability relative to the rational expectations benchmark. In this section, we show this finding is robust to alternative processes of diagnostic expectation formation (subsection 4.1) and to other types of collateral constraints (subsection 4.2). These alternative specifications are interesting on their own because they identify additional channels of interaction between extrapolative expectations and financial frictions. Lastly, in subsection 4.3, we investigate the extent to which differences in production technologies contribute to the amplification effects of diagnostic expectations on financial instability.

4.1 Generic Processes in Diagnostic Expectation Formation

In the baseline model, agents rely on a weighted average of past disturbances to form expectations about future disturbances. In addition, agents use those expectations to estimate any moment of every other future random variable. We now consider a more general process of diagnostic expectation formation, in which agents rely on a generic Ito path $\{dX_s\}_{s<t}$ to form expectations about a generic Ito variable dY_t.

Proposition 3. *If agents rely on Ito path $\{dX_s\}_{s<t}$ to form diagnostic expectations about Ito variable dY_t, the implied diagnostic expectation operator over disturbance dZ_t is*

$$\hat{E}_t\left[dZ_t\right] = \hat{\mu}\frac{\omega_t}{\sigma_{Y,t}Y_t}dt \ , \tag{39}$$

where $\sigma_{Y,t} \in \mathbb{R}$ is the diffusion of the variable and where sentiment $\omega_t \in \mathbb{R}$ is given by

$$\omega_t = \int_0^t e^{-\delta(t-s)}dX_s \ . \tag{40}$$

This proposition shows how a generic expectation process maps into a forecast operator for disturbances and an Ito process for sentiment. This result thus allows us to consider alternative specifications for diagnostic expectations, for instance, one in which agents use past portfolio returns to form expectations about future investment returns, as in Barberis et al. (2015).

In what follows, we consider an intuitive case in which agents rely on past forecast errors to form expectations about future returns. Formally, let $dX_s = \frac{dR_{1,s} - \hat{E}_s[dR_{1,s}]}{Std_s[dR_{1,s}]}$ be a process for risk-adjusted forecast errors (RAFE) and let $dY_t = dR_{1,t}$ be the return of the productive technology. Then, applying results from proposition 3, one gets the following forecast operator for disturbances and law of motion of sentiment.

Corollary 5. *If $dX_s = \frac{dR_{1,s} - \hat{E}_s[dR_{1,s}]}{Std_s[dR_{1,s}]}$ and $dY_t = dR_{1,t}$, then*

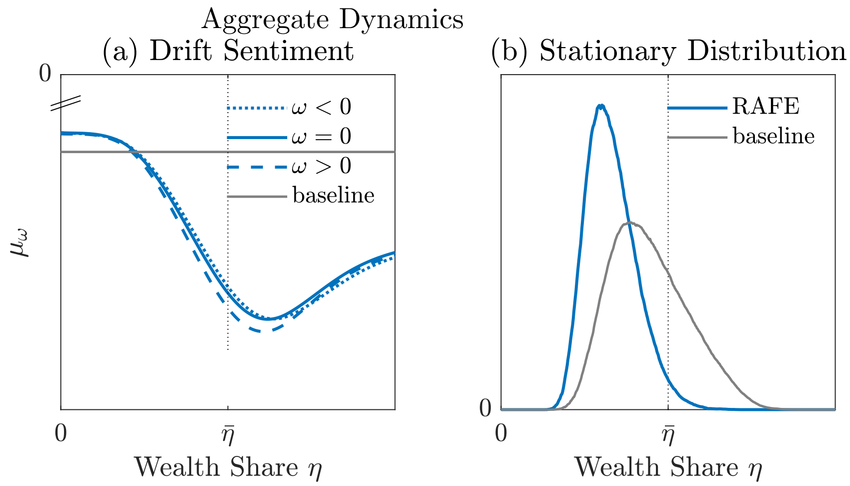
$$\hat{E}_t\left[dZ_t\right] = \hat{\mu}\omega_t dt \ , \tag{41}$$

and

$$d\omega_t = \left(-\delta + \frac{\hat{\mu}}{\sigma_{q,t} + \sigma_1}\right)\omega_t dt + dZ_t \ . \tag{42}$$

The operator in this corollary is the same as in the baseline specification because forecast errors are deflated by risk. The law of motion of sentiment is also mean-reverting as in the baseline, but under the RAFE specification, the drift term of sentiment is endogenous. In particular, the drift is inversely related to the volatility of the asset price. This relationship creates an additional interaction between diagnostic expectations and financial frictions. Specifically, everything else being the same, the greater the volatility of the asset price, the more unstable extreme values of sentiment are (in the sense that extreme sentiment tends to revert quicker to the unconditional mean). Numerical simulations reported in Figure 5 show this relationship further contributes to financial instability. This happens because asset price volatility tends to positively co-move with sentiment and the wealth share (panel a).

Figure 5: RAFE-based Specification for Diagnostic Expectations



Notes: The figure plots the drift of sentiment as a function of the wealth share for different values of sentiment (panel a) and the limiting marginal distribution of the wealth share (panel b). Grey lines refer to the economy with the baseline specification for diagnostic expectations (i.e., section 3.4), whereas blue lines refer to the economy with the specification based on risk-adjusted forecast errors (RAFE). In the RAFE specification, we set $\delta = 7.5$ to match the same correlation between sentiment and the wealth share as in the baseline.

4.2 Endogenous Limit on Leverage

In the baseline model, the collateral constraint creates an exogenous limit on leverage, $\phi_t \leq \lambda$. The literature—notably, Gertler and Kiyotaki (2010) and Gertler and Karadi (2011)—has instead considered a constraint that generates an endogenous leverage limit $\phi_t \leq \nu v_t$, where $\nu \geq 1$ is a parameter and $v_t \geq 1$ is the marginal value of net worth. This constraint is derived from an agency problem, where upon default, financiers lose access to their company, whose value is V_t. In this subsection, we consider this alternative specification and examine its possible implications for financial stability.

An endogenous leverage limit νv_t creates an additional interaction between diagnostic expectations and financial frictions. In particular, everything else being the same, higher sentiment ω_t improves perceived rents $\alpha_{1,t}\phi_t > 0$, increases marginal value v_t, and thus relaxes leverage constraint $\phi_t \leq \nu v_t$. Lower sentiment naturally does the opposite. This additional interaction has two consequences on the allocation of the asset. First, in the financially constrained regime, higher sentiment increases the share of the asset allocated to the productive technology. Second, outside the precautionary regime, higher sentiment reduces the threshold state that separates the two financial regimes.

To study how these effects shape the equilibrium outcome and influence financial stability, we set $\nu = 2.7$, to target for comparability the same average leverage as in the baseline specification. Table 2 compares key equilibrium moments between the two specifications. The main takeaway is that endogenous leverage limit νv_t does not significantly change the equilibrium outcome or affect financial stability. This happens mainly because rents $\alpha_{1,t}\phi_t$ are weakly correlated with sentiment ω_t and because the average value of the rents is low. These two features combined render value v_t not sensitive to sentiment, as reflected by an average elasticity $E[\varepsilon_{v,\omega}] = -0.4\%$, where $\varepsilon_{v,\omega} \equiv \frac{\partial v}{\partial \omega}\frac{\omega}{v}$. These results thus reveal the amplifying effects of diagnostic expectations on financial instability primarily operate through sentiment-based fluctuations in asset prices, not through sentiment-driven variations in leverage limits.

4.3 Differences between Technologies

Lastly, the baseline model proposes three sources of differences between the technologies: productivity A_j, reinvestment opportunity $\mathcal{I}_j(\iota_j)$, and exposure to capital-quality risk σ_j. This section investigates the sensitivity of the positive results to each of these elements. To do so, we derive the equilibrium for three alternative specifications, each of which shuts down one of these differences. (Figure 6).

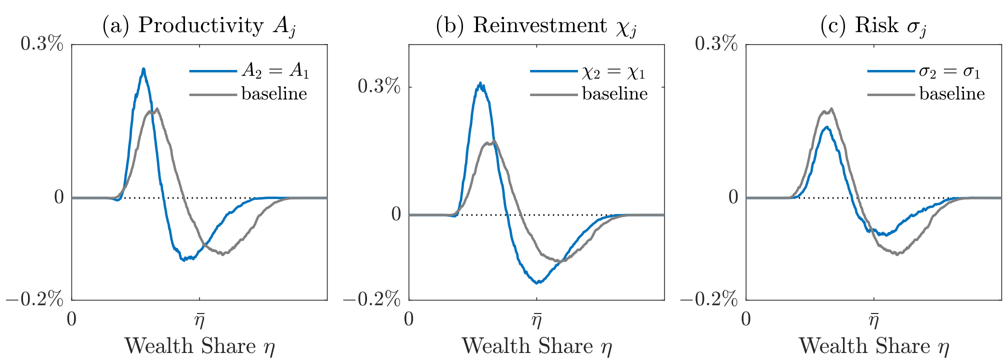
Closing either the productivity gap or the reinvestment gap, that is, $A_2 = A_1$ or $\chi_2 = \chi_1$, does not substantially affect the contribution of diagnostic expectations to financial instability. By contrast, closing the gap in risk exposure, that is, $\sigma_2 = \sigma_1$, eliminates most of it. These results thus further highlight the joint importance of counter-cyclical forecast errors and pro-cyclical aggregate risk for the positive contribution of diagnostic expectations to financial instability.

Table 2: Exogenous vs. Endogenous Collateral Constraint

Moment	Exogenous $\phi_t \leq \lambda$		Endogenous $\phi_t \leq \nu v_t$	
	$\hat{\mu} = 0$	$\hat{\mu} > 0$	$\hat{\mu} = 0$	$\hat{\mu} > 0$
Unconditional average:				
Allocation Asset κ	0.879	0.830	0.864	0.820
Reinvestment $\kappa \cdot \iota_1$	0.255	0.272	0.255	0.254
Price q	0.543	0.568	0.541	0.554
Leverage ϕ	3.765	3.828	3.704	3.695
Wealth share η	0.238	0.221	0.240	0.226
Elasticity value $\varepsilon_{v,\omega}$	0	-0.004	0	-0.004
Unconditional frequency:				
Precautionary regime	0	0.002	0	0.003
Constrained regime	0.583	0.689	0.801	0.861
Unconstrained regime	0.417	0.309	0.199	0.046

Notes: The table reports selected unconditional moments for the baseline specification with exogenous collateral constraint and for the specification with endogenous constraint as in Gertler and Kiyotaki (2010) and Gertler and Karadi (2011)

Figure 6: Differences between Technologies



Notes: The figure plots the difference between the limiting marginal distributions of the wealth share under diagnostic expectations and rational expectations. Grey lines refer to the economy with the baseline specification (i.e., section 3.4), whereas blue lines refer to the economies with the alternative specifications presented in section 4.3.

5 Normative Analysis

The positive analysis highlighted how diagnostic expectations intensify financial instability relative to the rational expectations benchmark. In this section, we study the normative implications of the resulting additional instability. To do so, we characterize a socially optimal allocation and compare its properties with the equilibrium allocations presented in section 3.

5.1 The Socially Optimal Allocation

To characterize this allocation, we consider a social welfare problem that is consistent with the incentive constraints of private agents and satisfies the resource constraints of the competitive equilibrium. In addition, for tractability, we impose the following three restrictions. First, the social welfare problem has a Markov structure with the same state variables as in the competitive equilibrium. Second, the social planner—who determines the allocation—does not have commitment, meaning social welfare is maximized state by state, taking the future paths of the socially optimal allocation as given. Lastly, the planner evaluates social welfare using an expectation weight $\tilde{\mu}$ that lies in the interval $[0, \hat{\mu}]$. Intuitively, under the first two restrictions, the planner can solve any coordination problem among private agents that may arise in a (Markov) competitive equilibrium within a time instant or a short time period, but she cannot solve coordination problems that may arise at more distant time horizons. The last restriction imposes reasonable expectations on the planner in the sense that her expectations can be more rational but not more diagnostic than those of private agents.

The definition below formally specifies the socially optimal allocation.

Definition 2. *The socially optimal allocation is the solution to the optimization problem in the following dynamic program:*

$$
\begin{aligned}
r\tilde{u} \;=\; \max_{\{\iota_1,\kappa\}} \Bigg\{ & \kappa \left\{ A_1 - \iota_1 + \left[\mathcal{I}_1(\iota_1) + \sigma_1 \tilde{\mu}\omega\right] \tilde{u} \right\} + (1-\kappa) A_2 + \frac{\partial \tilde{u}}{\partial \omega} \left(-\delta\omega + \tilde{\mu}\omega + \kappa\sigma_1\right) \\
& + \frac{\partial \tilde{u}}{\partial \eta} \left(\mu_\eta \eta + \sigma_\eta \eta \tilde{\mu}\omega + \sigma_\eta \eta \kappa \sigma_1\right) + \frac{1}{2} \frac{\partial^2 \tilde{u}}{(\partial\omega)^2} + \frac{\partial^2 \tilde{u}}{\partial\omega\partial\eta} \sigma_\eta \eta + \frac{1}{2} \frac{\partial^2 \tilde{u}}{(\partial\eta)^2} \left(\sigma_\eta \eta\right)^2 \Bigg\} ,
\end{aligned} \tag{43}
$$

with

$$
\iota_1 \in [0, A_1] \;\; \text{and} \;\; \kappa \in [0, \min\{\lambda\eta, 1\}] , \tag{44}
$$

where drift μ_η and diffusion σ_η are given by

$$\mu_\eta = \frac{1}{1-(\frac{\kappa}{\eta}-1)\varepsilon_{q,\eta}}\left\{\left[\frac{A_1-\iota_1}{q}+\mathcal{I}_1(\iota_1)+\sigma_q\sigma_1\right]\frac{\kappa}{\eta}-\kappa\mathcal{I}_1(\iota_1)-\sigma_q\kappa\sigma_1+\right. \tag{45}$$
$$+\frac{1}{q}\left[-\frac{\partial q}{\partial\omega}\delta\omega+\frac{1}{2}\frac{\partial^2 q}{(\partial\omega)^2}+\frac{\partial^2 q}{\partial\omega\partial\eta}\sigma_\eta\eta+\frac{1}{2}\frac{\partial^2 q}{(\partial\eta)^2}(\sigma_\eta\eta)^2-rq\right]\left(\frac{\kappa}{\eta}-1\right)+$$
$$\left.+(\sigma_q+\kappa\sigma_1)\left[(\sigma_q+\kappa\sigma_1)-\frac{\kappa}{\eta}(\sigma_q+\sigma_1)\right]-\left(\theta-\frac{\gamma}{\eta}\right)\right\},$$
$$\sigma_\eta = \frac{\frac{\kappa}{\eta}\left(\frac{1}{q}\frac{\partial q}{\partial\omega}+\sigma_1\right)-\left(\frac{1}{q}\frac{\partial q}{\partial\omega}+\kappa\sigma_1\right)}{1-\left(\frac{\kappa}{\eta}-1\right)\varepsilon_{q,\eta}}, \tag{46}$$

respectively, with

$$\sigma_q = \frac{\left(\frac{\kappa}{\eta}-\kappa\right)\varepsilon_{q,\eta}\sigma_1+\frac{1}{q}\frac{\partial q}{\partial\omega}}{1-\left(\frac{\kappa}{\eta}-1\right)\varepsilon_{q,\eta}}. \tag{47}$$

Mapping q is consistent with the following three mutually exclusive relationships:

$$\begin{array}{lllll} \textit{Relationship \#1:} & \lambda\eta\geq 1\,, & \kappa=1\,, & \alpha_1=0\,, & \alpha_2<0\,; \\ \textit{Relationship \#2:} & \lambda\eta\geq 1\,, & \kappa\in[0,1)\,, & & \alpha_2=0\,; \\ \textit{Relationship \#3:} & \lambda\eta<1\,, & \kappa\in[0,\lambda\eta]\,, & & \alpha_2=0\,; \end{array}\quad , \tag{48}$$

with

$$\alpha_1\equiv\frac{1}{dt}\hat{E}\left[dR_1|\cdot\right]-r+(\sigma_q+\sigma_1)\sigma_v\,,\quad \alpha_2\equiv\frac{1}{dt}\hat{E}\left[dR_2|\cdot\right]-r\,. \tag{49}$$

Mapping v satisfies

$$0=\alpha_1\frac{\kappa}{\eta}+\mu_v+\hat{\mu}\omega\sigma_v+\frac{\theta}{v}-\theta\,. \tag{50}$$

In the above notations, mapping $\tilde{u}\geq 0$ is the present discounted value of consumption per unit of the asset under expectation weight $\tilde{\mu}\in[0,\hat{\mu}]$. Mappings $\{q,v\}$ and objects $\{\mu_\eta,\sigma_\eta,\sigma_q,\mu_v,\sigma_v,\alpha_1,\alpha_2\}$ are "shadow" variables, corresponding to the respective variables in a decentralization of the socially optimal allocation as a competitive equilibrium.[16]

In the optimization problem, controls $\{\iota_1,\kappa\}$ are set state by state, to maximize the utility of households—as measured under the expectation weight of the planner. Restric-

[16]The socially optimal allocation can be decentralized as a competitive equilibrium using Pigouvian taxes or subsidies on reinvestment and the productive technology.

tions (44) follow from the resource constraints and the collateral constraint. Mappings $\{\tilde{u}, q, v\}$ and their partial derivatives with respect to the states are taken as given. The reason is that those objects are determined by the future paths of the allocation and thus are not influenced by the current allocation.

By contrast, in the dynamic program, the mappings and their derivatives are endogenous. Expression (48) for mapping q determines three mutually exclusive relationships between share κ and private valuations α_1 and α_2. These relationships feature two differences relative to those in expression (21). First, the planner can set $\kappa > 0$ while $\alpha_1 < 0$, which means the asset can be allocated to the productive technology even when the expected excess return of that technology is negative according to private valuations. Second, the planner can also set $\kappa < \lambda\eta$ while $\alpha_1 > 0$, which implies the collateral constraint can be slack even when the expected excess return of the productive technology is positive according to private valuations.

Proposition 4. *The socially optimal reinvestment rate solves*

$$\mathcal{I}_1'(\iota_1) = \frac{1 + \frac{1}{1-(\phi-1)\varepsilon_{q,\eta}} \frac{1}{q} \frac{\partial \tilde{u}}{\partial \eta}}{\tilde{u} + \frac{1-\eta}{1-(\phi-1)\varepsilon_{q,\eta}} \frac{\partial \tilde{u}}{\partial \eta}} . \tag{51}$$

The socially optimal share maximizes the RHS in (43). The candidate solutions are $\kappa = 0$, $\kappa = \min\{\lambda\eta, 1\}$ *and any interior* $\kappa \in (0, \min\{\lambda\eta, 1\})$ *that solves*

$$\begin{aligned} 0 \;=\; & \left[\frac{A_1 - \iota_1 - A_2}{\tilde{u}} + \mathcal{I}_1(\iota_1) + (\sigma_{\tilde{u}} + \tilde{\mu}\omega)\,\sigma_1\right] + \varepsilon_{\tilde{u},\eta}\left[\frac{\partial \mu_\eta}{\partial \kappa} + (\tilde{\mu}\omega + \kappa\sigma_1)\,\frac{\partial \sigma_\eta}{\partial \kappa}\right] \\ & + \frac{1}{\tilde{u}}\left(\frac{\partial^2 \tilde{u}}{(\partial \eta)^2}\sigma_\eta \eta + \frac{\partial^2 \tilde{u}}{\partial \eta \partial \omega}\right)\frac{\partial \sigma_\eta}{\partial \kappa}\eta , \end{aligned} \tag{52}$$

where $\frac{\partial \mu_\eta}{\partial \kappa}$ *and* $\frac{\partial \sigma_\eta}{\partial \kappa}$ *are the partial derivatives of* μ_η *and* σ_η *with respect to* κ*, respectively.*

These optimality conditions follow from the first-order derivatives in (43) with respect to ι_1 and κ. Comparing these conditions with their counterparts of the competitive equilibrium—that is, (17) and (21)—highlights the following two differences between the socially optimal and the equilibrium allocations. First, the social planner internalizes the collective contribution of individual decisions to aggregate variables and aggregate dynamics, whereas individual agents in the competitive equilibrium do not. In the conditions for reinvestment, for instance, this difference is reflected by the second terms in the numerator and the denominator of the RHS in (51), which are not present in (17). Second, the social planner bases her decisions on the social value of the asset as perceived by her own expectations, namely, $\tilde{u}$, whereas private agents in the competitive equilibrium take decisions based on the asset price, q.

All in all, the optimality conditions in the proposition together with definition 2 analytically characterize the socially optimal allocation and its associated mappings $\{\tilde{u}, v, q\}$. The characterization can be reduced to a system of second-order PDEs for the mappings in states $\{\omega, \eta\}$.

Proposition 5. *The socially optimal allocation and its associated mappings* $\{\tilde{u}, v, q\}$ *is analytically characterized by a system of second-order PDEs for the mappings in the state* $\{\omega, \eta\}$*.*

5.2 The Equilibrium Outcome under the Socially Optimal Allocation

We now contrast the socially optimal allocation with the equilibrium allocation. We report results gradually as in section 3, but we omit the first economy (rational expectations and no financial frictions) because its equilibrium is already first-best. To ease exposition, we only consider planners with expectation weights $\tilde{\mu} = 0$ and $\tilde{\mu} = \hat{\mu}$, who can be regarded as *paternalistic* and *benevolent*, respectively. Planners with intermediate degrees of diagnostic expectations naturally favor allocations in between the ones these two planners implement.

5.2.1 Diagnostic Expectations but No Financial Frictions

Consider first the economy in subsection 3.2. The following corollary from proposition 4 describes the socially optimal allocation.

Corollary 6. *In the economy with diagnostic expectations and without financial frictions, the socially optimal allocation is first-best efficient according to the expectation weight of the planner. If the planner is benevolent, the socially optimal allocation is the same as the equilibrium allocation. If the planner is paternalistic, the socially optimal allocation is the same as the equilibrium allocation of the economy presented in subsection 3.1.*

In this economy, the planner can implement the allocation that, according to her own expectations, attains the first-best. This is because the economy has no friction. Naturally, for a benevolent planner, the equilibrium allocation with sentiment-driven economic cycles (section 3.2) is already first-best efficient. By contrast, for a paternalistic planner, the equilibrium allocation of the economy with rational expectations (section 3.1) is the desired first-best outcome.

Under paternalism, the socially optimal allocation is thus insulated from sentiment. Put differently, the aggregate quantity of the asset is allocated to the productive technology even when sentiment $\omega < \bar{\omega}$ is low, and reinvestment is invariant to fluctuations in sentiment even when sentiment $\omega \geq \bar{\omega}$ is high. Therefore, the socially optimal allocation eliminates fluctuations in aggregate output, reinvestment, and economic growth rates, but relative to the competitive equilibrium, it intensifies fluctuations in the asset price. The latter happens because the sensitivity of the price to sentiment is larger if the asset is always exposed to the shock—as under the socially optimal allocation—than what it is if the asset is exposed only when sentiment is high—as under the equilibrium allocation.

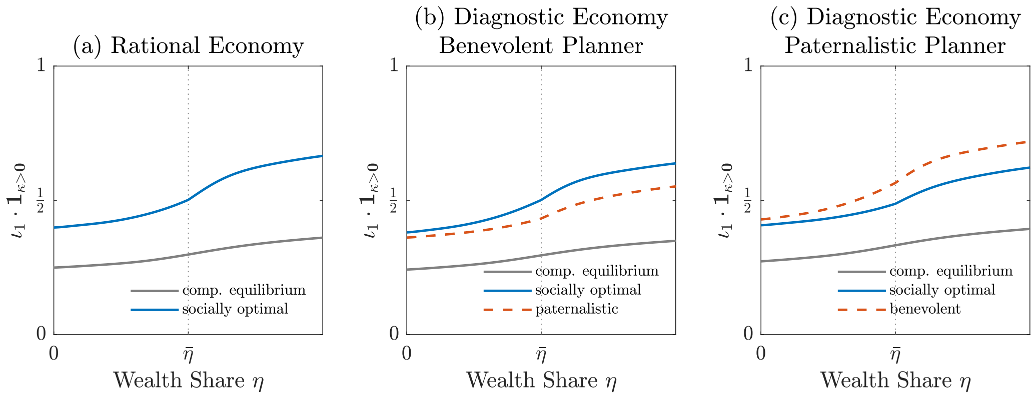
5.2.2 Financial Frictions but Rational Expectations

Consider now the economy in subsection 3.3. Because of financial frictions, the planner cannot implement the allocation that, according to her rational expectations, attains the first-best. Notwithstanding, in general, the planner can improve social welfare over the competitive equilibrium. This is because the collateral constraint together with non-contingent debt depress the asset price—and thus also reinvestment—excessively relative to what is socially desirable (Figure 7).

Specifically, relative to the competitive equilibrium, the socially optimal allocation features higher reinvestment throughout the cycle. These higher rates speeds up the average recapitalization of financiers (i.e., drift $\mu_\eta \eta$), which increases the relative frequency of larger wealth shares in the stationary distribution. However, at least under the baseline parameter values, the socially optimal allocation does not alter the allocation of the asset between the technologies. That is, $\kappa = \min\{\lambda\eta, 1\}$ is optimal for the planner.[17] The latter result is modified, nonetheless, once we allow for diagnostic expectations.

[17] In a similar economy with financial frictions and rational expectations, Brunnermeier and Sannikov (2014) also find null to negligible welfare gains from altering the asset allocation relative to the competitive equilibrium. By contrast, Van der Ghote (2021) finds large welfare gains, but his economy features concavity in preferences over consumption, which creates gains from reducing consumption volatility and smoothing consumption over time.

Figure 7: Socially Optimal Reinvestment Rates



Notes: The figure plots the reinvestment rates of the competitive equilibrium allocation (grey lines) and the socially optimal allocation (blue lines) for the economy with rational expectations (panel a), the economy with diagnostic expectations and a benevolent planner (panel b), and the economy with diagnostic expectations and a paternalistic planner (panel c). For the latter two economies, the socially optimal reinvestment rates of the paternalistic and the benevolent planner, respectively, are also plotted. All of the reinvestment rates are deflated by the first-best value in the corresponding economy.

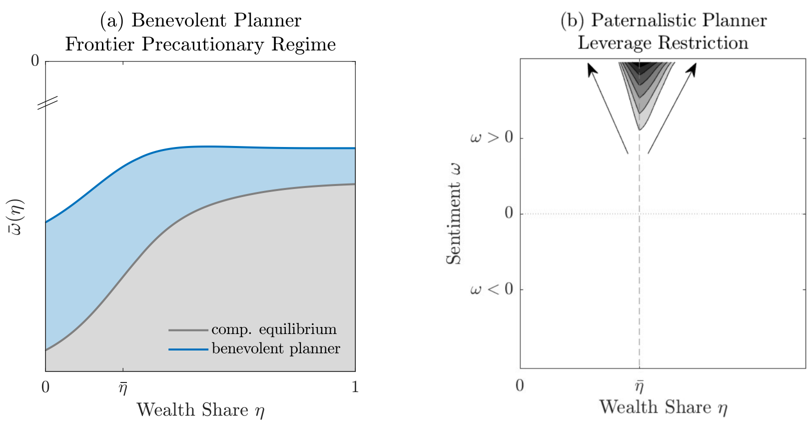
5.2.3 Diagnostic Expectations and Financial Frictions

Finally, consider the economy with both diagnostic expectations and financial frictions presented in section 2. In this economy, the socially optimal allocation shares the key properties of its counterparts in subsections 5.2.1 and 5.2.2. Notably, as in subsection 5.2.1, the allocation depends on the expectations of the planner. Moreover, as in subsection 5.2.2, the allocation in general improves social welfare over the competitive equilibrium, but without attaining the first best.

These two properties are reflected in the socially optimal reinvestment rate (Figure 7). Relative to the competitive equilibrium, reinvestment is higher under both paternalism and benevolence, but in neither case does reinvestment attain the first-best. When the wealth share is sufficiently high, moreover, reinvestment is higher under benevolence than under paternalism. The reason is that a benevolent planner perceives sentiment as fundamental information for setting the allocation, whereas a paternalistic planner seeks to insulate the allocation from sentiment.

The interplay between diagnostic expectations and financial frictions creates additional considerations for the socially optimal allocation. Specifically, relative to the competitive

Figure 8: Socially Optimal Asset Allocation



Notes: This figure illustrates socially optimal restrictions of the allocation of the asset to the productive technology relative to the competitive equilibrium presented in section 3.4. Panel (a) reports the occurrence of the precautionary regime for the competitive equilibrium (grey area) and the socially optimal allocation when the planner is benevolent (blue area). Panel (b) reports restrictions implemented by a paternalistic planner in the non-precautionary regime. A darker shade means the planner imposes stronger restrictions on the share κ relative to the upper bound $\min\{\lambda\eta, 1\}$ that applies in a competitive equilibrium. The white color means no reduction in the share below $\kappa = \min\{\lambda\eta, 1\}$.

equilibrium, the share of the asset allocated to the productive technology is lower in some regions of the state space (Figure 8). These regions depend, in turn, on whether the planner is benevolent or paternalistic. In particular, if the planner is benevolent, the share is lower when sentiment is moderately low. Put formally, the precautionary regime expands. The reason is that in that region, the planner believes allocating the asset to the productive technology excessively deteriorates the expected recovery rate of the wealth share. If the planner is paternalistic, by contrast, the share is lower when financial amplification effects peak (i.e., around threshold state $\bar{\eta}$ and when sentiment is moderately high). This happens because in that region, the planner is particularly concerned with the stronger financial amplification effects arising from the interactions between diagnostic expectations and financial frictions.

Overall, the interplay between financial frictions and diagnostic beliefs motivates additional restrictions on financial risk-taking relative to an economy with rational expectations. The nature of the restrictions depends on the degree of diagnosticity in the expectations of the planner: a paternalistic planner imposes leverage restrictions during booms, while a benevolent imposes leverage restrictions during economic downturns.

6 Conclusion

This paper examines the joint implications of diagnostic expectations and external financing frictions for financial stability and financial regulation in an environment with a real risk-taking channel. We find that interactions between those two elements exacerbate instability in financial markets relative to the rational expectations benchmark, a result that aligns closely with classical writings on the Financial Instability Hypothesis. As a consequence, the socially optimal regulation imposes additional restrictions on leverage and risk-taking relative to those derived in an economy under rational expectations, regardless of the degree of diagnosticity in the expectations of the planner. This analysis has only considered an expectations deviation from the full information rational expectations (FIRE) benchmark. Investigating also the effects of imperfect information on financial stability and financial regulation remains for future research.

References

Barberis, Nicholas, Robin Greenwood, Lawrence Jin, and Andrei Shleifer.

2015. “X-CAPM: An extrapolative capital asset pricing model.” *Journal of financial economics*, 115(1): 1–24.

Bernanke, Ben, and Mark Gertler. 1989. “Agency Costs, Net Worth, and Business Fluctuations.” *The American Economic Review*, 79(1): 14–31.

Bernanke, Ben S., Mark Gertler, and Simon Gilchrist. 1999. “Chapter 21 The financial accelerator in a quantitative business cycle framework.” In . Vol. 1 of *Handbook of Macroeconomics*, 1341–1393. Elsevier.

Bianchi, Francesco, Cosmin L Ilut, and Hikaru Saijo. 2021. “Diagnostic Business Cycles.” National Bureau of Economic Research Working Paper 28604.

Bordalo, Pedro, Nicola Gennaioli, and Andrei Shleifer. 2018. “Diagnostic expectations and credit cycles.” *The Journal of Finance*, 73(1): 199–227.

Bordalo, Pedro, Nicola Gennaioli, Andrei Shleifer, and Stephen J Terry. 2019. “Real credit cycles.” *Harvard University.*

Bordalo, Pedro, Nicola Gennaioli, Yueran Ma, and Andrei Shleifer. 2020. “Overreaction in macroeconomic expectations.” *American Economic Review*, 110(9): 2748–82.

Brunnermeier, Markus K., and Yuliy Sannikov. 2014. “A macroeconomic model with a financial sector.” *American Economic Review*, 104(2): 379–421.

Dávila, Eduardo, and Ansgar Walther. 2021. “Prudential policy with distorted beliefs.” National Bureau of Economic Research.

Dell’Ariccia, Giovanni, Deniz Igan, and Luc UC Laeven. 2012. “Credit booms and lending standards: Evidence from the subprime mortgage market.” *Journal of Money, Credit and Banking*, 44(2-3): 367–384.

Di Tella, Sebastian. 2019. “Optimal Regulation of Financial Intermediaries.” *American Economic Review*, 109(1): 271–313.

Farhi, Emmanuel, and Iván Werning. 2020. “Taming a Minsky Cycle.” *Working paper. Harvard University.*

Fontanier, Paul. 2022. “Optimal Policy for Behavioral Financial Crises.” *Working paper. Yale University.*

Gertler, Mark, and Nobuhiro Kiyotaki. 2010. "Financial intermediation and credit policy in business cycle analysis." In *Handbook of Monetary Economics*. Vol. 3, 547–599. Elsevier.

Gertler, Mark, and Peter Karadi. 2011. "A model of unconventional monetary policy." *Journal of Monetary Economics*, 58(1): 17–34.

He, Zhiguo, and Arvind Krishnamurthy. 2019. "A Macroeconomic Framework for Quantifying Systemic Risk." *American Economic Journal: Macroeconomics*, 11(4): 1–37.

Kindleberger, Charles P. 1978. *Manias, panics, and crashes: A history of financial crises.* Palgrave Macmillan.

Kiyotaki, Nobuhiro, and John Moore. 1997. "Credit Cycles." *Journal of Political Economy*, 105(2): 211–248.

Krishnamurthy, Arvind, and Wenhao Li. 2020. "Dissecting mechanisms of financial crises: Intermediation and sentiment." National Bureau of Economic Research.

L'Huillier, Jean-Paul, Gregory Phelan, and Hunter Wieman. 2022. "Technology shocks and predictable Minsky cycles." *Available at SSRN 4298974*.

L'Huillier, Jean-Paul, Sanjay R Singh, and Donghoon Yoo. 2021. "Incorporating diagnostic expectations into the New Keynesian framework." *Available at SSRN 3910318*.

Maggiori, Matteo. 2017. "Financial intermediation, international risk sharing, and reserve currencies." *American Economic Review*, 107(10): 3038–71.

Maxted, Peter. 2023. "A Macro-Finance Model with Sentiment." *The Review of Economic Studies*.

Mian, Atif, and Amir Sufi. 2011. "House prices, home equity-based borrowing, and the US household leverage crisis." *American Economic Review*, 101(5): 2132–56.

Minsky, H.P. 1986. *Stabilizing an Unstable Economy. Twentieth Century Fund report*, Yale University Press.

Minsky, Hyman P. 1977. "The financial instability hypothesis: An interpretation of Keynes and an alternative to "standard" theory." *Challenge*, 20(1): 20–27.

Phelan, Gregory. 2016. "Financial Intermediation, Leverage, and Macroeconomic Instability." *American Economic Journal: Macroeconomics*, 8(4): 199–224.

Shleifer, Andrei, and Robert W. Vishny. 1992. "Liquidation Values and Debt Capacity: A Market Equilibrium Approach." *The Journal of Finance*, 47(4): 1343–1366.

Tversky, Amos, and Daniel Kahneman. 1983. "Extensional versus intuitive reasoning: The conjunction fallacy in probability judgment." *Psychological review*, 90(4): 293.

Van der Ghote, Alejandro. 2021. "Interactions and coordination between monetary and macroprudential policies." *American Economic Journal: Macroeconomics*, 13(1): 1–34.

Online Appendix

Evaluating the Financial Instability Hypothesis: A Positive and Normative Analysis of Leveraged Risk-Taking and Extrapolative Expectations

The Appendix has two parts. The first part proves the propositions and corollaries stated in the text as well as derives the planner's problem in Definition 2. The second part describes the numerical method used to solve the PDEs.

1 Proofs of Propositions and Corollaries

Lemma 1 *At any given time t, households are indifferent among any consumption rate c_t. Moreover, they choose reinvestment rate $i_{2,t}$ and asset holding $k_{2,t}$ as follows:*

$$\mathcal{I}_2'(\iota_{2,t}) = \frac{1}{q_t} \ , \tag{1}$$

and

$$q_t k_{2,t} \begin{bmatrix} = 0 & \text{if } \alpha_{2,t} < 0 \\ \in [0, +\infty) & \text{if } \alpha_{2,t} = 0 \end{bmatrix} \ , \tag{2}$$

where the estimated risk-adjusted excess return to allocate the asset to the unproductive technology over holding debt, that is, $\alpha_{2,t} \leq 0$, is given by

$$\alpha_{2,t} \equiv \frac{1}{dt} \hat{E}_t \left[dR_{2,t} \right] - r \leq 0 \ . \tag{3}$$

Proof. Households maximize the present discounted value of consumption

$$B_t \equiv \max_{c_t, \iota_{2,t}, k_{2,t} \geq 0} \hat{E}_t \int_t^{+\infty} e^{-r(s-t)} c_s ds \ , \tag{4}$$

subject to the law of motion of wealth,

$$dw_s = dR_{2,s} q_s k_{2,s} + r(w_s - q_s k_{2,s}) ds - c_s ds + \tau_s ds \ . \tag{5}$$

Let's postulate that

$$B_t = w_t - e^{rt} \int_0^t e^{-rs} \tau_s ds \ . \tag{6}$$

Substituting (4) into (6) and rearranging, one gets the following condition:

$$e^{-rt} w_t + \int_0^t e^{-rs} \left(c_s - \tau_s \right) ds = \hat{E}_t \int_0^{+\infty} e^{-rs} c_s ds \ . \tag{7}$$

The RHS in this equation is the conditional expectation of a random variable. Thus, the drift of the RHS is null. From applying Ito's Lemma to the LHS and equalizing the resulting drift process to zero, one gets the following Hamilton-Jacobi-Bellman (HJB) equation:

$$r w_t = \max_{c_t, \iota_{2,t}, k_{2,t} \geq 0} \left\{ c_t - \tau_t + \left[\frac{1}{dt} \hat{E}_t \left[dR_{2,t} \right] - r \right] q_t k_{2,t} + r w_t - c_t + \tau_t \right\} \ , \tag{8}$$

Note that any c_t is optimal. The optimal $\iota_{2,t}$ and $k_{2,t}$ are

$$\mathcal{I}_2'(\iota_{2,t}) = \frac{1}{q_t} \ , \tag{9}$$

and

$$k_{2,t} \left[\begin{array}{ll} = +\infty & \text{if } \frac{1}{dt} \hat{E}_t \left[dR_{2,t} \right] - r > 0 \\ \in [0, +\infty) & \text{if } \frac{1}{dt} \hat{E}_t \left[dR_{2,t} \right] - r = 0 \\ = 0 & \text{if } \frac{1}{dt} \hat{E}_t \left[dR_{2,t} \right] - r < 0 \end{array} \right. \ . \tag{10}$$

The HJB equation thus reduces to

$$0 = \left[\frac{1}{dt} \hat{E}_t \left[dR_{2,t} \right] - r \right] q_t k_{2,t} \ , \tag{11}$$

where $\iota_{2,t}$ and $k_{2,t}$ are given by (9) and (10), respectively—which under restriction $\alpha_{2,t} \leq 0$, verifies the postulate. ■

Lemma 2 *At any given time t, financiers choose reinvestment rate $\iota_{1,t}$ and asset holding $k_{1,t}$ as follows:*

$$\mathcal{I}_1'(\iota_{1,t}) = \frac{1}{q_t} \ , \tag{12}$$

and

$$\frac{q_t k_{1,t}}{n_t} \left[\begin{array}{ll} = 0 & \text{if } \alpha_{1,t} < 0 \\ \in [0, \lambda] & \text{if } \alpha_{1,t} = 0 \\ = \lambda & \text{if } \alpha_{1,t} > 0 \end{array} \right. \ , \tag{13}$$

where the estimated risk-adjusted excess return to allocate the asset to the productive technology over holding debt, namely, $\alpha_{1,t} \in \mathbb{R}$, *is given by*

$$\alpha_{1,t} \equiv \frac{1}{dt}\hat{E}_t\left[dR_{1,t}\right] - r + \left(\sigma_{q,t} + \sigma_1\right)\sigma_{v,t} \; . \tag{14}$$

The marginal value of net worth, v_t, *satisfies*

$$0 = \alpha_{1,t}\frac{q_t k_{1,t}}{n_t} + \mu_{v,t} + \hat{\mu}\omega_t\sigma_{v,t} + \frac{\theta}{v_t} - \theta \; . \tag{15}$$

Proof. Financiers maximize the present discount value of dividend payouts

$$V_t \equiv \max_{\iota_{1,s}, k_{1,s} \geq 0} \hat{E}_t \int_t^{\infty} \theta e^{-(r+\theta)(s-t)} n_s ds \; , \tag{16}$$

subject to the law of motion of net worth,

$$dn_s = dR_{1,s} q_s k_{1,s} - r(q_s k_{1,s} - n_s)ds \; , \tag{17}$$

and collateral constraint $q_s k_{1,s} \leq \lambda n_s$, with $n_s \geq 0$.

Note that value $V_t = v_t n_t$ satisfies

$$e^{-(r+\theta)t} v_t n_t + \int_0^t \theta e^{-(r+\theta)s} n_s ds = \hat{E}_t \int_0^{\infty} \theta e^{-(r+\theta)s} n_s ds \; . \tag{18}$$

The RHS of this equation is the conditional expectation of a random variable. Thus, the drift of the RHS is null. Applying Ito's Lemma to the LHS and equalizing the resulting drift process to zero, one gets the following HJB equation:

$$\begin{aligned}(r+\theta)\, v_t \quad &= \quad \max_{\iota_{1,t}, \phi_t \geq 0} \left\{\theta + \left[\mu_{v,t} + \hat{\mu}\omega_t\sigma_{v,t} + \frac{1}{dt}\hat{E}_t\left[dR_{1,t}\right]\phi_t - r\left(\phi_t - 1\right) + \sigma_{v,t}\left(\sigma_{q,t} + \sigma\right)\phi_t\right] v_t\right\} \\ &\qquad \text{subject to} : \phi_t \leq \lambda \; .\end{aligned} \tag{19}$$

where $\phi_t \equiv q_t k_{1,t}/n_t$. The optimal $\iota_{1,t}$ and ϕ_t are

$$\mathcal{I}_1'\left(\iota_{1,t}\right) = \frac{1}{q_t} \; . \tag{20}$$

and

$$\phi_t \left[\begin{array}{ll} = \lambda & \text{if } \alpha_{1,t} > 0 \\ \in [0, \lambda] & \text{if } \alpha_{1,t} = 0 \\ = 0 & \text{if } \alpha_{1,t} < 0 \end{array} \right. . \tag{21}$$

Substituting (20) and (21) into (19), one gets the following equation:

$$\alpha_{1,t}\phi_t + \mu_{v,t} + \hat{\mu}\omega_t\sigma_{v,t} + \frac{\theta}{v_t} - \theta = 0 \ . \tag{22}$$

■

Proposition 1 *Let $\eta_t \equiv n_t/q_t k_t \in [0,1]$ be the aggregate net worth of financiers as a share of total wealth and let $\kappa_t \equiv k_{1,t}/k_t \in [0,1]$ be the aggregate share of the asset allocated to the productive technology. Then, the equilibrium outcome is partitioned into the following three regimes,*

$$\begin{array}{llll} \textit{1. Financially unconstrained regime:} & \kappa_t = 1 \leq \lambda\eta_t \ , & \alpha_{1,t} = 0 \ , & \alpha_{2,t} < 0 \ ; \\ \textit{2. Financially constrained regime:} & \kappa_t = \lambda\eta_t \in [0,1] \ , & \alpha_{1,t} > 0 \ , & \alpha_{2,t} = 0 \ ; \\ \textit{3. Precautionary regime:} & \kappa_t = 0 \ , & \alpha_{1,t} < 0 \ , & \alpha_{2,t} = 0 \ ; \end{array} \tag{23}$$

The equilibrium allocation can be summarized as $\{\iota_{1,t}, \iota_{2,t}, \kappa_t\}$, and can be characterized by $\{(1), (3), (12), (14), (22), (23)\}$. The equilibrium utility of households per unit of the asset, namely, $u_t > 0$, satisfies

$$\begin{aligned} 0 \ = \ & \kappa_t \{A_1 - \iota_{1,t} + [\mathcal{I}_1(\iota_{1,t}) + \sigma_1\hat{\mu}\omega_t] \, u_t\} + \\ & + (1-\kappa_t)\{A_2 - \iota_{2,t} + [\mathcal{I}_2(\iota_{2,t}) + \sigma_2\hat{\mu}\omega_t] \, u_t\} + \hat{E}_t [du_t] - ru_t. \end{aligned} \tag{24}$$

Proof. Expressions $\{(1), (3)\}$ characterize the optimality conditions of households and expressions $\{(12), (14), (22)\}$ characterize the optimality conditions of financiers. Expression (23) ensures that market clearing for the asset is consistent with individual optimality. Specifically, if $\alpha_{2,t} < 0$, then $\alpha_{1,t} = 0$ must hold, which requires $\kappa_t = 1$. If $\alpha_{2,t} = 0$, then either $\alpha_{1,t} < 0$ or $\alpha_{1,t} > 0$ must hold. In the first case, $\kappa_t = 0$ is required, while in the second, $\kappa_t = \lambda\eta_t$ is required. Variables $\{\iota_{1,t}, \iota_{2,t}, \kappa_t\}$ together with $c_t/k_t = (A_1 - \iota_{1,t})\,\kappa_t + (A_2 - \iota_{2,t})\,(1-\kappa_t)$ ensure that market clearing for the good holds. Market clearing for debt automatically holds because of Walras Law.

Let U_t be the utility of households under the equilibrium allocation. Then,

$$U_t \equiv \hat{E}_t \int_t^{+\infty} e^{-r(s-t)} \left[\kappa_s \left(A_1 - \iota_{1,s}\right) + \left(1-\kappa_s\right)\left(A_2 - \iota_{2,s}\right)\right] k_s ds \ . \tag{25}$$

Utility $\hat{U}_t$ can be expressed as

$$\begin{aligned} &e^{-rt}U_t + \int_0^t e^{-rs} \left[\kappa_s \left(A_1 - \iota_{1,s}\right) + \left(1-\kappa_s\right)\left(A_2 - \iota_{2,s}\right)\right] k_s ds = \\ &= \hat{E}_t \int_0^{+\infty} e^{-rs} \left[\kappa_s \left(A_1 - \iota_{1,s}\right) + \left(1-\kappa_s\right)\left(A_2 - \iota_{2,s}\right)\right] k_s ds \ . \end{aligned} \tag{26}$$

The RHS in this equation is the conditional expectation of a random variable. Thus, the drift of the RHS is null. Applying Ito's Lemma to the LHS and equalizing the resulting drift process to zero, one gets the following HJB equation:

$$0 = \kappa_t \left(A_1 - \iota_{1,t}\right) + \left(1-\kappa_t\right)\left(A_2 - \iota_{2,t}\right) + \hat{E}_t\left[dU_t\right] - rU_t \ . \tag{27}$$

We postulate that $U_t = u_t k_t$, where $u_t > 0$ is an Ito process with disturbance dZ_t. The above HJB equation can then be reduced to

$$\begin{aligned} &0 = \kappa_t \left\{A_1 - \iota_{1,t} + \left[\mathcal{I}_1(\iota_{1,t}) + \sigma_1 \hat{\mu}\omega_t\right] u_t\right\} + \\ &+ \left(1-\kappa_t\right)\left\{A_2 - \iota_{2,t} + \left[\mathcal{I}_2(\iota_{2,t}) + \sigma_2 \hat{\mu}\omega_t\right] \ u_t\right\} + \hat{E}_t\left[du_t\right] - ru_t \ . \end{aligned} \tag{28}$$

■

Proposition 2 *The Markov equilibrium can be analytically characterized as the solution to a system of second-order PDEs for* $\{q, v\}$ *in* $\{\eta, \omega\}$.

Proof. This section derives the system of partial differential equations (PDEs) that analytically characterizes the Markov equilibrium. To do so, we consider more general specifications for diagnostic expectations, the collateral constraint, and the production technologies than the baseline specification in Section 2.3 of the paper. Specifically, (*i*) drift μ_ω can be any function of the state space that does not depend on drift μ_η; (*ii*) diffusion σ_ω can be any exogenous function of the state space; (*iii*) expectation weight $\hat{\mu}$ can also be any exogenous function of the state space; (*iv*) leverage limit λ can be either a parameter or a linear function of value v; and (*v*) return function $\mathcal{I}_2(\iota_2) \geq 0$ or volatily $\sigma_2 \geq 0$ can be positive. This more general specification suffices to characterize the Markov equilibrium in all of the specifications in the paper. In the remainder of the section, we omit time subscript t.

The equation that determines price q is

$$\begin{array}{ll} \alpha_1 = 0 & \text{if } \kappa = 1 \\ \alpha_2 = 0 & \text{otherwise} \end{array} , \tag{29}$$

or equivalently,

$$\begin{array}{ll} \frac{A_1-\iota_1}{q} + \mu_q + \mathcal{I}_1\left(\iota_1\right) + \left(\sigma_q + \sigma_1\right)\hat{\mu}\omega + \sigma_q\sigma_1 - r + \left(\sigma_q + \sigma_1\right)\sigma_v = 0 & \text{if } \omega \geq \bar{\omega} \text{ and } \eta \geq \bar{\eta} \\ \frac{A_2-\iota_2}{q} + \mu_q + \mathcal{I}_2(\iota_2) + \left(\sigma_q + \sigma_2\right)\hat{\mu}\omega + \sigma_q\sigma_2 - r = 0 & \text{otherwise} \end{array} . \tag{30}$$

The equation that determines value v is

$$\alpha_1\phi + \mu_v + \hat{\mu}\omega\sigma_v + \frac{\theta}{v} - \theta = 0 \; . \tag{31}$$

Ito's Lemma implies that for $x \in \{q, v\}$

$$\mu_x = \frac{1}{x}\left[\frac{\partial x}{\partial\omega}\mu_\omega\omega + \frac{\partial x}{\partial\eta}\mu_\eta\eta + \frac{1}{2}\frac{\partial^2 x}{(\partial\omega)^2}(\sigma_\omega\omega)^2 + \frac{\partial^2 x}{\partial\omega\partial\eta}\sigma_\omega\omega\sigma_\eta\eta + \frac{1}{2}\frac{\partial^2 x}{(\partial\eta)^2}\left(\sigma_\eta\eta\right)^2\right], \tag{32}$$

$$\sigma_x = \frac{1}{x}\left[\frac{\partial x}{\partial\omega}\sigma_\omega\omega + \frac{\partial x}{\partial\eta}\sigma_\eta\eta\right] , \tag{33}$$

where recall that

$$\begin{aligned} \mu_\eta \;=\;& \left[\frac{A_1-\iota_1}{q} + \mathcal{I}_1(\iota_1) + \sigma_q\sigma_1\right]\phi + \left(\mu_q - r\right)\left(\phi - 1\right) - \mu_k \\ & -\sigma_q\sigma_k + \left(\sigma_q + \sigma_k\right)\left[\left(\sigma_q + \sigma_k\right) - \phi\left(\sigma_q + \sigma_1\right)\right] - \left(\theta - \frac{\gamma}{\eta}\right) , \end{aligned} \tag{34}$$

$$\sigma_\eta \;=\; \phi\left(\sigma_q + \sigma_1\right) - \left(\sigma_q + \sigma_k\right) , \tag{35}$$

with

$$\mu_k = \kappa\mathcal{I}_1(\iota_1) + \left(1 - \kappa\right)\mathcal{I}_2(\iota_2) \; , \tag{36}$$

$$\sigma_k = \kappa\sigma_1 + \left(1 - \kappa\right)\sigma_2 \; . \tag{37}$$

According to (32) and (33), objects $\{\mu_q, \sigma_q\}$ depend on $\{\mu_\eta, \sigma_\eta\}$, but according to (34) and (35), objects $\{\mu_\eta, \sigma_\eta\}$ in turn depend on $\{\mu_q, \sigma_q\}$. To eliminate this circularity, we substitute (34) and (35) into (32) and (33). We obtain

$$\mu_\eta = \frac{1}{1-(\phi-1)\,\varepsilon_{q,\eta}}\left\{\left[\frac{A_1-\iota_1}{q}+(1-\eta)\left[\mathcal{I}_1(\iota_1)+\sigma_q\sigma_1\right]\right]\phi+\right. \tag{38}$$
$$+\left(\mu_\omega\varepsilon_{q,\omega}+\xi_{q,\eta/\omega}-r\right)(\phi-1)-(1-\phi\eta)\left[\mathcal{I}_2(\iota_2)-\sigma_k\sigma_2\right]+$$
$$\left.-\left[\sigma_q+\sigma_1\phi\eta+(1-\phi\eta)\sigma_2\right]\left[(\phi-1)\,\sigma_q+(1-\eta)\,\sigma_1\phi\right]-\left(\theta-\frac{\gamma}{\eta}\right)\right\},$$

$$\sigma_\eta = \frac{(\phi-1)\,\sigma_\omega\varepsilon_{q,\omega}+(1-\eta)\,\sigma_1\phi-(1-\phi\eta)\sigma_2}{1-(\phi-1)\,\varepsilon_{q,\eta}}, \tag{39}$$

where

$$\varepsilon_{q,\eta} \equiv \frac{\partial q}{\partial \eta}\frac{\eta}{q}, \quad \varepsilon_{q,\omega} \equiv \frac{\partial q}{\partial \omega}\frac{\omega}{q}, \tag{40}$$

$$\xi_{q,\eta/\omega} \equiv \frac{1}{2}\frac{\partial^2 q}{(\partial\omega)^2}(\sigma_\omega\omega)^2+\frac{\partial^2 q}{\partial\eta\partial\omega}\sigma_\eta\eta\sigma_\omega\omega+\frac{1}{2}\frac{\partial^2 q}{(\partial\eta)^2}(\sigma_\eta\eta)^2. \tag{41}$$

Thus, the system of equations,

$$\{(30),\ (31),\ (32),\ (33),\ (37),\ (38),\ (39)\}, \tag{42}$$

with

$$\iota_j = \mathcal{I}_j'^{-1}\left(\frac{1}{q}\right), \tag{43}$$

$$\kappa = \phi\eta \text{ with } \phi = \min\left\{\lambda, \frac{1}{\eta}\right\}\mathbf{1}_{\omega\geq\bar{\omega}}, \tag{44}$$

$$\bar{\omega}(\eta) = \{\omega<0 : \alpha_1(\omega,\eta)=\alpha_2(\omega,\eta)=0\}, \tag{45}$$

$$\bar{\eta}(\omega) = \{\eta\in[0,1] : \lambda(\omega,\eta)\,\eta=1\}, \tag{46}$$

determines a second-order PDEs for $\{q,v\}$ in $\{\omega,\eta\}$.

We impose the following boundary conditions to the PDEs:

$$\lim_{\eta\to1}\sigma_q = 0, \quad \lim_{\eta\to1}\frac{\partial\sigma_q}{\partial\eta}=0, \quad \lim_{\eta\to1}\sigma_v=0, \quad \lim_{\eta\to1}\frac{\partial\sigma_v}{\partial\eta}=0. \tag{47}$$

These conditions ensure that diffusions σ_q and σ_v vanish smoothly as the aggregate net worth of financiers approaches total wealth. ■

Corollary 1 *In the economy with rational expectations and without financial frictions, neither sentiment ω nor wealth share η influence the equilibrium outcome. The asset price is a constant that satisfies*

$$\alpha_1 = 0 \Leftrightarrow \frac{A_1 - \iota_1}{q} + \mathcal{I}_1(\iota_1) - r = 0 \ , \ \textit{with} \ \ \mathcal{I}_1'(\iota_1) = \frac{1}{q} \ . \tag{48}$$

Value $v = 1$ is also a constant. The aggregate quantity of the asset is allocated to the productive technology, that is, $\kappa = 1$. The social value of the asset equals the asset price, that is, $u = q$.

Proof. Under RE, the productive technology yields higher return than the unproductive one, accordingly the Equilibrium relationship #3 from (23) cannot occur. In the absence of financial frictions, the allocation of the asset to the productive technology is not restricted by the collateral constraint, hence the Equilibrium relationship #2 cannot occur. Accordingly, the conditions Equilibrium relationship #1 characterize the equilibrium, that is $\alpha_1 = 0$ with $\kappa = 1$. It derives that $v = 1$, since financiers earn no rent on the asset, and

$$\alpha_1 = \frac{1}{dt} E\left[dR_1\right] - r = 0.$$

Finally, the price of the asset q is constant and satisfies

$$\frac{A_1 - \iota_1}{q} + I_1(\iota_1) - r = 0,$$

where ι_1 satisfies (20). ■

Corollary 2 *In the economy with diagnostic expectations and without financial frictions, sentiment ω is the only relevant state that affects the equilibrium outcome. A threshold state $\bar{\omega} < 0$ exists such that*

$$\begin{array}{ll} \textit{if}\ \omega < \bar{\omega} \Rightarrow & \kappa = 0 \ , \ \ \alpha_1 < 0 \ , \ \ \alpha_2 = 0 \ ; \\ \textit{if}\ \omega > \bar{\omega} \Rightarrow & \kappa = 1 \ , \ \ \alpha_1 = 0 \ , \ \ \alpha_2 < 0 \ ; \end{array} . \tag{49}$$

The threshold state $\bar{\omega} < 0$ is the solution to

$$\alpha_1 = \alpha_2 = 0 \Rightarrow \frac{A_1 - \iota_1 - A_2}{q} + \mathcal{I}_1(\iota_1) + (\sigma_q + \hat{\mu}\omega)\,\sigma_1 = 0 \ . \tag{50}$$

Proof. In the absence of financial frictions, Equilibrium relationship #2 cannot occur. Accordingly, the economy alternates between Equilibrium relationships #1 and #3, depending on the value of sentiment ω, i.e., depending on the perceived relative returns to each technology, as indicated in (49). The sentiment threshold state $\bar{\omega}$ is such that the perceived return to each technology is the same, i.e., $\alpha_1 = \alpha_2$. Using (3) and (14), one gets (50). ■

Corollary 3 *In the economy with rational expectations and financial frictions, wealth share η is the only relevant state that affects the equilibrium outcome. A threshold state $\bar{\eta} \in (0,1)$ exists such that*

$$\begin{array}{lll} \text{if } \eta < \bar{\eta} \Rightarrow & \kappa = \lambda\eta < 1 \ , & \alpha_1 > 0 \ , \ \alpha_2 = 0 \ ; \\ \text{if } \eta > \bar{\eta} \Rightarrow & \kappa = 1 \ , & \alpha_1 = 0 \ , \ \alpha_2 < 0 \ ; \end{array} . \tag{51}$$

The threshold state $\bar{\eta} \in (0,1)$ is the solution to

$$\lambda\eta = 1 \Leftrightarrow \eta = \frac{1}{\lambda} \ . \tag{52}$$

Proof. Under rational expectations, only Equilibrium relationships #1 and #2 can occur, since the productive technology is correctly perceived as providing higher returns. Accordingly, the economy alternates between the financially constrainted and financially unconstrained regime, depending on the wealth share η of financiers. The cut-off value $\bar{\eta}$ naturally satisfies (52). ■

Corollary 4 *In the economy with diagnostic expectations and financial frictions, both sentiment ω and wealth share η affect the equilibrium outcome. Thresholds $\bar{\omega} < 0$ and $\bar{\eta} \in (0,1)$, partition the state space as follows:*

$$\begin{array}{llll} \text{if } \omega < \bar{\omega} \Rightarrow & \kappa = 0 \ , & \alpha_1 < 0 \ , \ \alpha_2 = 0 \ ; \\ \text{if } \omega > \bar{\omega} \text{ and } \eta < \bar{\eta} \Rightarrow & \kappa = \lambda\eta \ , & \alpha_1 > 0 \ , \ \alpha_2 = 0 \ ; \\ \text{if } \omega > \bar{\omega} \text{ and } \eta > \bar{\eta} \Rightarrow & \kappa = 1 \ , & \alpha_1 = 0 \ , \ \alpha_2 < 0 \ ; \end{array} \tag{53}$$

Threshold process $\bar{\omega}$ is the solution to

$$\alpha_1 = \alpha_2 = 0 \Rightarrow \frac{A_1 - \iota_1 - A_2}{q} + \mathcal{I}_1(\iota_1) + (\sigma_q + \hat{\mu}\omega)\,\sigma_1 + (\sigma_q + \sigma_1)\,\sigma_v = 0 \ . \tag{54}$$

Threshold state $\bar{\eta}$ is the solution to

$$\lambda\eta = 1 \Leftrightarrow \eta = \frac{1}{\lambda} \ . \tag{55}$$

Proof. With both diagnostic expectations and financial frictions, the economy alternates between the three Equilibrium relationships. The characterization of cut-off states $\bar{\omega}$ and $\bar{\eta}$ follows from the proofs of Propositions 6 and 7. ■

Proposition 3 *If agents rely on Ito path $\{dX_s\}_{s<t}$ to form diagnostic expectations about Ito variable dY_t, the implied diagnostic expectation operator over disturbance dZ_t is*

$$\hat{E}_t\left[dZ_t\right] = \hat{\mu}\frac{\omega_t}{\sigma_{Y,t}Y_t}dt \ , \tag{56}$$

where $\sigma_{Y,t} \in \mathbb{R}$ is the diffusion of the variable and where sentiment $\omega_t \in \mathbb{R}$ is given by

$$\omega_t = \int_0^t e^{-\delta(t-s)}dX_s \ . \tag{57}$$

Corollary 5 *If $dX_s = \frac{dR_{1,s}-\hat{E}_s[dR_{1,s}]}{Std_s[dR_{1,s}]}$ and $dY_t = dR_{1,t}$, then*

$$\hat{E}_t\left[dZ_t\right] = \hat{\mu}\omega_t dt \ , \tag{58}$$

and

$$d\omega_t = \left(-\delta + \frac{\hat{\mu}}{\sigma_{q,t}+\sigma_1}\right)\omega_t dt + dZ_t \ . \tag{59}$$

Proof. Let ω_t be a sentiment operator tied to a Ito process $\{dX_s\}$, i.e.,

$$\omega_t = \int_0^t e^{-\delta(t-s)}dX_s. \tag{60}$$

A diagnostic operator over a generic Ito process dY_t is defined as:

$$\hat{E}_t\left[dY_t\right] \equiv E_t\left[d\hat{Y}_t\right], \quad \text{with} \quad d\hat{Y}_t \equiv \hat{\mu}\omega_t dt + dY_t \tag{61}$$

Let's define the expectation operator $\check{E}_t[dZ_t]$ as

$$\check{E}_t[dZ_t] \equiv E_t[d\hat{Z}_t], \text{ with } d\hat{Z}_t \equiv \hat{\mu}\frac{\omega_t}{\sigma_{Y,t}Y_t}dt + dZ_t \ . \tag{62}$$

Then

$$\check{E}_t[dY_t] = \check{E}_t[\mu_{Y,t}Y_t dt + \sigma_{Y,t}Y_t dZ_t] = \mu_{Y,t}Y_t dt + \sigma_{Y,t}Y_t\check{E}_t[dZ_t] = \left(\mu_{Y,t}Y_t + \hat{\mu}\omega_t\right)dt \ . \tag{63}$$

Accordingly, $\check{E}_t[dY_t] = \hat{E}_t[dY_t]$, and the implied diagnostic expectation operator over dZ_t is

$$\hat{E}_t\left[dZ_t\right] = \check{E}_t[dZ_t] = \hat{\mu}\frac{\omega_t}{\sigma_{Y,t}Y_t}dt \ . \tag{64}$$

Applying these results to the specific case $dX_s = \frac{dR_{1,s}-\hat{E}_s[dR_{1,s}]}{Std_s[dR_{1,s}]}$ and $dY_t = dR_{1,t}$, one gets the expressions presented in Corollary 2. ■

Proposition 4 *The socially optimal reinvestment rate solves*

$$\mathcal{I}_1'(\iota_1) = \frac{1+\frac{1}{1-(\phi-1)\varepsilon_{q,\eta}}\frac{1}{q}\frac{\partial\tilde{u}}{\partial\eta}}{\tilde{u}+\frac{1-\eta}{1-(\phi-1)\varepsilon_{q,\eta}}\frac{\partial\tilde{u}}{\partial\eta}} \ . \tag{65}$$

The socially optimal share κ maximizes the RHS in (1). The candidate solutions are $\kappa = 0$, $\kappa = \min\{\lambda\eta, 1\}$, and any interior $\kappa \in (0, \min\{\lambda\eta, 1\})$ that solves

$$\begin{aligned} 0 \quad = \quad & \left[\frac{A_1-\iota_1-A_2}{\tilde{u}}+\mathcal{I}_1(\iota_1)+\left(\sigma_{\tilde{u}}+\tilde{\mu}\omega\right)\sigma_1\right]+\varepsilon_{\tilde{u},\eta}\left[\frac{\partial\mu_\eta}{\partial\kappa}+\left(\tilde{\mu}\omega+\kappa\sigma_1\right)\frac{\partial\sigma_\eta}{\partial\kappa}\right]+ \\ & +\frac{1}{\tilde{u}}\left(\frac{\partial^2\tilde{u}}{(\partial\eta)^2}\sigma_\eta\eta+\frac{\partial^2\tilde{u}}{\partial\eta\partial\omega}\right)\frac{\partial\sigma_\eta}{\partial\kappa}\eta \ , \end{aligned} \tag{66}$$

where $\frac{\partial\mu_\eta}{\partial\kappa}$ and $\frac{\partial\sigma_\eta}{\partial\kappa}$ are the partial derivatives of μ_η and σ_η with respect to κ, respectively.

Proof. This proof lay outs and solves the problem of the planner. The present discounted value of consumption under expectation weight $\tilde{\mu} \in [0, \hat{\mu}]$ is

$$\tilde{U}_t \equiv \tilde{E}_t\int_t^{+\infty} e^{-r(s-t)}\left[\kappa_s\left(A_1-\iota_{1,s}\right)+\left(1-\kappa_s\right)\left(A_2-\iota_{2,s}\right)\right]k_s ds \ , \tag{67}$$

where expectation operator $\tilde{E}_t\left[\cdot\right]$ is

$$\tilde{E}_t[dZ_t] \equiv E_t[d\tilde{Z}_t], \text{ with } d\tilde{Z}_t \equiv \tilde{\mu}\omega_t dt + dZ_t \ . \tag{68}$$

Note that the term in brackets in the integrand follows from resource constraint

$$c_t = y_t = \kappa_t\left(A_1-\iota_{1,t}\right)+\left(1-\kappa_t\right)\left(A_2-\iota_{2,t}\right) \ . \tag{69}$$

Utility $\tilde{U}_t$ can be expressed as

$$
\begin{aligned}
& e^{-rt}\tilde{U}_t + \int_0^t e^{-rs}\left[\kappa_s\left(A_1 - \iota_{1,s}\right) + \left(1-\kappa_s\right)\left(A_2 - \iota_{2,s}\right)\right] k_s ds \\
& = \tilde{E}_t \int_0^{+\infty} e^{-rs}\left[\kappa_s\left(A_1 - \iota_{1,s}\right) + \left(1-\kappa_s\right)\left(A_2 - \iota_{2,s}\right)\right] k_s ds \; .
\end{aligned}
\tag{70}
$$

The RHS in this equation is the conditional expectation of a random variable. Thus, the drift of the RHS is null. Applying Ito's Lemma to the LHS and equalizing the resulting drift process to zero, one gets the following HJB equation:

$$
0 = \kappa_t\left(A_1 - \iota_{1,t}\right) + \left(1-\kappa_t\right)\left(A_2 - \iota_{2,t}\right) + \tilde{E}_t\left[d\tilde{U}_t\right] - r\tilde{U}_t \; . \tag{71}
$$

We postulate that $U_t = u_t k_t$, where $u_t > 0$ is an Ito process with disturbance dZ_t. The above HJB equation can then be reduced to

$$
\begin{aligned}
& 0 = \kappa_t\left\{A_1 - \iota_{1,t} + \left[\mathcal{I}_1(\iota_{1,t}) + \sigma_1\tilde{\mu}\omega_t\right]\tilde{u}_t\right\} + \\
& + \left(1-\kappa_t\right)\left\{A_2 - \iota_{2,t} + \left[\mathcal{I}_2(\iota_{2,t}) + \sigma_2\tilde{\mu}\omega_t\right]\tilde{u}_t\right\} + \tilde{E}_t\left[d\tilde{u}_t\right] - r\tilde{u}_t \; .
\end{aligned}
\tag{72}
$$

In what follows, we restrict attention to a Markov structure with same state variables as in the competitive equilibrium. Thus, we omit time subscript $_t$ from now on. In addition, we consider the parametrization of the baseline specification (Section 2.3 of the paper). Equation (72) can then be expressed as

$$
\begin{aligned}
r\tilde{u} \;=\; & \kappa\left\{A_1 - \iota_1 + \left[\mathcal{I}_1(\iota_1) + \sigma_1\tilde{\mu}\omega\right]\tilde{u}\right\} + (1-\kappa)\,A_2 + \frac{\partial\tilde{u}}{\partial\omega}\left(-\delta\omega + \tilde{\mu}\omega + \kappa\sigma_1\right) + \\
& + \frac{\partial\tilde{u}}{\partial\eta}\left(\mu_\eta\eta + \sigma_\eta\eta\tilde{\mu}\omega + \sigma_\eta\eta\kappa\sigma_1\right) + \frac{1}{2}\frac{\partial^2\tilde{u}}{\left(\partial\omega\right)^2} + \frac{\partial^2\tilde{u}}{\partial\omega\partial\eta}\sigma_\eta\eta + \frac{1}{2}\frac{\partial^2\tilde{u}}{\left(\partial\eta\right)^2}\left(\sigma_\eta\eta\right)^2\Bigg\} \; ,
\end{aligned}
\tag{73}
$$

where

$$\mu_\eta = \frac{1}{1-(\frac{\kappa}{\eta}-1)\varepsilon_{q,\eta}}\left\{\left[\frac{A_1-\iota_1}{q}+\mathcal{I}_1(\iota_1)+\sigma_q\sigma_1\right]\frac{\kappa}{\eta}-\kappa\mathcal{I}_1(\iota_1)-\sigma_q\kappa\sigma_1+\right. \tag{74}$$
$$+\frac{1}{q}\left[-\frac{\partial q}{\partial\omega}\delta\omega+\frac{1}{2}\frac{\partial^2 q}{(\partial\omega)^2}+\frac{\partial^2 q}{\partial\omega\partial\eta}\sigma_\eta\eta+\frac{1}{2}\frac{\partial^2 q}{(\partial\eta)^2}(\sigma_\eta\eta)^2-rq\right]\left(\frac{\kappa}{\eta}-1\right)+$$
$$\left.+(\sigma_q+\kappa\sigma_1)\left[(\sigma_q+\kappa\sigma_1)-\frac{\kappa}{\eta}(\sigma_q+\sigma_1)\right]-\left(\theta-\frac{\gamma}{\eta}\right)\right\},$$

$$\sigma_\eta = \frac{\frac{\kappa}{\eta}\left(\frac{1}{q}\frac{\partial q}{\partial\omega}+\sigma_1\right)-\left(\frac{1}{q}\frac{\partial q}{\partial\omega}+\kappa\sigma_1\right)}{1-\left(\frac{\kappa}{\eta}-1\right)\varepsilon_{q,\eta}}, \tag{75}$$

with

$$\sigma_q = \frac{\left(\frac{\kappa}{\eta}-\kappa\right)\varepsilon_{q,\eta}\sigma_1+\frac{1}{q}\frac{\partial q}{\partial\omega}}{1-\left(\frac{\kappa}{\eta}-1\right)\varepsilon_{q,\eta}}. \tag{76}$$

The above formulae for $\{\mu_\eta,\sigma_\eta,\sigma_q\}$ follow from evaluating $\{(33),\ (38),\ (39)\}$ at the baseline parametrization.

The problem of the planner is then

$$r\tilde{u} = \max_{\{\iota_1,\kappa\}}\left\{\kappa\left\{A_1-\iota_1+\left[\mathcal{I}_1(\iota_1)+\sigma_1\tilde{\mu}\omega\right]\tilde{u}\right\}+(1-\kappa)A_2+\frac{\partial\tilde{u}}{\partial\omega}(-\delta\omega+\tilde{\mu}\omega+\kappa\sigma_1)+\right. \tag{77}$$
$$\left.+\frac{\partial\tilde{u}}{\partial\eta}(\mu_\eta\eta+\sigma_\eta\eta\tilde{\mu}\omega+\sigma_\eta\eta\kappa\sigma_1)+\frac{1}{2}\frac{\partial^2\tilde{u}}{(\partial\omega)^2}+\frac{\partial^2\tilde{u}}{\partial\omega\partial\eta}\sigma_\eta\eta+\frac{1}{2}\frac{\partial^2\tilde{u}}{(\partial\eta)^2}(\sigma_\eta\eta)^2\right\},$$

with

$$\iota_1\in[0,A_1]\ \text{and}\ \kappa\in[0,\min\{\lambda\eta,1\}], \tag{78}$$

where $\{\mu_\eta,\sigma_\eta,\sigma_q\}$ are given by $\{(74),\ (75),\ (76)\}$.

The first-order condition with respect to ι_1 implies that

$$\mathcal{I}_1'(\iota_1) = \frac{1+\frac{1}{1-(\phi-1)\varepsilon_{q,\eta}}\frac{1}{q}\frac{\partial\tilde{u}}{\partial\eta}}{\tilde{u}+\frac{1-\eta}{1-(\phi-1)\varepsilon_{q,\eta}}\frac{\partial\tilde{u}}{\partial\eta}}. \tag{79}$$

Note that the problem is concave in ι_1.

The first-order condition with respect to κ implies that

$$\left[\frac{A_1-\iota_1-A_2}{\tilde{u}}+\mathcal{I}_1(\iota_1)+(\sigma_{\tilde{u}}+\tilde{\mu}\omega)\,\sigma_1\right]+\varepsilon_{\tilde{u},\eta}\left[\frac{\partial\mu_\eta}{\partial\kappa}+(\tilde{\mu}\omega+\kappa\sigma_1)\,\frac{\partial\sigma_\eta}{\partial\kappa}\right]+$$
$$+\frac{1}{\tilde{u}}\left(\frac{\partial^2\tilde{u}}{\partial\omega\partial\eta}+\frac{\partial^2\tilde{u}}{(\partial\eta)^2}\sigma_\eta\eta\right)\frac{\partial\sigma_\eta}{\partial\kappa}\eta \gtreqless 0\;, \tag{80}$$

with $\kappa=0$ if inequality "$<$" holds and $\kappa=\min\{\lambda\eta,1\}$ if the other inequality does so.

Note that $\frac{\partial\mu_\eta}{\partial\kappa}$ and $\frac{\partial\sigma_\eta}{\partial\kappa}$ are the partial derivatives with respect to κ of the RHS on expressions (74) and (75), respectively. Diffusion $\sigma_{\tilde{u}}$ is

$$\sigma_{\tilde{u}}=\frac{1}{\tilde{u}}\left[\frac{\partial\tilde{u}}{\partial\omega}+\frac{\partial\tilde{u}}{\partial\eta}\sigma_\eta\eta\right]. \tag{81}$$

■

Proposition 5 *The socially optimal allocation and its associated mappings* $\{\tilde{u},v,q\}$ *is analytically characterized by a system of second-order PDEs for the mappings in the state* $\{\omega,\eta\}$.

Proof. The equations that determine price q and value v are

$$\begin{cases}\frac{A_1-\iota_1}{q}+\mu_q+\mathcal{I}_1(\iota_1)+(\sigma_q+\sigma_1)\,\hat{\mu}\omega+\sigma_q\sigma_1-r+(\sigma_q+\sigma_1)\,\sigma_v=0 & \text{if } \kappa=1\\ \frac{A_2}{q}+\mu_q+\sigma_q\hat{\mu}\omega-r=0 & \text{otherwise}\end{cases}. \tag{82}$$

and

$$\alpha_1\frac{\kappa}{\eta}+\mu_v+\hat{\mu}\omega\sigma_v+\frac{\theta}{v}-\theta=0\;, \tag{83}$$

respectively.

The system of equations,

$$\{(32)\,,\;(33)\,,\;(86)\,,\;(74)\,,\;(75)\,,\;(79)\,,\;(80)\,,\;(82)\,,\;(83)\}\;, \tag{84}$$

thus determines a second-order PDEs for $\{\tilde{u},q,v\}$ in $\{\omega,\eta\}$.

We impose the following boundary conditions to the PDEs:

$$\lim_{\eta\to1}\sigma_x=0\;,\;\;\lim_{\eta\to1}\frac{\partial\sigma_x}{\partial\eta}=0\;. \tag{85}$$

for $x\in\{\tilde{u},q,v\}$. ■

Corollary 6 *In the economy with diagnostic expectations and without financial frictions, the socially optimal allocation is first-best efficient according to the expectation weight of the planner. If the planner is benevolent, the socially optimal allocation is the same as the equilibrium allocation. If the planner is paternalistic, the socially optimal allocation is the same as the equilibrium allocation of the economy presented in subsection 3.1.*

Proof. If the planner is benevolent, the socially optimal allocation is the same as the equilibrium allocation.

Let's postulate that $\partial\tilde{u}/\partial\eta = 0$. Thus,

$$r\tilde{u} = \max_{\{\iota_1,\kappa\}} \left\{ \kappa \left\{ A_1 - \iota_1 + \left[\mathcal{I}_1(\iota_1) + \sigma_1 \hat{\mu}\omega \right] \tilde{u} \right\} + (1-\kappa) A_2 + \right. \\ \left. + \frac{\partial \tilde{u}}{\partial \omega} \left(-\delta\omega + \tilde{\mu}\omega + \kappa\sigma_1 \right) + \frac{1}{2} \frac{\partial^2 \tilde{u}}{(\partial\omega)^2} \right\}. \tag{86}$$

Let's also postulate that $\tilde{u} = q$. Then, the equilibrium allocation solves the optimization problem, which verifies the postulates.

If the planner is paternalistic, the socially optimal allocation is the same as the equilibrium allocation of the economy presented in subsection 3.1.

Let's postulate that $\partial\tilde{u}/\partial\eta = \partial\tilde{u}/\partial\omega = 0$. Thus,

$$r\tilde{u} = \max_{\{\iota_1,\kappa\}} \left\{ \kappa \left\{ A_1 - \iota_1 + \mathcal{I}_1(\iota_1)\tilde{u} \right\} + (1-\kappa) A_2 \right\}.$$

Let's also postulate that $\tilde{u}$ equals the asset price of the economy presented in subsection 3.1. Then, the equilibrium allocation of that economy solves the optimization problem, which verifies the postulates. ■

2 Numerical Solution Method

To solve the PDEs we use spectral methods. Specifically, we interpolate $\{q, v\}$ or $\{\tilde{u}, q, v\}$ with linear combinations of Chebyshev polynomials of the first kind. We evaluate the interpolation at the Chebyshev nodes. We use a nonlinear solver to find the coefficients associated with the polynomials in the linear combination. As initial guess for the solver, we use the values of $\{q, v\}$ or $\{\tilde{u}, q, v\}$ in the economy of Section 3.1 of the paper.