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Differentiation in Risk Profiles

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Differentiation in Risk Profiles

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Abstract

This paper offers a model of vertical product differentiation in derivatives markets. Two dealers that choose their risk profile offer insurance to clients who differ in risk aversion. For given risk profiles, a unique price equilibrium exists in which the dealer with the lower risk profile has larger profits. Under plausible conditions, market discipline in the choice of risk profiles emerges: the first mover chooses a low risk profile, and the second mover follows at an optimal distance. The result serves as a reference point when considering the effects of introducing a central counterparty (CCP) that removes the quality dimension of competition.

JEL Classification Numbers: G12, G23, G28, L13, L15

Keywords: OTC Markets, Derivatives, Central Clearing, Imperfect Competition, Vertical Product Differentiation

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1 Introduction

Markets for risk transfer play a key role in the economy. They have been characterized by a hub-and-spoke structure with large banks at the core and numerous heterogeneous clients in the periphery (Abad et al., 2016; Hau et al., 2021). Since the financial crisis exposed the weaknesses of opaque and highly interconnected derivatives markets, there have been substantial changes in the market structure. Most notably, introducing a central node for clearing in derivatives markets – a *central counterparty (CCP)* – has gained importance.[1] Membership in CCPs is selective and typically only a few large banks are members. Most market participants instead access clearing as clients of members, which retains the hub-and-spoke structure and high level of concentration in centrally-cleared markets. On the one hand, a CCP can support financial stability through enforced margining (Biais et al., 2016), netting (Duffie and Zhu, 2011) as well as transparency for better regulatory oversight. At the same time, it changes the structure at the core of a highly concentrated market, and our understanding of how competition works therein – even before the push towards central clearing – is limited. In this paper, I study the mechanics of oligopolistic competition in risk transfer markets, and to that end present a model of differentiation in risk profiles.

The model naturally maps key features of risk transfer markets. In the model, two dealers that choose their risk profile (i.e., their own default probability) sell insurance to clients who wish to hedge against a common macro risk. Clients are differentiated in their level of risk aversion. Competition occurs in two stages: dealers sequentially choose their publicly observable risk profile before they engage in simultaneous price competition. This setup reflects three key characteristics of the market: Firstly, there is a hub-and-spoke structure with numerous clients that differ in their risk aversion and seek insurance from a small set of dealers. Secondly, risk aversion is the key driving force. Thirdly, there is competition between few risk-neutral dealers, and as part of the competition they choose how much effort to exert to ensure their solvency (e.g., setting aside capital or having balanced books). That is, they choose their risk profile, consistent with observed heterogeneity in credit default swaps (CDS) premia. Figure 1, for example, shows for eleven large dealers at the end of 2008 Q1 the heterogeneity in CDS premia.[2] The resulting model resembles models of vertical product differentiation with risk profiles akin to (inverse) qualities.

The key insight of the analysis is that when protection traded in risk transfer markets is seen as a good that can be differentiated in the price and the risk profile of its seller, competition in these two dimensions gives rise to a *market discipline in the choice of risk profiles.* This serves as a benchmark when conceptualizing the introduction of a CCP. When dealers are members of

1 A CCP replaces a contract between two market participants with two contracts that each have the CCP on one end. It thereby insulates the contracting parties from the risk that the counterparty defaults.

2 Following Duffie and Zhu (2011), participants of the ICE Trust for which data is available are considered (`https://www.theice.com/publicdocs/globalmarketfacts/docs/factsheets/ICE_CDS_Fact_Sheet.pdf`).

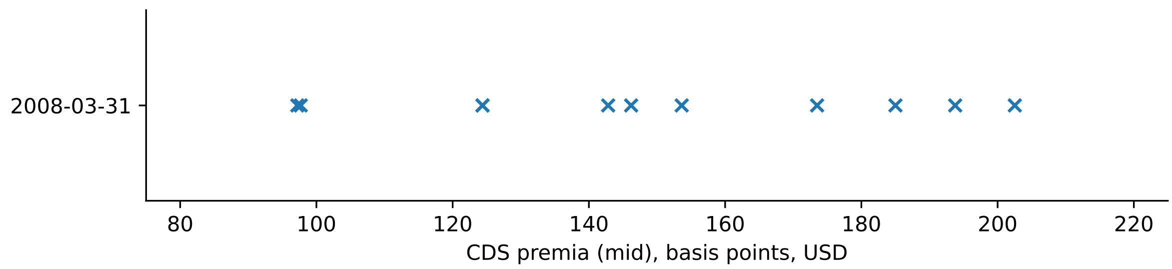

Figure 1: Heterogeneity in CDS premia. Source: Capital IQ, Fixed Income CDS, 5y senior.

a CCP, from the perspective of the client both offer the same probability of contract continuity. This is due to porting, that is, in case one member defaults, the portfolios of the clients of the defaulted member get ported to another solvent member (Braithwaite and Murphy, 2020). From the perspective of the client, there is thus no difference in risk profiles between dealers anymore and the market force disciplining the choice of risk profiles is absent. CCPs set membership criteria and mandate standards in risk profiles, however. The benchmark analyzed in this paper suggests that the risk profile thresholds mandated by CCPs should be as least as low as the risk profiles that arise endogenously from market discipline in the absence of a CCP.

The main results of the model are as follows. Firstly, in equilibrium dealers differentiate in terms of risk profile. The market is segmented with more risk-averse clients buying from the dealer with the lower risk profile at a higher price, and dealers make positive profits. On the contrary, if risk profiles are exogenously set to some same level, pure price competition prevails.

Secondly, the first mover is under pressure to choose a low risk profile. The mechanic behind the result is intuitive: In equilibrium, the dealer with the lower risk profile (i.e., that offers the product of higher quality) has a larger profit than the other dealer rendering the leadership position in quality more attractive. The first mover wants to occupy the leadership position in quality and faces the risk that the second mover takes over in the next period. As a result, the first mover chooses a sufficiently high quality to exclude this possibility.

Thirdly, there is a push-and-pull effect on the second mover's quality choice. For each pair of risk profiles, there exists a unique price equilibrium, and price equilibria are characterized almost fully by the difference in risk profiles. I analyze properties of subgame-perfect equilibria in risk profiles and identify two conditions under which one can fully characterize equilibrium risk profiles. Firstly, the profit of the second mover as a function of the second mover's risk profile must have a unique maximum, and, secondly, the profit of the first mover as a function of the second mover's risk profile needs to be increasing. If these conditions hold, for any (low) risk profile that the first mover chooses, the second mover follows at an optimal distance. In particular, the second mover chooses neither the same nor the maximal quality, but an optimally-distanced one, i.e., the first mover exerts push and pull forces on the second mover's quality choice. While the conditions cannot hold in general, I conjecture that they hold for a large range of parameters, and present a numerical example in which they hold.

Fourthly, in a numerical example with plausible parameter values, I explicitly characterize the subgame-perfect Nash equilibria. In this example, pressure on the first mover to choose a low risk profile is so large that the chosen default probability is close to zero. The second mover follows at a distance that is much smaller than the maximal possible differentiation.

Taken together, upward pressure in quality choice for the first mover and a push-and-pull effect on the second mover's quality choice can be seen as *market discipline in the choice of risk profiles*. When models of vertical product differentiation are used to study consumer goods, the notion of market discipline is not of particular interest because for consumer goods quality does not embed an aspect of stability of the system. In the insurance and derivatives context, however, the level of costly effort undertaken by individual market participants to ensure low levels of own default probability is connected to financial stability. Hence, a market force that provides an incentive to ensure low levels of counterparty risk *beyond regulation* is relevant for an assessment of the market microstructure.

Relation to the Literature. The paper is related to several strands of the literature. Firstly, an endogenously arising market discipline in quality choice is, to the best of my knowledge, new compared to the existing literature on vertical product differentiation, pioneered by Gabszewicz and Thisse (1979, 1980) and Shaked and Sutton (1982). In the standard model in Tirole (1988), that closely follows Shaked and Sutton (1982), two firms compete in quality (chosen first) and price (chosen second) for consumers that differ in their valuation of quality. The key mechanic is that for any two pairs of quality choices, firms choose prices in such a way that the resulting market shares remain unchanged. This eliminates a quantity effect and with only a price effect left, firms soften price competition as much as possible by choosing maximally differentiated qualities. In the present model, market discipline instead of maximal differentiation arises for three reasons. Firstly, I break the symmetry between firms not by assigning roles upfront, but by making the quality choice sequential. Then the risk of being overtaken creates upward pressure on the first mover's quality choice. Secondly, consumers' utility, which captures risk aversion, is non-linear. Thirdly, as a simplifying version of the assumption that there is no full market coverage, prices are capped. In the appendix, I revisit the standard model and show that removing the assumption of full market coverage and quality-invariant costs already gives rise to push-and-pull factors as well as upward pressure in the standard model. Different from Moorthy (1988, 1991), who lifts the same assumptions and numerically computes and compares outcomes using quadratic costs, I use a general convex cost function and derive the push factor directly from profit-maximizing incentives.

The second strand of the literature the paper contributes to is a growing literature on derivatives markets and central clearing. Seminal contributions have examined netting benefits (Duffie and Zhu, 2011), transparency (Acharya and Bisin, 2014), and the role of margins (Biais et al., 2012, 2016). Recently, the focus has shifted towards the question of how loss-sharing mechanisms of CCPs should optimally be designed (Cucic, 2022; Huang and Zhu, 2021; Kuong and Maurin, 2020; Wang et al., 2022), whether loss-sharing rules have heterogeneous effects across different types of market participants (Kubitza et al., 2021) and incentives of a profit-maximizing CCP (Huang, 2019). Carapella and Monnet (2020) study the effect of central clearing on the entry decision of dealers in derivatives markets. The idea is that, if more dealers enter as a result of the regulation, there is more intense competition and a resulting lower level of spreads may alter incentives to invest in efficient technologies ex-ante. A key difference to my model is that in Carapella and Monnet (2020) all agents are risk-neutral and the focus is on search frictions for dealers that intermediate derivatives.

The rest of the paper is organized as follows. Section 2 introduces the model framework. Section 3 derives key results on market segmentation. Section 4 illustrates the setup. Section 5 shows uniqueness and existence of price equilibria for any pair of quality choices. Section 6 examines properties of subgame-perfect equilibria in quality. Section 7 presents a numerical example. Section 8 contains a further discussion and section 9 concludes.

2 Model

2.1 Setup

There is a continuum of protection buyers, also called clients, with a hedging need and there are two protection sellers, also called dealers, with the capability to sell derivatives.

Timing. There are five points in time, $t \in \{0, 1, 2, 3, 4\}$. In the first three periods, the protection sellers engage in competition in price and quality. Specifically, in $t = 0$ protection seller 1 chooses a risk profile, i.e., own default probability b_1. Upon observing protection seller 1's risk profile, protection seller 2 chooses own default probability b_2.[3] In $t = 2$ they simultaneously choose fees $\gamma_i, i \in \{1, 2\}$, for establishing client-dealer relationships. Upon observing the protection sellers' choices (b_1, γ_1) and (b_2, γ_2), protection buyers decide from whom to buy in $t = 3$. Lastly, protection buyers' endowment risk materializes and payments are exchanged in $t = 4$.

Protection buyers/ Clients. Protection buyers face an uncertain endowment risk $\tilde{x} \in \{\overline{\theta}, \underline{\theta}\}$ with $\underline{\theta} < 0 < \overline{\theta}$. $\tilde{x}$ materializes in $t = 4$ taking the value $\underline{\theta}$ with probability p and $\overline{\theta}$ with probability $(1 - p)$. Suppose $E[\tilde{x}] = 0$.[4] The endowment risk is the *same* across all protection buyers. Protection buyers are risk-averse with utility function $u_a : \mathbb{R} \to \mathbb{R}$, $u_a \in \mathcal{C}^2$ that exhibits constant absolute risk aversion (CARA). Specifically,

$$u_a(x) = \frac{1}{a}\left(1 - \exp(-ax)\right) \qquad a \gneq 0. \tag{1}$$

The limit case $a = 0$ yields $u_0(x) = x$, i.e., risk neutrality for all payments. An increase in a corresponds to being more risk-averse. Protection buyers are characterized by their degree of absolute risk aversion a and a is assumed to be uniformly distributed over the interval $[\underline{a}, \overline{a}]$, $\underline{a} > 0$. (1) parameterizes the degree of absolute risk aversion, while satisfying the following two desirable normalizations: for all positive a, $u_a(0) = 0$ and $u'_a(0) = 1$. The second normalization achieves that, up to a first-order approximation, for small payments the utility coincides with the size of the payment – independent of the degree of risk aversion. This ensures that differences in risk aversion determine different attitudes towards large negative outcomes, but are irrelevant for very small payments.

Protection sellers/ Dealers. Protection sellers are risk-neutral and maximize profits. Protection seller $i \in \{1, 2\}$ defaults with probability b_i in the bad endowment state $\underline{\theta}$ and with probability 0 in the good endowment state $\overline{\theta}$. He faces costs c for offering the derivative. For now, the costs do not vary with the risk profile, and let $c = 0$.

3 One can think of default probabilities being publicly observable via CDS premia or ratings by rating agencies. They should be based on available information with particular emphasis on the measures an institution undertakes to ensure its solvency such as setting aside capital, having balanced trading books, etc. Another possibility how at least the ballpark of an institution's default probability can be common knowledge is through rumors in the market. For example, there were rumors on Lehman's insolvency weeks before it actually collapsed.

4 Otherwise $E[\tilde{x}]$ is a certain payment and consider the random variable $\tilde{x} - E[\tilde{x}]$ instead of $\tilde{x}$.

Derivative contract. Dealers offer to fully insure clients against their endowment risk in exchange for a fixed payment γ. A contract (b, γ), sold by protection seller with default probability b, is called *derivative* (b, γ), and a protection buyer with risk aversion parameter a derives the following utility from it:

$$U_a(b, \gamma) := (1 - bp)u_a(-\gamma) + bpu_a(\underline{\theta}). \tag{2}$$

From the perspective of the protection buyer, the derivative contract swaps the uncertain state-contingent endowment against a fixed payment of γ, unless the protection seller defaults which happens with probability (bp). In this case the protection buyer is left with the original bad endowment.

As shown in Appendix B1, full insurance is in fact the outcome of the following optimal contracting problem. Suppose clients, after deciding from whom to buy, choose trade volumes, i.e., payments (y, z) *from* the protection buyer *to* the protection seller with [5]

$$y \text{ due if } \tilde{x} = \overline{\theta} \text{ and the protection seller survives,}$$
$$z \text{ due if } \tilde{x} = \underline{\theta} \text{ and the protection seller survives.}$$

The payments (y, z) that maximize the expected utility of a protection buyer subject to the profit constraint of the protection seller, are such that the risk-averse protection buyer receives a state-independent amount, namely $(-\gamma)$, unless the counterparty defaults.

Figure 2 summarizes the timing of events as described.

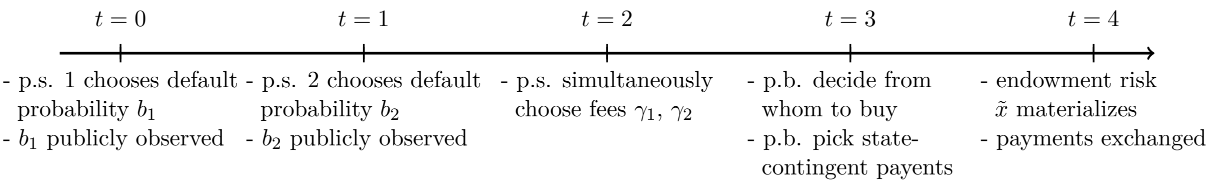

Figure 2: Timeline.

5 All payments are due in $t = 4$. This includes γ, which, although set ex-ante, is also exchanged in $t = 4$ and hence only due if the protection seller survives. $y, z < 0$ indicate that funds flow in the opposite direction, i.e. *from* the protection seller *to* the protection buyer.

2.2 Assumptions

The setup is kept simple and general. There will be no closed-form solution and, using implicit characterizations, one can show the economic forces at play directly. In order to focus on the question of interest, we need a few assumptions that restrict the setup to scenarios in which risk aversion is sufficiently relevant.

Assumption A1.

$$p < \frac{1}{3}.$$

This restricts range of probabilities for the bad endowment. One should think of the bad endowment $\underline{\theta}$ as a large negative number, and as an event that rarely occurs, hence p small.

Assumption A2.

$$\text{For } i \in \{1, 2\} : b_i \in [0, b^{max}] \text{ with } b^{max} \leq \frac{1}{3}.$$

Since this is no model of frequent defaults, b_i is bounded from above by $1/2$ a priori. Assumption A2 is slightly more restrictive. In the numerical example provided in section 7 the constraint is non-binding in equilibrium.

Assumption A3.

$$\text{For } i \in \{1, 2\} : \gamma_i \in [0, \gamma^{max}] \text{ with } \gamma^{max} \leq \frac{1}{3}(-\underline{\theta}).$$

As a consequence of assumption A3 in combination with assumption A4, $-a(\underline{\theta} + \gamma_i) > 2$ for all a and for all admissible fees $\gamma_i \in [0, \gamma^{max}]$.[6] This demands that the difference between the fee and the absolute value of the bad endowment, $(\gamma_i - (-\underline{\theta}))$, is still large enough such that risk aversion is relevant.

Note that $\gamma^{max} \leq (-\underline{\theta})$ by construction, since otherwise the fee exceeds the bad endowment. One can calculate the fee $(\gamma_a^{exit}(b))$ at which protection buyer with risk aversion a is indifferent between the insurance contract offered by protection seller with risk profile b and no insurance (see section B2 in the appendix for details). A priori, there is no market for prices exceeding $\gamma_{\overline{a}}^{exit}(0)$, i.e., the fee above which the most risk-averse client is unwilling to buy insurance even if offered at the highest quality. In the numerical example, this fee is lower than $(-\underline{\theta})/2$, while the fee at which the least risk-averse client is unwilling to buy insurance of highest quality is below the bound provided in Assumption A3. There, the constraint is also non-binding in equilibrium.

6 Note that $-a(\underline{\theta} + \gamma_i) > 2 \Leftrightarrow \gamma^{max} < 2/(-\underline{a}) - \underline{\theta}$. The RHS holds, since by assumption A3 $\gamma^{max} < (-\underline{\theta})/3$ and $2/(\underline{a}) - \underline{\theta} > (-\underline{\theta})/3 \Leftrightarrow (-\underline{a})\underline{\theta} > 3$, which is ensured by assumption A4.

Assumption A4.

$$\underline{a}(-\underline{\theta}) > \log\left[\frac{1-\frac{1}{8}}{\exp(-2)-\frac{1}{8}}\right] \approx 4.4.$$

Assumption A4 imposes a lower bound on the degree of risk aversion times the absolute value of the bad endowment, $a(-\underline{\theta})$ for all $a \in [\underline{a}, \overline{a}]$. It demands that even for the least risk-averse protection buyer, $\exp(-a\underline{\theta})$ is so large that the utility from the bad endowment, $u_a(\underline{\theta}) = 1/a \cdot (1 - \exp(-a\underline{\theta}))$, is large and negative. Demanding that risk aversion plays a role for all protection buyers is a condition on both the range of a and $\underline{\theta}$. For any large $\underline{\theta}$, one can find a small a such that assumption A4 is violated. Intuitively, for any large payment without limitations on a, one can find protection buyers whose utility is sufficiently close to a risk-neutral one (i.e. a close to 0) such that risk aversion barely kicks in. Assumption A4 rules out such almost risk-neutral protection buyers – relative to the bad endowment.

Assumptions A1 - A4 are always assumed. The remaining two assumptions are needed for the existence of protection seller 1's reaction function and assumed from section 6 onward.

Assumption A5.

$$\overline{a} \leq \frac{3}{2}\,\underline{a}.$$

Assumption A5 restricts the heterogeneity in risk aversion among clients. The system is highly sensitive to changes in risk aversion, so some restriction on the range of risk aversion seems warranted.

Assumption A6.

$$d_1\Pi_1\left(\gamma_1^{\underline{a}}(\gamma_2^*(\gamma^{max})), \gamma_2^*(\gamma^{max})\right) \geq 0.$$

The functions $\gamma_1^{\underline{a}}$ and γ_2^* will be formally introduced later. Intuitively, assumption A6 demands that at a point at which dealer 1 "owns" the entire market, dealer 1 has no incentive to decrease fees. The assumption is required, because a negative market share at negative prices also leads to positive turnover – a case certainly not of interest.

3 Market Segmentation

Until section 6, we take dealers' risk profiles as given and investigate Nash equilibria in prices in $t = 3$. To fix roles, protection seller 1 defaults with a lower probability or, in other words, offers the product of higher quality. That is, let $\Delta b := b_2 - b_1 > 0$.

We first characterize the protection buyer who is indifferent between two derivatives (b_1, γ_1) and (b_2, γ_2). The key idea is that the degree of risk aversion, a, translates into an "intensity in taste for quality". Let $\vec{b} := (b_1, b_2)$ and $\vec{\gamma} := (\gamma_1, \gamma_2)$ denote the pairs of risk profiles and fees.

Lemma 1 **(Characterization of the Indifferent Client).** *For given $\vec{\gamma}$ and $\vec{b}$ with $\Delta b > 0$, a protection buyer with degree of risk aversion a is indifferent between the two contracts (b_1, γ_1) and (b_2, γ_2) if*

$$g(a, \vec{\gamma}) := \frac{\exp(-a\Delta\gamma) - 1}{\exp(-a(\underline{\theta} + \gamma_2)) - 1} = \frac{p\Delta b}{1 - b_1 p} =: \tilde{g}(\vec{b}). \tag{3}$$

Proof. See Appendix A1. □

For any two derivatives with $b_2 > b_1$, if there is a solution to (3), then $\gamma_1 > \gamma_2$.[7] That is, the protection seller that offers the product of higher quality sets the higher price.[8]

The following main result of this section establishes that there is at most one protection buyer, characterized by some a^*, who is indifferent between derivatives (b_1, γ_1) and (b_2, γ_2) and segments the market.

Proposition 1 **(Market Segmentation).** *For given $\vec{b}$ and $\vec{\gamma}$, there is at most one $a^*(\vec{\gamma})$ satisfying*

$$g(a^*(\vec{\gamma}), \vec{\gamma}) = \tilde{g}(\vec{b}) = \frac{p\Delta b}{1 - b_1 p}. \tag{4}$$

Such an $a^*(\vec{\gamma}) \in [\underline{a}, \overline{a}]$ indeed exists, if

$$g(\overline{a}, \vec{\gamma}) \leq \frac{p\Delta b}{1 - b_1 p} \leq g(\underline{a}, \vec{\gamma}). \tag{5}$$

In this case, protection buyer a will choose protection seller 1 iff

$$a \geq a^*(\vec{\gamma}). \tag{6}$$

7 To see this, note that with $\Delta b > 0$, the RHS of (3) is positive. The denominator of the LHS of (3) is positive, which necessitates $\Delta\gamma < 0$.

8 A priori, there may be protection buyers who prefer no insurance over a derivative contract (b, γ). The fee at which a protection buyer is indifferent between (b, γ) and no insurance is increasing in the level of risk aversion (see Appendix B2 for the derivation and discussion). In other words, for a fixed b and as the fee increases, the protection buyer that first exits the market is the least risk-averse.

Proof. See Appendix A2. ☐

The idea of the proof is to show that $g(a, \vec{\gamma})$ is strictly decreasing in a, while the RHS of (4) is fixed. Hence, there can be at most one solution, and (5) indeed guarantees existence of a unique indifferent client.

Subsequently, protection buyers in the market segment $[a^*(\vec{\gamma}), \overline{a}]$ buy from protection seller 1, while protection seller 2 receives the market share $[\underline{a}, a^*(\vec{\gamma})]$, as depicted in Figure 3.

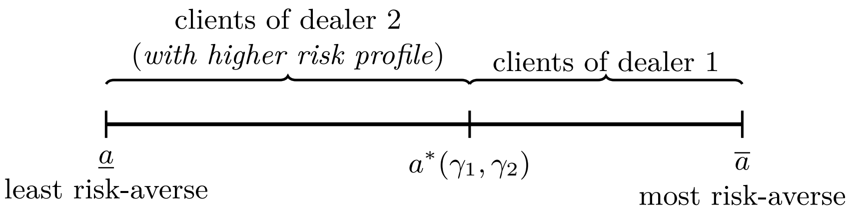

Figure 3: Market segmentation for two given derivatives (b_1, γ_1) and (b_2, γ_2) with $b_2 > b_1$.

Importantly, if risk profiles are fixed at some same level, pure price competition drives profits to zero.

Corollary 1 (Zero Profits with Equal Risk Profiles). *If $\Delta b = 0$, a client can be indifferent only if $\Delta\gamma := \gamma_2 - \gamma_1 = 0$. That is, if the dealers' risk profiles coincide, pure price competition drives prices to marginal costs (which are set to zero here).*

Formally, for a given vector of default probabilities, $\vec{b}$, a^* is defined via $g(a^*(\vec{b}, \vec{\gamma}), \vec{\gamma}) = \tilde{g}(\vec{b})$ on the set

$$\mathcal{G}_{[\underline{a},\overline{a}]} := \{\vec{\gamma} \,|\, 0 \le \gamma_2 < \gamma_1 \le \gamma^{max} \text{ and } g(\underline{a}, \vec{\gamma}) \le \tilde{g}(\vec{b}) \le g(\overline{a}, \vec{\gamma})\}. \tag{7}$$

Let $\mathcal{G}_0 := \{\, 0 \le \gamma_2 < \gamma_1 \le \gamma^{max}\}$. Then protection sellers' profits read

$$\Pi_1(\gamma_1, \gamma_2) = \begin{cases} (\overline{a} - a^*(\gamma_1, \gamma_2))\, \gamma_1 & \text{on } \mathcal{G}_{[\underline{a},\overline{a}]} \\ (\overline{a} - \underline{a})\, \gamma_1 & \text{on } \mathcal{G}_0 \setminus \mathcal{G}_{[\underline{a},\overline{a}]} \text{ if } \tilde{g}(\vec{b}) \le g(\underline{a}, \vec{\gamma}) \\ 0 & \text{on } \mathcal{G}_0 \setminus \mathcal{G}_{[\underline{a},\overline{a}]} \text{ if } g(\overline{a}, \vec{\gamma}) \le \tilde{g}(\vec{b}) \end{cases} \tag{8}$$

$$\Pi_2(\gamma_1, \gamma_2) = \begin{cases} (a^*(\gamma_1, \gamma_2) - \underline{a})\, \gamma_2 & \text{on } \mathcal{G}_{[\underline{a},\overline{a}]} \\ 0 & \text{on } \mathcal{G}_0 \setminus \mathcal{G}_{[\underline{a},\overline{a}]} \text{ if } \tilde{g}(\vec{b}) \le g(\underline{a}, \vec{\gamma}) \\ (\overline{a} - \underline{a})\, \gamma_2 & \text{on } \mathcal{G}_0 \setminus \mathcal{G}_{[\underline{a},\overline{a}]} \text{ if } g(\overline{a}, \vec{\gamma}) \le \tilde{g}(\vec{b}) \end{cases} \tag{9}$$

In the following we restrict attention to the set $\mathcal{G}_{[\underline{a},\overline{a}]}$.

4 Illustration

In this section, we graphically illustrate the setup. In order to succinctly formulate properties and later results, we introduce some further notation. Define

$$\tilde{A} : [\underline{a}, \overline{a}] \times [0, -\underline{\theta})^2 \to \mathbb{R}, \quad (a, \vec{\gamma}) \mapsto \exp(-a\Delta\gamma) \tag{10}$$

$$\text{and} \quad \tilde{B}_i : [\underline{a}, \overline{a}] \times [0, -\underline{\theta}) \to \mathbb{R}, \quad (a, \gamma_i) \mapsto \exp(-a(\underline{\theta} + \gamma_i)) \tag{11}$$

and let

$$A(\vec{\gamma}) := \tilde{A}(a^*(\vec{\gamma}), \vec{\gamma}), \quad \text{and} \quad B_i(\vec{\gamma}) := \tilde{B}_i(a^*(\vec{\gamma}), \gamma_i) \tag{12}$$

be the two functions, defined on $[0, -\underline{\theta})^2$, one obtains when inserting the indifferent client $a^*(\vec{\gamma})$ into (10) and (11). Whenever clear from the context, the explicit dependence on $\vec{\gamma}$ is omitted.

Lemma 1 has characterized the indifferent client $a^*(\vec{\gamma})$ implicitly. There is a second characterization of the indifferent client, symmetric to the one in Lemma 1, and exploiting this symmetry will be key in the sequel.

Lemma 2 (**Symmetric Characterization of the Indifferent Client**). *The protection buyer, a, that is indifferent between two derivatives (b_1, γ_1) and (b_2, γ_2) has a second characterization*

$$h(a, \vec{\gamma}) := \frac{1 - \exp(-(-a\Delta\gamma))}{\exp(-a(\underline{\theta} + \gamma_1)) - 1} = \frac{p\Delta b}{1 - b_2 p}. \tag{13}$$

Proof. See Appendix A3. □

Since the RHSs of (3) and (13) are constant, we infer that the respective LHSs, i.e.

$$g(a^*(\vec{\gamma}), \vec{\gamma}) = \frac{A(\vec{\gamma}) - 1}{B_2(\vec{\gamma}) - 1} \quad \text{and} \quad h(a^*(\vec{\gamma}), \vec{\gamma}) = \frac{1 - \frac{1}{A(\vec{\gamma})}}{B_1(\vec{\gamma}) - 1}, \tag{14}$$

are constants, and call them g and h respectively. Finally, define

$$\xi_2 := (\underline{\theta} + \gamma_2), \quad \varphi_1 := \xi_2 B_1 \quad \text{and} \quad \tau_1 := (\Delta\gamma - g\varphi_1), \tag{15}$$

$$\text{as well as} \quad \xi_1 := (\underline{\theta} + \gamma_1), \quad \varphi_2 := \xi_1 B_2, \quad \text{and} \quad \tau_2 := (\Delta\gamma - h\varphi_2). \tag{16}$$

The following lemma shows that both firms indeed loose market share when increasing fees.

Lemma 3 (**Market Shares**). *The indifferent client is increasing in γ_1 and decreasing in γ_2, namely*

$$d_1 a^* = \frac{a^*}{\tau_1} > 0 \tag{17}$$

$$d_2 a^* = \frac{-a^*}{\tau_2} < 0. \tag{18}$$

For the slope of a contour line $\{(\gamma_1, \gamma_2) | \, a^*(\gamma_1, \gamma_2)$ *constant*$\}$ *we have*

$$\frac{-d_2 a^*}{d_1 a^*} =: \alpha < 1. \tag{19}$$

Proof. See Appendix A4. □

Figure 4 visualizes the set up with the admissible fees $[0, \gamma^{max}]$ of dealer 1 and 2 on the x- and y-axis respectively. With dealer 1 the dealer with the lower risk profile, fees lie below

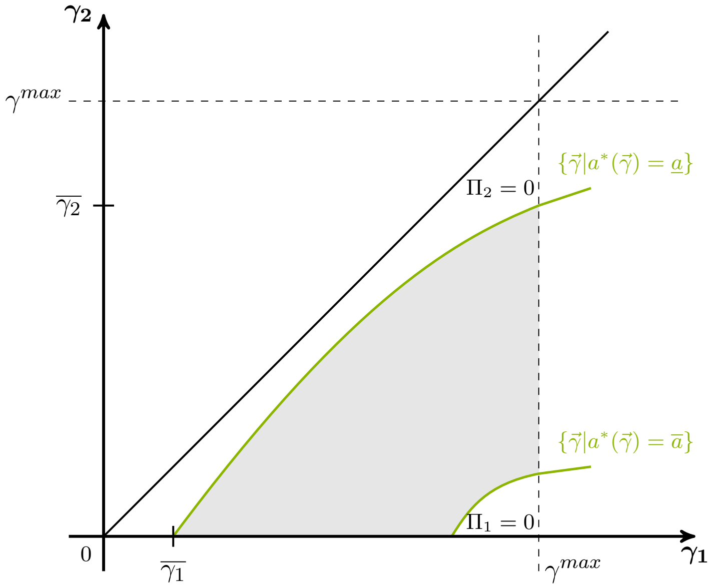

Figure 4: Illustration of the set $\mathcal{G}_{[\underline{a}, \overline{a}]}$.

the diagonal. The green line just below the diagonal depicts the pairs of fees for which the indifferent client a^* takes value $\underline{a}$. We parameterize these pairs by defining for $\gamma_2 \in [0, \gamma^{max}]$ $\gamma_1^{\underline{a}}(\gamma_2)$ such that $a^*(\gamma_1^{\underline{a}}(\gamma_2), \gamma_2) = \underline{a}$. From Lemma 3 we know that contour lines of a^* qualitatively take this shape. Above this line the unsafer dealer has no market share and subsequently no profits. Denote by $\overline{\gamma_1}$ and $\overline{\gamma_2}$ the intercepts of this line with the x- and y-axis respectively (see Lemma 7 in the appendix for a formal definition). Ensuring that the setup is interesting, i.e., that dealer 1 does not a priori get the entire market, requires $\overline{\gamma_2} > 0$, which is exactly assumption A4. Lemma 7 in the appendix formalizes the above.

5 Equilibrium in Prices

For a given pair of risk profiles, this section establishes existence and uniqueness of a Nash equilibrium in prices in $t = 3$. We first derive the reaction function for protection seller 2.

Proposition 2 (Dealer 2's Reaction Function). *Suppose some fixed $\vec{b}$ with $\Delta b > 0$. Then,*

i) for any $\gamma_1 \in [0, \gamma^{max}]$, there is a unique best response in fees for dealer 2, $\gamma_2^*(\gamma_1)$. For $\gamma_1 \in (\overline{\gamma}_1, \gamma^{max})$, γ_2^* is in the interior of $\mathcal{G}_{[\underline{a},\overline{a}]}$ and uniquely characterized via $d_2\Pi_2 = 0$.

ii) for $\gamma_1 \in [\overline{\gamma}_1, \gamma^{max}]$, γ_2^* is a smooth function and strictly increasing in γ_1.

iii) $d_1\gamma_2^* < 1/\alpha^*$ with $\alpha^* := \alpha(\gamma_1, \gamma_2^*(\gamma_1))$, i.e., α evaluated on dealer 2's reaction function.

Proof. See Appendix A6. □

The strategy of the proof is standard: One shows $d_2^2\Pi_2 < 0$ and existence follows, since profits are a continuous function that are zero at the boundaries of the interval.

For the other protection seller, existence of a reaction function is not straight-forward, since protection seller 1's profit function is not necessarily concave. In fact, parameter restrictions ensuring concavity are not compatible with the existing set of assumptions that require risk aversion to have enough bite. Without concavity of dealer 1's profit function, points that satisfy the first-order condition need not correspond to best responses.

Instead, we prove an auxiliary lemma (Lemma 4 in the appendix) for a smooth real-valued function f on some interval $[a, b]$ with $df(a) > 0$: If there exists a point in the interval below which local extrema may only be local mimina and above which local extrema may only be local maxima, then f has a global maximum. Assumptions A5 and A6 ensure that we can use this lemma to obtain dealer 1's best responses for the relevant interval, that is, for $\gamma_2 \in [0, \gamma_2^*(\gamma^{max})]$.

Proposition 3 (Dealer 1's Reaction Function). *Suppose assumptions A1 - A6. Suppose some fixed $\vec{b}$ with $\Delta b > 0$. Then,*

i) for any $\gamma_2 \in [0, \gamma_2^*(\gamma^{max})]$, there is a unique best response in fees for dealer 1, $\gamma_1^{\otimes}(\gamma_2)$. $\gamma_1^{\otimes}$ is uniquely characterized via

$$d_1\Pi_1(\gamma_1^{\otimes}(\gamma_2), \gamma_2) = 0 \qquad or \qquad (\gamma_1^{\otimes}(\gamma_2) = \gamma^{max} \, and \, \forall \mu \geq \gamma_1^{a}(\gamma_2) : d_1\Pi_1(\mu, \gamma_2) > 0).$$

ii) $\gamma_1^{\otimes}$ is a continuous function, smooth except at finitely many points. These exception points are well-behaved (see appendix for details).

iii) $d_2\gamma_1^{\otimes} < \alpha^{\otimes}$ with $\alpha^{\otimes} := \alpha(\gamma_1^{\otimes}(\gamma_2), \gamma_2)$, i.e., α evaluated on dealer 1's reaction function.

Proof. See Appendix A7. □

Proposition 4 (Existence and Uniqueness). *Suppose assumptions A1 - A6. Suppose some fixed $\vec{b}$ with $\Delta b > 0$. Then there exists a unique Nash equilibrium in prices.*

Proof. See Appendix A11. □

The intuition of the proof is as follows: Since dealer 2's reaction function, $\gamma_2^*(\gamma_1)$, is strictly increasing, there exists an inverse function, $\gamma_1^{*-1}(\gamma_2)$. Since $\gamma_1^{*-1}(0) = \gamma_1^{\overline{a}}(0)$ and $\gamma_1^{*-1}(\gamma^{max}) = \gamma^{max}$, dealer 1's reaction function and the inverse reaction function of dealer 2 must cross (see Figure 5). Formally, we apply Brouwer's Fixed Point Theorem for existence. From the bounds on $d_2\gamma_1^{\otimes}$ and $d_1\gamma_2^*$ in Propositions 3 and 2, respectively, it follows that there can be at most one intersection.

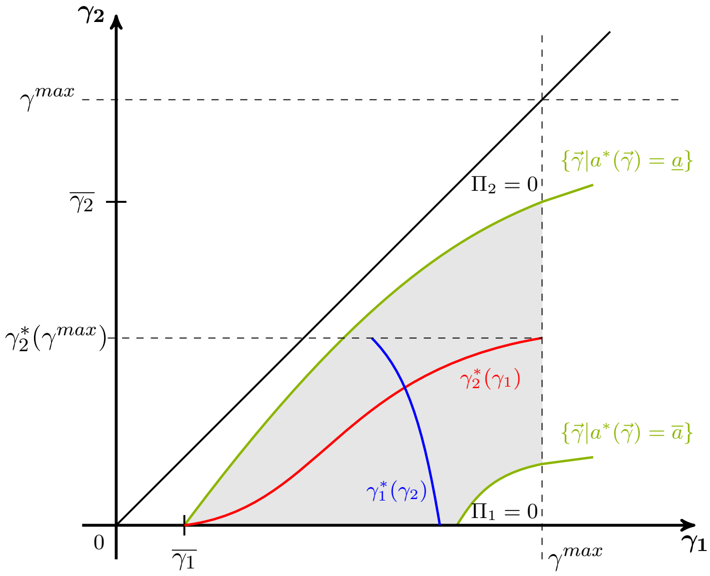

Figure 5: Price equilibrium

For quality choices $\vec{b} = (b_1, b_2)$, let $\vec{\gamma}^{\square}(\vec{b})$ be the corresponding price equilibrium. As shown in Proposition 4, the price equilibrium exists and is unique, hence $\vec{\gamma}^{\square}(\vec{b})$ is well-defined. In Appendix B4, we show that price equilibria are smooth functions in qualities. For a function $f(\vec{b}, \vec{\gamma})$ define

$$f^{\square}(\vec{b}) := f\left(\vec{\gamma}^{\square}(\vec{b}), \vec{b}\right). \tag{20}$$

6 Quality Choices

From now on we impose assumptions A1 - A6 to ensure existence of a unique price equilibrium for every pair of qualities $\vec{b} = (b_1, b_2)$. We start the analysis of subgame-perfect equilibria in qualities by noting two key properties. Firstly, the dealer with the lower risk profile has larger profits.

Proposition 5 (Lower Risk Profile More Attractive). *At any Nash equilibrium in prices,*

i) the dealer with the lower risk profile (quality-leader) has larger profits, $\Pi_1 > \Pi_2$*,*

ii) the dealer with the lower risk profile has a larger market share, $(\overline{a} - a^*) > (a^* - \underline{a})$*.*

Proof. See Appendix A12. □

Secondly, the difference in qualities matters for the price equilibrium.

Proposition 6 (Difference in Qualities Matters).

i) Risk profiles (b_1^0, b_2^0) *and* (b_1, b_2) *with* $\tilde{g}(b_1, b_2) = \tilde{g}(b_1^0, b_2^0)$ *lead to the same price equilibrium.*

ii) On the set of risk profiles $\{b \in [0, b^{max}]^2 | b_1 < b_2\}$*, price equilibria (and subsequently profits) are constant on straight lines with slope* $(1 - \tilde{g})$*.*

Proof. See Appendix A13. □

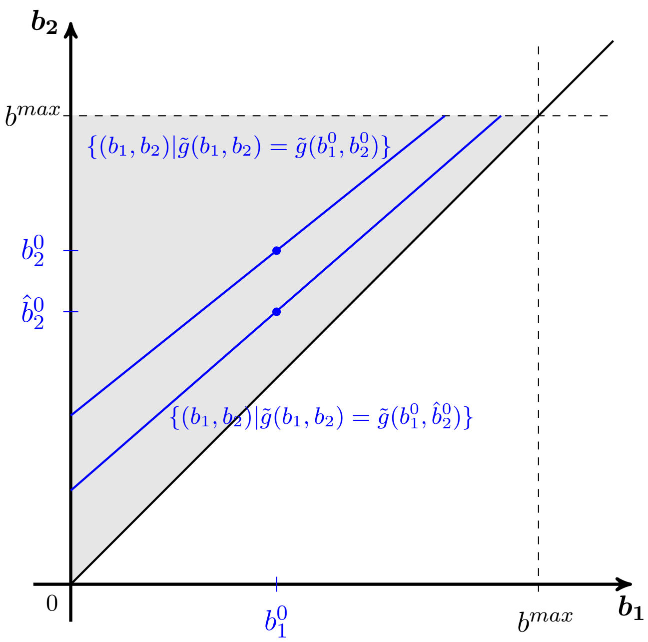

Figure 6: Risk profiles that lead to the same price equilibrium lie on straight lines

Figure 6 visualizes the result with risk profiles of dealer 1 and 2 on the x- and y-axis respectively. Since dealer 1 is the one with the lower risk profile, default probabilities lie above the diagonal. For (b_1^0, b_2^0) (or $(b_1^0, \hat{b}_2^0)$) the upper (or lower) blue line depicts all pairs of risk profiles that lead to the same value of $\tilde{g}$ and subsequently the same price equilibrium.

Proposition 7 (Upward Pressure in Qualities for First Mover). *Let (b_1^*, b_2^*) be a subgame-perfect Nash equilibrium in qualities.*

i) Then,

$$b_1^* < \left(\frac{8}{15}\right) b^{max}. \tag{21}$$

ii) Suppose the following two conditions hold

$$\Pi_2^{\square}(0, b_2) \text{ as a function of } b_2 \text{ has a unique maximum smaller than } b^{max}, \tag{N1}$$
$$\Pi_1^{\square}(0, b_2) \text{ as a function of } b_2 \text{ is increasing in } b_2. \tag{N2}$$

Let $\Pi_2^{\square,\tilde{g}^*}$ be the unique maximum of $\Pi_2^{\square}(0, b_2)$ as a function of b_2. Then,

$$b_1^* < \bar{b}_1 \text{ for any } \bar{b}_1 \text{ such that for some } \tilde{g}',\ \Pi_1^{\square,\tilde{g}'}(0, \bar{b}_1) = \Pi_2^{\square,\tilde{g}^*}. \tag{22}$$

Proof. See Appendix A14. □

The intuition is as follows: Consider a Nash equilibrium in qualities, $b_1 < b_2$. Then it must not be profitable for the second mover (protection seller 2) to take over the lead position in qualities. Proposition 6 characterizes the quality choices that lead to the same price equilibrium as straight lines. Thus, a profitable deviation of the second mover is to choose a quality which leads to the same price equilibrium, but with reversed roles. (21) rules out this profitable deviation. (22) tightens the bound if conditions (N1) and (N2) hold.

We now turn to the second mover.

Proposition 8 (Push-and-Pull Effect for Second Mover). *Suppose conditions* (N1) *and* (N2) *hold. Let $\Pi_2^{\square,\tilde{g}^*}$ be the unique maximum of $\Pi_2^{\square}(0, b_2)$, assumed at $\tilde{b_2}$. Let $\bar{b}_1$ be the minimum of all $\bar{b}_1$ defined as in Proposition 7. Then the subgame-perfect quality equilibria are all pairs (b_1^*, b_2^*) such that*

$$b_1^* \in [0, \bar{b}_1] \tag{23}$$
$$b_2^* = (1 - \tilde{g}(0, \tilde{b_2}))b_1^* + \tilde{g}(0, \tilde{b_2})/p. \tag{24}$$

The first mover's choice of risk profile thus exerts a push-and-pull effect on the second mover's *choice of risk profile.*

Proof. Direct consequence of Propositions 6 and 7. □

The intuition of the push-and-pull effect on the second mover is illustrated in Figure 7. The blue line depicts $\tilde{g}$ such that the second mover's profit is maximized (ensured by condition (N1)). The first mover needs to choose b_1 in the marked interval, since otherwise he risks loosing the leadership position in qualities. For any risk profile $b_1 < \bar{b}_1$ chosen by dealer 1, dealer 2's choice is pinned down by the blue line, i.e., the second mover follows at an optimal distance.

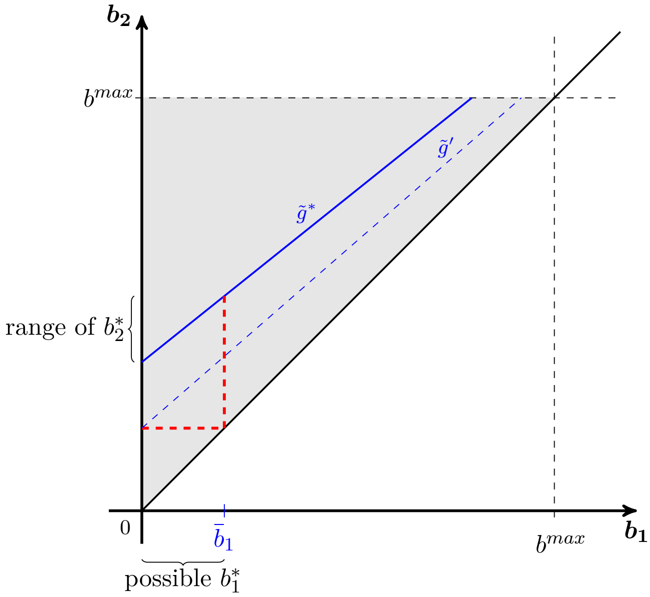

Figure 7: Equilibrium risk profiles

While conditions (N1) and (N2) cannot hold in general, e.g., picking b^{max} sufficiently small may violate (N1), I conjecture that they do for a wide range of parameters. They hold in a numerical example for plausible parameter values as shown in the following section.

7 Numerical Example

Parameter Values. Consider the model for a specific set of parameters, namely

$$\underline{\theta} = -100 \cdot 10^6 \tag{25}$$

$$p = 0.03 \tag{26}$$

$$\underline{a}(-\underline{\theta}) = 4.5 \tag{27}$$

$$\overline{a} = \frac{3}{2}\underline{a} \tag{28}$$

$$\gamma^{max} = 33 \cdot 10^6 \tag{29}$$

$$b^{max} = \frac{1}{3} \tag{30}$$

(25) and (26) correspond to a scenario with a large rare loss, e.g., a 100 million loss from a sudden movement in exchange rates that occurs every 33 years on average. (27), (28), (29) and (30) are chosen in the simplest way such that assumptions A4, A5, A3 and A2, respectively, are satisfied.

Based on Proposition 6, we first consider $b_1 = 0$.

Solving for the price equilibrium for $(0, b_2)$ *for some fixed* b_2. For $(0, b_2)$, we numerically solve for the indifferent client as a function of fees (γ_1, γ_2). For $b_2 = 0.15$, Figure 8 shows the resulting profit functions for both protection sellers.

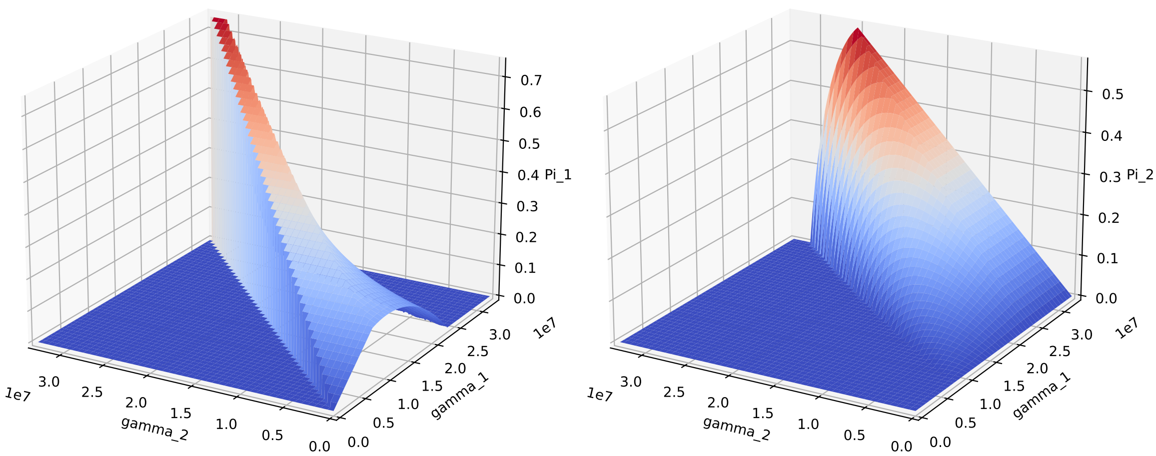

Figure 8: Profit functions of protection seller 1 (LHS) and 2 (RHS) for $b_2 = 0.15$

Equilibrium profits for $(0, b_2)$ *as a function of* b_2. We then solve for price equilibria (and subsequently profits) for a range of b_2. Figure 9 shows the resulting equilibrium profits for both protection sellers as a function of b_2.

In particular, protection seller 2's profit as a function of b_2 has a unique interior maximum,

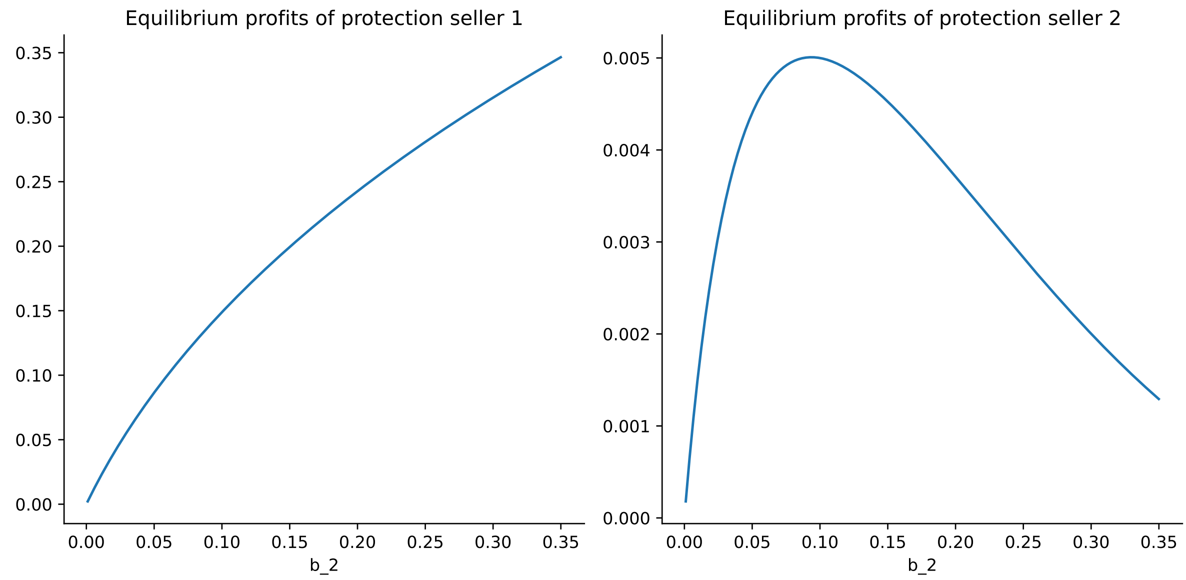

Figure 9: Equilibrium profits of protection seller 1 (LHS) and 2 (RHS) as a function of b_2.

while protection seller 1's profit as a function of b_2 is increasing. That is, conditions (N1) and (N2) hold.

Equilibrium qualities. We then calculate $\bar{b}_1 \approx 0.0023$, hence the resulting quality equilibria are

$$b_1^* \in [0, 0.0023] \tag{31}$$

$$b_2^* = 0.9972\, b_1^* + 0.0937 \tag{32}$$

Fees set in (any) equilibrium are depicted in Figure 10.

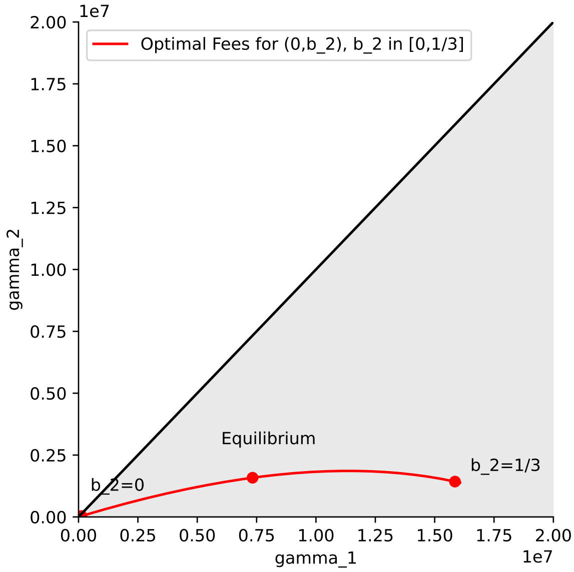

Figure 10: Equilibrium fees.

8 Discussion

Relation to Existing Models of Vertical Product Differentiation. The structure of the present model resembles models of vertical product differentiation with risk profiles akin to (inverse) qualities. In the standard model in Tirole (1988), maximal differentiation in qualities emerges. This section discusses differences between the models and results.

In the standard model in Tirole (1988), that closely follows Shaked and Sutton (1982), two firms compete in quality (chosen first) and price (chosen second) for consumers that differ in their valuation of quality. The key mechanic is that for any two pairs of quality choices, firms choose prices in such a way that the resulting market shares remain unchanged. This eliminates a quantity effect and with only a price effect left, firms soften price competition as much as possible by choosing maximally differentiated qualities. The result of maximal differentiation in qualities in the standard model hinges on three assumption: firstly, it is assumed that the market is always fully covered, secondly, costs are quality-invariant and, thirdly, consumers' utility is linear.

In my model, there is no maximal differentiation in qualities, but, instead, for the first mover there is the need to choose a high quality and for the second mover there exists a push-and-pull effect. There are three key differences between my model and the standard model. Firstly, I break the symmetry between firms not by assigning roles upfront (as in the standard model), but by making the quality choice sequential. This gives rise to the threat of loosing the leadership position in quality for the first mover. Secondly, consumers' utility, which captures risk aversion in the present setup, is non-linear. As outlined above, in the standard model the market shares remain invariant at price equilibria for varying quality pairs. This is no longer the case in my model due to client's non-linear utility. Thirdly, prices are capped.

In Online Appendix C, I revisit the standard model and show that market discipline in quality choice also emerges in the standard model when one aligns two differences (sequential quality choice, no full market coverage) between the models. Specifically, with sequential quality choice and in the absence of the assumptions that the market is always fully covered and that costs are quality-invariant, upward pressure on the first mover's quality choice as well as a push-and-pull factor emerge then as well.

9 Conclusion

In this paper, I provide a simple model of differentiation in risk profiles to better understand competition in risk transfer markets. Two risk-neutral dealers offer insurance to a continuum of clients that are heterogeneous in their risk aversion. Derivative contracts are differentiated along two dimensions: the price and the default probability (risk profile) of their sellers. The key insight from the model is that in this case, competition in these two dimensions gives rise to market discipline in the choice of risk profiles by dealers. Such a market force that incentivizes efforts to ensure low levels of own default probability beyond regulation is relevant for the assessment of the market microstructure and stability of the market. The insight may serve as a benchmark when one conceptualizes the introduction of a CCP. In a market with a CCP, there is no difference in risk profiles across sellers from the perspective of the client, and thus the quality dimension of the competition is absent.

While one can reasonably conceptualize a CCP in the context of the model, a CCP is not formally introduced. Hence, many aspects of a CCP (e.g. loss sharing mechanisms, margins, default probability of the CCP, etc) are outside of the scope of the current model. Keeping the current framework that maps key characteristics of the market such as a two-tiered market structure and risk aversion as key driver, while modeling a CCP in more detail is left for future research.

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A Appendix: Proofs

A1 Proof of Lemma 1

For the indifferent protection buyer we have

$$\begin{aligned}
U_a(b_1,\gamma_1) &= U_a(b_2,\gamma_2) && \text{(A3)}\\
\Leftrightarrow (1-b_1p)u_a(-\gamma_1)+b_1pu_a(\underline{\theta}) &= (1-b_2p)u_a(-\gamma_2)+b_2pu_a(\underline{\theta}) && \text{(A4)}\\
\Leftrightarrow u_a(-\gamma_1)-u_a(-\gamma_2)+p\,[b_2u_a(-\gamma_2)-b_1u_a(-\gamma_1)] &= p\Delta b u_a(\underline{\theta}) && \text{(A5)}\\
\Leftrightarrow [u_a(-\gamma_1)-u_a(-\gamma_2)]\,(1-b_1p) &= p\Delta b\,[u_a(\underline{\theta})-u_a(-\gamma_2)] && \text{(A6)}\\
\Leftrightarrow \frac{u_a(-\gamma_1)-u_a(-\gamma_2)}{u_a(\underline{\theta})-u_a(-\gamma_2)} &= \frac{p\Delta b}{1-b_1p} && \text{(A7)}\\
\Leftrightarrow \frac{\exp(-a\Delta\gamma)-1}{\exp(-a(\underline{\theta}+\gamma_2))-1} &= \frac{p\Delta b}{1-b_1p}. && \text{(A8)}
\end{aligned}$$

□

A2 Proof of Proposition 1

ad i). The proof proceeds by showing that $\partial_a g < 0$. Suppose this was true. Then the LHS of (3) is monotonically decreasing, while the RHS of (3) is fixed, yielding at most one solution.

Claim. $\partial_a g < 0$.

Proof of claim. For the derivative of the function g with respect to a we get

$$\begin{aligned}
\frac{\partial g(a)}{\partial a} &= \frac{-\Delta\gamma\exp(-a\Delta\gamma)\,(\exp(-a(\underline{\theta}+\gamma_2))-1)}{(\exp(-a(\underline{\theta}+\gamma_2))-1)^2} && \text{(A9)}\\
&\quad+\frac{(\exp(-a\Delta\gamma)-1)\,(\underline{\theta}+\gamma_2)\exp(-a(\underline{\theta}+\gamma_2))}{(\exp(-a(\underline{\theta}+\gamma_2))-1)^2}\\
&= \frac{1}{(\exp(-a(\underline{\theta}+\gamma_2))-1)^2}\Big[\exp(-a\Delta\gamma)\Big(-\Delta\gamma\,(\exp(-a(\underline{\theta}+\gamma_2))-1) && \text{(A10)}\\
&\quad+(\underline{\theta}+\gamma_2)\exp(-a(\underline{\theta}+\gamma_2))\Big)-(\underline{\theta}+\gamma_2)\exp\left(-a(\underline{\theta}+\gamma_2)\right)\Big]\\
&= \frac{1}{(\exp(-a(\underline{\theta}+\gamma_2))-1)^2} && \text{(A11)}\\
&\quad\Big[\exp(-a\Delta\gamma)\Big(\exp\left(-a(\underline{\theta}+\gamma_2)\right)(\underline{\theta}+\gamma_1)+\Delta\gamma\Big)-(\underline{\theta}+\gamma_2)\exp(-a(\underline{\theta}+\gamma_2))\Big]\\
&= \underbrace{\frac{\exp(-a\Delta\gamma)}{(\exp(-a(\underline{\theta}+\gamma_2))-1)^2}}_{>0} && \text{(A12)}\\
&\quad\Big[\underbrace{\Delta\gamma}_{<0}+\underbrace{\exp\left(-a(\underline{\theta}+\gamma_1)\right)}_{>0}\underbrace{\Big(\exp(-a\Delta\gamma)(\underline{\theta}+\gamma_1)-(\underline{\theta}+\gamma_2)\Big)}_{:=f(a)}\Big]
\end{aligned}$$

using that

$$\exp(-a(\underline{\theta}+\gamma_2)) = \exp(-a(\underline{\theta}+\gamma_1))\exp(-a\Delta\gamma). \tag{A13}$$

Then

$$f(a) < 0 \Rightarrow \frac{\partial g(a)}{\partial a} < 0. \tag{A14}$$

We have

$$\begin{aligned} f(a) = \exp(-a\Delta\gamma)(\underline{\theta}+\gamma_1) - (\underline{\theta}+\gamma_2) < 0 && \text{(A15)}\\ \Leftrightarrow \exp(-a\Delta\gamma)(\underline{\theta}+\gamma_1) < (\underline{\theta}+\gamma_2) && \text{(A16)}\\ \Leftrightarrow \frac{\exp(-a(\underline{\theta}+\gamma_2))}{\exp(-a(\underline{\theta}+\gamma_1))}(\underline{\theta}+\gamma_1) < (\underline{\theta}+\gamma_2) && \text{(A17)}\\ \Leftrightarrow \frac{\exp(-a(\underline{\theta}+\gamma_2))}{(\underline{\theta}+\gamma_2)} < \frac{\exp(-a(\underline{\theta}+\gamma_1))}{(\underline{\theta}+\gamma_1)}. && \text{(A18)} \end{aligned}$$

For $x < 0$ the function

$$h(x) := \frac{\exp(-ax)}{x} \tag{A19}$$

is negative and

$$h'(x) = h(x)\left[-a - \frac{1}{x}\right] > 0 \;\;\Leftrightarrow\;\; a + \frac{1}{x} > 0 \;\;\Leftrightarrow\;\; a(-x) > 1. \tag{A20}$$

For $x = \underline{\theta} + \gamma$ this is true from assumption A3. Since $\underline{\theta} + \gamma_2 < \underline{\theta} + \gamma_1$, (A18) indeed holds and proves the claim.

ad ii). With $g(\cdot, \vec{\gamma})$ strictly decreasing, existence under (5) follows immediately.

ad iii). A protection buyer with risk aversion parameter a chooses protection seller 1 if

$$\begin{aligned} U_a(b_1,\gamma_1) > U_a(b_2,\gamma_2) && \text{(A21)}\\ \Leftrightarrow (1-b_1p)u_a(-\gamma_1) + b_1pu_a(\underline{\theta}) > (1-b_2p)u_a(-\gamma_2) + b_2pu_a(\underline{\theta}) && \text{(A22)}\\ \Leftrightarrow [u_a(-\gamma_1) - u_a(-\gamma_2)](1-b_1p) > p\Delta b \underbrace{(u_a(\underline{\theta}) - u_a(-\gamma_2))}_{<0} && \text{(A23)}\\ \Leftrightarrow \frac{u_a(-\gamma_1) - u_a(-\gamma_2)}{u_a(\underline{\theta}) - u_a(-\gamma_2)} < \frac{p\Delta b}{1-b_1p} && \text{(A24)}\\ \Leftrightarrow g(a) < g(a^*) && \text{(A25)}\\ \Leftrightarrow a > a^*(\gamma_1,\gamma_2). && \text{(A26)} \end{aligned}$$

□

A3 Proof of Lemma 2

The idea is to proceed analogously to the proof of Proposition 1, but add and subtract $b_2u_a(-\gamma_1)$ instead of $b_1u_a(-\gamma_2)$. Namely, for the indifferent protection buyer we have

$$\begin{aligned} U_a(b_1,\gamma_1) = U_a(b_2,\gamma_2) && \text{(A27)}\\ \Leftrightarrow u_a(-\gamma_1) - u_a(-\gamma_2) + p\,[b_2u_a(-\gamma_2) - b_1u_a(-\gamma_1)] = p\Delta b u_a(\underline{\theta}) && \text{(A28)}\\ \Leftrightarrow [u_a(-\gamma_1) - u_a(-\gamma_2)](1-b_2p) = p\Delta b\,[u_a(\underline{\theta}) - u_a(-\gamma_1)] && \text{(A29)}\\ \Leftrightarrow \frac{u_a(-\gamma_1) - u_a(-\gamma_2)}{u_a(\underline{\theta}) - u_a(-\gamma_1)} = \frac{p\Delta b}{1-b_2p} && \text{(A30)} \end{aligned}$$

$$\Leftrightarrow \frac{1-\exp(-(-a\Delta\gamma))}{\exp(-a(\underline{\theta}+\gamma_1))-1} = \frac{p\Delta b}{1-b_2 p}. \tag{A31}$$

□

A4 Proof of Lemma 3

We show three claims from which the Lemma directly follows.

Claim 1. $a^* = \tau_1(d_1 a^*)$ with $\tau_1, (d_1 a^*) > 0$.

Claim 2. $a^* = \tau_2(-d_2 a^*)$ with $\tau_2, (-d_2 a^*) > 0$.

Claim 3. $(-d_2 a^*)/(d_1 a^*) =: \alpha = (1 - gB_1) = 1/(1 + hB_2) = \tau_1/\tau_2 < 1$.

Proof of claim 1. For the function $g(a^*(\vec{\gamma}), \vec{\gamma})$, as defined in (3), we have from Proposition 1

$$0 = d_1 g = \partial_1 g|_{a=a^*} + \partial_a g|_{a=a^*} \cdot d_1 a^* \tag{A32}$$

$$\Leftrightarrow d_1 a^* = \frac{-\partial_1 g|_{a=a^*}}{\partial_a g|_{a=a^*}}. \tag{A33}$$

In the following write $\partial_i g$ shorthand for $\partial_i g|_{a=a^*}$. We have

$$\partial_1 g = a^* \frac{A}{B_2 - 1} > 0. \tag{A34}$$

and from Proposition 1 we know that $\partial_a g < 0$. Hence, in light of (A33), we have $d_1 a^* > 0$.

Further, note that the expression for $\partial_a g$, derived in the proof of Proposition 1, can be written in short-hand notation as follows

$$\partial_a g = \frac{A}{(B_2-1)}\left[-\Delta\gamma + (\underline{\theta}+\gamma_2)\underbrace{\frac{(A-1)}{(B_2-1)}}_{=g}\underbrace{\frac{B_2}{A}}_{=B_1}\right] \overset{(A34)}{=} \frac{\partial_1 g}{a^*}\left[-\Delta\gamma + g\varphi_1\right]. \tag{A35}$$

Inserted into (A32) this yields

$$0 = \partial_1 g + \frac{\partial_1 g}{a^*}(-\Delta\gamma + g\varphi_1) d_1 a^* \tag{A36}$$

$$= \underbrace{\frac{\partial_1 g}{a^*}}_{>0}\left[a^* + (-\Delta\gamma + g\varphi_1) d_1 a^*\right]. \tag{A37}$$

Hence

$$a^* = (\Delta\gamma - g\varphi_1)\underbrace{d_1 a^*}_{>0}, \tag{A38}$$

and subsequently

$$\tau_1 = (\Delta\gamma - g\varphi_1) > 0. \tag{A39}$$

Proof of claim 2. Analogously, for the function $h(a^*(\vec{\gamma}), \vec{\gamma})$, as defined in (13), we have

$$0 = d_2 h = \partial_2 h|_{a=a^*} + \partial_a h|_{a=a^*} \cdot d_2 a^* \tag{A40}$$

$$\Leftrightarrow d_2 a^* = \frac{-\partial_2 h|_{a=a^*}}{\partial_a h|_{a=a^*}}. \tag{A41}$$

Similar to before we write $\partial_i h$ shorthand for $\partial_i h|_{a=a^*}$. Then we have

$$\partial_2 h = (-a^*)\frac{1}{A(B_1-1)} < 0, \tag{A42}$$

and

$$\begin{aligned}
\partial_a h &= (-\Delta\gamma)\frac{1}{A(B_1-1)} + (\underline{\theta}+\gamma_1)\frac{(1-\frac{1}{A})B_1}{(B_1-1)^2} && \text{(A43)}\\
&= \frac{1}{A(B_1-1)^2}\left[\Delta\gamma - \Delta\gamma B_1 - (\underline{\theta}+\gamma_1)B_1 + (\underline{\theta}+\gamma_1)AB_1\right] && \text{(A44)}\\
&= \underbrace{\frac{1}{A(B_1-1)^2}}_{>0}\left[\underbrace{\Delta\gamma}_{<0} + \underbrace{B_1}_{>0}\left(A(\underline{\theta}+\gamma_1) - (\underline{\theta}+\gamma_2)\right)\right]. && \text{(A45)}
\end{aligned}$$

From the proof of Proposition 1 we know that $A(\underline{\theta}+\gamma_1) - (\underline{\theta}+\gamma_2)$ is negative, hence $\partial_a h < 0$. Then from (A41) we get $d_2 a^* < 0$.

For the remaining part, note that $AB_1 = B_2$ and hence (A43) can also be written as

$$\begin{aligned}
\partial_a h &= \frac{1}{A(B_1-1)}\left[-\Delta\gamma + (\underline{\theta}+\gamma_1)AB_1\frac{(1-\frac{1}{A})B_1}{(B_1-1)}\right] && \text{(A46)}\\
&= \frac{\partial_a h}{a^*}\left[\Delta\gamma - \varphi_2 h\right]. && \text{(A47)}
\end{aligned}$$

Inserted into (A40) this yields

$$0 = \underbrace{\frac{\partial_2 h}{a^*}}_{<0}\left[a^* + (\Delta\gamma - \varphi_2 h)d_2 a^*\right]. \tag{A48}$$

Hence,

$$a^* = -(\Delta\gamma - \varphi_2 h)\underbrace{d_2 a^*}_{<0}, \tag{A49}$$

and subsequently

$$\tau_2 = (\Delta\gamma - \varphi_2 h) > 0. \tag{A50}$$

Proof of claim 3. We first establish that

$$(1 - gB_1) = \frac{B_1 - 1}{B_2 - 1} = \frac{1}{(1 + hB_2)}. \tag{A51}$$

This follows, since from the definition

$$\begin{aligned}
1 - gB_1 &= 1 - \frac{A-1}{B_2-1}\frac{B_2}{A} = \frac{B_2 - A}{A(B_2-1)} = \frac{B_1-1}{B_2-1} && \text{(A52)}\\
1 + hB_2 &= 1 + \frac{1-\frac{1}{A}}{B_1-1}B_2 = \frac{B_1 - 1 + B_2 - \frac{B_2}{A}}{B_1-1} = \frac{B_2-1}{B_1-1}. && \text{(A53)}
\end{aligned}$$

In light of (17) and (18) we have

$$\begin{aligned}
\alpha &= \frac{\Delta\gamma - g\varphi_1}{\Delta\gamma - h\varphi_2} && \text{(A54)}\\
&= \frac{(\underline{\theta}+\gamma_2) - (\underline{\theta}+\gamma_1) - g\varphi_1}{(\underline{\theta}+\gamma_2) - (\underline{\theta}+\gamma_1) - h\varphi_2} && \text{(A55)}\\
&= \frac{(\underline{\theta}+\gamma_2)(1-gB_1) - (\underline{\theta}+\gamma_1)}{-(\underline{\theta}+\gamma_1)(1+hB_2) + (\underline{\theta}+\gamma_2)} && \text{(A56)}\\
&= \frac{(1-gB_1)\left((\underline{\theta}+\gamma_2) - \frac{1}{(1-gB_1)}(\underline{\theta}+\gamma_1)\right)}{(\underline{\theta}+\gamma_2) - (1+hB_2)(\underline{\theta}+\gamma_1)} && \text{(A57)}\\
&= (1-gB_1), && \text{(A58)}
\end{aligned}$$

which concludes the proof.

□

A5 Auxiliary Properties

Proposition 9. *As always, we consider the set $\mathcal{G}_{[\underline{a},\overline{a}]}$. Then the following properties hold*

$$\begin{aligned}
d_2^2 a^* &= \frac{(-d_2a^*)}{\tau_2}\left[2 + h\varphi_2 a^*(1-\frac{\xi_2}{\tau_2})\right] < 0 && \text{(A59)}\\
d_1d_2a^* &= (d_1a^*)^2\frac{\alpha}{a^*}\left[a^*\xi_2\frac{h\varphi_2}{\tau_2} - 2\right] > 0 && \text{(A60)}\\
d_1^2a^* &= \left[\frac{2}{a^*} - g\varphi_1\xi_2\frac{\alpha}{\tau_1}\right] && \text{(A61)}\\
d_2^2\Pi_2 &= (d_2a^*)\left[2 + \frac{\gamma_2}{\tau_2}\left(\frac{(a^*\xi_1)h\varphi_2}{\tau_1} - 2\right)\right] < 0 && \text{(A62)}\\
d_1d_2\Pi_2 &= (d_1a^*)\left[1 + \frac{\gamma_2}{\tau_2}\left(\frac{(a^*\xi_2)h\varphi_2}{\tau_2} - 2\right)\right] > 0 && \text{(A63)}\\
d_1^2\Pi_1 &= (-d_1a^*)\left[2 + \frac{\gamma_1}{\tau_1}\left(2 - \frac{a^*\xi_2 g\varphi_1}{\tau_2}\right)\right] && \text{(A64)}\\
d_1d_2\Pi_1 &= (-d_2a^*)\left[1 + \frac{\gamma_1}{\tau_1}\left(2 - \frac{a^*\xi_2 h\varphi_2}{\tau_2}\right)\right] && \text{(A65)}\\
d_1^2\Pi_1 + \frac{1}{\alpha}d_1d_2\Pi_1 &< 0,\ \textit{hence}\ d_1^2\Pi_1 \neq 0 \vee d_1d_2\Pi_1 \neq 0. && \text{(A66)}
\end{aligned}$$

Proof.
ad $d_2^2a^*$. Note that

$$\frac{\xi_1}{\tau_1} - \frac{\xi_2}{\tau_2} = \frac{1}{\tau_1}\underbrace{[\xi_1 - \alpha\xi_2]}_{=-\tau_1} = -1 \tag{A67}$$

We have,

$$\begin{aligned}
d_2^2a^* &= d_2\left(-\frac{a^*}{\tau_2}\right) && \text{(A68)}\\
&= -\frac{d_2a^*}{\tau_2} + a^*\frac{1}{\tau_2^2}d_2\tau_2 && \text{(A69)}\\
&= -\frac{d_2a^*}{\tau_2}\left[1 + d_2\tau_2\right] && \text{(A70)}
\end{aligned}$$

$$= -\frac{d_2 a^*}{\tau_2}\left[1 + 1 + h\xi_1 B_2(a^* + (d_2 a^*)\xi_2)\right] \tag{A71}$$

$$= -\frac{d_2 a^*}{\tau_2}\left[2 + h\varphi_2 a^*\left(1 - \frac{\xi_1}{\tau_2}\right)\right] \tag{A72}$$

$$= -\frac{d_2 a^*}{\tau_2}\left[2 + h\varphi_2 a^*\frac{\xi_1}{\tau_1}\right] \tag{A73}$$

$$= \underbrace{(d_2 a^*)}_{<0}\frac{1}{\tau_2^2}\left[-2\tau_2 + h\varphi_2(-a^*\xi_1)\frac{1}{\alpha}\right] \tag{A74}$$

$d_2^2 a^*$ is negative iff

$$-a^*\xi_1 > 2\underbrace{\frac{\tau_2}{-h\varphi_2}}_{\in(0,1)}\underbrace{\alpha}_{<1}, \tag{A75}$$

which is ensured by $-\underline{a}\xi_1 > 2$ from assumption A3.

ad $d_1 d_2 a^*$. We have

$$d_1\varphi_2 = B_2 + \xi_1 d_1 B_2 \tag{A76}$$

$$= B_2 + \xi_1 B_2(-d_2 a^*)\xi_2 \tag{A77}$$

$$= B_2\left((1 - \xi_1\xi_2 d_1 a^*\right) \tag{A78}$$

Then

$$d_1 d_2 a^* = -d_1\left(\frac{a^*}{\tau_2}\right) \tag{A79}$$

$$= -\frac{d_1 a^*\tau_2 - a^*(-1 - hB_2(1 - \xi_1\xi_2 d_1 a^*)}{\tau_2^2} \tag{A80}$$

$$= \frac{-d_1 a^*\left[\tau_2 - a^*\xi_1\xi_2 hB_2\right] + a^*(1 + hB_2)}{\tau_2^2} \tag{A81}$$

$$= \underbrace{\frac{-d_1 a^*}{\tau_2^2}}_{<0}\underbrace{\left[\tau_2 - a^*\xi_1\xi_2 hB_2 + \tau_1(1 + hB_2)\right]}_{=:W} \tag{A82}$$

Hence, $d_1 d_2 a^* > 0$ if the expression in brackets is negative. This is indeed the case, since

$$W = 2\Delta\gamma + hB_2\left[\tau_1 - \xi_1 - a^*\xi_1\xi_2\right] - g\varphi_1 \tag{A83}$$

$$= \Delta\gamma(2 + hB_2) - \xi_1 hB_2 \underbrace{(1 + a^*\xi_2)}_{=-1+(2+a^*(\underline{\theta}+\gamma_2))} - g\varphi_1(1 + hB_2) \tag{A84}$$

$$= hB_2\left[\xi_1 + \Delta\gamma\right] - g\xi_2 B_1(1 + hB_2) - \xi_1 hB_2(2 + a^*\xi_2) + 2\Delta\gamma \tag{A85}$$

$$= \xi_2\underbrace{(hB_2 - gB_1(1 + hB_2))}_{=\frac{\Delta B}{B_1-1}-\frac{\Delta B}{B_2-1}\frac{B_1-1}{B_2-1}=0} - \xi_1 hB_2(2 + a^*\xi_2) + 2\Delta\gamma \tag{A86}$$

$$= \underbrace{-\xi_1}_{>0} hB_2\underbrace{(2 + a^*\xi_2)}_{<0} + \underbrace{2\Delta\gamma}_{<0} \tag{A87}$$

$$< 0, \tag{A88}$$

which together yields

$$
\begin{aligned}
d_1 d_2 a^* &= \frac{(d_1 a^*) h\varphi_2}{\tau_2^2}\left(a^*\xi_2 + 2\frac{\tau_2}{(-h\varphi_2)}\right) && \text{(A89)}\\
&= \frac{(d_1 a^*) h\varphi_2}{\tau_2^2}\left[a^*\xi_2 + 2\frac{\tau_2}{(-h\varphi_2)}\right] && \text{(A90)}\\
&= (d_1 a^*)^2 \frac{\alpha}{a^*}\left[a^*\xi_2 \frac{h\varphi_2}{\tau_2} - 2\right]. && \text{(A91)}
\end{aligned}
$$

ad $d_1^2 a^*$*.* We know

$$
d_1\varphi_1 = \varphi_1\xi_2\alpha(d_1 a^*), \tag{A92}
$$

hence

$$
\begin{aligned}
d_1^2 a^* &= d_1\left(\frac{a^*}{\tau_1}\right) && \text{(A93)}\\
&= \frac{d_1 a^*}{\tau_1} + a^* d_1\left(\frac{1}{\tau_1}\right) && \text{(A94)}\\
&\overset{\text{(A92)}}{=} \frac{d_1 a^*}{\tau_1} - a^*\frac{1}{(\tau_1)^2}\left[-1 + g\varphi_1\xi_2\alpha(d_1 a^*)\right] && \text{(A95)}\\
&= \frac{d_1 a^*}{\tau_1}\left[2 - g\varphi_1\xi_2\alpha(d_1 a^*)\right] && \text{(A96)}\\
&= (d_1 a^*)\frac{a^*}{\tau_1}\left[\frac{2}{a^*} - g\varphi_1\xi_2\alpha\frac{d_1 a^*}{a^*}\right] && \text{(A97)}\\
&= (d_1 a^*)^2\left[\frac{2}{a^*} - g\varphi_1\xi_2\frac{\alpha}{\tau_1}\right]. && \text{(A98)}
\end{aligned}
$$

ad $d_2^2\Pi_2$*.* Using (A60) and (A59),

$$
\begin{aligned}
d_2^2\Pi_2 &= 2d_2 a^* + \gamma_2 d_2^2 a^* && \text{(A99)}\\
&= (d_2 a^*)\left[2 + \frac{\gamma_2}{\tau_2^2}\Big((-h\varphi_2)a^*(\tau_2 - \xi_2) - 2\tau_2\Big)\right] < 0 && \text{(A100)}
\end{aligned}
$$

We use $\xi_1/\tau_1 - \xi_2/\tau_2 = (-1)$ to simplify to

$$
d_2^2\Pi_2 = (d_2 a^*)\left[2 + \frac{\gamma_2}{\tau_2}\Big(\frac{(a^*\xi_1)h\varphi_2}{\tau_1} - 2\Big)\right]. \tag{A101}
$$

ad $d_1 d_2\Pi_2$*.* Using (A60) and (A59),

$$
\begin{aligned}
d_1 d_2\Pi_2 &= (d_1 a^*) + (d_1 d_2 a^*)\gamma_2 && \text{(A102)}\\
&= (d_1 a^*)\Big[1 + \frac{\gamma_2}{\tau_2^2}\underbrace{\Big((-a^*\xi_2)(-h\varphi_2) - 2\tau_2\Big)}_{:=E}\Big] > 0, && \text{(A103)}
\end{aligned}
$$

since $E > 0$ by assumption (A4). Again this further simplifies to

$$
d_1 d_2\Pi_2 = (d_1 a^*)\Big[1 + \frac{\gamma_2}{\tau_2}\Big(\frac{(a^*\xi_2)h\varphi_2}{\tau_2} - 2\Big)\Big]. \tag{A104}
$$

ad $d_1^2\Pi_1$. Using (A61),

$$\begin{aligned} d_1^2\Pi_1 &= d_1\left[(\overline{a}-a^*)-(d_1a^*)\gamma_1\right] && \text{(A105)}\\ &= -2(d_1a^*)-\gamma_1(d_1^2a^*) && \text{(A106)}\\ &= -d_1a^*\left[2+\gamma_1(d_1a^*)\left(\frac{2}{a^*}-g\varphi_1\xi_2\frac{\alpha}{\tau_1}\right)\right] && \text{(A107)}\\ &= -d_1a^*\left[2+\frac{\gamma_1}{\tau_1}\left(2-\frac{a^*\xi_2 g\varphi_1}{\tau_2}\right)\right]. && \text{(A108)} \end{aligned}$$

ad $d_1d_2\Pi_1$. Using (A60),

$$\begin{aligned} d_2d_1\Pi_1 &= d_2\left[(\overline{a}-a^*)-(d_1a^*)\gamma_1\right] && \text{(A109)}\\ &= (-d_2a^*)-\gamma_1(d_1a^*)^2\frac{\alpha}{a^*}\left[a^*\xi_2\frac{h\varphi_2}{\tau_2}-2\right] && \text{(A110)}\\ &= (-d_2a^*)\left[1+\frac{\gamma_1}{\tau_1}\left(2-\frac{a^*\xi_2 h\varphi_2}{\tau_2}\right)\right]. && \text{(A111)} \end{aligned}$$

ad $d_1^2\Pi_1+\frac{1}{\alpha}d_1d_2\Pi_1$. Using (A60) and (A62),

$$d_1^2\Pi_1+\frac{1}{\alpha}(d_1d_2\Pi_1) = (-d_1a^*)\left[2+\frac{\gamma_1}{\tau_1}\left(2-\frac{(a^*\xi_2)g\varphi_1}{\tau_2}\right)\right]+(d_1a^*)\left[1+\frac{\gamma_1}{\tau_1}\left(2-\frac{(a^*\xi_2)h\varphi_2}{\tau_2}\right)\right] \tag{A112}$$

$$\begin{aligned} &= (-d_1a^*)+(d_1a^*)\frac{\gamma_1}{\tau_1}\left[\frac{(a^*\xi_2)g\varphi_1}{\tau_2}-\frac{(a^*\xi_2)h\varphi_2}{\tau_2}\right] && \text{(A113)}\\ &= \underbrace{(-d_1a^*)}_{<0}+(d_1a^*)\frac{\gamma_1}{\tau_1}\underbrace{\frac{(-a^*\xi_2)}{\tau_2}}_{2/\tau_2>0}\underbrace{(h\varphi_2-g\varphi_1)}_{<0} && \text{(A114)}\\ &< 0, && \text{(A115)} \end{aligned}$$

with $(h\varphi_2-g\varphi_1)<0$, since

$$g\varphi_1-h\varphi_2 = hB_2\left[\underbrace{\frac{gB_1}{hB_2}}_{=\alpha}\xi_2-\xi_1\right] = hB_2\tau_2>0 \tag{A116}$$

where the last equality follows, since

$$\tau_2-\xi_2 = \Delta\gamma-h\varphi_2-(\xi_1+\Delta\gamma) = (-\xi_1)(1+hB_2) = \frac{-\xi_1}{\alpha}. \tag{A117}$$

$\square$

A6 Proof of Proposition 2

Auxiliary properties are proven in Appendix A5. We first prove the following central claim.

Claim. The following notation is used: For a function $f(\vec{\gamma})$ let $f^*(\gamma_1) := f(\gamma_1, \gamma_2^*(\gamma_1))$. Then,

$$d_1\gamma_2^* = \frac{(d_1 d_2 \Pi_2)^*}{(-d_2^2\Pi_2)^*}. \tag{A118}$$

Proof of claim. By definition, $0 \equiv (d_2\Pi_2)^*$ and thus

$$0 = d_1((d_2\Pi_2)^*) = (d_1 d_2\Pi_2)^* + (d_2^2\Pi_2)^* d_1\gamma_2^* \tag{A119}$$

$$\Leftrightarrow d_1\gamma_2^* = \frac{(d_1 d_2 \Pi_2)^*}{(-d_2^2\Pi_2)^*}. \tag{A120}$$

ad i). From (A62) we have concavity of Π_2, which ensures uniqueness of a solution. For existence, note that $\gamma_2 \mapsto \Pi_2(\gamma_1, \gamma_2)$ as continuous function on a compact interval, assumes its maximum. But $\Pi_2(\gamma_1, 0) = \Pi_2(\gamma_1, \gamma^{max}) = 0$, hence the maximum is assumed in the interior.

ad ii). For $\gamma_2^* \in \mathcal{C}^\infty$, we make use of the implicit function theorem. We know $d_2\Pi_2 \in \mathcal{C}^\infty$ and $d_2^2\Pi_2 < 0$. Hence, from the implicit function theorem the mapping

$$\gamma_1 \mapsto \gamma_2^*(\gamma_1) = \arg_{\gamma_2}\{d_2\Pi_2(\gamma_1, \gamma_2) = 0\} \tag{A121}$$

is smooth. Monotonicity of γ_2^* follows from the claim together with (A63) and (A62).

ad iii). Using (A62) and (A63),

$$\alpha^* d_1\gamma_2^* = \alpha^* \frac{(d_1 d_2\Pi_2)^*}{(-d_2^2\Pi_2)^*} \tag{A122}$$

$$= \alpha^* \frac{(d_1 a^*)\left[1 + \frac{\gamma_2}{\tau_2}\left(\frac{(a^*\xi_2)h\varphi_2}{\tau_2} - 2\right)\right]}{(-d_2 a^*)\left[2 + \frac{\gamma_2}{\tau_2}\left(\frac{(a^*\xi_1)h\varphi_2}{\tau_1} - 2\right)\right]} \tag{A123}$$

$$= \frac{1 + \frac{\gamma_2}{\tau_2}\left(\frac{(a^*\xi_2)h\varphi_2}{\tau_2} - 2\right)}{2 + \frac{\gamma_2}{\tau_2}\left(\frac{(a^*\xi_1)h\varphi_2}{\tau_1} - 2\right)}. \tag{A124}$$

In the numerator

$$\frac{(a^*\xi_2)h\varphi_2}{\tau_2} - 2 = (-a^*\xi_2)\frac{(-h\varphi_2)}{\tau_2} - 2 > 0, \tag{A125}$$

since $a^*(-\xi_2) > 2$ from assumption A3, and in the denominator

$$\frac{(a^*\xi_1)h\varphi_2}{\tau_1} - 2 = (a^*h\varphi_2)\left(1 - \frac{\xi_2}{\tau_2}\right) - 2 = \underbrace{\frac{(a^*\xi_2)h\varphi_2}{\tau_2} - 2}_{>0} + \underbrace{(-h\varphi_2)a^*}_{>0} > 0, \tag{A126}$$

hence, we get $\alpha^* d_1\gamma_2^* < 1$.

□

A7 Proof of Proposition 3

The proof proceeds by showing a basic lemma from real analysis (Lemma 4) and then proving its applicability in the present context (Lemma 5 and Lemma 6). The lemmata are presented upfront and proven in the subsequent appendices.

Notation. $\gamma_1^{\overline{a}}(\gamma_2)$ is defined similar to $\gamma_1^{\underline{a}}(\gamma_2)$. In particular, $\gamma_1^{\overline{a}}(\gamma_2)$ is defined as $a^*(\gamma_1^{\overline{a}}(\gamma_2), \gamma_2) = \overline{a}$ if there is a solution in $\mathcal{G}_{[\underline{a},\overline{a}]}$, and as $\gamma_1^{\overline{a}}(\gamma_2) = \gamma^{max}$ otherwise.

Lemma 4. *Let f be a smooth function on some interval $[a,b] \subset \mathbb{R}$. If there exists a $\mu \in [a,b]$ such that*

$$\forall x < \mu : \quad df(x) = 0 \Rightarrow d^2 f(x) > 0 \tag{S1}$$

$$\forall x > \mu : \quad df(x) = 0 \Rightarrow d^2 f(x) < 0 \tag{S2}$$

$$df(a) > 0, \tag{S3}$$

then f has a global maximum τ and $\forall x < \tau : df(x) > 0$ and $\forall x > \tau : df(x) < 0$.

Lemma 5. *Consider a fixed γ_2 for which*

$$d_1\Pi_1(\gamma_1^{\underline{a}}(\gamma_2), \gamma_2) > 0. \tag{T3}$$

If assumption A5 holds, there exists a $\mu \in [\gamma_1^{\underline{a}}(\gamma_2), \gamma_1^{\overline{a}}(\gamma_2)]$ such that for all $\gamma_1 \in [\gamma_1^{\underline{a}}(\gamma_2), \gamma_1^{\overline{a}}(\gamma_2)]$

$$\gamma_1 < \mu \Rightarrow \quad (d_1\Pi_1 = 0 \Rightarrow d_1^2\Pi_1(\gamma_1) > 0) \tag{T1}$$

$$\gamma_1 > \mu \Rightarrow \quad (d_1\Pi_1 = 0 \Rightarrow d_1^2\Pi_1(\gamma_1) < 0). \tag{T2}$$

If $\mu = \gamma_1^{\overline{a}}(\gamma_2)$, then $\gamma_1^{\overline{a}}(\gamma_2) = \gamma^{max}$.

Lemma 6. *Assumption A6 implies that, for all $\gamma_2 \in [0, \gamma_2^*(\gamma^{max})]$,*

$$d_1\Pi_1(\gamma_1^{\underline{a}}(\gamma_2), \gamma_2) > 0. \tag{A127}$$

ad i). Lemma 5 and Lemma 6 show that for any $\gamma_2 \in [0, \gamma_2^*(\gamma^{max})]$ we can make use of Lemma 4. Then we know from Lemma 4 that for all $\gamma_2 \in [0, \gamma_2^*(\gamma^{max})]$, a) $\Pi_1(\cdot, \gamma_2)$ has a unique global maximum $\tau \in [\gamma_1^{\underline{a}}(\gamma_2), \gamma_1^{\overline{a}}(\gamma_2)]$, b) $\tau = \operatorname{argmin}_\mu\{d_1\Pi_1 = 0 \vee \mu = \gamma^{max}\}$, i.e., τ is either the unique solution to $d_1\Pi_1(\tau) = 0$, or $\tau = \gamma^{max}$, and, c) for all $\gamma_1 < \tau$: $d_1\Pi_1(\gamma_1) > 0$ and for all $\gamma_1 > \tau$: $d_1\Pi_1(\gamma_1) < 0$.

ad ii). We consider $\gamma_1^{\otimes}$ separately on

$$\mathcal{N}_1 := \{\gamma_2 \in [0, \gamma_2^*(\gamma^{max})] | \gamma_1^{\otimes} < \gamma^{max}\} \tag{A128}$$

$$\mathcal{N}_2 := \{\gamma_2 \in [0, \gamma_2^*(\gamma^{max})] | \gamma_1^{\otimes} = \gamma^{max}\} \tag{A129}$$

and first show continuity on $[0, \gamma_2^*(\gamma^{max})] = \mathcal{N}_1 \,\dot{\cup}\, \mathcal{N}_2$.

On $\mathcal{N}_1$, we already know from (A65) that $d_1^2\Pi_1 \neq 0 \vee d_1 d_2 \Pi_1 \neq 0$. Hence $\{d_1\Pi_1 = 0\}$ is a smooth curve. Then we make a case distinction.

1) If $d_1^2\Pi_1 \neq 0$, we know from the implicit function theorem that one can parameterize $\{d_1\Pi_1 = 0\}$ via $\gamma_1^{\otimes}(\gamma_2)$. In particular, such a parameterization is smooth.

2) At a point $q = (\gamma_1^{\otimes}(\gamma_2), \gamma_2)$ with $d_1^2\Pi_1(q) = 0$, we have $d_1 d_2 \Pi_1(q) \neq 0$, hence one can parameterize $\{d_1\Pi_1 = 0\}$ locally via $\gamma_2^{\otimes}(\gamma_1)$. $\gamma_2^{\otimes}(\gamma_1)$ has to strictly increase in some neighborhood around q or strictly decrease in some neighborhood around q, since otherwise the inverse couldn't exist. Hence, $\gamma_2^{\otimes}$ is bijective on some neighborhood U of $\gamma_1^{\otimes}(\gamma_2)$ and V of $\gamma_2^{\otimes}$. Then $\gamma_1^{\otimes}$ is monotone on U and continuous.

Hence, $\gamma_1^{\otimes}$ is continuous on $\mathcal{N}_1$, $\mathcal{N}_1$ is open and $\gamma_1^{\otimes}$ is also continuous on the closure of $\mathcal{N}_1$, $\overline{\mathcal{N}}_1$. Since the complement of $\mathcal{N}_1$ is $\mathcal{N}_2$, $\mathcal{N}_2$ is closed. On $\mathcal{N}_2$, $\gamma_1^{\otimes}$ is a constant function and as such continuous on $\mathcal{N}_2$. Thus, since $\gamma_1^{\otimes}$ is continuous on $\overline{\mathcal{N}}_1$ and on $\mathcal{N}_2$, it is continuous on $[0, \gamma_2^*(\gamma^{max})]$.

It remains to show smoothness and that exception points are isolated.

Again, we consider $\mathcal{N}_1$ first. If $d_1^2\Pi_1 \neq 0$, the above argument has already shown smoothness. At a point $q = (\gamma_1^{\otimes}(\gamma_2), \gamma_2)$ with $d_1^2\Pi_1(q) = 0$, we have $d_1 d_2 \Pi_1(q) \neq 0$, hence one can parameterize $\{d_1\Pi_1 = 0\}$ locally via $\gamma_2^{\otimes}(\gamma_1)$. $d_1^2\Pi_1(\gamma_1, \gamma_2^{\otimes}(\gamma_1))$ is an analytic function, i.e., the Taylor expansion converges at every point with positive radius of convergence. From complex analysis (see e.g. Theorem 4.8 in Shakarchi and Stein (2003)) we know that, if the zeros of the function accumulate, then $d_1^2\Pi_1(\gamma_1, \gamma_2^{\otimes}(\gamma_1)) \equiv 0$ on some open neighborhood U of γ_1. But this is a contradiction: Consider the image $V := \{(\gamma_1, \gamma_2^{\otimes}(\gamma_1)) | \gamma_1 \in U\} \subset \{d_1\Pi_1 = 0\}$. There, $d_1^2\Pi_1 = 0 \wedge d_1 d_2 \Pi_1 \neq 0$ everywhere. So the tangent to $\{d_1\Pi_1 = 0\}$ may not have a component in d_2-direction. But this means that $\gamma_2^{\otimes}(\gamma_1)$ is constant on U – a contradiction to the argument in the proof of i). This proves the claim on $\overline{\mathcal{N}}_1$.

On $\mathcal{N}_2$, $\gamma_1^{\otimes}$ is constant and thus smooth everywhere.

In addition, exception points are well-behaved:
Claim. Let γ_2^0 be a point at which $\gamma_1^{\otimes}$ is non-differentiable, i.e. $d_1^2\Pi_1(\gamma_1^{\otimes}(\gamma_2^0), \gamma_2^0) = 0$. Then,

i) $d_2\gamma_1^{\otimes}$ converges to minus infinity in γ_2^0.

ii) $\gamma_1^{\otimes}$ decreases in a neighborhood of γ_2^0.

Proof of claim.
Consider a point $q = (\gamma_1^{\otimes}(\gamma_2^0), \gamma_2^0)$ with $d_1^2\Pi_1(q) = 0$, and a locally inverse function $\gamma_2^{\otimes}$ of the parameterization $\gamma_1^{\otimes}$ in a neighborhood V. Since $d_1\Pi_1(q) = 0$, $d_1\gamma_2^{\otimes}(\gamma_1^{\otimes}) = 0$ and $\lim_{\gamma_1 \to \gamma_1^{\otimes}} d_1\gamma_2^{\otimes}(\gamma_1) = 0$. From monotonicity of $\gamma_2^{\otimes}(\gamma_1)$, either for all $\gamma_2^0 \neq \gamma_2 \in V$, $d_1\gamma_2^{\otimes} > 0$ or for all $\gamma_2^0 \neq \gamma_2 \in V$, $d_1\gamma_2^{\otimes} < 0$. Since $d_2\gamma_1^{\otimes} = 1/d_1\gamma_2^{\otimes}$, either $\lim_{\gamma_2 \to \gamma_2^{\otimes}} d_2\gamma_1^{\otimes} = \infty$ or $\lim_{\gamma_2 \to \gamma_2^{\otimes}} d_2\gamma_1^{\otimes} = -\infty$, but the second case is ruled out by the part iii).

ad iii). $d_2\gamma_1^{\otimes}$ is only defined for $\gamma_1^{\otimes} < \gamma^{max}$ and $d_1^2\Pi_1 \neq 0$. Hence, let $\gamma_1^{\otimes} < \gamma^{max}$ and $d_1^2\Pi_1 \neq 0$. We first prove a preliminary claim.

Claim. Consider a continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ and vectors $(1, a)$ and $(1, b)$ with $0 < a < b \in \mathbb{R}$. Consider a point $p \in \mathbb{R}^2$ with $Df(p) \neq 0$. If in p the directional derivatives $D_{(1,a)}f$ and $D_{(1,b)}f$ have the same sign, then all directional derivatives $D_{(1,x)}f$ with $a \leq x \leq b$ have the same sign in p. If in p one of the directional derivatives, $D_{(1,a)}f, D_{(1,b)}f$, is equal and the other unequal to zero, then for all $x \in (a, b)$ $D_{(1,x)}f \neq 0$ and has the same sign.

Proof of claim. We have $Df(p) \neq 0$, hence the gradient $\mathrm{grad}(f) = (d_1 f, d_2 f)$ does not vanish at p. Hence,

$$D_{(1,x)}f = \langle (1, x), \mathrm{grad}(f) \rangle = d_1 f + x\, d_2 f \tag{A130}$$

is a linear function in x. Subsequently, if $d_1 f + x\, d_2 f$ (as a function in x) has the same sign for $x = a$ and $x = b$, it has the same sign for all $x \in (a, b)$. This also holds in case one of the two directional derivatives $D_{(1,a)}f, D_{(1,b)}f$ are zero. This proves the claim.

$d_2\gamma_1^{\otimes}$ is defined by $D_{(d_2\gamma_1^{\otimes},1)}(d_1\Pi_1) = 0$. Thus, it is to show that for $\kappa > \alpha$, $D_{(\kappa,1)}(d_1\Pi_1) \neq 0$.

Since $D_{(\kappa,1)}f = \kappa D_{(1,1/\kappa)}f$, this is equivalent to showing that for $1/\kappa \in (0, 1/\alpha)$, $D_{(1,1/\kappa)}(d_1\Pi_1) \neq 0$. With the above claim it thus remains to show that

$$d_1^2\Pi_1^{\otimes} = D_{(1,0)}(d_1\Pi_1) \leq 0 \tag{A131}$$

$$\text{and} \qquad D_{(1,1/\alpha)}(d_1\Pi_1) < 0. \tag{A132}$$

(A132) holds, since $D_{(1,1/\alpha)}(d_1\Pi_1) = d_1^2\Pi_1 + \frac{1}{\alpha}(d_1d_2\Pi_1) < 0$ from (A65). (A131) holds since $\gamma_1^{\otimes}(\gamma_2)$ is a local minimum.

□

A8 Proof of Lemma 4

Condition (S1) requires that for $x < \mu$, f only has local minima. Condition (S2), on the other hand, requires that for $x > \mu$, f only has local maxima. Hence, f is increasing on the interval $[a, \mu]$, since otherwise from condition (S3) there was a local maxima below μ. On the interval $[b, \mu]$ there can be at most one local maximum τ, since otherwise there would be another local minima in between - contradiction.

Subsequently, f is increasing on $[a, \mu]$ and decreasing on $[\mu, b]$. Hence, τ is a global maximum and from monotonicity we have $\forall x < \tau : df(x) \geq 0$ and $\forall x > \tau : df(x) \leq 0$. But $df(x)$ must not be zero for $x \neq \tau$, since otherwise from (S1) and (S2) at that point there would be another local extremum, which would entail another extremum in between - contradiction.

□

A9 Proof of Lemma 5

If $d_1\Pi_1 = 0$,

$$d_1^2\Pi_1 \geq 0 \tag{A133}$$

$$\Leftrightarrow -(2d_1a^* + \gamma_1 d_1^2 a^*) \geq 0 \tag{A134}$$

$$\Leftrightarrow 2d_1a^* \leq -\gamma_1(d_1a^*)^2\left[\frac{2}{a^*} - g\varphi_1\frac{\xi_2\alpha}{\tau_1}\right] \tag{A135}$$

$$\overset{d_1\Pi_1=0}{\Leftrightarrow} 2 \leq -\gamma_1\frac{(\overline{a} - a^*)}{\gamma_1}\left[\frac{2}{a^*} - g\varphi_1\frac{\xi_2\alpha}{\tau_1}\right] \tag{A136}$$

$$\Leftrightarrow 2a^* \leq (\overline{a} - a^*)\Big[-2 + g\varphi_1\xi_2 \underbrace{a^*\frac{\alpha}{\tau_1}}_{=(-d_2a^*)}\Big] \tag{A137}$$

$$\Leftrightarrow 2\overline{a} \leq (\overline{a} - a^*)g\varphi_1\xi_2(-d_2a^*), \tag{A138}$$

where we used $d_1\Pi_1 = 0 \Leftrightarrow d_1a^* = \frac{(\overline{a}-a^*)}{\gamma_1}$ as well as (A61).

Define the RHS of (A138) as

$$R(\gamma_1, \gamma_2) := (\overline{a} - a^*)g\varphi_1\xi_2(-d_2a^*). \tag{A139}$$

Claim. It suffices to show $d_1R < 0$.
Proof of claim. If $d_1R < 0$, there can be at most one μ with $2\overline{a} = R(\mu, \gamma_2)$ and for this μ (T1) and (T2) hold. In case there is no μ with $2\overline{a} = R(\mu, \gamma_2)$, we distinguish the following cases:

1) If there is an interior local maximum, at this interior local maximum we must have $d_1\Pi_1^2 < 0$. Hence $2\overline{a} > R$ on the entire interval and $\mu = \gamma_1^{\underline{a}}(\gamma_2)$ satisfies the condition.

2) If there is no interior local maximum, Π_1 increases on the entire interval by Assumption A6 and Lemma 6, and $\mu = \gamma^{max}$ satisfies the condition. In that case also $\gamma_1^{\overline{a}}(\gamma_2) = \gamma^{max}$, because otherwise $\Pi_1(\gamma_1^{\overline{a}}(\gamma_2), \gamma_2) = 0$ would contradict monotonicity of Π_1.

3) By Assumption A6 and Lemma 6 there can be no interior local minima.

Claim. $d_1 R < 0$.
Proof of claim. Using (A64) in Proposition 3,

$$\begin{aligned} d_1 R &= (-d_1 a^*)\frac{R}{(\overline{a}-a^*)} + d_1\varphi_1\frac{R}{\varphi_1} - (d_1 d_2 a^*)\frac{R}{(-d_2 a^*)} && \text{(A140)} \\ &= R(d_1 a^*)\cdot\left[-\frac{1}{(\overline{a}-a^*)} + \alpha\xi_2 - \frac{1}{\alpha}\frac{(d_1 d_2 a^*)}{(d_1 a^*)^2}\right] && \text{(A141)} \\ &= R(d_1 a^*)\cdot\left[-\frac{1}{(\overline{a}-a^*)} + \alpha\xi_2 - \frac{1}{\alpha}\frac{\alpha}{a^*}\left(a^*\xi_2\frac{h\varphi_2}{\tau_2} - 2\right)\right] && \text{(A142)} \\ &= \underbrace{R(d_1 a^*)}_{>0}\cdot\left[-\frac{1}{(\overline{a}-a^*)} + \frac{2}{a^*} - \xi_2(\alpha + \frac{h\varphi_2}{\tau_2})\right] && \text{(A143)} \end{aligned}$$

Subsequently

$$\begin{aligned} d_1 R < 0 &\Leftrightarrow \frac{2\overline{a}-3a^*}{(\overline{a}-a^*)a^*} < \xi_2\left(\alpha + \frac{h\varphi_2}{\tau_2}\right) && \text{(A144)} \\ &\Leftrightarrow \underbrace{\frac{2\overline{a}-3a^*}{(\overline{a}-a^*)}}_{<2} < \underbrace{a^*\xi_2}_{<(-2)}\left(\underbrace{\alpha}_{\in(0,1)} + \underbrace{\frac{h\varphi_2}{\tau_2}}_{<(-1)}\right). && \text{(A145)} \end{aligned}$$

Assumption A5 ensures that the LHS of (A145) is negative, and, thus, under assumption A5 (A145) holds.

□

A10 Proof of Lemma 6

From assumption A6 we have $d_1\Pi_1(\gamma_1^{\underline{a}}(\gamma_2), \gamma_2) > 0$ for $\gamma_2 = \gamma_2^*(\gamma^{max})$. From (A65), we know

$$D_{(1,1/\alpha)}(d_1\Pi_1) = d_1^2\Pi_1 + \frac{1}{\alpha}(d_1 d_2 \Pi_1) < 0. \tag{A146}$$

Hence, $d_1\Pi_1(\gamma_1^{\underline{a}}(\gamma_2), \gamma_2)$ increases along $\{a^* = \underline{a}\}$ as γ_2 decreases, and, thus, $d_1\Pi_1(\gamma_1^{\underline{a}}(\gamma_2), \gamma_2) > 0$ for all $\gamma_2 \in [0, \gamma_2^*(\gamma^{max})]$.

□

A11 Proof of Proposition 4

ad Existence. We consider dealer 2's reaction functions

$$\begin{aligned} \gamma_2^* : [\gamma_1^{\underline{a}}(0), \gamma^{max}] &\to [0, \gamma_2^*(\gamma^{max})] && \text{(A147)} \\ \gamma_1 &\mapsto \gamma_2^*(\gamma_1) && \text{(A148)} \end{aligned}$$

and dealer 1

$$\gamma_1^{\otimes} : [0, \gamma_2^*(\gamma^{max})] \to [\gamma_1^{\underline{a}}(0), \gamma^{max}] \tag{A149}$$
$$\gamma_2 \mapsto \gamma_1^{\otimes}(\gamma_2) \tag{A150}$$

From Propositions 3 and 2 we know that γ_2^* and $\gamma_1^{\otimes}$ are continuous functions. Subsequently,

$$(\gamma_2^* \circ \gamma_1^{\otimes}) : [0, \gamma_2^*(\gamma^{max})] \to [0, \gamma_2^*(\gamma^{max})] \tag{A151}$$

is a continuous self-mapping on a nonempty, compact and convex set and, hence, by Brouwer's fixed point theorem (rf Mas-Colell et al. (1995, p. 952)) there exists a fixed point. By construction a fixed point either satisfies both FOCs or lies at the boundary.

ad Uniqueness. Since dealer 2's reaction function γ_2^* is strictly increasing, there exists an inverse function, denoted by γ_1^{*-1}. From part iii) of Proposition 3 we have for dealer 2's reaction function $d_1\gamma_2^* < 1/\alpha^*$, hence, for its inverse function

$$d_2\gamma_1^{*-1} > \alpha^*. \tag{A152}$$

At the same time, we know from Proposition 2 that for dealer 1's reaction function

$$d_2\gamma_1^{\otimes} < \alpha^{\otimes}. \tag{A153}$$

Consider the mapping

$$\gamma_2 \mapsto \gamma_1^{*-1}(\gamma_2) \mapsto a^*(\gamma_1^{*-1}(\gamma_2), \gamma_2). \tag{A154}$$

Then $a^*(\gamma_1^{*-1}(\cdot), \cdot)$ as a function of γ_2 is increasing in γ_2, since

$$0 < d_2a^*(\gamma_1^{*-1}(\gamma_2), \gamma_2) = (d_1a^*)(d_2\gamma_1^*) + d_2a^* \Leftrightarrow d_2(\gamma_1^*) > \frac{(-d_2a^*)}{(d_1a^*)} = \alpha, \tag{A155}$$
$$\Leftrightarrow \frac{1}{d_1(\gamma_2^*)} > \alpha, \tag{A156}$$

which holds by Proposition 2 part iii).

Likewise, one can consider the analogous mapping using dealer 1's reaction function

$$\gamma_2 \mapsto \gamma_1^{\otimes}(\gamma_2) \mapsto a^*(\gamma_1^{\otimes}(\gamma_2), \gamma_2). \tag{A157}$$

Then $a^*(\gamma_1^{\otimes}(\cdot), \cdot)$ as a function of γ_2 is decreasing in γ_2, since

$$0 > d_2a^*(\gamma_1^{\otimes}(\gamma_2), \gamma_2) = (d_1a^*)(d_2\gamma_1^{\otimes}) + d_2a^* \Leftrightarrow d_2(\gamma_1^{\otimes}) < \frac{(-d_2a^*)}{(d_1a^*)} = \alpha, \tag{A158}$$

which holds by Proposition 3 part iii).

Since a^* values at a point must coincide at a point at which the two function intersect, there can be at most one intersection.

□

A12 Proof of Proposition 5

First, at a Nash equilibrium $\vec{\gamma}$ one has $d_1\Pi_1(\vec{\gamma}) \geq 0 = d_2\Pi_2(\vec{\gamma})$ with $d_1\Pi_1(\vec{\gamma}) > 0$ only if $\gamma_1 = \gamma^{max}$. Note furthermore that

$$d_1\Pi_1 \geq 0 \Leftrightarrow (\overline{a} - a^*) - \gamma_1 d_1 a^* \geq 0 \tag{A159}$$
$$d_2\Pi_2 = 0 \Leftrightarrow (a^* - \underline{a}) + \gamma_2 d_2 a^* = 0. \tag{A160}$$

Using Lemma 3 part iii), it thus follows that at a point $\vec{\gamma}$ with $d_1\Pi_1(\vec{\gamma}) \geq 0 = d_2\Pi_2(\vec{\gamma})$ we have

$$1 > \alpha = \frac{-d_2 a^*}{d_1 a^*} \geq \frac{-d_2 a^*}{(\overline{a} - a^*)}\gamma_1 = \frac{(a^* - \underline{a})}{(\overline{a} - a^*)}\frac{\gamma_1}{\gamma_2} = \frac{\Pi_2}{\Pi_1}\frac{\gamma_1^2}{\gamma_2^2} > \frac{\Pi_2}{\Pi_1}, \tag{A161}$$

where the last inequality follows since $\Delta\gamma < 0 \Leftrightarrow \gamma_1/\gamma_2 > 1$. Hence, (A161) yields $(a^* - \underline{a}) < (\overline{a} - a^*)$ and $\Pi_2 < \Pi_1$.

□

A13 Proof of Proposition 6

The optimization problem for a given vector of default probabilities $\vec{b}^0$ depends only on $\tilde{g}(\vec{b}^0) = p(b_2^0 - b_1^0)/(1 - b_1^0 p)$. Hence, vectors of default probabilities with the same $\tilde{g}$ yield the same Nash equilibria.

Claim. For a given pair of default probabilities $(b_1^0, b_2^0) = \vec{b}^0$ with $\tilde{g}(\vec{b}^0)$, the set of default probabilities $\vec{b}$ with the same $\tilde{g}$ is

$$\left\{ \left(b_1^0 - \alpha, b_2^0 - (1 - \tilde{g}(b_1^0, b_2^0))\alpha \right) | \alpha \in \left[b_1^0 - \frac{1}{3}, b_1^0 \right] \right\}. \tag{A162}$$

Proof of claim. We have

$$\partial_{b_2}\tilde{g}\big|_{\vec{b}^0} = \frac{p}{1 - b_1^0 p} \tag{A163}$$
$$\partial_{b_1}\tilde{g}\bigg|_{\vec{b}^0} = \frac{-p(1 - b_1^0 p) + p(b_2^0 - b_1^0)p}{(1 - b_1^0 p)^2} \tag{A164}$$
$$= -p\frac{(1 - b_2^0 p)}{(1 - b_1^0 p)^2} \tag{A165}$$
$$= -\frac{p}{(1 - b_1^0 p)}[1 - \underbrace{\frac{p\Delta b}{(1 - b_1^0 p)}}_{=\tilde{g}(\vec{b}^0)}] \tag{A166}$$
$$-\frac{\partial_{b_1}\tilde{g}}{\partial_{b_2}\tilde{g}}\bigg|_{\vec{b}^0} = (1 - \tilde{g}(\vec{b}^0)) \in (0, 1) \tag{A167}$$

Hence, from the implicit function theorem we know that sets $\{\vec{\gamma} | \tilde{g}(\vec{\gamma}) = c\}$ are submanifolds that have (for a given c) the same slope $(1 - \tilde{g})$ at each point. Hence they are straight lines.

□

A14 Proof of Proposition 7

ad i). Suppose $b_1^0 < b_2^0$ is a subgame-perfect Nash equilibrium in qualities. In that case one must not be able to find a profitable deviation for the unsafer dealer, that is, no b_2^1 with $b_2^1 < b_1^0 < b_2^0$

such that the profit when taking the lead position in quality, exceeds the profit when choosing the optimal quality as unsafer dealer, i.e. no b_2^1 with $\Pi_1(b_2^1, b_1^0) > \Pi_2(b_1^0, b_2^0)$. From Proposition 6 we know that pairs of default probabilities (b_1, b_2) with

$$(b_1, b_2) = (b_1^0 - z, b_2^0 - (1 - \tilde{g}(b_1^0, b_2^0))z), \tag{A168}$$

$z \in [b_1^0 - \frac{1}{3}, b_1^0]$, lead to the same Nash equilibria in prices. Hence, if the unsafer dealer, protection seller 2, has the option to choose a quality $b_2^1 < b_1^0$ with

$$(b_2^1, b_1^0) = \left(\frac{1}{(1-\tilde{g})} \left[(1-\tilde{g})b_1^0 - (b_2^0 - b_1^0) \right], b_1^0 \right) \tag{A169}$$

it leads to the same Nash equilibrium in prices, but with reversed roles. By Proposition 5, we know that the protection seller 2 has a larger profit than before, hence, this is a profitable deviation. This deviation is infeasible if

$$b_2^0 - b_1^0 > (1 - \tilde{g}(b_1^0, b_2^0))b_1^0 \tag{A170}$$

$$\Leftrightarrow b_2^0 > (2 - \tilde{g}(b_1^0, b_2^0))b_1^0 \tag{A171}$$

$$\Leftrightarrow b_1^0 < \underbrace{\frac{1}{(2-\tilde{g})}}_{<2-1/8 \text{ from Lemma 7}} \underbrace{b_2^0}_{<b^{max}}. \tag{A172}$$

ad ii). Suppose $\tilde{g}^*$ is the $\tilde{g}$ that maximizes protection seller 2's profit. It is well-defined from condition (N1). Call this maximum $\Pi_2^{\square,\tilde{g}^*}$.

Then similar to part i), dealer 1 must choose b_1 in such a way that it is not profitable for dealer 2 to take over the leadership position in quality. This is the case if $\Pi_2^{\square,\tilde{g}^*}$ exceeds any profit dealer 2 can capture as quality-leader. Since from condition (N2), the profit of the quality-leader is increasing in the lower quality, this is true for b_1 below $\bar{b}_1$ with $\bar{b}_1$ such that for some $\tilde{g}'$ $\Pi_1^{\square,\tilde{g}'}(0, \bar{b}_1) = \Pi_2^{\square,\tilde{g}^*}$.

$\square$

B Appendix: Additional Results

B1 Optimal Choice of State-Contingent Payments

This section shows that the derivative (b, γ) is the outcome of the optimal contracting problem described in the text. Consider a protection buyer who is deciding whether to buy a derivative (b, γ). Upon entering the derivative contract, the client agrees to pay a fixed rate γ for establishing the client-dealer relationship, before the volume of the derivative is determined endogenously and the dealer offers the actuarially fair price. In particular, the protection buyer chooses payments (y, z) to maximize expected utility

$$(1-p)u(\overline{\theta}-y)+p(1-b)u(\underline{\theta}-z)+bpu(\underline{\theta}) \tag{B3}$$

subject to the constraint

$$(1-p)y+p(1-b)z-\left[\gamma-\frac{bp\underline{\theta}}{(1-bp)}\right](1-bp)\geq 0 \tag{B4}$$

$$\Leftrightarrow (1-p)y+p(1-b)z\geq \gamma(1-bp)-bp\underline{\theta}. \tag{B5}$$

(B4) and (B5) offer two views on the constraint. (B5) demands that the expected cash flows to the protection seller (LHS) must be at least as high as the expected fee already agreed upon minus the expected endowment if the protection seller survives. To see the latter part note that

$$E\left[\tilde{x}|\text{dealer survives}\right]P[\text{dealer survives}]=(1-p)\overline{\theta}+p(1-b)\underline{\theta}\overset{E[\tilde{x}]=0}{=}-bp\underline{\theta} \tag{B6}$$

$$\Leftrightarrow E\left[\tilde{x}|\text{dealer survives}\right]=\frac{-bp\underline{\theta}}{(1-bp)}>0. \tag{B7}$$

The risk-averse protection buyer passes the risky endowment to the protection seller unless the protection seller defaults.

(B4) offers an alternative explanation. Let γ^{nom} be the expression in brackets, i.e.

$$\gamma^{nom}:=\gamma-\frac{bp\underline{\theta}}{(1-bp)}. \tag{B8}$$

Then the third term on the LHS of (B4) is the "nominal" fee per client-dealer relationship, γ^{nom}, times the survival probability of the protection seller, since only in that case the payment is actually exchanged. It is subtracted because this fee for establishing the client-dealer relationship has already been agreed upon, so the dealer already "mentally set it aside" and subsequently wants to break even in $t=3$. Compared to γ, from the definition we have $\gamma=\gamma^{nom}+bp\underline{\theta}/(1-bp)<\gamma^{nom}$. In view of (B7) the adjustment term, $bp\underline{\theta}/(1-bp)$, is precisely the expected endowment conditional on the survival of the protection seller. Since it is positive, the protection buyer claims this extra revenue for himself, rendering γ the "true" fees for the protection seller. In the formulation of the protection seller's constraint in (B4) one assumes that the protection seller chooses "true" fees γ instead of "nominal" ones γ^{nom}. This reparametrization will make subsequent calculations tractable as we will see, while simplifying the intuition.

Proposition 10. *For a given (b, γ), the protection buyer optimally chooses*

$$y^{*}(b,\gamma)=\gamma+\frac{p(1-b)\overline{\theta}-p\underline{\theta}}{(1-bp)} \tag{B9}$$

$$z^*(b,\gamma) = \gamma - \frac{(1-p)}{(1-bp)}\overline{\theta} - \frac{b(1-bp)-(1-p)}{(1-b)(1-bp)}\underline{\theta}. \tag{B10}$$

Let $r^*(b,\gamma)$ be the payoff a protection buyer is left with in an optimal derivative contract unless the counterparty defaults (residual endowment), i.e. $r^*(b,\gamma) := \overline{\theta} - y^*(b,\gamma) = \underline{\theta} + z^*(b,\gamma)$. Then, as one would expect from risk aversion, $r^*(b,\gamma)$ does not depend on the endowment state, namely

$$r^*(b,\gamma) = -\gamma. \tag{B11}$$

Intuition. Let us rewrite the constraint (B4) under equality,

$$(1-p)y + p(1-b)z = \gamma^{nom}(1-bp) \tag{B12}$$

$$\Leftrightarrow (1-p)(y-\gamma^{nom}) + p(1-b)(z-\gamma^{nom}) = 0. \tag{B13}$$

The risk-averse protection buyer chooses payments (y, z) to equalize his outcome across states, i.e. payments (y, z) such that

$$\overline{\theta} - y = \underline{\theta} - z \tag{B14}$$

$$\Leftrightarrow \qquad y = \overline{\theta} + k \;\text{ and }\; z = \underline{\theta} + k \quad \text{for some } k \in \mathbb{R}. \tag{B15}$$

Then the derivative contract can be interpreted as follows: the protection seller offsets the endowment for the protection buyer in each state in exchange for a fixed payment k, leaving the protection buyer with $(-k)$ unless the protection seller defaults. In other words, $-k = \overline{\theta} - y = \underline{\theta} - z$ is the *residual endowment* of the protection buyer. Plugging (B15) into (B13) yields

$$(1-p)(\overline{\theta} + k - \gamma^{nom}) + p(1-b)(\underline{\theta} + k - \gamma^{nom}) = 0 \tag{B16}$$

$$\Leftrightarrow \underbrace{\left[(1-p)\overline{\theta} + p(1-b)\underline{\theta}\right]}_{=-bp\underline{\theta}} + (1-bp)(k-\gamma^{nom}) = 0 \tag{B17}$$

$$\Leftrightarrow k = \gamma^{nom} + \frac{bp\underline{\theta}}{(1-bp)}. \tag{B18}$$

Hence, $-k > -\gamma^{nom}$, i.e. the protection buyer pays *less* than the nominal profit per contract. As explained above, this is because the expected endowment conditional on the protection seller's survival is positive and the protection buyer claims this extra revenue for himself, rendering the "true" profits $k = \gamma = \gamma^{nom} + bp\underline{\theta}/(1-bp)$.

Proof. The protection buyer solves the following optimization problem

$$\max_{y,z} \left\{ (1-p)u(\overline{\theta} - y) + p(1-b)u(\underline{\theta} - z) + bpu(\underline{\theta}) \,\middle|\, (1-p)y + p(1-b)z = \gamma(1-bp) - bp\underline{\theta} \right\} \tag{B19}$$

which is equivalent to the unconstrained problem

$$\max_{y} \left\{ (1-p)u(\overline{\theta} - y) + p(1-b)u\left(\underline{\theta} - \frac{(1-bp)}{p(1-b)}\gamma + \frac{b}{(1-b)}\underline{\theta} + \frac{(1-p)}{p(1-b)}y\right) + bpu(\underline{\theta}) \right\}.$$

With $\Delta\theta := \overline{\theta} - \underline{\theta}$, the resulting first-order condition reads

$$0 \;=\; -(1-p)u'(\overline{\theta} - y) + p(1-b)u'\left(\underline{\theta} - \frac{(1-bp)}{p(1-b)}\gamma + \frac{b}{(1-b)}\underline{\theta} + \frac{(1-p)}{p(1-b)}y\right)\frac{(1-p)}{p(1-b)}$$

$$
\begin{aligned}
\Leftrightarrow u'(\overline{\theta}-y) &= u'\left(\underline{\theta}-\frac{(1-bp)}{p(1-b)}\gamma+\frac{b}{(1-b)}\underline{\theta}+\frac{(1-p)}{p(1-b)}y\right) \\
\Leftrightarrow \Delta\theta - y\left[1+\frac{(1-p)}{p(1-b)}\right] &= -\gamma\frac{(1-bp)}{p(1-b)}+\frac{b}{(1-b)}\underline{\theta} \\
\Leftrightarrow y\frac{(1-bp)}{p(1-b)} &= \Delta\theta+\gamma\frac{(1-bp)}{p(1-b)}-\frac{bp}{p(1-b)}\underline{\theta} \\
\Leftrightarrow y &= \frac{p(1-b)}{(1-bp)}\Delta\theta+\gamma-\frac{bp}{(1-bp)}\underline{\theta} \\
\Leftrightarrow y &= \gamma+\frac{p(1-b)(\overline{\theta}-\underline{\theta})-bp\underline{\theta}}{(1-bp)} \qquad \text{(B20)} \\
\Leftrightarrow y^*(b,\gamma) &= \gamma+\frac{p(1-b)\overline{\theta}-p\underline{\theta}}{(1-bp)}. \qquad \text{(B21)}
\end{aligned}
$$

With (B21) plugged into

$$
z^*(b,\gamma) = \frac{1-bp}{p(1-b)}\gamma-\frac{pb}{p(1-b)}\underline{\theta}-\frac{(1-p)}{p(1-b)}y^*(b,\gamma) \qquad \text{(B22)}
$$

from the constraint in (B19), some simple rearranging yields the formula for $z^*(b,\gamma)$. Using (B20) we confirm that

$$
\begin{aligned}
\overline{\theta}-y^*(b,\gamma) &= -\gamma+\frac{\overline{\theta}-bp\overline{\theta}-\left[p(1-b)(\overline{\theta}-\underline{\theta})-bp\underline{\theta}\right]}{(1-bp)} \qquad \text{(B23)} \\
&= -\gamma+\frac{(1-p)\overline{\theta}+p\underline{\theta}}{(1-bp)} \qquad \text{(B24)} \\
&\overset{E[\tilde{x}]=0}{=} -\gamma \qquad \text{(B25)}
\end{aligned}
$$

as well as

$$
\begin{aligned}
\underline{\theta}-z^*(b,\gamma) &= \underline{\theta}-\left(\gamma-\frac{(1-p)}{(1-bp)}\overline{\theta}-\frac{b(1-bp)-(1-p)}{(1-b)(1-bp)}\underline{\theta}\right) \qquad \text{(B26)} \\
&= -\gamma+\frac{p\underline{\theta}+(1-p)\overline{\theta}}{(1-bp)} \qquad \text{(B27)} \\
&\overset{E[\tilde{x}]=0}{=} -\gamma. \qquad \text{(B28)}
\end{aligned}
$$

□

B2 Market Coverage

A derivative contract (b,γ) is called *feasible for a* if protection buyer a prefers the contract to none. This translates into the following condition

$$
\begin{aligned}
pu_a(\underline{\theta})+(1-p)u_a(\overline{\theta}) &\leq (1-bp)u_a(-\gamma)+bpu_a(\underline{\theta}) \qquad \text{(B29)} \\
\Leftrightarrow bp\left[u_a(-\gamma)-u_a(\underline{\theta})\right]+u_a(\overline{\theta})-u_a(-\gamma) &\leq p\left[u_a(\overline{\theta})-u_a(-\gamma)+u_a(-\gamma)-u_a(\underline{\theta})\right] \qquad \text{(B30)} \\
\Leftrightarrow (1-p)\left[u_a(\overline{\theta})-u_a(-\gamma)\right] &\leq p(1-b)\left[u_a(-\gamma)-u_a(\underline{\theta})\right] \qquad \text{(B31)}
\end{aligned}
$$

(B31) admits an intuitive interpretation: Protection buyer a prefers the contract to no insurance, if the expected utility gain from avoiding the bad endowment in case the seller does not default (RHS) outweighs the expected utility loss from the fee if the good endowment materializes

(LHS).[9]

The following proposition characterizes the protection buyer that is indifferent between derivative contract (b, γ) and no insurance.

Proposition 11. *Protection buyer a is indifferent between (b, γ) and no insurance, if*

$$\gamma = \gamma_a^{exit}(b) := (-\underline{\theta}) - \frac{1}{a} \ln \left(\frac{K(b) + 1}{K(b) + \exp(-a(\overline{\theta} - \underline{\theta}))} \right) \tag{B32}$$

with $K(b) = (1 - b)p/(1 - p)$. $\gamma_a^{exit}(b)$ is strictly increasing in a and decreasing in b.

Proof. In light of (B31), a protection buyer a is indifferent between buying contract (b, γ) and no insurance, if

$$\frac{u_a(\overline{\theta}) - u_a(-\gamma)}{u_a(-\gamma) - u_a(\underline{\theta})} = \frac{p}{1-p}(1 - b) \tag{B33}$$

$$\Leftrightarrow \frac{\exp(-a\overline{\theta}) - \exp(a\gamma)}{\exp(a\gamma) - \exp(-a\underline{\theta})} = K(b) \tag{B34}$$

$$\Leftrightarrow \frac{\exp(-a(\overline{\theta} + \gamma)) - 1}{1 - \exp(-a(\underline{\theta} + \gamma))} = K(b) \tag{B35}$$

$$\Leftrightarrow \frac{\exp(-a\Delta\theta)\exp(-a(\underline{\theta} + \gamma)) - 1}{1 - \exp(-a(\underline{\theta} + \gamma))} = K(b) \tag{B36}$$

$$\Leftrightarrow \exp(-a(\underline{\theta} + \gamma)) = \frac{K(b) + 1}{K(b) + \exp(-a\Delta\theta)} \tag{B37}$$

$$\Leftrightarrow \gamma = \gamma_a^{exit}(b) := (-\underline{\theta}) - \frac{1}{a} \ln \left(\frac{K(b) + 1}{K(b) + \exp(-a\Delta\theta)} \right), \tag{B38}$$

with $K(b) := (1 - b)p/(1 - p)$ and $\Delta\theta := (\overline{\theta} - \underline{\theta})$.

ad $\gamma_a^{exit}(b)$ increasing in a. We have

$$\frac{\partial \gamma_a^{exit}}{\partial a} = \frac{1}{a} \left[\frac{1}{a} \log \left(\frac{K(b) + 1}{K(b) + \exp(-a\Delta\theta)} \right) - \frac{\exp(-a\Delta\theta)}{K(b) + \exp(-a\Delta\theta)} \Delta\theta \right]. \tag{B39}$$

With

$$y := \frac{1 - \exp(-a\Delta\theta)}{K(b) + \exp(-a\Delta\theta)} \tag{B40}$$

this reads

$$\frac{\partial \gamma_a^{exit}}{\partial a} = \frac{1}{a} \left[\frac{1}{a} \ln(1 + y) + \left(y - \frac{1}{K(b) + \exp(-a\Delta\theta)} \right) \Delta\theta \right] \tag{B41}$$

$$= \frac{1}{a} \left[\frac{1}{a} y \left(\frac{\log(1 + y)}{y} + a\Delta\theta \right) - \frac{1}{K(b) + \exp(-a\Delta\theta)} \Delta\theta \right] \tag{B42}$$

$$= \left(\frac{1}{a} \right)^2 \frac{1}{K(b) + \exp(-a\Delta\theta)} \left[(1 - \exp(-a\Delta\theta)) \left(\frac{\log(1 + y)}{y} + a\Delta\theta \right) - a\Delta\theta \right] \tag{B43}$$

9 Note that from (B31) we also know that for any feasible contract $(\underline{\theta} + \gamma) < 0$. (Since $-\gamma < 0 < \overline{\theta}$, the LHS of (B31) is positive, hence, the RHS needs to be positive as well.) Indeed, we already restricted attention to $\gamma < (-\underline{\theta})$ by assumption A3.

$$= \left(\frac{1}{a}\right)^2 \frac{1}{K(b)+\exp(-a\Delta\theta)} \left[(1-\exp(-a\Delta\theta))\frac{\log(1+y)}{y} - \exp(-a\Delta\theta)a\Delta\theta\right] \tag{B44}$$

With $x := a\Delta\theta$ this expression is positive if and only if

$$\frac{\exp(x)-1}{x} > \frac{y}{\log(1+y)} \tag{B45}$$

$$\Leftrightarrow \log(1+y) > y\frac{x}{\exp(x)-1} \tag{B46}$$

$$\Leftrightarrow \log\left(\frac{K(b)+1}{K(b)+\exp(-x)}\right) > \frac{x}{\exp(x)}\frac{1}{K(b)+exp(-x)}. \tag{B47}$$

For $x=0$ the LHS and RHS are 0. For $x>0$ the derivative w.r.t. x of the LHS reads

$$\frac{\partial LHS}{\partial x} = \frac{\exp(-x)}{K(b)+\exp(-x))}, \tag{B48}$$

while the derivative of the RHS reads

$$\frac{\partial RHS}{\partial x} = \frac{\exp(-x)}{K(b)+\exp(-x)} \underbrace{\left[(1-x)+\frac{1}{K(b)+\exp(-x)}\frac{x}{\exp(x)}\right]}_{<1}. \tag{B49}$$

To see why the expression in brackets is smaller one, note that

$$(1-x)+\frac{1}{K(b)+\exp(-x)}\frac{x}{\exp(x)} < 1 \Leftrightarrow \frac{1}{K(b)+\exp(-x)} < \exp(x) \Leftrightarrow 0 < K(b)\exp(x),$$

which always holds and proves the claim.

ad $\gamma_a^{exit}(b)$ increasing in b. Follows directly, since

$$\frac{\partial\gamma_{\underline{a}}^{exit}}{\partial K(b)} = \frac{1-\exp(-a\Delta\theta)}{(1+K(b))(K(b)+\exp(-a\Delta\theta))} > 0 \tag{B50}$$

$$\text{and } \partial_b K(b) < 0. \tag{B51}$$

□

The result is intuitive: the fee at which a protection buyer is indifferent between the contract and no insurance is higher the more risk-averse he is. The next corollary follows as a direct consequence.

Corollary 2. *i) For fixed default probability b_i, a derivative contract (b_i, γ_i) is feasible for protection buyer a if $\gamma_i < \gamma_a^{exit}(b_i)$.*

ii) Let $a^{exit}(b_i, \gamma_i)$ be the protection buyer that is indifferent between contract (b_i, γ_i) and no insurance. For γ_i outside of $[\gamma_{\underline{a}}^{exit}(b_i), \gamma_{\overline{a}}^{exit}(b_i)]$, a^{exit} lies outside of the interval $[\underline{a}, \overline{a}]$ and is set to the respective boundary. Then protection buyers with $a < a^{exit}(b_i, \gamma_i)$ prefer no insurance.

iii) If the fee set by the unsafer dealer, γ_2, is smaller than $\gamma_{\underline{a}}^{exit}(b_2)$, then $a^{exit} < \underline{a}$ and there is full market coverage.

In the analysis, I restrict attention to the case in which the market is fully covered.[10]

B3 Formal Results on the Illustration

Lemma 7. *i) For $\gamma_2 \in [0, \gamma^{max}]$ define*

$$\gamma_1^{\underline{a}}(\gamma_2) \text{ such that } a^*(\gamma_1^{\underline{a}}(\gamma_2), \gamma_2) = \underline{a} \tag{B52}$$
$$\gamma_1^{\overline{a}}(\gamma_2) \text{ such that } a^*(\gamma_1^{\overline{a}}(\gamma_2), \gamma_2) = \overline{a}. \tag{B53}$$

Then $\gamma_1^{\underline{a}} < \gamma_1^{\overline{a}}$ and

$$\gamma_1^{\underline{a}} \leq \gamma^{max} \quad \Leftrightarrow \quad \gamma_2 \leq \overline{\gamma_2} \tag{B54}$$
$$\text{with } \overline{\gamma_2} := \arg_\gamma\{a^*(\gamma^{max}, \gamma) = \underline{a}\} = (-\underline{\theta}) - \frac{1}{\underline{a}} \log\left[\frac{1 - \tilde{g}(\vec{b})}{\exp(-2) - \tilde{g}(\vec{b})}\right]. \tag{B55}$$

ii) Analogously, for $\gamma_1 \in [0, \gamma^{max}]$ define

$$\gamma_2^{\underline{a}}(\gamma_1) \text{ such that } a^*(\gamma_1, \gamma_2^{\underline{a}}(\gamma_1)) = \underline{a} \tag{B56}$$
$$\gamma_2^{\overline{a}}(\gamma_1) \text{ such that } a^*(\gamma_1, \gamma_2^{\overline{a}}(\gamma_1)) = \overline{a} \tag{B57}$$

Then $\gamma_2^{\underline{a}} > \gamma_2^{\overline{a}}$ and

$$\gamma_2^{\underline{a}} \geq 0 \quad \Leftrightarrow \quad \gamma_1 \geq \overline{\gamma_1} \tag{B58}$$
$$\text{with } \overline{\gamma_1} := \arg_\gamma\{a^*(\gamma, 0) = \underline{a}\} = \frac{1}{\underline{a}} \log\left[1 + \tilde{g}(\vec{b})\left(\exp(\underline{a}(-\underline{\theta})) - 1\right)\right]. \tag{B59}$$

iii) As one would expect from the picture $\gamma_2 \leq \overline{\gamma_2}$ iff $\gamma_1 \geq \overline{\gamma_1}$.

iv) Protection seller 1 gets the entire market if

$$\overline{\gamma_2} \leq 0 \quad \Leftrightarrow \quad \underline{a}(-\underline{\theta}) \leq \log\left[\frac{1 - \tilde{g}(\vec{b})}{\exp(-2) - \tilde{g}(\vec{b})}\right]. \tag{B60}$$

With $\tilde{g}(\vec{b}) < 1/8$ from assumption A1 and A2,

$$\underline{a}(-\underline{\theta}) > \log\left[\frac{1 - \frac{1}{8}}{\exp(-2) - \frac{1}{8}}\right] \approx 4.4, \tag{B61}$$

ensures that the setup is interesting. This is exactly assumption A4.

Proof. ad i). First of all, we show that for a fixed $\gamma_2 \in [0, \gamma^{max}]$ such $\gamma_1^{\underline{a}}(\gamma_2), \gamma_1^{\overline{a}}(\gamma_2)$ indeed exist. Whenever clear form the context we suppress the dependence on γ_2. Note that for $a \in [\underline{a}, \overline{a}]$ $g(a, \gamma_2, \gamma_2) = 0$, while $\lim_{\gamma\to\infty} g(a, \gamma, \gamma_2) = \lim_{\gamma\to\infty} \frac{1}{c_1}(\exp(a\gamma)c_2 - 1) = \infty$ with

10 Later we will introduce $\gamma_2^*(\gamma^{max})$, that is, dealer 2's best response to the largest possible fee set by dealer 1. Dealer 2's reaction function is increasing. Hence $\gamma_2^*(\gamma^{max})$ is the largest fee possibly set by protection seller in equilibrium, and if $\gamma_2^*(\gamma^{max}) \leq \gamma_{\underline{a}}^{exit}(b_2)$ there is full market coverage anyways. Otherwise, dealer 2's reaction function remains unaltered until $\gamma_{\underline{a}}^{exit}(b_2)$. Above that point, dealer 2 potentially looses market share "from below" when increasing fees, which may induce him to set fees as best responses. Hence, we expect the reaction function to change above $\gamma_{\underline{a}}^{exit}(b_2)$, but it should leave the core of the analysis unchanged.

$c_1 := \exp(-a(\underline{\theta} + \gamma_2))$ and $c_2 := \exp(-a\gamma_2)$ independent of γ. Hence, from continuity such $\gamma_1^{\underline{a}}, \gamma_1^{\overline{a}}$ exist and, since $\partial_1 g > 0$, they are also unique.

Claim. $\gamma_1^{\underline{a}} < \gamma_1^{\overline{a}}$

Proof of claim. Since $\partial_a g < 0$ we have $\tilde{g}(\vec{b}) = g(\overline{a}, \gamma_1^{\overline{a}}, \gamma_2) = g(\underline{a}, \gamma_1^{\underline{a}}, \gamma_2) > g(\overline{a}, \gamma_1^{\underline{a}}, \gamma_2)$. With $\partial_1 g > 0$ this implies $\gamma_1^{\underline{a}} < \gamma_1^{\overline{a}}$.

For the last part of the statement we have

$$\gamma_1^{\underline{a}} \leq \gamma^{max} \tag{B62}$$

$$\Leftrightarrow g(\underline{a}, \gamma^{max}, \gamma_2) \geq \tilde{g}(\vec{b}) \tag{B63}$$

$$\frac{\exp(-\underline{a}(\underline{\theta} + \gamma_2)) \exp(-2) - 1}{\exp(-\underline{a}(\underline{\theta} + \gamma_2)) - 1} \geq \tilde{g}(\vec{b}) \tag{B64}$$

$$\exp(-\underline{a}(\underline{\theta} + \gamma_2)) \left(\exp(-2) - \tilde{g}(\vec{b})\right) \geq 1 - \tilde{g}(\vec{b}) \tag{B65}$$

$$-\underline{a}(\underline{\theta} + \gamma_2) \geq \log\left[\frac{1 - \tilde{g}(\vec{b})}{\exp(-2) - \tilde{g}(\vec{b})}\right] \tag{B66}$$

$$\gamma_2 \leq (-\underline{\theta}) - \frac{1}{\underline{a}} \log\left[\frac{1 - \tilde{g}(\vec{b})}{\exp(-2) - \tilde{g}(\vec{b})}\right]. \tag{B67}$$

Note that we use $\tilde{g}(\vec{b}) < \exp(-2)$ here, which is ensured by assumptions A1 and A2.

ad ii). The argument for existence is analogous to before, so is the argument for $\gamma_2^{\underline{a}} > \gamma_2^{\overline{a}}$ except that now $\partial_2 g < 0$. For the last part we have

$$\gamma_2^{\underline{a}} \geq 0 \tag{B68}$$

$$\Leftrightarrow g(\underline{a}, \gamma_1, 0) \geq \tilde{g}(\vec{b}) \tag{B69}$$

$$\Leftrightarrow \frac{\exp(\underline{a}\gamma_1) - 1}{\exp(\underline{a}(-\underline{\theta})) - 1} \geq \tilde{g}(\vec{b}) \tag{B70}$$

$$\Leftrightarrow \exp(\underline{a}\gamma_1) \geq 1 + \tilde{g}(\vec{b}) \left(\exp(\underline{a}(-\underline{\theta})) - 1\right) \tag{B71}$$

$$\Leftrightarrow \gamma_1 \geq \overline{\gamma_1} =: \frac{1}{\underline{a}} \log\left[1 + \tilde{g}(\vec{b}) \left(\exp(\underline{a}(-\underline{\theta}) - 1\right)\right]. \tag{B72}$$

ad iii). We have

$$\overline{\gamma_2} \geq 0 \tag{B73}$$

$$\Leftrightarrow (-\underline{\theta}) - \frac{1}{\underline{a}} \log\left[\frac{1 - \tilde{g}(\vec{b})}{\exp(-2) - \tilde{g}(\vec{b})}\right] \geq 0 \tag{B74}$$

$$\Leftrightarrow \log\left[\frac{1 - \tilde{g}(\vec{b})}{\exp(-2) - \tilde{g}(\vec{b})}\right] \leq \underline{a}(-\underline{\theta}). \tag{B75}$$

At the same time

$$\overline{\gamma_1} \leq \gamma^{max} \tag{B76}$$

$$\Leftrightarrow 2 + \log\left[1 + \tilde{g}(\vec{b}) \left(\exp(\underline{a}(-\underline{\theta})) - 1\right)\right] \leq \underline{a}(-\underline{\theta}) \tag{B77}$$

$$\Leftrightarrow \log\left[\exp(2) \left(1 + \tilde{g}(\vec{b}) \left(\exp(\underline{a}(-\underline{\theta})) - 1\right)\right)\right] \leq \underline{a}(-\underline{\theta}) \tag{B78}$$

$$\Leftrightarrow \exp(2)\left(1+\tilde{g}(\vec{b})\left(\exp(\underline{a}(-\underline{\theta}))-1\right)\right) \leq \exp(\underline{a}(-\underline{\theta})) \tag{B79}$$

$$\Leftrightarrow \exp(2)\left(1-\tilde{g}(\vec{b})\right) \leq \exp(\underline{a}(-\underline{\theta}))\left(1-\exp(2)\tilde{g}(\vec{b})\right) \tag{B80}$$

$$\Leftrightarrow \frac{1-\tilde{g}(\vec{b})}{\exp(-2)-\tilde{g}(\vec{b})} \leq \exp(\underline{a}(-\underline{\theta})) \tag{B81}$$

$$\Leftrightarrow \overline{\gamma_2} \geq 0. \tag{B82}$$

ad iv). Since the LHS of (B75) is increasing in $\tilde{g}(\vec{b})$ and under Assumption A2 $\tilde{g}(\vec{b}) < 1/8$,

$$\underline{a}(-\underline{\theta}) > \log\left[\frac{1-\frac{1}{8}}{\exp(-2)-\frac{1}{8}}\right] \approx 4.4 \tag{B83}$$

ensures $\overline{\gamma_2} \geq 0$ for all admissible parameters and hence renders the setup interesting. □

B4 Price Equilibria are Smooth Functions of Qualities

Proposition 12 (**Price Equilibrium Smooth Function in Qualities).** *Without loss of generality let* $b_1 = 0$. *Let*

$$\mathcal{D}_1 := \left\{b_2 | \gamma_1^{\square}(b_2) \lneq \gamma^{max}\right\} \tag{B84}$$

be the set of b_2 *that lead to price equilibria in the interior. Let*

$$\mathcal{D}_2 := \left\{b_2 | \gamma_1^{\square}(b_2) = \gamma^{max}, (d_1\Pi_1)^{\square} \gneq 0\right\} \tag{B85}$$

be the set of b_2 *that lead to price equilibria in which protection seller 1 chooses the highest admissible price. The price equilibrium* $\vec{\gamma}^{\square}(\vec{b})$ *as a function of quality choices* $\vec{b}$ *is a smooth function on* $\mathcal{D}_1$ *and* $\mathcal{D}_2$.

Proof. Let

$$\mathcal{M} := \{(b_2, \vec{\gamma}) | b_2 \in (0, b^{max}], 0 \leq \gamma_2 < \gamma_1 \leq \gamma^{max}\} \tag{B86}$$

and

$$\mathcal{L}_1 := \{d_2\Pi_2 = 0\} \cap \{d_1\Pi_1 = 0\} \subset \mathcal{M} \tag{B87}$$

$$\mathcal{L}_2 := \{d_2\Pi_2 = 0\} \cap \{\gamma_1 = \gamma^{max}\} \subset \mathcal{M}. \tag{B88}$$

Then we know that price equilibria are a subset of $\mathcal{L} := \mathcal{L}_1 \cup \mathcal{L}_2$, and, that $\mathcal{L}_1$ consists of price equilibria.

Claim 1. $\mathcal{L}_1$ is a smooth submanifold of $\mathcal{M}$ and $\vec{\gamma}^{\square}$ is smooth on $\mathcal{D}_1$.
Proof of claim 1. $\mathcal{L}_1$ is the intersection of nullsets of smooth functions

$$\vec{f} := \begin{pmatrix} d_2\Pi_2 \\ d_1\Pi_1 \end{pmatrix}. \tag{B89}$$

The intersection of two nullsets $\{\vec{f} = 0\}$ is smooth if $rank(Df) = 2$. If $d_1^2\Pi_1 \neq 0$,

$$\det(D_{\vec{\gamma}}f) = \det\begin{pmatrix} d_1 d_2\Pi_2 & d_2^2\Pi_2 \\ d_1^2\Pi_1 & d_2 d_1\Pi_1 \end{pmatrix} \tag{B90}$$

$$= -(d_2^2\Pi_2)(d_1^2\Pi_1)\left[1 - \frac{(d_1 d_2 \Pi_2)}{(d_2^2\Pi_2)}\frac{(d_2 d_1 \Pi_1)}{(d_1^2\Pi_1)}\right] \tag{B91}$$

$$= -(d_2^2\Pi_2)(d_1^2\Pi_1)\left[1 - \underbrace{(d_2\gamma_1^{\otimes})}_{<1/\alpha}\underbrace{(d_1\gamma_2^{*})}_{<\alpha}\right] \tag{B92}$$

$$\neq 0. \tag{B93}$$

If $d_1^2\Pi_1 = 0$, then $d_1 d_2 \Pi_1 \neq 0$, and thus

$$\det(D_{\vec{\gamma}} f) = (d_1 d_2 \Pi_2)(d_2 d_1 \Pi_1) \neq 0. \tag{B94}$$

As shown in Proposition 4, for any $\vec{b}$ there is exactly one price equilibrium $\vec{\gamma}^{\square}(\vec{b})$ such that $(\vec{b}, \vec{\gamma}^{\square}(\vec{b})) \in \mathcal{L}$. This defines a function

$$\vec{\gamma}^{\square} : (0, b^{max}) \to \mathcal{L} \tag{B95}$$

$$b_2 \mapsto \vec{\gamma}^{\square}(0, b_2)) \tag{B96}$$

with $\vec{\gamma}^{\square} : \mathcal{D}_i \to \mathcal{L}_i$ for $i \in \{1, 2\}$. Hence, from the Implicit Function Theorem, $\vec{\gamma}^{\square}|_{\mathcal{D}_1}$ is the smooth parameterization of the submanifold $\mathcal{L}_1$.

Claim 2. $\mathcal{L}_2$ is a smooth submanifold of $\mathcal{M}$ and $\vec{\gamma}^{\square}$ is smooth on $\mathcal{D}_2$.
Proof of claim 2. The proof proceeds analogously, but now $\mathcal{L}_2$ is the intersection of nullsets of

$$\vec{g} := \begin{pmatrix} d_2\Pi_2 \\ \gamma_1 - \gamma^{max} \end{pmatrix}, \tag{B97}$$

with

$$\det(D_{\vec{\gamma}} g) = \det \begin{pmatrix} d_1 d_2 \Pi_2 & 1 \\ d_2^2 \Pi_2 & 0 \end{pmatrix} = d_2^2 \Pi_2 < 0. \tag{B98}$$

$\square$

C Online Appendix: Standard Model of Vertical Product Differentiation Revisited

This section clarifies which assumption in the standard model of vertical product differentiation need to be relaxed to yield endogenous market discipline. I revisit the standard model (see e.g. Tirole (1988, section 7.5.1)) and lift the assumptions of full market coverage and quality-invariant costs. The section then shows a refined principle of product differentiation and in how far upward pressure on qualities emerges.

C1 Setup

Agents. There are two firms that produce the same good, but of different qualities $s_i, i \in 1, 2$ taken from some interval $[\underline{s}, \overline{s}]$, $\underline{s} \geq 0$. There is a continuum of consumers who each demand one unit of the good. Consumers differ in their preference for quality captured by a taste parameter θ. Specifically, a consumer with taste parameter θ derives linear utility $U(p, s) = \theta s - p$ from a good of quality s sold at price p. The taste parameter is assumed to be uniformly distributed over some interval $[\underline{\theta}, \overline{\theta}]$, $\underline{\theta} \geq 0$.

Timing. There are three points in time, $t \in \{0, 1, 2\}$. At date 0, firms simultaneously choose qualities s_i. In $t = 1$, firms simultaneously choose prices p_i upon the publicly observed quality decisions in the previous period. Lastly, consumers decide from whom to buy in $t = 2$. Figure 11 summarizes the simple timing of events.

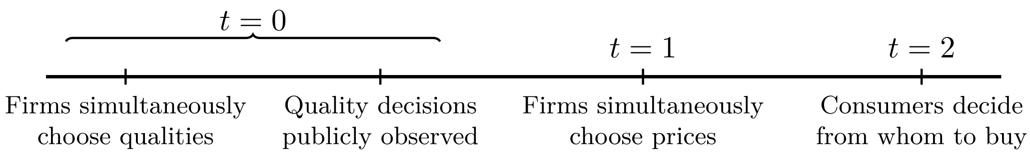

Figure 11: Timeline

If the firms choose the same level of quality, their products can potentially only differ in the price. Since consumers prefer a lower price, competition solely in prices drives the profit margins (or markups) to zero. In order to soften price competition, firms have an incentive to differentiate their products in quality. Since firms are ex-ante symmetric and do not choose the same qualities in equilibrium, if (s_1^*, s_2^*) is an equilibrium in qualities, so is (s_2^*, s_1^*). Without loss of generality we assume that firm 1 is the *low-quality* firm while firm 2 is the *high-quality* firm, that is, suppose $\Delta s := s_2 - s_1 > 0$.[11] I am interested in subgame-perfect Nash equilibria.

11 In the presence of multiple equilibria, a coordination issue emerges and one needs to break the symmetry between the two firms somehow. Here, the symmetry is broken by assigning the role of quality-leader ex-ante.

C2 Maximal Differentiation under Full Market Coverage and Constant Costs

We briefly review the driving forces at play under the standard assumptions.[12] The standard model assumes that per-unit costs c are the same for all qualities. Additionally the following restrictions on parameters are imposed:

$$\overline{\theta} = \underline{\theta} + 1 \tag{C0}$$

$$\overline{\theta} > 2\underline{\theta} \tag{A1}$$

$$c + \frac{1}{3}(\overline{s} - \underline{s})(\overline{\theta} - 2\underline{\theta}) \leq \underline{\theta}\underline{s}. \tag{A2}$$

Since (C0) and (A1) together imply $\underline{\theta} \in [0, 1)$, they can be understood as demanding that, relative to $\underline{\theta}$, there is sufficient consumer heterogeneity. As will become clear from the prices derived below, the LHS of (A2) is the highest price the low-quality firm might set in equilibrium. The RHS is the lowest possible valuation a consumer can have for the low-quality product. Hence, (A2) ensures that all consumers buy the good (*full market coverage*).

The standard result states that given quality choices $s_1 < s_2$ made in $t = 0$, the prices

$$p_1(s_1, s_2) = c + \frac{1}{3}\Delta s(\overline{\theta} - 2\underline{\theta}) \quad \text{and} \quad p_2(s_1, s_2) = c + \frac{1}{3}\Delta s(2\overline{\theta} - \underline{\theta}) \tag{C3}$$

form a Nash equilibrium in $t = 1$. In $t = 0$, there are two pure-strategy Nash equilibria in the choice of qualities and both exhibit maximal product differentiation. Specifically, for $s_1 < s_2$, firm 1 chooses the lowest possible quality $\underline{s}$ and firm 2 chooses the highest possible quality $\overline{s}$. Reversing the role of the two firms yields the other equilibrium.

The intuition of the result is as follows: In $t = 1$, when qualities $s_1 < s_2$ are already chosen, the consumer who is indifferent between the two firms is characterized by a taste parameter $\hat{\theta}$ such that $\hat{\theta}s_1 - p_1 = \hat{\theta}s_2 - p_2$, hence $\hat{\theta} = (p_2 - p_1)/\Delta s$. Firm 1 receives the consumers with θ below the threshold $\hat{\theta}$, while firm 2 receives those with $\theta > \hat{\theta}$. Firm's profits Π_1 and Π_2 take the form

$$\Pi_1(p_1, p_2) = \underbrace{(p_1 - c)}_{\text{profit margin}} \cdot \underbrace{\left[\frac{(p_2 - p_1)}{\Delta s} - \underline{\theta}\right]}_{\text{market share}}, \qquad \Pi_2(p_1, p_2) = (p_2 - c) \cdot \left[\overline{\theta} - \frac{(p_2 - p_1)}{\Delta s}\right]. \tag{C4}$$

In $t = 1$, each firm chooses a price, taking the price of the other firm as given, in order to maximize profits. In $t = 0$, each firm takes into account the Nash equilibrium in prices in the next period, which gives rise to profits as a function of quality choices, specifically $\Pi_1(s_1, s_2) = \frac{1}{9}\Delta s(\overline{\theta} - 2\underline{\theta})^2$ and $\Pi_2(s_1, s_2) = \frac{1}{9}\Delta s(2\overline{\theta} - \underline{\theta})^2$. As profits are increasing in the quality differential, firm 1 chooses the lowest possible quality, while firm 2 chooses the highest possible quality. Note that as a direct consequence the quality-leader enjoys the larger profits - an important observation for later.

12 as in section 7.5.1 in Tirole (1988)

The driving forces behind the result of maximal product differentiation are twofold. Firstly, assumption (A2) ensures that the entire market is always covered. Whatever quality choices firms make in $t = 0$ under (A2), they will always be able to optimally respond with their price choices in such a way that the indifferent consumer is left unchanged.[13] This implies that the quantity effect cancels out and only the margin effect is left. For firm 1, for example, we have

$$\frac{\partial \Pi_1(s_1)}{\partial s_1} = \underbrace{\frac{\partial (p_1(s_1) - c)}{\partial s_1}}_{\text{margin effect}} \underbrace{[\hat{\theta}(s_1) - \underline{\theta}]}_{>0} + (p_1(s_1) - c) \underbrace{\frac{\partial [\hat{\theta}(s_1) - \underline{\theta}]}{\partial s_1}}_{=0,\ \text{quantity effect}} . \tag{C5}$$

Since prices positively depend on the amount of product differentiation, both firms have an incentive to implement maximal product differentiation. Crucial for this result is that there is no upper limit on the price. For both firms it is optimal to increase prices in response to more product differentiation, keeping the indifferent consumer and as a result the market shares constant. Especially for the high-quality firm which charges the higher price, this means that potentially very large (also relative to costs) prices are set without the risk of loosing customers. Secondly, higher quality is not associated with higher costs.

C3 No Full Market Coverage and Costs Varying with Quality

Let's consider the following generalized setup. Suppose costs are increasing in quality, that is, suppose there is a smooth "cost" function $c : \mathbb{R}_+ \to \mathbb{R}_+$ with $c' \geq 0$ and $c'' \geq 0$ where the argument is thought of as quality. A firm incurs higher costs when choosing a higher quality, and, at a higher level of quality, increasing quality even further is even more costly.

We lift the assumption that the entire market is covered, i.e. we do not assume (C0), (A1) and (A2) anymore. In the absence of (A2), the symmetry between the two firms vanishes, since firm 1 needs to take into account that at too unfavorable quality and price choices, some consumers might not buy at all. Specifically, a consumer θ_0 is indifferent between not buying at all and buying from the low-quality firm if $p_1 = \theta_0 s_1$. Firm 1 faces only the market segment from θ_0 upwards, which alters its optimization problem to

$$\max_{p_1} \left\{ (p_1 - c(s_1)) \left[\frac{(p_2 - p_1)}{\Delta s} - \max \left\{ \underline{\theta}, \frac{p_1}{s_1} \right\} \right] \right\} . \tag{C6}$$

In order to avoid cumbersome case distinctions that do not seem to carry further intuition, we ensure that $p_1/s_1 \geq \underline{\theta}$ by assuming $\underline{\theta} = 0$.

Attention is restricted to pairs of qualities (s_1, s_2) that satisfy the following assumptions.

Assumption C1. $c(s_1)/s_1 < \overline{\theta}/2$

13 Formally, this can be seen when we insert equilibrium prices into the formula for the indifferent consumer and obtain $\hat{\theta}(s_1, s_2) = \frac{1}{3}(\underline{\theta} + \overline{\theta})$, independent of s_1, s_2.

Assumption C2. $c(s_2)/s_2 < 2\overline{\theta}$

Assumption C3.

$$\frac{\Delta c}{\Delta s} := \frac{c(s_2) - c(s_1)}{\Delta s} \in \left(2\frac{c(s_1)}{s_1} - \overline{\theta}, 2\overline{\theta} - \frac{c(s_2)}{s_2}\right) \tag{C7}$$

Assumption C3 ensures that the markups of both firms are positive. In particular, as will become clear from the equilibrium prices derived below, firm 1's markup will be positive if and only if $\Delta c/\Delta s > 2c(s_1)/s_1 - \overline{\theta}$, while firms 2's markup will be positive if and only if $\Delta c/\Delta s < 2\overline{\theta} - c(s_2)/s_2$. Assumption C3 is a condition on the difference in costs relative to the difference in quality chosen by the two firms. It means that some combinations of (s_1, s_2) kick one firm out of the market, which makes it plausible how a firm may exert a "pull effect" on the quality decisions of the other firm, as shown below. Assumptions C1 and C2 mandate that the upper and lower boundary of the admissible interval in assumption C3 are positive and negative, respectively. Since $\Delta c/\Delta s$ is positive, assumption C2 is a necessary condition, while assumption C1 is only a sufficient condition for positive profit margins of firm 2 and 1 respectively.[14] [15] Assumptions C1 - C3 can be ensured by a large enough $\overline{\theta}$, hence sufficient consumer heterogeneity.

Refined Principle of Product Differentiation

The Nash equilibrium in prices takes the following form.

Proposition 13. *Given quality choices (s_1, s_2) that satisfy assumptions C1 - C3, the following is a Nash equilibrium in prices in $t = 1$:*

$$p_1(s_1, s_2) = \frac{s_1}{3s_2 + \Delta s}\left[c(s_2) + 2\frac{s_2}{s_1}c(s_1) + \overline{\theta}\Delta s\right] \tag{C8}$$

$$= c(s_1) + \frac{s_1}{3s_2 + \Delta s}\left[\Delta c + \Delta s\left(-2\frac{c(s_1)}{s_1} + \overline{\theta}\right)\right] \tag{C9}$$

$$p_2(s_1, s_2) = \frac{s_2}{3s_2 + \Delta s}\left[2c(s_2) + c(s_1) + 2\overline{\theta}\Delta s\right] \tag{C10}$$

$$= c(s_2) + \frac{s_2}{3s_2 + \Delta s}\left[-\Delta c + \Delta s\left(-\frac{c(s_2)}{s_2} + 2\overline{\theta}\right)\right] \tag{C11}$$

Proof. The idea of the proof is analogous to the proof of the standard result presented above in the text. The details are presented in online appendix C5. $\square$

As before, we are interested in whether the quality-leader has higher profits than the low-quality

14 If $c(0)$ is normalized to zero, the function $x \mapsto c(x)/x$ is increasing for positive x, since for $x > 0$ we have $\frac{\partial}{\partial x}\left(\frac{c(x)}{x}\right) = \frac{1}{x}\left[c'(x) - \frac{c(x)-c(0)}{(x-0)}\right] \geq 0$ from convexity. But we do not make this assumption here in general as it would rule out fixed costs.

15 Constant costs imply $\Delta c/\Delta s = 0$, hence, satisfy assumption C3 under assumptions C1 and C2.

firm. The following corollary shows that this is the case as long as $\Delta c/\Delta s$ lies closer to the lower than to the upper boundary of the admissible interval.

Corollary 3. *i) Firm 2 enjoys larger profit margins than firm 1, i.e.* $p_1 - c(s_1) < p_2 - c(s_2)$ *if and only if*

$$s_1\left[\frac{\Delta c}{\Delta s} - \left(2\frac{c(s_1)}{s_1} - \overline{\theta}\right)\right] < s_2\left[\left(2\overline{\theta} - \frac{c(s_2)}{s_2}\right) - \frac{\Delta c}{\Delta s}\right].$$

ii) Firm 2 enjoys larger market shares than firm 1, i.e. $\hat{\theta} - \theta_0 < \overline{\theta} - \hat{\theta}$ *if and only if*

$$\left[\frac{\Delta c}{\Delta s} - \left(2\frac{c(s_1)}{s_1} - \overline{\theta}\right)\right] < \left[\left(2\overline{\theta} - \frac{c(s_2)}{s_2}\right) - \frac{\Delta c}{\Delta s}\right]. \tag{B4}$$

iii) If firm 2 has the higher market share, i.e. if (B4) *is satisfied, it also has the higher profit margin and, as a result, higher profits.*

Proof. Follows directly from plugging in the respective formulas. □

When both firms anticipate the equilibrium in prices for given quality choices, one can express profits as a function of quality choices:

$$\Pi_1(s_1,s_2) = \Delta s\frac{s_2}{s_1}\left[\frac{s_1}{3s_2+\Delta s}\left(\frac{c(s_2)-c(s_1)}{\Delta s} - \left(2\frac{c(s_1)}{s_1} - \overline{\theta}\right)\right)\right]^2 \tag{C12}$$

$$\Pi_2(s_1,s_2) = \Delta s\left[\frac{s_2}{3s_2+\Delta s}\left(2\overline{\theta} - \frac{c(s_2)}{s_2} - \frac{c(s_2)-c(s_1)}{\Delta s}\right)\right]^2. \tag{C13}$$

In the original setup, profits were increasing in the quality differential. Here, in (C12) as well as in (C13), the first factor increases as products become more differentiated, but the effect on the expressions in brackets is unclear. Hence, an interior Nash equilibrium in qualities may be possible. Specifying conditions on the functional form of $c(\cdot)$ that ensure existence of an interior Nash equilibrium does not promise interesting economic results because of lenghty and tedious expressions, and I do not have a general existence proof. The following result, however, derives properties of a Nash equilibrium in qualities and shows a *refined principle of product differentiation*.

Proposition 14. *a) At any point* (s_1, s_2) *that satisfies assumption C1 - C3*

ii) if marginal costs for extra quality are small for firm 1, firm 1 wants to increase quality. Specifically,

$$c'(s_1) < 2\frac{c(s_1)}{s_1} - \overline{\theta} \qquad \Rightarrow \qquad \frac{\partial \Pi_1(s_1,s_2)}{\partial s_1} > 0. \tag{C14}$$

iiii) For firm 2, if marginal costs for extra quality are large, decreasing quality increases profits. Specifically,

$$2\overline{\theta} - \frac{c(s_2)}{s_2} < c'(s_2) \qquad \Rightarrow \qquad \frac{\partial \Pi_2(s_1, s_2)}{\partial s_2} < 0. \tag{C15}$$

b) For a sequence of (s_1, s_2) where each pair of qualities satisfies assumptions C1 - C3 and stays distinct while converging to some s_0, i.e. Δs going to zero, we have

$$\lim_{s_1,s_2 \to s_0} \frac{\partial \Pi_1(s_1, s_2)}{\partial s_1} = -\frac{1}{9}\left(c'(s_0) - 2\frac{c(s_0)}{s_0} + \overline{\theta}\right)^2 \leq 0, \tag{C16}$$

$$\lim_{s_1,s_2 \to s_0} \frac{\partial \Pi_2(s_1, s_2)}{\partial s_2} = \frac{1}{9}\left(2\overline{\theta} - \frac{c(s_0)}{s_0} - c'(s_0)\right)^2 \geq 0. \tag{C17}$$

Proof. See online appendix C6. □

The following observation follows. The threshold $c(s_1)/s_1 - \overline{\theta}$ in (C14) indeed also depends on s_1. It can be meaningfully interpreted, since by assumption C3, $\Delta c/\Delta s$ needs to lie above this threshold. Analogously for the threshold in (C15).

Proposition 4 part b) shows that, if qualities are very close together, i.e. when Δs is small, firms want to differentiate qualities. In other words, the same effect as in the original model prevails, but now it is only an "infinitesimal" effect as it holds for small differences in quality. At the same time, Proposition 4 part a) demonstrates that high or low marginal costs for firm 2 or 1, respectively, can be the driver behind a tendency to move qualities closer together. From Proposition 4 part aii) the quality-leader wants to provide only as much quality as "necessary", while from part ai) the low-quality firm provides "as much quality as feasible" with respect to the increasing marginal costs of quality. Together the forces from part a) and b) act like pull and push factors keeping the qualities of the two firms somewhat close together, but never equal, as illustrated in Figure 12.

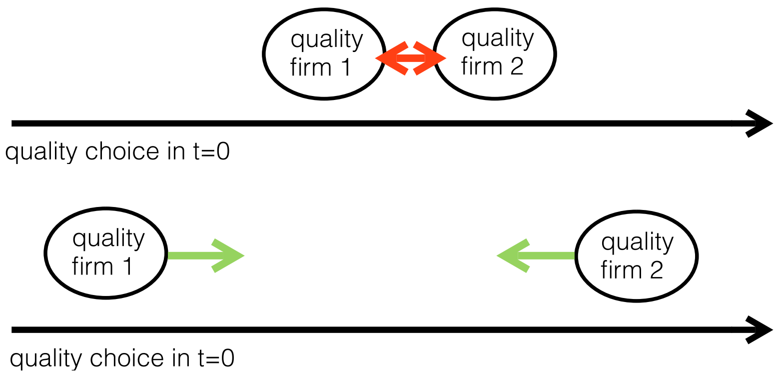

Figure 12: Without full market coverage and without quality-invariant costs there are push *and* pull factors keeping the quality choices somewhat close, but never equal.

C4 Upward Pressure on Qualities

Two questions arise naturally. Firstly, since the low-quality firm now experiences competition from above (the high-quality firm) and below (the option not to buy), does that exert a pull effect on the quality choice of firm 1? Secondly, when the leadership position in quality is the more attractive one, can the threat to be overtaken by the other firm induce the quality-leader to set high qualities whatsoever? The interplay of these forces would produce upward pressure on qualities.

The following proposition and subsequent discussion clarifies in how far there may be a pull effect on the quality choice of the low-quality firm.

Proposition 15. *At any point* (s_1, s_2) *that satisfies assumptions C1 - C3, if*

$$K := \overline{\theta}\,\underbrace{(s_2 - 2s_1)}_{=:A} + \underbrace{\left(2s_2 - \Delta s\frac{s_1}{s_2}\right)}_{>0}\underbrace{\left[\frac{c(s_1)}{s_1} - c'(s_1)\right]}_{=:B} + \underbrace{\Delta s\frac{s_1}{s_2}}_{>0}\underbrace{\left[\frac{\Delta c}{\Delta s} - c'(s_1)\right]}_{=:C} + \underbrace{2\Delta s}_{>0}\underbrace{(-c'(s_1))}_{=:D}$$

is non-negative, then $\partial\Pi_1/\partial s_1 > 0$ *and subsequently the point can not be an equilibrium.*

Proof. See online appendix C7. □

We discuss the consequences for the special case of constant costs $c \in \mathbb{R}_+$, quadratic costs and the general case. For constant costs, K reduces to $\overline{\theta}(s_2 - 2s_1) + (2s_2 - (\Delta s)s_1/s_2)\, c/s_1$. For $s_2 \geq 2s_1$ this expression is positive, subsequently the point can not be an equilibrium. This admits the following interpretation: In order for an equilibrium to exist, the low-quality firm needs to choose s_1 sufficiently close to the quality of firm 2, i.e. larger than $0.5\, s_2$ (*pull effect*).[16] For quadratic costs, which play a prominent role in the literature on the subject, say $c(s) = \tau s^2$, K reduces to $K = \overline{\theta}(s_2 - 2s_1) - \tau s_1 \underbrace{(5s_2 - 3s_1)}_{>0}$. So $K \geq 0$ requires $(s_2 - 2s_1) > 0$ and is fulfilled if $0 \leq \tau < \overline{\theta}(s_2 - 2s_1)/(5s_2 - 3s_1)$. This again has an intuitive interpretation when we think of the costs $c(s) = \tau s^2$ as a quadratic "error term" to zero costs with "intensity" τ. A non-negative K requires that the condition $s_2 \geq 2s_1$, which precludes an equilibrium for zero costs, still suffices to preclude existence for quadratic costs provided the "intensity" τ of the "error term" is below some threshold.

For the general case, K consists of "drivers" A, B, C and D, as defined above, with positive weights. For a fixed s_2, each driver is monotone in s_1 and the level of s_1 determines whether the corresponding driver increases or decreases K, i.e. whether it exerts upward pressure or

16 In the case of constant costs, one can easily show that firm 2 chooses the maximal quality. This is intuitive, as higher quality is not associated with higher costs in this case. The simplification of constant costs helps show the key idea of a "pull" effect exerted on the low-quality firm most clearly, but it also eliminates the force that previously counteracted the quality-leader's incentive to choose the extreme quality.

not. Specifically, A is positive iff $s_1 < 1/2\, s_2$, B is positive iff s_1 is smaller than s_0 with s_0 such that $c'(s_0) = c(s_0)/s_0$, C is positive for $s_1 \neq s_2$ and D is always negative.

That the quality-leader exerts a "pull effect" on the low-quality firm upwards rather than the other way around is intuitive also from a different point of view. Already in the original model the quality-leader enjoys greater profits. Albeit the fact that the low-quality firm will subsequently choose the lowest quality there, this indicates that there is room for a race for the "pole position in quality", as also noted in Tirole (1988, p. 297). Corollary 3 shows that this result persists in the generalized setup under the condition that the relation $\Delta c/\Delta s$ may not be too large. Specifically, if $\Delta c/\Delta s$ lies closer to the lower than to the upper boundary of the admissible interval, the lead position in quality is the more attractive one and the quality-leader will try to keep this "pole position". It seems plausible that the quality-leader is aware of the risk of being overtaken by the other firm at too low quality choices. Then the risk of being overtaken may exert upward pressure on the quality choices when moving qualities closer together. This is shown formally in the sequel.

To capture this, suppose we break the symmetry between the two firms not, as done so far, by assigning the roles of quality-leader and quality-follower ex-ante, but instead by making the quality choice sequential. We call the new setup *sequential game without assigned roles* and assume firm 2 has a first mover advantage in the choice of quality. Specifically, we introduce an additional time period $t = (-1)$ in which firm 2 chooses its quality, while firm 1, upon observing firm 2's decision, continues to choose its quality in $t = 0$. The rest remains as before.

In $t = 0$, firm 1 can either "adapt" by actually becoming the quality-follower or overtake firm 2's leadership position by choosing a higher quality. We ensure assumptions C1 - C3 and (B4) for all quality pairs by assuming that for all s in $[\underline{s}, \overline{s}]$

$$\frac{c(s)}{s} < \frac{\overline{\theta}}{2} \tag{B1'}$$

$$c'(s) \in \left(2 \sup_t \frac{c(t)}{t} - \overline{\theta}, 2\overline{\theta} - \inf_t \frac{c(t)}{t}\right) \tag{B3'}$$

$$\overline{\theta} - 2 \inf_t \frac{c(t)}{t} + c'(s) < 2\overline{\theta} - \sup_t \frac{c(t)}{t} - c'(s). \tag{B4'}$$

(B1') - (B4') relate marginal costs of a further quality improvement to $\overline{\theta}$, the marginal willingness to pay of the most quality-sensitive consumer for a quality improvement. Note that with $c(s_2) - c(s_1) = \int_{s_1}^{s_2} c'(t)dt$, (B3') yields assumption C3 for all $s \in [\underline{s}, \overline{s}]$, while (B4') ensures condition (B4) for all qualities. Conditions (B1'), (B3') and (B4') can be ensured if $\overline{\theta}$ is large enough.[17]

17 In the same spirit as in the original model, this can be interpreted as a condition on sufficient consumer heterogeneity, and thereby neatly connects to the set of assumptions made in the original model. There, (C0) and (A1) demand sufficient consumer heterogeneity while (A2) demands full market coverage; here, only sufficient consumer heterogeneity is needed.

Hence, the quality-leader always enjoys larger profits, which enables us to derive the following proposition.

Proposition 16. *A necessary condition for some (s_1, s_2) to be a subgame-perfect Nash equilibrium in the sequential game without assigned roles, is that*

$$s_2 > \frac{4}{5}\bar{s}. \tag{C18}$$

Proof. As before, the main idea is presented below in the text, while some calculations are relegated to online appendix C8. □

The intuition of the result is as follows: Suppose $s_1 < s_2$ is a Nash equilibrium in the sequential game without assigned roles. In that case one must not be able to find a profitable deviation for the quality-follower, that is, no s_3 with $s_2 < s_3 \leq \bar{s}$ such that the profit when taking the lead position in quality, exceeds the profit when choosing the optimal quality as quality-follower, that is no s_3 $\Pi_2(s_2, s_3) > \Pi_1(s_1, s_2)$. As shown in the appendix, $s_3 = (s_2^2 + s_1 s_2 - s_1^2)/s_2$ is such an profitable deviation, which is infeasible if $(4/5)\bar{s} < s_2$.

Proposition 16 shows that in the sequential game without assigned roles, a necessary condition for a Nash equilibrium to exist is that the quality-leader chooses a quality at least as high as 80% of the maximal quality, as illustrated in Figure 13. In other words, the threat of being overtaken and loosing the leadership position in quality induces the first mover to pick a high quality even in an environment where costs are increasing and convex in the level of quality.

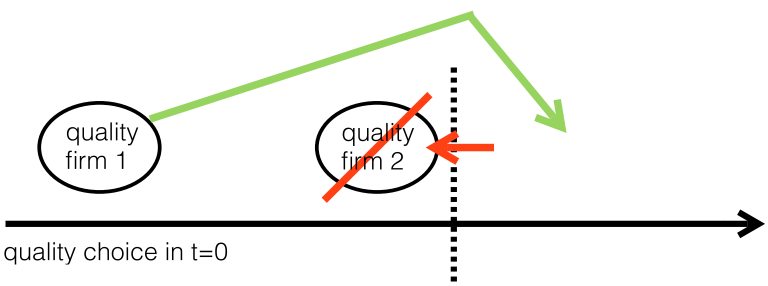

Figure 13: The first mover wants to keep the leadership position in quality, exerting upward pressure on the qualities.

The interplay between a pull effect on the quality choice of the low-quality firm and pressure on the high-quality firm not to leave too much room quality-wise above, gives rise to upward pressure on the quality choices.

C5 Proof of Proposition 13

The full maximization problem reads

$$\max_{p_1} \Pi_1(p_1, p_2) = \max_{p_1} \left\{ (p_1 - c(s_1)) \left[\frac{(p_2 - p_1)}{\Delta s} - \frac{p_1}{s_1} \right] \right\} \tag{C19}$$

$$\max_{p_2} \Pi_2(p_1, p_2) = \max_{p_2} \left\{ (p_2 - c(s_2)) \left[\overline{\theta} - \frac{(p_2 - p_1)}{\Delta s} \right] \right\}, \tag{C20}$$

with the additional conditions

$$(p_1 - c(s_1)) \geq 0 \qquad \text{positive profit margin of firm 1} \tag{Bi}$$
$$(p_2 - c(s_2)) \geq 0 \qquad \text{positive profit margin of firm 2} \tag{Bii}$$
$$\frac{p_2 - p_1}{\Delta s} \geq \frac{p_1}{s_1} \qquad \text{positive market share of firm 1} \tag{Biii}$$
$$\overline{\theta} \geq \frac{p_2 - p_1}{\Delta s} \qquad \text{positive market share of firm 2} \tag{Biv}$$
$$\frac{p_1}{s_1} \geq \underline{\theta} \qquad \text{firm 1's market share takes the form } \frac{(p_2 - p_1)}{\Delta s} - \frac{p_1}{s_1} \tag{Bv}$$
$$\frac{p_2 - p_1}{\Delta s} \geq \frac{p_2}{s_2} \qquad \text{firm 2's market share takes the form } \overline{\theta} - \frac{(p_2 - p_1)}{\Delta s}. \tag{Bvi}$$

I first solve the unconstrained maximization problem and then verify that the (unique) solution satisfies (Bi) - (Bvi). Solving the reaction functions

$$p_1 = R_1(p_2) := \frac{1}{2} \left[p_2 \frac{s_1}{s_2} + c(s_1) \right] \tag{C21}$$
$$p_2 = R_2(p_1) := \frac{1}{2} \left[p_1 + c(s_2) + \overline{\theta} \Delta s \right] \tag{C22}$$

yields the formula for the prices.

It remains to check whether conditions (Bi) - (Bvi) hold. (Bi) and (Bii) are ensured by (C3) as argued in the text. Since plugging in the respective formulas directly yields

$$\hat{\theta} - \theta_0 = \frac{s_2}{(3s_2 + \Delta s)} \left[\frac{\Delta c}{\Delta s} - \left(2\frac{c(s_1)}{s_1} - \overline{\theta} \right) \right] \tag{C23}$$
$$\overline{\theta} - \hat{\theta} = \frac{s_2}{(3s_2 + \Delta s)} \left[\left(2\overline{\theta} - \frac{c(s_2)}{s_2} \right) - \frac{\Delta c}{\Delta s} \right], \tag{C24}$$

(C3) ensures (Biii) and (Biv). (Bv) follows directly from the assumption $\underline{\theta} = 0$, since prices are positive. It remains to show (Bvi), which is a little more cumbersome. As a first step note that (Bvi) follows if we know that

$$\frac{p_2}{p_1} \geq \frac{s_2}{s_1}, \tag{C25}$$

since then

$$p_2 s_1 \geq p_1 s_2 \tag{C26}$$
$$\Leftrightarrow p_2 s_1 - p_2 s_2 + p_2 s_2 \geq p_1 s_2 \tag{C27}$$
$$\Leftrightarrow -p_2 \Delta s + s_2 \Delta p \geq 0 \tag{C28}$$
$$\Leftrightarrow \frac{\Delta p}{\Delta s} \geq \frac{p_2}{s_2}. \tag{C29}$$

It remains to show that (C25) holds. To that end we have

$$\frac{p_2}{p_1} \geq \frac{s_2}{s_1} \tag{C30}$$

$$\Leftrightarrow \frac{\frac{s_2}{3s_2+\Delta s}\left[2c(s_2)+c(s_1)+2\overline{\theta}\Delta s\right]}{\frac{s_1}{3s_2+\Delta s}\left[c(s_2)+2\frac{s_2}{s_1}c(s_1)+\overline{\theta}\Delta s\right]} \geq \frac{s_2}{s_1} \tag{C31}$$

$$\Leftrightarrow \frac{2c(s_2)+c(s_1)+2\overline{\theta}\Delta s}{c(s_2)+2\frac{s_2}{s_1}c(s_1)+\overline{\theta}\Delta s} \geq 1 \tag{C32}$$

$$\Leftrightarrow c(s_2)+c(s_1)\underbrace{\left(1-2\frac{s_2}{s_1}\right)}_{=-\frac{(s_2+\Delta s)}{s_1}}+\overline{\theta}\Delta s \geq 0 \tag{C33}$$

$$\Leftrightarrow c(s_2)-\frac{s_2}{s_1}c(s_1)+\frac{c(s_1)}{s_1}(-\Delta s+2\Delta s)+\Delta s\left[\overline{\theta}-2\frac{c(s_1)}{s_1}\right] \geq 0 \tag{C34}$$

$$\Leftrightarrow \frac{\Delta c}{\Delta s} \geq 2\frac{c(s_1)}{s_1}-\overline{\theta}, \tag{C35}$$

which is ensured by (C3).

□

C6 Proof of Proposition 14

Part a) follows immediately, if we know the following expressions for the derivatives of the profits. With α and β the expressions inside the squared brackets in (C12) and (C13), namely

$$\alpha(s_1,s_2) := \frac{s_1}{3s_2+\Delta s}\left(\frac{\Delta c}{\Delta s}-\left(2\frac{c(s_1)}{s_1}-\overline{\theta}\right)\right) \tag{C36}$$

$$\beta(s_1,s_2) := \frac{s_2}{3s_2+\Delta s}\left(2\overline{\theta}-\frac{c(s_2)}{s_2}-\frac{\Delta c}{\Delta s}\right). \tag{C37}$$

we claim that

$$\begin{aligned}
\frac{\partial \Pi_1(s_1,s_2)}{\partial s_1} &= -\frac{s_2^2}{s_1^2}\alpha^2+2\alpha\frac{s_2}{s_1}\Delta s\frac{\partial \alpha(s_1,s_2)}{\partial s_1} && \text{(C38)}\\
&= \underbrace{\frac{s_2}{s_1}\frac{\alpha}{(3s_2+\Delta s)^2}}_{>0}\Bigg[(2\Delta s(3s_2+\Delta s)+3s_1s_2)\underbrace{\left(\frac{\Delta c}{\Delta s}-c'(s_1)\right)}_{\geq 0 \text{ from convexity}} && \text{(C39)}\\
&\quad +s_2(3s_2+\Delta s)\left(2\frac{c(s_1)}{s_1}-\overline{\theta}-c'(s_1)\right)+4s_2\Delta s\underbrace{\left(2\overline{\theta}-\frac{c(s_2)}{s_2}-c'(s_1)\right)}_{>0 \text{ from (C3)}}\Bigg],\\
\frac{\partial \Pi_2(s_1,s_2)}{\partial s_2} &= \beta^2+2\beta\Delta s\frac{\partial \beta(s_1,s_2)}{\partial s_2} && \text{(C40)}\\
&= \underbrace{\frac{\beta}{(3s_2+\Delta s)^2}}_{>0}\Bigg[(3s_2+\Delta s)(s_2+2\Delta s)\underbrace{\left(\frac{\Delta c}{\Delta s}-c'(s_2)\right)}_{\leq 0 \text{ from convexity}} && \text{(C41)}\\
&\quad +4s_1\Delta s\underbrace{\left(2\frac{c(s_1)}{s_1}-\overline{\theta}-\frac{\Delta c}{\Delta s}\right)}_{<0 \text{ from (C3)}}+(3s_2+\Delta s)s_2\left(2\overline{\theta}-\frac{c(s_2)}{s_2}-c'(s_2)\right)\Bigg].
\end{aligned}$$

To show this, note that for firm 1 the derivative can be written as follows

$$\frac{\partial \Pi_1(s_1,s_2)}{\partial s_1} = -\frac{s_2^2}{s_1^2}\alpha^2+2\alpha\frac{s_2}{s_1}\Delta s\frac{\partial \alpha(s_1,s_2)}{\partial s_1} \tag{C42}$$

$$\stackrel{\text{Def } \alpha}{=} \frac{s_2}{s_1}\frac{\alpha}{(3s_2+\Delta s)^2}\Bigg[-\frac{s_2}{s_1}(3s_2+\Delta s)s_1\left(\frac{\Delta c}{\Delta s}-\left(2\frac{c(s_1)}{s_1}-\overline{\theta}\right)\right)$$
$$+2\Delta s\frac{\partial\alpha(s_1,s_2)}{\partial s_1}(3s_2+\Delta s)^2\Bigg]. \tag{C43}$$

For the derivative of $\alpha(s_1,s_2)$ w.r.t. s_1 it proves helpful to use two versions of the formula for α when applying the product rule, namely

$$\alpha(s_1,s_2) := \frac{1}{3s_2+\Delta s}\left(s_1\frac{\Delta c}{\Delta s}-\left(2c(s_1)-s_1\overline{\theta}\right)\right) \tag{C44}$$
$$= \frac{1}{3s_2+\Delta s}\left(\frac{s_1}{\Delta s}c(s_2)-\frac{s_2+\Delta s}{\Delta s}c(s_1)+s_1\overline{\theta}\right). \tag{C45}$$

Then

$$\begin{aligned}
\frac{\partial\alpha(s_1,s_2)}{\partial s_1}(3s_2+\Delta s)^2 &= \left[\overline{\theta}+\frac{s_2}{(\Delta s)^2}c(s_2)-\frac{s_2}{(\Delta s)^2}c(s_1)-\frac{s_2+\Delta s}{\Delta s}c'(s_1)\right](3s_2+\Delta s) \\
&\quad +s_1\overline{\theta}-2c(s_1)+s_1\frac{\Delta c}{\Delta s} \\
&= 4\overline{\theta}s_2-2c(s_1)+\frac{(s_2+\Delta s)(3s_2+\Delta s)}{\Delta s}\left[\frac{\Delta c}{\Delta s}-c'(s_1)\right] \\
&\quad -(3s_2+\Delta s)\frac{\Delta c}{\Delta s}+s_1\frac{\Delta c}{\Delta s} \\
&= \frac{(s_2+\Delta s)(3s_2+\Delta s)}{\Delta s}\left[\frac{\Delta c}{\Delta s}-c'(s_1)\right] \\
&\quad +\underbrace{\left[-(3s_2+\Delta s)+s_1+2\Delta s\right]}_{=-2s_2}\frac{\Delta c}{\Delta s}+2s_2\left(2\overline{\theta}-\frac{c(s_2)}{s_2}\right) \\
&= \frac{(s_2+\Delta s)(3s_2+\Delta s)}{\Delta s}\left[\frac{\Delta c}{\Delta s}-c'(s_1)\right]+2s_2\left[2\overline{\theta}-\frac{c(s_2)}{s_2}-\frac{\Delta c}{\Delta s}\right].
\end{aligned}$$

Hence together with (C43)

$$\frac{\partial\Pi_1(s_1,s_2)}{\partial s_1} = \frac{s_2}{s_1}\frac{\alpha}{(3s_2+\Delta s)^2}\Bigg[s_2(3s_2+\Delta s)\left(2\frac{c(s_1)}{s_1}-\overline{\theta}-\frac{\Delta c}{\Delta s}\right) \tag{C46}$$
$$+2(s_2+\Delta s)(3s_2+\Delta s)\left(\frac{\Delta c}{\Delta s}-c'(s_1)\right)$$
$$+(2\Delta s)2s_2\left(2\overline{\theta}-\frac{c(s_2)}{s_2}-\frac{\Delta c}{\Delta s}\right)\Bigg]$$
$$= \frac{s_2}{s_1}\frac{\alpha}{(3s_2+\Delta s)^2}\Bigg[s_2\Delta s\left(2\frac{c(s_1)}{s_1}-\overline{\theta}\right)+s_2\Delta s\left(2\overline{\theta}-\frac{c(s_2)}{s_2}\right) \tag{C47}$$
$$-s_2(3s_2+\Delta s)\frac{\Delta c}{\Delta s}+2s_2\left[3s_2\left(2\frac{c(s_1)}{s_1}-\overline{\theta}\right)+3\Delta s\left(2\overline{\theta}-\frac{c(s_2)}{s_2}\right)\right]$$
$$-4s_2\Delta s\frac{\Delta c}{\Delta s}-2(s_2+\Delta s)(3s_2+\Delta s)c'(s_1)$$
$$+2s_2(3s_2+\Delta s)\frac{\Delta c}{\Delta s}+2\Delta s(3s_2+\Delta s)\frac{\Delta c}{\Delta s}\Bigg]$$
$$= \frac{s_2}{s_1}\frac{\alpha}{(3s_2+\Delta s)^2}\Bigg[2\Delta s(3s_2+\Delta s)\left(\frac{\Delta c}{\Delta s}-c'(s_1)\right) \tag{C48}$$
$$-2s_2(3s_2+\Delta s)c'(s_1)+s_2(3s_2+\Delta s)\left(2\frac{c(s_1)}{s_1}-\overline{\theta}\right)$$

$$
\begin{aligned}
& +s_2\left(3\Delta s+\Delta s\right)\left(2\overline{\theta}-\frac{c(s_2)}{s_2}\right)-4s_2\Delta s\frac{\Delta c}{\Delta s}+s_2(3s_2+\Delta s)\frac{\Delta c}{\Delta s}\Bigg] \\
= \; & \frac{s_2}{s_1}\frac{\alpha}{(3s_2+\Delta s)^2}\Bigg[2\Delta s(3s_2+\Delta s)\left(\frac{\Delta c}{\Delta s}-c'(s_1)\right) \qquad \text{(C49)} \\
& +s_2(3s_2+\Delta s)\left(2\frac{c(s_1)}{s_1}-\overline{\theta}-c'(s_1)\right) \\
& +s_2(3s_2+\Delta s)\left(2\overline{\theta}-\frac{c(s_2)}{s_2}-c'(s_1)\right) \\
& +\underbrace{(4s_2\Delta s-s_2(3s_2+\Delta s)}_{=-3s_1s_2}\left(2\overline{\theta}-\frac{c(s_2)}{s_2}-\frac{\Delta c}{\Delta s}\right)\Bigg] \\
= \; & \frac{s_2}{s_1}\frac{\alpha}{(3s_2+\Delta s)^2}\Bigg[2\Delta s(3s_2+\Delta s)\left(\frac{\Delta c}{\Delta s}-c'(s_1)\right) \qquad \text{(C50)} \\
& +s_2(3s_2+\Delta s)\left(2\frac{c(s_1)}{s_1}-\overline{\theta}-c'(s_1)\right) \\
& +s_2(3s_2+\Delta s)\left(2\overline{\theta}-\frac{c(s_2)}{s_2}-c'(s_1)\right) \\
& -3s_1s_2\left(2\overline{\theta}-\frac{c(s_2)}{s_2}-c'(s_1)\right)+3s_1s_2\left(\frac{\Delta c}{\Delta s}-c'(s_1)\right)\Bigg] \\
= \; & \frac{s_2}{s_1}\frac{\alpha}{(3s_2+\Delta s)^2}\Bigg[\left(2\Delta s(3s_2+\Delta s)+3s_1s_2\right)\left(\frac{\Delta c}{\Delta s}-c'(s_1)\right) \qquad \text{(C51)} \\
& +s_2(3s_2+\Delta s)\left(2\frac{c(s_1)}{s_1}-\overline{\theta}-c'(s_1)\right) \\
& +\underbrace{\left(3s_2^2+s_2\Delta s-3s_1s_2\right)}_{=4s_2\Delta s}\left(2\overline{\theta}-\frac{c(s_2)}{s_2}-c'(s_1)\right)\Bigg].
\end{aligned}
$$

For firm 2 the proof follows analogous steps but now

$$
\begin{aligned}
\frac{\partial\Pi_2(s_1,s_2)}{\partial s_2} \; &= \; \beta^2+2\beta\Delta s\frac{\partial\beta(s_1,s_2)}{\partial s_2} \qquad \text{(C52)} \\
&\overset{\text{Def }\beta}{=} \; \frac{\beta}{(3s_2+\Delta s)^2}\Bigg[(3s_2+\Delta s)s_2\left(2\overline{\theta}-\frac{c(s_2)}{s_2}-\frac{\Delta c}{\Delta s}\right) \\
&\quad +2\Delta s\frac{\partial\beta(s_1,s_2)}{\partial s_2}(3s_2+\Delta s)^2\Bigg],
\end{aligned}
$$

the two versions of β read

$$
\begin{aligned}
\beta(s_1,s_2) &:= \frac{1}{3s_2+\Delta s}\left(2s_2\overline{\theta}-c(s_2)-s_2\frac{c(s_2)-c(s_1)}{\Delta s}\right) \qquad \text{(C53)} \\
&= \frac{1}{3s_2+\Delta s}\left(2s_2\overline{\theta}+\frac{s_2}{\Delta s}c(s_1)-\frac{s_2+\Delta s}{\Delta s}c(s_2)\right), \qquad \text{(C54)}
\end{aligned}
$$

for the derivative of β w.r.t. s_2 we have

$$
\begin{aligned}
\frac{\partial\beta(s_1,s_2)}{\partial s_2}(3s_2+\Delta s)^2 \; &= \; \left[2\overline{\theta}-\frac{s_1}{(\Delta s)^2}c(s_1)+\frac{s_1}{(\Delta s)^2}c(s_2)-\frac{s_2+\Delta s}{\Delta s}c'(s_2)\right](3s_2+\Delta s) \\
&\quad -4\left(2\overline{\theta}s_2-c(s_2)-s_2\frac{\Delta c}{\Delta s}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{(s_2+\Delta s)(3s_2+\Delta s)}{\Delta s}c'(s_2) + \underbrace{\frac{s_1(3s_2+\Delta s)+4s_2\Delta s}{\Delta s}}_{=\frac{(s_2+\Delta s)(3s_2-\Delta s)}{\Delta s}}\left(\frac{\Delta c}{\Delta s}\right) \\
&\quad -2\overline{\theta}s_1 + 4c(s_2) \\
&= \frac{(s_2+\Delta s)(3s_2+\Delta s)}{\Delta s}\left(\frac{\Delta c}{\Delta s} - c'(s_2)\right) - 2\frac{(s_2+\Delta s)\Delta s}{\Delta s}\left(\frac{\Delta c}{\Delta s}\right) \\
&\quad -2\overline{\theta}s_1 + 4c(s_2) \\
&= \frac{(s_2+\Delta s)(3s_2+\Delta s)}{\Delta s}\left(\frac{\Delta c}{\Delta s} - c'(s_2)\right) \\
&\quad -2s_1\left(\overline{\theta} + \frac{\Delta c}{\Delta s}\right)\underbrace{+2(2s_1)\frac{\Delta c}{\Delta s} - 4s_2\frac{\Delta c}{\Delta s} + 4c(s_2)}_{=-4\Delta c+4c(s_2)=4c(s_1)} \\
&= \frac{(s_2+\Delta s)(3s_2+\Delta s)}{\Delta s}\left(\frac{\Delta c}{\Delta s} - c'(s_2)\right) + 2s_1\left(2\frac{c(s_1)}{s_1} - \overline{\theta} - \frac{\Delta c}{\Delta s}\right)
\end{aligned}
$$

and together with (C53) this yields

$$
\begin{aligned}
\frac{\partial \Pi_2(s_1,s_2)}{\partial s_2} &= \frac{\beta}{(3s_2+\Delta s)^2}\Bigg[(3s_2+\Delta s)s_2\left(2\overline{\theta} - \frac{c(s_2)}{s_2} - \frac{\Delta c}{\Delta s}\right) && \text{(C55)} \\
&\quad -2(s_2+\Delta s)(3s_2+\Delta s)\left(c'(s_2) - \frac{\Delta c}{\Delta s}\right) + (2\Delta s)2s_1\left(2\frac{c(s_1)}{s_1} - \overline{\theta} - \frac{\Delta c}{\Delta s}\right)\Bigg] \\
&= \frac{\beta}{(3s_2+\Delta s)^2}\Bigg[(3s_2+\Delta s)s_2\left(2\overline{\theta} - \frac{c(s_2)}{s_2} - c'(s_2)\right) && \text{(C56)} \\
&\quad -s_2(3s_2+\Delta s)\frac{\Delta c}{\Delta s} + s_2(3s_2+\Delta s)c'(s_2) \\
&\quad +(2s_2+2\Delta s)(3s_2+\Delta s)\left(\frac{\Delta c}{\Delta s} - c'(s_2)\right) + 4s_1\Delta s\left(2\frac{c(s_1)}{s_1} - \overline{\theta} - \frac{\Delta c}{\Delta s}\right)\Bigg] \\
&= \frac{\beta}{(3s_2+\Delta s)^2}\Bigg[(3s_2+\Delta s)s_2\left(2\overline{\theta} - \frac{c(s_2)}{s_2} - c'(s_2)\right) && \text{(C57)} \\
&\quad +(3s_2+\Delta s)(s_2+2\Delta s)\left(\frac{\Delta c}{\Delta s} - c'(s_2)\right) + 4s_1\Delta s\left(2\frac{c(s_1)}{s_1} - \overline{\theta} - \frac{\Delta c}{\Delta s}\right)\Bigg].
\end{aligned}
$$

For the limits in part b) note that for firm 2, if the limit exists, (C38) implies

$$
\lim_{s_1,s_2\to s_0}\frac{\partial \Pi_1(s_1,s_2)}{\partial s_1} = -\lim_{s_1,s_2\to s_0}\alpha(s_1,s_2)^2 + \lim_{s_1,s_2\to s_0}2\alpha\Delta s\frac{\partial \alpha(s_1,s_2)}{\partial s_1}, \tag{C58}
$$

with

$$
\begin{aligned}
\lim_{s_1,s_2\to s_0}\alpha(s_1,s_2) &= \frac{1}{3}\left(c'(s_0) - 2\frac{c(s_0)}{s_0} + \overline{\theta}\right) =: K_1 \\
\lim_{s_1,s_2\to s_0}\frac{\partial \alpha(s_1,s_2)}{\partial s_1} &= \lim_{s_1,s_2\to s_0}\Bigg[\underbrace{\frac{(s_2+\Delta s)(3s_2+\Delta s)}{\Delta s}\left(\frac{\Delta c}{\Delta s} - c'(s_1)\right)}_{\to 3s_2^2\cdot 0} + 2s_2\underbrace{\left(2\overline{\theta} - \frac{c(s_2)}{s_2} - \frac{\Delta c}{\Delta s}\right)}_{\to 2\overline{\theta} - \frac{c(s_0)}{s_0} - c'(s_0)}\Bigg] \\
&= 2s_0\left(2\overline{\theta} - \frac{c(s_0)}{s_0} - c'(s_0)\right) =: K_2.
\end{aligned}
$$

Plugged into (C58) this yields

$$
\lim_{s_1,s_2\to s_0}\frac{\partial \Pi_1(s_1,s_2)}{\partial s_1} = -K_1^2 + 2K_1K_2\lim_{s_1,s_2\to s_0}\Delta s = -K_1^2. \tag{C59}
$$

Analogously for firm 2 we know from (C40) that, if the limit exists,

$$\lim_{s_1,s_2\to s_0}\frac{\partial\Pi_2(s_1,s_2)}{\partial s_2}=\lim_{s_1,s_2\to s_0}\beta(s_1,s_2)^2+\lim_{s_1,s_2\to s_0}2\beta\Delta s\frac{\partial\beta(s_1,s_2)}{\partial s_2},\tag{C60}$$

with

$$\lim_{s_1,s_2\to s_0}\beta(s_1,s_2)=\frac{1}{3}\left(2\overline{\theta}-\frac{c(s_0)}{s_0}-c'(s_0)\right)=:K_3,$$

$$\lim_{s_1,s_2\to s_0}\frac{\partial\beta(s_1,s_2)}{\partial s_2}=\lim_{s_1,s_2\to s_0}\Bigg[\underbrace{\frac{(s_2+\Delta s)(3s_2+\Delta s)}{\Delta s}\left(\frac{\Delta c}{\Delta s}-c'(s_2)\right)}_{\to 3s_2^2\cdot 0}+2s_1\underbrace{\left(2\frac{c(s_1)}{s_1}-\overline{\theta}-\frac{\Delta c}{\Delta s}\right)}_{\to 2\frac{c(s_0)}{s_0}-\overline{\theta}-c'(s_0)}\Bigg]$$

$$=2s_0\left(2\frac{c(s_0)}{s_0}-\overline{\theta}-c'(s_0)\right)=:K_4.$$

Plugged into (C60) this yields

$$\lim_{s_1,s_2\to s_0}\frac{\partial\Pi_2(s_1,s_2)}{\partial s_2}=K_3^2+2K_3K_4\lim_{s_1,s_2\to s_0}\Delta s=K_3^2,\tag{C61}$$

which concludes the proof.

□

C7 Proof of Proposition 15

The proposition is a direct consequence of the following claim.

Claim. $\partial\Pi_1/\partial s_1$ can be bounded from below as follows

$$\frac{\partial\Pi_1(s_1,s_2)}{\partial s_1}\geq\underbrace{\frac{s_2}{s_1}\frac{\alpha}{(3s_2+\Delta s)^2}}_{>0}\Bigg[\underbrace{(3s_2+\Delta s)}_{>0}K+\underbrace{s_1\Delta s\left(2\overline{\theta}-\frac{c(s_2)}{s_2}-\frac{\Delta c}{\Delta s}\right)}_{>0}\Bigg]\tag{C62}$$

with K as defined in the proposition.

Proof of claim. For the lower bound of $\partial\Pi_1/\partial s_1$, we start with (C49) to obtain

$$\begin{aligned}\frac{\partial\Pi_1(s_1,s_2)}{\partial s_1}\overset{(C49)}{=}&\frac{s_2}{s_1}\frac{\alpha}{(3s_2+\Delta s)^2}\Bigg[2\Delta s(3s_2+\Delta s)\left(\frac{\Delta c}{\Delta s}-c'(s_1)\right)\\&+s_2(3s_2+\Delta s)\left(2\frac{c(s_1)}{s_1}-\overline{\theta}-c'(s_1)\right)\\&+s_2(3s_2+\Delta s)\left(2\overline{\theta}-\frac{c(s_2)}{s_2}-c'(s_1)\right)-3s_1s_2\left(2\overline{\theta}-\frac{c(s_2)}{s_2}-\frac{\Delta c}{\Delta s}\right)\Bigg]\\=&\frac{s_2}{s_1}\frac{\alpha}{(3s_2+\Delta s)^2}\Bigg[2\Delta s(3s_2+\Delta s)\left(\frac{\Delta c}{\Delta s}-c'(s_1)\right)\\&+s_2(3s_2+\Delta s)\left(2\frac{c(s_1)}{s_1}-\overline{\theta}-c'(s_1)\right)\\&+s_2(3s_2+\Delta s)\left(2\overline{\theta}-\frac{c(s_2)}{s_2}-c'(s_1)\right)\\&-(3s_2+\Delta s)\left(2\overline{\theta}-\frac{c(s_2)}{s_2}-\frac{\Delta c}{\Delta s}\right)+s_1\Delta s\left(2\overline{\theta}-\frac{c(s_2)}{s_2}-\frac{\Delta c}{\Delta s}\right)\Bigg]\end{aligned}\tag{C63}$$

$$= \frac{s_2}{s_1}\frac{\alpha}{(3s_2+\Delta s)^2}\left[(3s_2+\Delta s)K + s_1\Delta s\left(2\overline{\theta} - \frac{c(s_2)}{s_2} - \frac{\Delta c}{\Delta s}\right)\right] \tag{C64}$$

with

$$\begin{aligned}
K &= \overline{\theta}(s_2-2s_1) + c(s_2)\left(\frac{s_1}{s_2}-1\right) + 2\frac{s_2}{s_1}c(s_1) - 2(s_2+\Delta s)c'(s_1) + \Delta c\left(2+\frac{s_1}{\Delta s}\right) && \text{(C65)}\\
&= \overline{\theta}(s_2-2s_1) - 2(s_2+\Delta s)c'(s_1) && \text{(C66)}\\
&\quad + \frac{1}{s_2\Delta s}\left[c(s_2)\left(s_2^2 - s_1^2 + s_1s_2\right) + c(s_1)\left(2\frac{s_2^3}{s_1} - 4s_2^2 + s_1s_2\right)\right]\\
&= \overline{\theta}(s_2-2s_1) - 2(s_2+\Delta s)c'(s_1) && \text{(C67)}\\
&\quad + \frac{1}{s_2\Delta s}\Bigg[s_2^2\underbrace{\left(c(s_2)+2c(s_1)\left(\frac{s_2}{s_1}-2\right)\right)}_{\geq c(s_1)\left[1+2\frac{s_2}{s_1}-4\right]=c(s_1)\left[\frac{2\Delta s - s_1}{s_1}\right]} + c(s_2)s_1\Delta s + c(s_1)s_1s_2\Bigg]\\
&\geq \overline{\theta}(s_2-2s_1) - 2(s_2+\Delta s)c'(s_1) && \text{(C68)}\\
&\quad + \frac{1}{s_2\Delta s}\left[c(s_1)\frac{s_2^2}{s_1}(2\Delta s) + c(s_2)s_1\Delta s + c(s_1)(-s_2\Delta s)\right]\\
&= \overline{\theta}(s_2-2s_1) + 2s_2\left(\frac{c(s_1)}{s_1} - c'(s_1)\right) - 2c'(s_1)\Delta s + \frac{1}{s_2}\underbrace{(c(s_2)s_1 - c(s_1)s_2)}_{=c(s_2)s_1 - c(s_1)s_1 + c(s_1)s_1 - c(s_1)s_2} && \text{(C69)}\\
&= \overline{\theta}(s_2-2s_1) + 2s_2\left(\frac{c(s_1)}{s_1} - c'(s_1)\right) - 2c'(s_1)\Delta s + \frac{s_1\Delta s}{s_2}\left(\frac{\Delta c}{\Delta s} - \frac{c(s_1)}{s_1}\right) && \text{(C70)}\\
&= \overline{\theta}(s_2-2s_1) + \left(2s_2 - \Delta s\frac{s_1}{s_2}\right)\left[\frac{c(s_1)}{s_1} - c'(s_1)\right] + \Delta s\frac{s_1}{s_2}\left[\frac{\Delta c}{\Delta s} - c'(s_1)\right] + 2\Delta s\left(-c'(s_1)\right).
\end{aligned}$$

□

C8 Proof of Proposition 16

It remains to derive the profitable deviation $s_3 = (s_2^2 + s_1s_2 - s_1^2)/s_2$. To that end, let s_3 be some quality choice with $s_2 < s_3 \leq \overline{s}$. Then with $\Delta_{ij}s := (s_j - s_i)$ and $\Delta_{ij}c := c(s_j) - c(s_i)$ the following inequalities are equivalent

$$\begin{aligned}
\Pi_2(s_2,s_3) &> \Pi_1(s_1,s_2) && \text{(C71)}\\
\Leftrightarrow \Delta_{23}s\beta(s_2,s_3)^2 &> \Delta_{12}s\frac{s_2}{s_1}\alpha(s_1,s_2)^2\\
\Leftrightarrow \Delta_{23}s\left[\frac{s_3}{3s_3+\Delta_{23}s}\left(2\overline{\theta} - \frac{c(s_3)}{s_3} - \frac{\Delta_{23}c}{\Delta_{23}s}\right)\right]^2 &> \Delta_{12}s\frac{s_2}{s_1}\left[\frac{s_1}{3s_2+\Delta_{12}s}\left(\frac{\Delta_{12}c}{\Delta_{12}s} - 2\frac{c(s_1)}{s_1} + \overline{\theta}\right)\right]^2\\
\Leftrightarrow \left(\frac{\Delta_{23}s}{\Delta_{12}s}\right)\left(\frac{s_1}{s_2}\right)\left(\frac{s_3^2}{s_1^2}\right)\frac{(3s_2+\Delta_{12}s)^2}{(3s_3+\Delta_{23}s)^2} &> \left(\frac{\frac{c(s_2)-c(s_1)}{\Delta_{12}s} - 2\frac{c(s_1)}{s_1} + \overline{\theta}}{2\overline{\theta} - \frac{c(s_3)}{s_3} - \frac{c(s_3)-c(s_2)}{\Delta_{23}s}}\right)^2 && \text{(C72)}
\end{aligned}$$

with α and β for $s_i < s_j$ as defined in (C36) and (C37) at the beginning of Appendix C6. Suppose we can choose s_3 in the admissible interval such that

$$\begin{aligned}
\frac{(s_3-s_2)}{(s_2-s_1)} &= \frac{s_1}{s_2} && \text{(C73)}\\
\Leftrightarrow s_3 &= \frac{s_2^2 + s_1s_2 - s_1^2}{s_2}. && \text{(C74)}
\end{aligned}$$

With $s_1/s_2 \leq s_3/s_2$ and this particular choice of s_3 we have

$$(s_3 - s_2) \leq \frac{s_3}{s_2}(s_2 - s_1), \tag{C75}$$

which implies

$$\frac{3s_2 + (s_2 - s_1)}{3s_3 + (s_3 - s_2)} \geq \frac{s_2}{s_3}. \tag{C76}$$

Hence, for this specific choice of s_3, the LHS of (C72) reads

$$\left(\frac{s_1}{s_2}\right)\left(\frac{s_1}{s_2}\right)\left(\frac{s_3^2}{s_1^2}\right)\frac{(3s_2 + \Delta_{12}s)^2}{(3s_3 + \Delta_{23}s)^2} \geq \left(\frac{s_3^2}{s_2^2}\right)\left(\frac{s_2^2}{s_3^2}\right) = 1, \tag{C77}$$

while we know that the RHS of (C72) is smaller than 1 if and only if

$$2\overline{\theta} - \frac{c(s_3)}{s_3} - \frac{c(s_3) - c(s_2)}{(s_3 - s_2)} > \frac{c(s_2) - c(s_1)}{(s_2 - s_1)} - 2\frac{c(s_1)}{s_1} + \overline{\theta}. \tag{C78}$$

But (C78) holds, since from (B4') we know

$$\begin{gathered} 2\overline{\theta} - \frac{c(s_3)}{s_3} - c'(s_3) > c'(s_3) - 2\frac{c(s_1)}{s_1} + \overline{\theta} \\ \Leftrightarrow 2\overline{\theta} - \frac{c(s_3)}{s_3} - \frac{\Delta_{23}c}{(s_3 - s_2)} + 2\frac{c(s_1)}{s_1} - \overline{\theta} - \frac{\Delta_{12}c}{(s_2 - s_1)} > \underbrace{c'(s_3) - \frac{\Delta_{23}c}{(s_3 - s_2)}}_{>0} + \underbrace{c'(s_3) - \frac{\Delta_{12}c}{(s_2 - s_1)}}_{>0}. \end{gathered}$$

Hence, (C72) holds and this particular choice of s_3 is in fact a profitable deviation. When is this choice of s_3 infeasible? Suppose $s_2 < \overline{s}$. For $s_3 = s_2$ the LSH of (C73) is zero. As s_3 increases, the expression on the LHS increases. Hence, either (C73) holds for some s_3 - in which case we have found a profitable deviation - or $(\overline{s} - s_2) < s_1/s_2(s_2 - s_1)$. This deviation is infeasible if

$$(\overline{s} - s_2) < \frac{s_1}{s_2}(s_2 - s_1) = \frac{s_1}{s_2}\left(1 - \frac{s_1}{s_2}\right)s_2, \tag{C79}$$

which, since the RHS is smaller equal than $s_2/4$, holds if

$$\frac{4}{5}\overline{s} < s_2. \tag{C80}$$

□