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Dynamic Tax Evasion and Capital Misallocation in General Equilibrium

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Dynamic Tax Evasion and Capital Misallocation in General Equilibrium*

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Abstract

We study tax evasion in a dynamic macroeconomic model where utility-maximizing entrepreneurs use capital to produce or buy bonds, depending on their firm's stochastic productivity. The government provides productivity-enhancing public goods financed through taxes and bond issuance. Entrepreneurs can increase their income by evading taxes at the risk of being audited and fined. Lower productivity boosts evasion incentives, exacerbating capital misallocation because unproductive entrepreneurs accumulate wealth at their peers' expense. Consistently with OECD data, the model predicts a negative relation between tax evasion and productivity in the aggregate but heterogeneous signs and magnitudes across productivities. Public goods provision affects these outcomes ambiguously. (JEL: E25, E26, H23, H26)

Keywords: Dynamic tax evasion; financial frictions; general equilibrium; misallocation

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1 Introduction

Tax evasion is a pervasive global phenomenon. Although it is particularly pronounced in developing countries (Beck et al., 2014), it has sizeable effects on mature economies too. In the US, for instance, the IRS estimates the gap between total taxes owed and taxes paid to be above 15 percent (Slemrod, 2007). Among the EU economies, the estimates range between 8 and 30 percent (Murphy, 2019).

The primary motivations for this paper are two empirical regularities: the observed negative correlation between a country's shadow economy and its productivity (Loayza and Rigolini, 2006; La Porta and Shleifer, 2014; Dabla-Norris et al., 2019) and between the size of its informal sector and financial development (Beck et al., 2014; La Porta and Shleifer, 2014).[1,2] Notably, the data also show that the sign and magnitude of the former relationship vary substantially across the productivity distribution of firms. Figure 1 uses CompNet (2023) data to illustrate this phenomenon. It reports the relationship between (residual) increments in Total-Factor-Productivity (TFP) and variations in the World Bank's estimated shadow economy size for different percentiles of the firms' productivity distribution in 16 OECD countries from 2000 until 2020. While the correlation is weak (or even positive) for lower productivity percentiles, it becomes progressively negative and more robust when moving towards higher ones. This finding highlights the importance of considering firm heterogeneity when investigating the relationship between tax evasion and productivity.

Based on this empirical evidence, our paper develops a dynamic macroeconomic model of a production economy where the government provides public goods, and financially-constrained entrepreneurs can evade income taxes, depending on their firm's stochastic productivity. We use the model to explore how idiosyncratic tax evasion decisions interact with financial frictions and government expenditure, affecting the cross-sectional distribution of firm productivity and, as a result, aggregate tax evasion and TFP.

Similarly to other related studies (e.g., Franjo et al., 2022; Erosa et al., 2023), we show

[1] At the micro level, Fajnzylber et al. (2011) and Amin and Okou (2020) find that productivity gaps between firms that comply with taxes range between 25 to 50 percent; Dabla-Norris et al. (2008) provide evidence of the connection between financing constraints and informality, especially among small firms.

[2] Additional evidence corroborating these patterns across OECD countries over the last twenty years appears in Section 3.

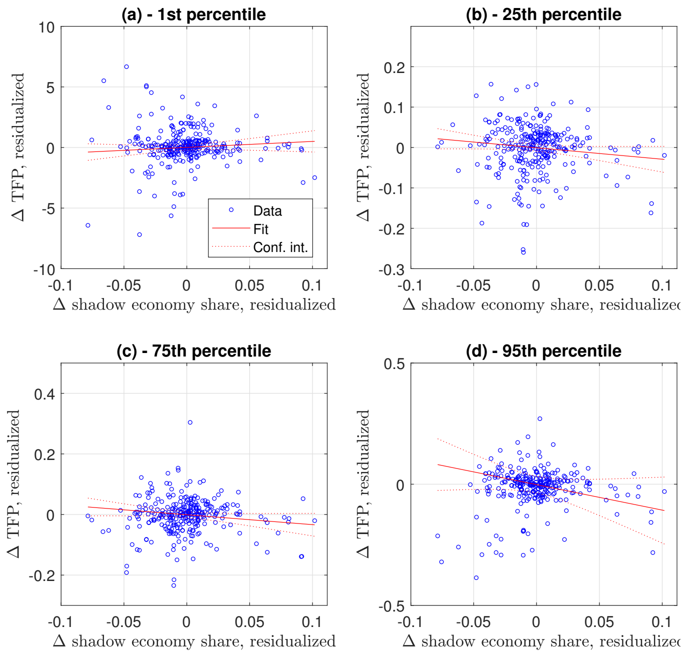

Figure 1: Correlation between country-level TFP and shadow economy shares within different firm-productivity percentiles; observations are residualized by using country dummies. Sources: the World Bank data and Schneider et al. (2010) (shadow economy); CompNet (2023) database (TFP by percentiles).

that tax evasion exacerbates the misallocation of capital across productivity levels due to financial frictions (see Moll, 2014). However, we differentiate from them by characterizing the general equilibrium mechanisms through which, jointly with public expenditure, tax evasion affects the cross-sectional distribution of firms' productivity, both in their steady states *and* transitional dynamics. Notably, the model can qualitatively replicate the empirical regularities motivating the paper.

Following Moll (2014), we build our model in continuous time. This choice allows us to improve analytical tractability relative to a discrete-time setting because we can characterize the whole transition dynamics of the equilibrium towards its steady state. The model features three main actors: a unit mass of entrepreneurs, each coupled with one firm, and the government.[3]

Similarly to Barro (1990), the government provides the economy with productivity-

[3]Parsimoniously-modelled "hand-to-mouth" workers are the fourth type of agent in the economy but do not play a significant role.

enhancing public goods. Its expenditures are financed by raising taxes and issuing risk-free bonds.[4] Each entrepreneur owns capital and the non-tradable ownership of one firm; firms are heterogeneous in their stochastic productivity levels. Entrepreneurs maximize the intertemporal utility from their consumption and choose whether to lend capital to their firms and produce (i.e., be "active") or purchase bonds issued by other firms and the government (i.e., be "passive"). Active entrepreneurs leverage their balance sheet consistently with a borrowing constraint and earn their firms' profits; passive ones finance other entrepreneurs' leverage and the government, earning no profits from their firms. The choice to be active depends on the stochastic productivity of their firms; hedging productivity fluctuations is impossible due to financial market incompleteness. Accordingly, each firm-entrepreneur couple can be viewed as a small-sized enterprise.[5] Entrepreneurs pay a flat tax on their income. They can relieve this burden by hiding a share of their income, which they select optimally by trading off the benefit with the risk of being audited and accordingly fined by the government.

In line with the classical tax evasion literature (e.g., Allingham and Sandmo, 1972; Andreoni, 1992; Lin and Yang, 2001), we characterize entrepreneurs' optimal tax evasion strategies in closed form. Unlike in those studies, in our paper, their evasion decisions are time-varying and depend negatively on their own firms' productivity, which squares nicely with the empirical findings in Gradstein et al. (2019).[6] We then solve the model for its competitive equilibrium and derive steady-state levels and the transition dynamics of macroeconomic aggregates and of the entrepreneurs' cross-sectional distribution. In doing so, we show analytically that the relationship between public debt and the economy's TFP is ambiguous, highlighting that public expenditure impacts entrepreneurs' cross-sectional distribution by affecting their production and tax evasion decisions.

[4] The crucial role of government spending in affecting the private sector has been highlighted by several works since the seminal contribution of Aschauer (1989). The interested reader can also refer to Agénor (2010) and Agénor and Neanidis (2015).

[5] Our focus on small firms is motivated by the empirical evidence that roughly half of the aggregate evasion consists of business profits under-reporting (Slemrod, 2019) and that about one-third of self-employed activities are misreported (Hurst et al., 2014).

[6] A very different mechanism that generates a similar result can be found in Gillman and Kejak (2014), who develops an endogenous growth model in which human capital enhances productivity and curbs tax evasion incentives.

To better understand the mechanisms at play, we calibrate the model and explore its dynamics numerically. More specifically, we study the non-linear response of macroeconomic aggregates and entrepreneurs' cross-sectional distribution following two types of unanticipated and temporary shocks. The former is a "financial" shock that tightens financial frictions by decreasing the borrowing capacity of all entrepreneurs; the latter is a "public expenditure" shock obtained by increasing the supply of public goods.

Coherently with our motivating evidence, the economy's response to financial shock entails a negative relationship between aggregate TFP and financial frictions and between frictions and tax evasion at the aggregate level. Importantly, the sign and magnitude of the relationship are substantially heterogeneous across enterprises, depending on their productivity level, accommodating the patterns of Figure 1. The mechanisms behind these results are the following.

Financial shocks, which further limit entrepreneurs' leverage capacity, redistribute net worth from high- to low-productivity enterprises, which are more prone to evade taxes. In the aggregate, this reallocation reduces the equilibrium returns on all investments (and thus the marginal cost of capital), encouraging more evasion across enterprises of all productivities. In turn, lower marginal costs reduce the productivity threshold above which being active is profitable. This second effect reduces the economy's TFP by "crowding" mid-productivity enterprises in the production process, allowing them to earn profits from their firms and thus curbing their evasion incentives.[7]

Financial shocks also affect the level of public debt, which exhibits a distinctive whiplash-shaped response. In the short run, public debt rises because lower aggregate TFP and a larger shadow economy hamper the government's fiscal capacity; in the medium run, it shrinks because lower aggregate productivity reduces debt service costs, overtaking the losses due to tax base erosion.

The economy's response to public expenditure shocks, i.e., temporary and unexpected increments in the supply of public goods, carries three notable effects. First, increasing public expenditure increases (mechanically) the economy's aggregate productivity and reduces the share of its shadow economy (as an endogenous consequence). Second, despite

[7] This result is reminiscent of the "selection effect" described in Di Nola et al. (2021).

their aggregate consequences, these productivity boosts carry heterogeneous effects across entrepreneurs with different productivities, depending on their net worth distribution. Similarly to González et al. (2022), we characterize this effect analytically as a "net worth channel". Third, milder financial frictions mitigate the positive effect of raising public expenditure. This counter-intuitive phenomenon occurs because public expenditure enhances productivity across all enterprises, distorting capital allocation towards low(er)-productivity ones ("threshold" channel; see also González et al., 2022).

The rest of the paper unfolds as follows. Section 2 connects the paper with the most closely related literature. Section 3 provides empirical evidence motivating the paper. Section 4 presents the model and characterizes its competitive equilibrium. Section 5 solves the model numerically and presents the main results. Section 6 concludes.

2 Related literature

Our paper builds on the literature studying optimal tax evasion in a dynamic context (e.g., Levaggi and Menoncin, 2013, 2016).[8] Unlike these papers, our model entails heterogeneous tax evasion policies across individuals and studies their interaction in a general equilibrium context.

In this respect, we relate to several studies investigating how tax evasion affects macroeconomic dynamics. For instance, an early paper by Chen (2003) analyses the link between tax evasion, public expenditure, and growth in a representative household economy. More recently, Ordonez (2014) and López (2017) study capital misallocation when firms can reduce their scale to remain undetected due to incomplete tax enforcement, showing that improving tax enforcement fosters government revenues at the cost of generating inefficiencies. We differentiate from these studies by focusing on the interaction between tax evasion and financing constraints in a setting where entrepreneurs' cross-sectional distribution arises endogenously. By doing so, we connect to the large literature on the role of frictions in generating capital misallocation; seminal contributions in the field are Cooley et al. (2004),

[8]The microeconomic theory concerned with the individual decision of avoiding taxes is long-dated and well-established, starting from the foundational contribution of Allingham and Sandmo (1972).

Cagetti and De Nardi (2006) Moll (2014), Buera and Moll (2015), and Bassetto et al. (2015), among others.

On the joint role of tax evasion and financial frictions in affecting macroeconomic aggregates in economies with and without heterogeneous agents, we are close to a few recent contributions by Di Nola et al. (2021), Franjo et al. (2022), and Erosa et al. (2023). Di Nola et al. (2021) point out that the welfare gains of tax evasion experienced by self-employed workers happen at the expense of firm-employed ones. Franjo et al. (2022) assume heterogeneous financial constraints between formal and informal firms, finding that their removal reduces the informal sector's size and tax evasion while increasing TFP. Erosa et al. (2023) show that eliminating payroll taxes reduces business informality, benefiting primarily large (and less credit-constrained) employers and that the productivity benefit from reducing frictions increases the economy's informality rate.

Our work contributes to this literature in at least two dimensions. First, building on the seminal framework of Moll (2014), we describe the heterogeneous effect of the interaction between financial frictions and tax evasion across the whole firm-productivity distribution without sacrificing analytical tractability. Second, in the same spirit of Barro (1990), we consider the role of public goods in fostering aggregate productivity. This choice allows us to point out a novel channel connecting government expenditures (and debt) to the level of aggregate tax evasion and the associated capital misallocation.

3 Motivating evidence

Before introducing the model, we provide empirical evidence of the correlations between productivity and tax evasion frictions and between tax evasion and financial frictions at the aggregate (i.e., country) level. Moreover, we highlight that the sign and magnitude of the first relationship change substantially across different percentiles of firms' productivity distribution, which has not yet been documented, up to our knowledge. Our data cover 23 countries from January 2000 to December 2020. Tax evasion data come from the World Bank data, and Schneider et al. (2010); TFP data come from the OECD dataset; the IMF financial development index proxies financial frictions.

We test the relation between the (de-trended) variations in a country's TFP and the share of its shadow economy estimating the following panel regression:

$$\Delta TFP_{i,t}(\%) = \beta_0 + \beta_1 \Delta \text{Shadow}_{i,t}(\%) + \mathbf{X}_{i,t}^T \mathbf{B} + u_i + \epsilon_{i,t}, \tag{1}$$

where $\Delta TFP_{i,t}(\%)$ and $\Delta\text{Shadow}_{i,t}(\%)$ denote percentage changes in total factor productivity and shadow economy share in country i and time t, respectively; $X_{i,t}$ is a vector of controls; u_i denotes a country-level fixed effect; and $\epsilon_{i,t}$ are HAC standard errors, clustered at the country level. The first column of Table 1 reports the estimated coefficients. The estimate of β_1 is negative and statistically significant at (at least) the 10 per cent level, confirming the negative relation between a country's productivity and the size of its shadow economy. Our paper's first objective is to develop a tractable theoretical framework that may explain this pattern.[9]

To assess the correlation between tax evasion and financial frictions, we estimate the following regression of the share of a country's shadow economy and the IMF financial development index:

$$\text{Shadow}_{i,t} = \gamma_0 + \gamma_1 IF_{i,t} + \mathbf{Y}_{i,t}^T \mathbf{C} + v_i + \xi_{i,t} \tag{2}$$

in which $IF_{i,t,}$ denotes the IMF financial development index in country i and time t; $Y_{i,t}$ is a vector of controls; v_i denotes a country-level fixed effect; and $\xi_{i,t}$ are HAC standard errors, clustered by country. The second column of Table 1 collects the estimates, showing that γ_1, the coefficient of the financial development level is negative and statistically significant at the 1 per cent level. This evidence is coherent with the primary mechanism that we model in our paper, i.e., that financial frictions hinder aggregate total factor productivity by generating a larger shadow economy.[10]

Empirical evidence also motivates our model's ability to generate heterogeneous effects of tax evasion across the entrepreneurs' productivity distribution. To highlight this aspect, we use CompNet (2023) data and estimate Eq. (1), isolating different percentiles of firms' pro-

[9]In Online Appendix B.2, we provide further country-level analysis, which document a negative covariation between TFP and the shadow economy in 21 countries out of 23.

[10]A similar analysis is performed country-by-country in Online Appendix B.2, verifying that the negative correlation holds in most OECD countries.

Dependent variable	ΔTFP (%)	ΔTFP (%)	Shadow (%)	Shadow (%)
ΔShadow (%)	-0.17*	-0.31***		
	(0.1)	(0.10)		
FI		-0.30	-6.97***	-6.99***
		(0.37)	(0.37)	(1.72)
Fixed effects	✓	✓	✓	✓
Controls	✗	✓	✗	✓
Observations	471	394	609	512
Adj. R-squared	0.01	0.05	0.52	0.58
Number of groups	23	20	29	25

Table 1: Linear fixed effect model of the relationship between variations in TFP and shadow economy shares and between shadow economy shares and financial development. Newey-West standard errors are in parentheses. Notes: *** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$; Controls: lnGDP, public debt/GDP, fin. dev. index, GDP growth, shadow economy share (%), ln public expenditure, output growth rate, public investment share.

ductivity distribution.[11] Our estimates are collected in Table 2. Coherently with the results based on aggregate data, TFP and the shadow economy variations appear to be negatively correlated. However, while the relation is weaker (or negligible) for lower productivity levels, it becomes progressively more robust, moving towards the right-hand side of the productivity distribution.

4 Model

Time is continuous and indexed $t \in [0, \infty)$. The economy features four actors: a unit mass of firms and their entrepreneurs, hand-to-mouth workers, and the government. Firms are risk-neutral and couple one-to-one with entrepreneurs, who are entitled to the profits of their activities. They face idiosyncratic productivity shocks and use capital, labour, and public goods to generate output. Entrepreneurs are risk averse and maximize the intertemporal utility of their consumption. They can supply capital to their firms by using their net worth endowment and leveraging their balance sheet (i.e., issuing bonds) up to their endogenous borrowing constraint. Else, they buy bonds to finance other entrepreneurs

[11] The data covers a subset of 14 OECD countries from 2000 to 2020; descriptive statistics appear in the Online Appendix B.2.

	Percentile						
ΔTFP (%)	1st	25th	50th	75th	90th	95th	Average
ΔShadow (%)	0.1	-0.19**	-0.19**	-0.2**	-0.53*	-0.59*	-0.30***
	(0.15)	(0.09)	(0.08)	(0.08)	(0.31)	(0.32)	(0.09)
Fixed effects	✓	✓	✓	✓	✓	✓	✓
Controls	✓	✓	✓	✓	✓	✓	✓
Observations	216	216	216	216	216	216	216
Adj. R-squared	0.01	0.08	0.08	0.07	0.03	0.02	0.03
Number of groups	14	14	14	14	14	14	14

Table 2: Linear regression of the variations in TFP and shadow economy shares for different TFP percentiles. Newey-West standard errors are in parentheses. Notes: *** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$; Controls: lnGDP, public debt/GDP, fin. dev. index, GDP growth, shadow economy share (%), ln public expenditure, output growth rate, public investment share.

and the government. Workers provide labour inelastically and immediately consume their wages. The government provides public goods by raising taxes on entrepreneurs' and workers' incomes and issuing bonds. Entrepreneurs can evade their taxes at the risk of being audited and accordingly fined by the government. Auditing revenues contribute to reducing the government's deficit.

We now discuss each actor in greater detail.

4.1 Firms, workers and production

Firms are heterogeneous in their TFP level $z_t \in \mathbb{Z} =: (0, z^{\max}]$. Similarly to Barro (1990), their production technology uses three factors as inputs: private capital k_t (or simply "capital"), labour l_t, and the public good $\bar{k}_t$. Firm-level output equals

$$y_t = A(z_t) k_t^{\alpha} \bar{k}_t^{\beta} l_t^{1-\alpha-\beta}, \tag{3}$$

in which α and β represent the output elasticities of capital and public goods, and A is an increasing function of z. As in Moll (2014), firm-level productivity is stochastic, and its log returns follow the Ornstein-Uhlenbeck (i.e., mean-reverting) process

$$d \ln z_t = -\nu \ln z_t dt + \sigma \sqrt{\nu} dW_t, \tag{4}$$

in which W_t denotes a standard Brownian motion that is independent across firms, ν captures the process auto-correlation, and σ captures idiosyncratic TFP volatility.[12]

At time zero, each firm couples with one entrepreneur, with whom she sticks forever. Financial markets are incomplete because entrepreneurs cannot trade in risky claims written on the productivity shocks of their firms. Accordingly, we interpret firm-entrepreneurs couples as small-sized enterprises. Entrepreneurs have a net worth endowment of n_t, which can be allocated in firms' capital to obtain the profits from their business activity. Capital depreciates at the exogenous rate δ.

After freely assessing their own firms' productivity (in the absence of moral hazard), entrepreneurs decide whether to engage in business activities (i.e., being active) or not. Active entrepreneurs provide their entire net worth to the firm as capital k_t and build up leverage by issuing bonds b_t on their firm's behalf. Their borrowing capacity is limited by the following constraint:[13]

$$k_t \leq \phi n_t, \tag{5}$$

where $\phi \geq 1$ is a constant that defines the economy's financial development. Jointly with market incompleteness, this constraint constitutes the only friction in the model.

Capital markets are perfectly competitive, and there is no aggregate risk. Therefore, the stock of capital financing firms yields the endogenous rate R_t; a no-arbitrage condition requires that bonds yield the risk-free rate $r_t = R_t - \delta$.

The labour side of the economy is modelled parsimoniously. Human capital l is supplied inelastically by "hand-to-mouth" workers. These agent have no endowment and holds no claims in firms. The government taxes their labour wages at the constant rate $\tau_l \in [0, 1]$.[14]

[12]An Ornstein-Uhlenbeck process is the continuous-time analogue of a $AR(1)$ process. It is possible to show that, over a unit time interval, $\nu = -\ln \mathbb{C}orr\left(z_{t+1}, z_t\right)$ (see Stokey, 2008). By applying Itô's lemma to $z = \exp \ln z$, one obtains that $dz = z\left[\nu\left(\sigma^2/2 - \ln z\right)dt + \sigma\sqrt{\nu}dW\right]$.

[13]Following Gertler and Kiyotaki (2010), the borrowing constraint can be micro-founded as the incentive-compatible constraint of entrepreneurs allowed to divert a fraction $1/\phi \in [0, 1]$ of their assets and default on their liabilities. The constraint dispels diversion incentives by ensuring that the value of diversion revenues is smaller than or equal to the continuation value of the enterprise.

[14]In Online Appendix B.1, we show that workers exhibit an endogenous "hand-to-mouth" behaviour under the assumption that their inter-temporal discount rate is large enough.

The public good $\bar{k}_t$ is provided by the government proportionally to capital, such that

$$\bar{k}_t = gk_t, \tag{6}$$

where g is a constant (henceforth, the public good "multiplier"). We interpret the supply in Eq. (6) as that of pure public goods, such as broadband and mobility infrastructures, benefiting firms proportionally to the level of their individual activity.

Taking factor prices and its own productivity as given, each firm solves

$$\max_{k_t, l_t} \{y_t - R_t k_t - l_t w_t\}, \tag{7}$$

subject to Eqs. (5) and (6). As we show in Appendix A.1, under the analytically convenient assumption that $A(z) = z^{\alpha+\beta}$, the optimal policies for a firm owned by an entrepreneur with net worth n_t are

$$k_t = \mathbb{I}_{z_t \geq \bar{z}_t} \phi n_t \quad \text{and,} \quad l_t = z_t k_t \frac{\pi_t}{w_t} \left(\frac{1-\epsilon}{\epsilon} \right), \tag{8}$$

where $\epsilon := \alpha + \beta$, $\pi_t := \epsilon g^{\beta/\epsilon} \left((1-\epsilon)/w_t \right)^{(1-\epsilon)/\epsilon}$. The endogenous threshold $\bar{z}_t = r_t/\pi_t$ denotes the productivity threshold level at which entrepreneurs are indifferent between financing their firm and buying bonds.

Eq. (8) tells us that optimal production strategies are of the bang-bang type. In other words, firms engage in production only if their TFP level z_t is large enough. In such cases, entrepreneurs supply them with capital leveraging their net worth up to the maximum limit. Conversely, when z_t is below the threshold, entrepreneurs allocate their whole net worth endowment in bonds.[15]

As a result of the policies in Eq. (8), individual firm's profits are linear in their productivity and entrepreneurs' net worth, generating pre-tax profit flows

$$\Pi_t = n_t \underbrace{\mathbb{I}_{z_t \geq \bar{z}} \left(z_t \pi_t - R_t \right)}_{:= \varphi_t}. \tag{9}$$

[15]Note that, as in Moll (2014), we do not model financial intermediation directly. Instead, we let inactive entrepreneurs finance other firms in a perfectly competitive rental (or bond) market.

4.2 Entrepreneurs

Entrepreneurs enjoy recursive utility (Duffie and Epstein, 1992) from their consumption; they have unit elasticity of inter-temporal substitution, relative risk aversion γ, and subjective discount rate ρ. Following a "perpetual youth" dynamics (Blanchard, 1985), they retire at the exogenous Poisson rate ω and have no bequest motif. At the same rate, an equal mass of new entrepreneurs enters the market to replace them. At retirement, the government collects a constant share $\tau_\omega \in [0, 1]$ of their net worth as a tax. The remainder is pooled and equally rebated across new entrants.[16]

Entrepreneurs choose their consumption c_t and allocate their net worth between capital k_t and bonds b_t, taking the production strategies of their firms as given. Capital yields $R_t - \delta$ and entitles entrepreneurs to receive their firm's profits according to Eq. (9); bonds are remunerated at the risk-free rate r_t. The government taxes all income at the flat rate $\tau_k \in [0, 1]$. Entrepreneurs can relieve their tax burden by concealing an income share e_t from the tax authority, exposing themselves to the possibility of being audited and fined by the government. As in Levaggi and Menoncin (2013), auditing events follow a continuum of Poisson processes J_t with intensity λ, which are independent across entrepreneurs. Upon being audited, evasion fines equal a constant share η of evaded revenues.

As a result of these assumptions, entrepreneurs' net worth evolves as

$$dn_t = (1 - \tau_k + \tau_k e_t) \underbrace{(b_t r_t + k_t(R_t - \delta) + \Pi_t)}_{\text{pre-tax earnings}} dt - \underbrace{e_t \eta (b_t r_t + k_t(R_t - \delta) + \Pi_t)}_{\text{evasion fine}} dJ_t - c_t dt, \quad (10)$$

and they face the following problem:

$$V_0 := \max_{\{c_t, e_t\}} \mathbb{E}_0 \left[\int_0^{\tau_\omega} (1-\gamma)\rho V_t \left(\log c_t - \frac{1}{1-\gamma} \log((1-\gamma)V_t) \right) dt \right], \quad (11)$$

subject to Eq. (10). As we show in Appendix A.2, the associated optimal controls are

$$c_t = n_t \rho, \text{ and } \quad e_t = \frac{1 - \left(\frac{\eta\lambda}{\tau_k}\right)^{\frac{1}{\gamma}}}{\eta\left(r_t + \varphi_t\right)}. \quad (12)$$

[16] The exogenous entry/exit process helps obtain a stationary steady state for a broader range of parameters when calibrating the model; it does not affect any other mechanism in the model.

Due to the unit elasticity of inter-temporal substitution, the optimal consumption is a constant share of entrepreneurs' net worth n_t, equal to the subjective discount rate ρ. The optimal evasion strategy is similar to that derived in Levaggi and Menoncin (2012) for a representative agent. However, it differs substantially in two aspects. First, it depends negatively on individual businesses' productivity (and thus profit) levels, which aligns with the empirical evidence of Gradstein et al. (2019). Second, it changes over time depending on the aggregate state of the economy through the levels of r_t, g, w_t, and $\bar{z}_t$ (see Eq. (9)).

4.3 Government

The government supplies public goods while collecting income taxes, auditing revenues (T_t), and issuing bonds to finance its activity. Accordingly, the aggregate stock of public debt B_t has dynamics:

$$dB_t = r_t B_t dt + \underbrace{gK_t dt - dT_t}_{\text{Primary deficit}}, \tag{13}$$

where gK_t is the total public good supply and K_t the aggregate capital stock.

The instantaneous tax-related revenues of the government T_t are the sum of capital taxes collected, fines, entrepreneurs' retirement and labour income taxes. By substituting Eq. (9) in Eq. (10) and aggregating across entrepreneurs, one obtains that

$$\frac{dT_t}{dt} = \underbrace{\int_0^\infty \int_{\mathbb{Z}} n \left[\tau_k (1-e) + \eta\lambda e\right] (r_t + \varphi_t)\, dF_t(n,z)}_{\text{Capital taxes plus auditing revenues}} + \underbrace{N_t \omega \tau_\omega}_{\text{Ret. tax}} + \underbrace{w_t \tau_l L_t}_{\text{Lab. tax}}, \tag{14}$$

where $F_t(n, z)$ denotes the joint distribution of firm-entrepreneurs' net worth and productivity at time t, N_t is the aggregate net worth of all enterpreneurs, and L_t is the aggregate labour supply.

4.4 Competitive equilibrium and aggregation

We now solve the model for its competitive equilibrium and characterize the transition dynamics towards its steady state. For this purpose, let $f_t(z, n)$ be the density function associated with $F_t(z, n)$. For ease of notation, we henceforth omit the time subscripts of micro-level quantities (such as n, k, and z) but maintain them to denote aggregate-level objects (such as r_t and w_t). Moreover, we normalize aggregate labour supply to one.

Definition 1. ***(Competitive equilibrium)*** *A competitive equilibrium is a set of macroeconomic aggregates (output, capital, and public debt), factor prices, consumption, and tax evasion strategies such that: the government supplies public capital, collects taxes, and performs tax auditing; firms maximize their profits; entrepreneurs maximize their inter-temporal utility; and all markets (capital, bonds, and labour) clear.*

In equilibrium, the net worth of all entrepreneurs equals aggregate capital plus public debt such that

$$\underbrace{\int_0^\infty \int_{\mathbb{Z}} k(n,z) f_t(n,z) dndz}_{:=K_t} + B_t = \underbrace{\int_0^\infty \int_{\mathbb{Z}} n_t f_t(n,z) dndz}_{:=N_t}. \tag{15}$$

Similarly, firms' labour demand satisfies

$$\int_0^\infty \int_{\mathbb{Z}} l(n,z) f_t(n,z) dndz = 1.$$

To characterize the equilibrium, we follow Moll (2014) and define the following function:

$$\theta_t(z) := \frac{\int_0^\infty n f_t(z,n) dn}{N_t} \geq 0, \tag{16}$$

which denotes the share of aggregate net worth held by entrepreneurs whose firm's productivity level equals z. Similarly to f_t, θ_t is a density function.[17] Its adoption simplifies the aggregation process significantly. Indeed, by substituting Eq. (8) into the market clearing condition in Eq. (15) and rearranging, we obtain:

$$\phi \underbrace{\left(1 - \int_0^{\bar{z}} \theta_t(z) dz\right)}_{:=\Theta_t} = 1 - \frac{B_t}{N_t}, \tag{17}$$

[17] This can be verified by substituting Eq. (16) in the definition of N_t and rearranging, which yields $\int_{\mathbb{Z}} N_t \theta_t(z) dz = \int_{\mathbb{Z}} \int_0^\infty n f_t(z,n) dndz = N_t$ and, thus, $\int_{\mathbb{Z}} \theta_t(z) dz = 1$. The analytical tractability provided by the use of $\theta_t(z)$ derives from the fact that, given entrepreneurs' optimal strategies in Eq. (12), the drift and jump terms of the state variable's dynamics in Eq. (10) are linear in n.

where Θ_t denotes the share of active entrepreneurs.

Eq. (17) pins the relationship between the productivity threshold $\bar{z}$, the magnitude of public debt relative to entrepreneurs' aggregate net worth, and the borrowing limit ϕ. It shows that the higher the leverage capacity of each entrepreneur, the higher the productivity threshold that sorts those who decide to be active from those who lend them capital. In the limit where $\phi \to \infty$, the most productive entrepreneur (whose $z = z^{\max}$) manages the whole stock of capital, and everyone else buys bonds. Accordingly, $B_t = N_t$, and heterogeneity plays no role. Conversely, when no leverage is allowed ($\phi = 1$), the share of inactive entrepreneurs equals the ratio between public debt and aggregate net worth. In all intermediate cases, the equilibrium is well-defined if $B_t \leq N_t$ for all t.

Equipped with the above relations, we can fully characterize all macroeconomic aggregates' levels and transition dynamics towards their steady states.

Proposition 1. ***(Macroeconomic aggregates: stocks and dynamics)*** *For a given triple $\{B_t, N_t, \theta_t\}$, the following holds:*

1. *Aggregate output equals*

$$Y_t = \underbrace{g^{\beta} X_t^{\epsilon}}_{:=TFP_t} K_t^{\epsilon}, \tag{18}$$

where $X_t := \mathbb{E}_t^{\theta}\left[z_t | \, \bar{z}\right]$ and $K_t = N_t - B_t$.

2. *Labour wages, risk-free, and capital rental rates satisfy*

$$w_t = (1-\epsilon) Y_t \text{ and } \quad r_t + \delta = R_t = \epsilon \bar{z} g^{\beta} \left(\phi Z_t K_t\right)^{\epsilon - 1}, \tag{19}$$

where $Z_t =: \int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz$.

3. *The growth rate of public debt and aggregate net worth have laws of motion*

$$\frac{d \ln B_t}{dt} = r_t + g\left(\frac{N_t}{B_t} - 1\right) - \frac{\tau_l (1-\epsilon) Y_t}{B_t} + \\ - \frac{N_t}{B_t}\left[\tau_k \left(r_t + \phi \Theta_t \left(\epsilon Y_t K_t^{-1} - R_t\right)\right) - \left(\frac{\tau_k}{\eta} - \lambda\right)\left[1 - \left(\frac{\eta\lambda}{\tau_k}\right)^{\frac{1}{\gamma}}\right] + \omega \tau_\omega\right], \tag{20}$$

$$\frac{d\ln N_t}{dt} = (1-\tau_k)\left[r_t + \phi\Theta_t\left(\epsilon Y_t K_t^{-1} - R_t\right)\right] + \left(\frac{\tau_k}{\eta} - \lambda\right)\left[1 - \left(\frac{\eta\lambda}{\tau_k}\right)^{\frac{1}{\gamma}}\right] - \rho - \omega\tau_\omega. \quad (21)$$

Proof. See Appendix A.3. □

The first part of Proposition 1 shows that, at each instant t, the economy's TFP is a non-linear function of the average productivity across active firms and the level of public expenditure. The second part reveals that wages are linear in output, which follows from our inelastic labour supply, and that the rental rate on capital crucially depends on both the productivity threshold $\bar{z}_t$ and the good public multiplier g. As we will see in Section 5, this has a critical role in explaining the endogenous relation between entrepreneurial tax evasion decisions, capital misallocation and, in turn, aggregate productivity. The third and last part of the proposition describes the transition dynamics of the model's state variables: the stock of public debt and entrepreneurs' aggregate net worth (or simply "savings"). As intuition suggests, the debt growth rate increases with the risk-free rate and with the average rate of tax evasion ($(\tau_k/\eta - \lambda)[1 - (\eta\lambda/\tau_k)^{1/\gamma}]$; see Eq. (21)). Conversely, it decreases in the aggregate amount of tax revenues ($\tau_k(r_t + \phi\Theta_t(\epsilon g^\beta Y_t/K_t - R_t)$; see Eq. (20))). The overall effect of public expenditure on the equilibrium level of debt is ambiguous. On the one hand, a higher public expenditure level proportionally increases debt growth because $(N_t/B_t - 1)g \geq 0$ (see the term to the right-hand side of Eq. 20). On the other hand, increasing public expenditure fosters entrepreneurs' productivity, output, and aggregate tax revenues (third and fourth term of Eq. (20)), thereby decreasing the growth rate of public debt.

Focusing on the dynamics of aggregate savings N_t, what stands out is that its growth rate is increasing in both after-tax capital revenues and the average evasion rates (first two terms in Eq. (21)), while it is decreasing in the consumption rate ρ and tax-adjusted death rate ω (the last two terms in Eq. (21)).

Another relevant aspect to stress is that public expenditure indirectly influences macroeconomic aggregates' dynamics by affecting the entrepreneurs' "entry" production decisions, which reflect factor prices and the distribution of net worth across productivity levels. While disentangling these forces in closed form is impossible, we explore them numerically in Section 5.3.

To complete the equilibrium's characterization, the following proposition derives the law of motion of the cross-sectional distribution of savings across enterprises with different productivity levels.

Proposition 2. ***(Cross-sectional distribution: transition dynamics)*** *For a given triple $\{B_t, N_t, \theta_t\}$, the net worth distribution across different productivity levels at time t obeys the following PDE:*

$$\frac{d\theta_t}{dt} = \underbrace{\left[(1-\tau_k)(r_t+\varphi_t) + \left(\frac{\tau_k}{\eta} - \lambda\right)\left[1 - \left(\frac{\eta\lambda}{\tau_k}\right)^{\frac{1}{\gamma}}\right] - \rho\right]\theta_t}_{\text{After-tax earnings minus auditing fines}} + \\ - \underbrace{\left[\frac{d\ln N_t}{dt} + \tau_\omega \omega\right]\theta_t}_{\text{Aggregate capital and retirement}} \underbrace{- \frac{\partial}{\partial z}\left[z\nu\left(\frac{1}{2} - \ln z\right)\theta_t\right] + \frac{\nu}{2}\frac{\partial^2}{\partial z^2}\left(z^2\theta_t\right)}_{\text{Productivity randomness}}. \quad (22)$$

with boundary and mass preservation conditions $\theta_t(0) = 0$ and $\int_{\mathbb{Z}} \theta_t(z)dz = 1$.

Proof. See Appendix A.4. □

According to Eq. (22), the time variations in entrepreneurs' cross-sectional distribution can be decomposed into three components. The first one captures that, for each productivity level z, the associated density mass grows at the rate of retained earnings, net of consumption and auditing fines. The second term tells us that, everything else constant, a higher growth rate in aggregate capital stock and death rate results in a lower net worth concentration. The third component incorporates that firm-level productivity evolves stochastically according to Eq. (4).

Equipped with the results in Proposition 1 and 2, we define the stationary steady state of our economy as a triple $\Omega_{ss} =: \{B_{ss}, N_{ss}, \theta_{ss}\}$ such that

$$\left.\frac{d\ln N_t}{dt}\right|_{\Omega_{ss}} = \left.\frac{d\ln B_t}{dt}\right|_{\Omega_{ss}} = \left.\frac{d\theta_t}{dt}\right|_{\Omega_{ss}} = 0.$$

5 Results

This section calibrates the model and solves it numerically for its steady-state equilibrium. Then, it studies the economy's transition dynamics in response to two unexpected temporary shocks hitting the economy at its steady state, on the financial frictions and public expenditure levels, respectively. According to our simulations, increasing financial frictions reduces aggregate TFP and fosters tax evasion, aligning with the motivating evidence reported in Section 3. This outcome is the result of two different but closely related forces. On the one hand, frictions exacerbate tax evasion by redistributing capital from high- to low-productivity (and highly evasive) entrepreneurs. On the other hand, lower average productivity generates lower returns on all investments (bonds or capital), encouraging more aggressive evasion strategies across all enterprises.

The mechanisms described above are also at play when considering a temporary unexpected shock on public goods supply. Our exercise shows that by increasing public goods and, thus, private capital productivity (an assumption in our model), the policymaker can reduce tax evasion in the aggregate (an endogenous outcome). However, we also find that milder financial frictions mitigate this positive feedback. To better frame this counter-intuitive result, we decompose the effect of a change in public expenditure on TFP analytically into a "threshold" channel, capturing the effect of public expenditure on entrepreneurs' entry decisions, and a "net-worth" channel, identifying its effect on the net worth distribution across entrepreneurs with different productivity levels.

5.1 Calibration and steady-state approximation

Calibrating the model requires some restrictions on its parameter domain. In particular, we set the capital tax rate (τ_k) and auditing parameters (η and λ) so that the optimal evasion rate remains in the interval $[0, 1]$ for all productivity levels z.

By using Eq. (12), the condition guaranteeing that tax evasion remains positive is $\eta\lambda \leq \tau_k$, which implies that evading taxes is viable if the gain from tax evasion (τ_k) is higher than its (expected) cost ($\lambda\eta$). Using the same equation, we have that $e_t \leq 1$ for all levels of z only if $1 - (\eta\lambda/\tau_k)^{1/\gamma} < \eta r_t$. In other words, entrepreneurs' optimal tax evasion is less or equal

Parameter	Meaning	Value	Source
β	Public good parameter	0.08	Glomm and Ravikumar (1997)
ϕ	Borrowing constraint	1.43	FRED & González et al. (2022)
λ	Evasion - auditing intensity	0.15	Bernasconi et al. (2020)
η	Evasion - fine rate	1.4	Bernasconi et al. (2020)
τ_k	Tax rate (capital)	0.23	Bernasconi et al. (2020)
τ_l	Tax rate (labour)	0.35	Bernasconi et al. (2020)
τ_ω	Tax rate (retirement)	0.25	Dividend tax, OECD data
ω	Retirement rate	0.11	González et al. (2022)
ν	Firm-level tfp (auto-correlation)	0.14	Gilchrist et al. (2014)
σ	Firm-level tfp (volatility)	0.3	González et al. (2022)
g	Public good supply	0.18	Public investments, OECD data
ρ	Discount rate	0.015	Standard
γ	Risk aversion	2.5	Standard
α	Capital share	0.33	Standard
δ	Capital depreciation	0.065	Standard

Table 3: Calibrated parameters.

to their revenues only if the risk-free rate r_t is sufficiently high. Unfortunately, we can not verify this restriction ex-ante because r_t is an endogenous object; we will therefore do that after solving the model numerically.

We select our benchmark parameters as reported in Table 3. Empirical estimates of the output elasticity to public good range between 0.06 (Ratner, 1983) and 0.2 (Binswanger and Rosenzweig, 1993).[18] Accordingly, we set $\beta = 0.08$. The financial friction parameter ϕ matches the corporate leverage (debt-to-net worth) of 43 per cent reported by FRED (see also González et al., 2022); the tax rate τ_k and auditing parameters η and λ are in line with the calibration exercise in Bernasconi et al. (2020) and the literature referred therein. The tax rate on labour income is the average wage tax across OECD countries. Following González et al. (2022), we set the entrepreneurs' retirement rate ω equal to the return on equity of OECD firms. Since we interpret retirement flows as shareholder payments, we set the "retirement" tax rate τ_ω to match the average OECD dividend tax rate.

The calibration of ν is based on Gilchrist et al. (2014), who estimate the yearly auto-correlation of firms' idiosyncratic productivity based on the AR process $x_{t+1} = \beta x_t + \epsilon_t$, and

[18] A review of the early literature on this topic appears in Glomm and Ravikumar (1997).

Friction parameter	TFP_t	Y_{ss}	N_{ss}	B_{ss}	$\bar{z}_{ss}$	Θ_{ss}	E_{ss}	R_{ss}	w_{ss}
$\phi = 1.43$	1.62	3.65	8.23	0.92	1.78	0.62	0.21	0.08	2.15
$\phi = 1.57$	1.71	3.81	8.66	1.06	2.06	0.55	0.20	0.09	2.25

Table 4: Macroeconomic aggregates in the steady state.

obtain $\beta \approx 0.87$. Accordingly, we set $\nu = -0.14$. The volatility parameter σ is in line with González et al. (2022).

The public good supply parameter g matches the public capital investments share in OECD data, which ranges between 3 and 51 percent (See online Appendix B.2). Within this interval, we choose a value of 15 percent and set

$$\text{Public inv. share} = 0.15 = \frac{gK}{K + gK} \Rightarrow g = \frac{0.15}{1 - 0.15} \approx 0.18.$$

Finally, the discount rate ρ, the relative risk aversion γ, the capital share α, and the depreciation rates δ take standard values in the literature. The time interval $dt \approx \Delta t = 1/12$ (i.e., we simulate the model monthly).

With the above parameters, we solve the model numerically for its steady state Ω_{ss}.[19] Table 4 collects the steady-state values of the model's key macroeconomic aggregates considering the baseline ($\phi = 1.43$) and a 10 per cent milder level of financial frictions ($\phi = 1.57$). Figure 2 displays the resulting stationary densities θ_{ss} and the cross-sections of entrepreneurs' tax evasion strategies e_{ss} as functions of z.

When comparing steady states, three aspects are worth noticing. First, milder (starker) financial frictions are associated with a higher (lower) threshold level $\bar{z}_{ss}$, meaning that more (less) productive but fewer (more) entrepreneurs participate in the production process. As a result, milder (starker) frictions lead to higher (lower) levels of TFP, output, and thus aggregate savings. Second, higher average productivity is associated with more public debt because of higher (lower) TFP, interest rates and, in turn, higher (lower) debt financing costs. Third, the model predicts a negative relationship between productivity and tax evasion,

[19]The numerical solution follows Achdou et al. (2022) and approximates the system of differential equations describing the equilibrium by coupling an up-wind scheme and an explicit Euler method. We collect the details in the Online Appendix B.3.

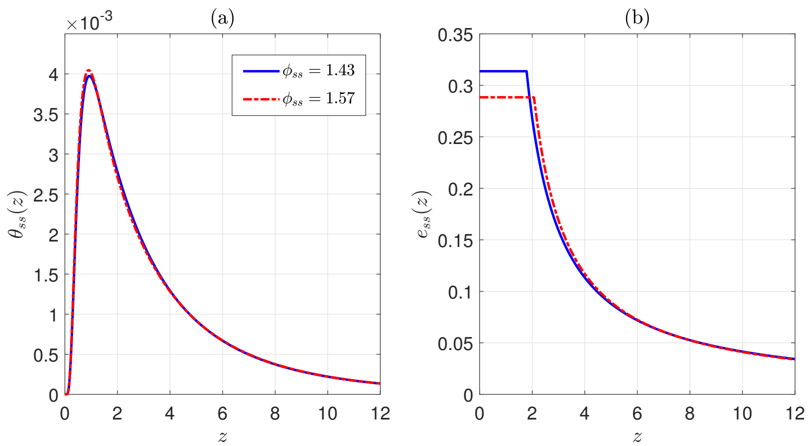

Figure 2: Net worth distribution and tax evasion strategies in the steady state.

which holds at both idiosyncratic and aggregate levels (see Table 4, Columns 1 and 7, and Figure 2, Panel (b)).[20]

5.2 Financial shocks

This section studies the effect of financial and real "MIT" (i.e., unexpected) shocks hitting the economy in its steady state. The shock materializes as a deterministic shift in the level of the leverage constraint parameters ϕ from its baseline level. After the shock, the parameter moves deterministically toward its long-run level ϕ_{ss} with dynamics $\dot{\phi}_t = \kappa\,(\phi_t - \phi_{ss})$, where $\kappa = 0.14$ defines the persistence of the shock.

Performing this analysis is equivalent to deriving a standard impulse-response function following a perturbation of the equilibrium in the neighbourhood of its steady state, whose (non-linear) transition dynamics obey the laws described in Propositions 1 and 2. This is true because, under our preferences (i.e., homothetic with unit EIS; see Eqs. (10) and (11)) and auditing process assumptions, entrepreneurs' optimal consumption and evasion policies do not depend on the slope of their value function but exclusively on the *current levels* of ϕ (see Eq. (12)) and other state variables.[21]

Figure 3 displays the impulse response functions of the model's macroeconomic aggregates

[20] Notice also that our calibration generates an aggregate tax evasion level of about 20 per cent, roughly in line with the average level observed in 2000-2020 across OECD countries, which averages 18.44 and ranges between 7 and 41.5 per cent. (see Table B.2 in Online Appendix B.2).

[21] A comprehensive discussion of the challenges involved into solving heterogeneous agents macro-finance models when that is not the case appears in Fernandez-Villaverde et al. (2023).

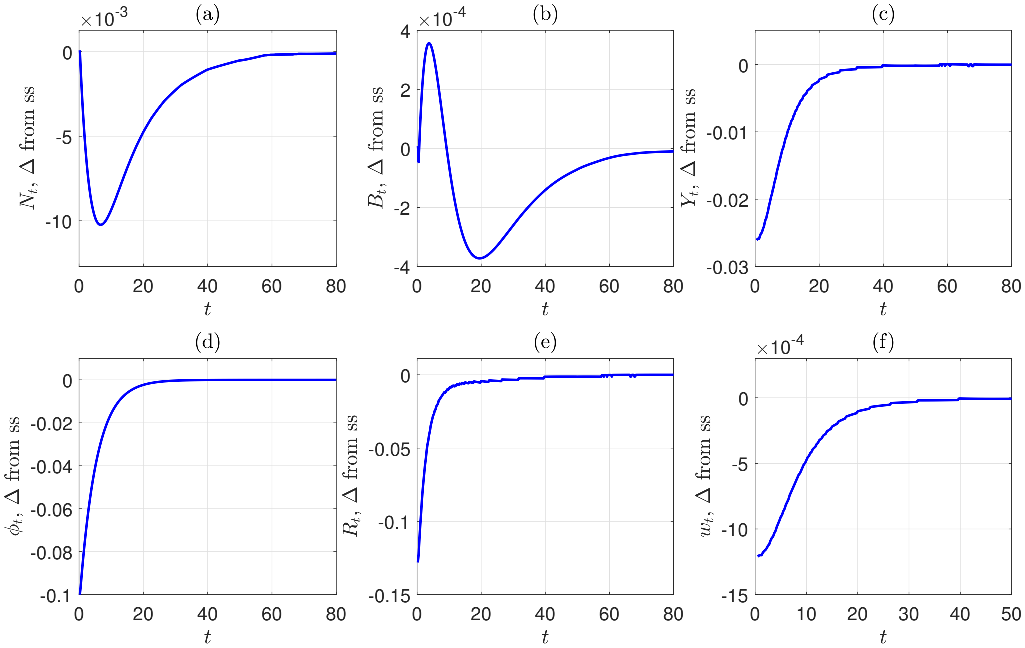

Figure 3: Impulse-response functions of key macroeconomic aggregates following 10 per cent unanticipated shocks to the level of ϕ.

starting from the high (blue) and low (red) initial levels of financial friction. As intuition suggests, increasing financial frictions decreases aggregate output (Panel (c)), hindering both factor prices (see Panels (e) and (f)) and entrepreneurs' aggregate net worth (Panel (a)). These dynamics contrast with the behaviour of public debt, which exceeds its steady-state level in the short run while shrinking below it in the medium term (Panel (b)). Notably, a higher initial level of friction yields a more significant response of output and return on capital and a milder impact on net worth, debt, and wages.

To disentangle the forces generating these outcomes, Figure 4 displays the effect of the shock on aggregate TFP, the threshold level $\bar{z}_t$, the mass of active entrepreneurs Θ_t, and the share of the shadow economy. In line with the results of Moll (2014), negative variations in output and factor prices take place jointly with reductions in TFP (Panel (a)), because financial frictions redistribute capital from active and high-productivity entrepreneurs to (previously) inactive and low-productivity ones. This redistribution decreases the marginal cost of capital, thereby reducing the threshold $\bar{z}_t$ (Panel (b)). Consequently, low-z entrepreneurs find it convenient to join the production process (Panel (c)), dampening the average

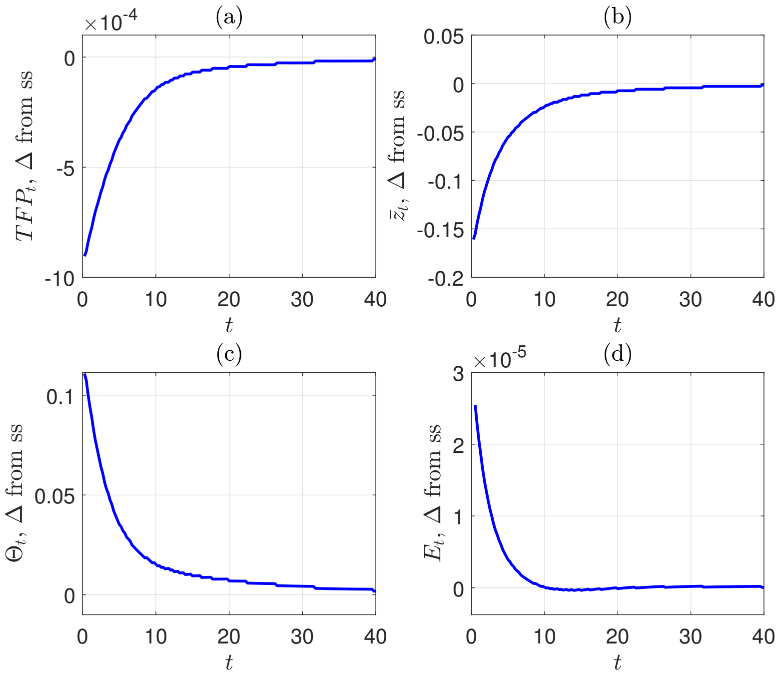

Figure 4: Impulse-response functions of the endogenous total factor productivity, its components, and aggregate tax evasion following 10 per cent MIT shocks to the level of ϕ.

productivity of the economy and, in turn, reducing aggregate net worth.

Our main result concerns the opposite relation between variations in aggregate TFP and tax evasion (Panels (a) and (d)), which replicates the motivating evidence reported in Section 3. This connection occurs as direct (idiosyncratic) and indirect (aggregate) forces are combined. At the microeconomic level, imposing higher financial frictions reallocates resources from high- to low-productivity enterprises, which have a higher propensity to evade taxes (see Eq. (12))). At the aggregate level, capital misallocation lower bonds and capital returns (remember that $r_t = R_t - \delta$), encouraging more aggressive evasion strategies across all entrepreneurs.

The impulse response of public debt exhibits a distinctive "whiplash" pattern, which can be elucidated as follows. In the short term, debt rises due to significant declines in cross-sectional and thus aggregate productivity, net worth, and tax evasion, eroding the government's tax revenues. In the medium term, debt decreases due to reduced financing costs and a broader tax base. The positive impact on tax revenues overtakes the previous effects, leading to a decline in B_t below its initial (steady-state) level.

To understand how financial shocks propagate across the distribution of entrepreneurs, Figure 5 visualizes the impulse response functions of their net worth density (Panel (a)) and the cross-section of their tax evasion strategies (Panel (b)). Coherently with our aggregate-level results, increasing financial frictions reduces the mass of high-z entrepreneurs while increasing the mass of their less-productive peers. This net worth reallocation effect is particularly pronounced in the neighbourhood of the lower bound $z = 0$ across the enterprises whose productivity is just above the threshold level $\bar{z}_t$, which were instead not viable prior to the shock.

When looking at cross-sectional tax evasion (Panel (b)), what stands out is that, on the one hand, the increase in financial frictions encourages tax evasion for both high- and low-productivity entrepreneurs. On the other hand, it mitigates evasion incentives for intermediate-z ones, whose mass is concentrated just above the productivity threshold $\bar{z}$. The first effect takes place because the redistribution of capital from high- to low-productivity entrepreneurs hinders the economy's TFP (see Figure 3, Panel (a)), thereby reducing the rental rate of capital (see Figure 3, Panel (e)). Ceteris paribus, this pattern provides additional evasion incentive across all z's, curbing the denominator of optimal tax evasion strategies in Eq. (12). The second effect materializes because, following the reduction in factor prices, businesses whose productivity level was in the lower neighbourhood of $z = \bar{z}_{ss}$ before the shock become viable (see Figure 3, Panels (e) and (f) and Figure 4, Panel (b)); that is, their businesses become profitable (see Eq. (12)). In summary, entrepreneurs whose increments in φ_t overtake reductions in r_t evade fewer taxes.

5.3 Public expenditure shocks

Having described the response of entrepreneurs' evasion decisions to changes in financial frictions, we now investigate how they interact with the government's public good provision. Similarly to the previous section's analysis, we look at the outcome of an unanticipated temporary shock which increases public expenditure by 10 percent.[22] We evaluate the shock in the presence of three levels of financial development: baseline ($\phi = 1.43$), baseline plus 10

[22]Once again, we assume that the level of g after the shock returns to its steady-state level $g_{ss} = 0.18$ with deterministic dynamics $dg_t = 0.14\,(g_t - g_{ss})\,dt$.

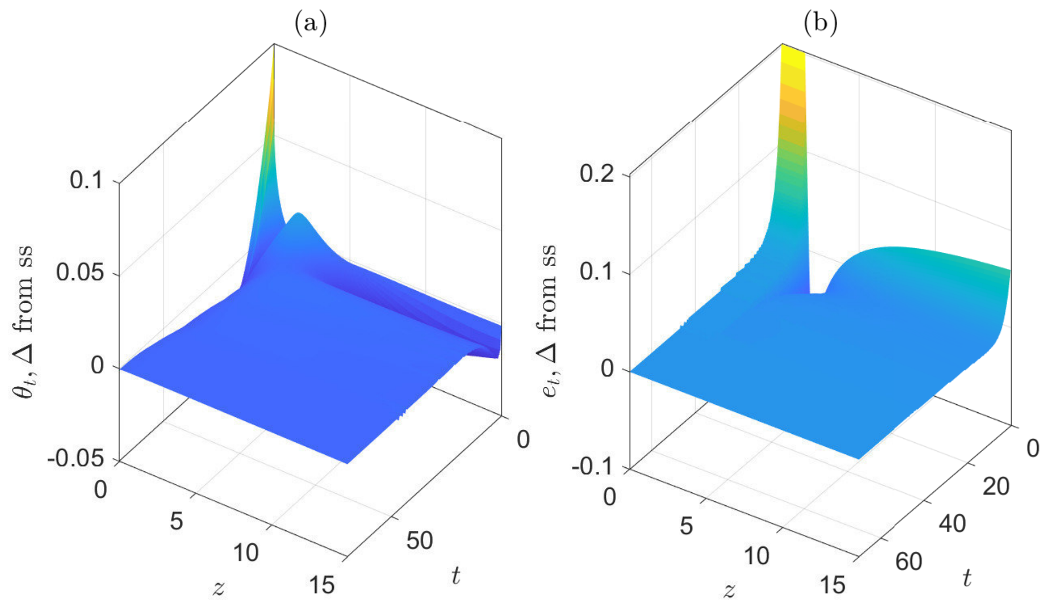

Figure 5: Impulse-response functions of the cross-sectional distribution of entrepreneurs and their tax evasion policies following 10 per cent MIT shocks to the level of ϕ.

percent ($\phi = 1.57$), and baseline plus 30 percent ($\phi = 1.86$).

As we show in Figure 7, the shock fosters the threshold level $\bar{z}_t$, thereby reducing the mass of active entrepreneurs (Panels (b) and (c)). As a result, increasing public expenditure curbs evasion incentives and reduces the size of the shadow economy (Panel (d)), even though the effect appears to be small. Notably, the magnitude of the effect is more pronounced the higher the financing frictions.

Increasing public expenditure has a substantial impact on macroeconomic aggregates, too. In particular, it leads to higher levels of aggregate net worth and public debt (see Figure 6, Panels (a) and (b)), because of boosted TFP in the aggregate. Productivity increments affect indeed directly the net worth and debt by expanding the government's primary deficit by an amount that is larger than its gains in tax revenues (see Figure 6, Panel (e)).

The dynamics of aggregate output display an unusual pattern, experiencing an initial sharp increase following the shock, then rapidly declining towards the steady-state level before rebounding and increasing again. This behaviour takes place because, following the shock, the debt grows at a faster rate than entrepreneurs' aggregate savings, thereby reducing the aggregate stock of capital that is available for production (remember that, according to

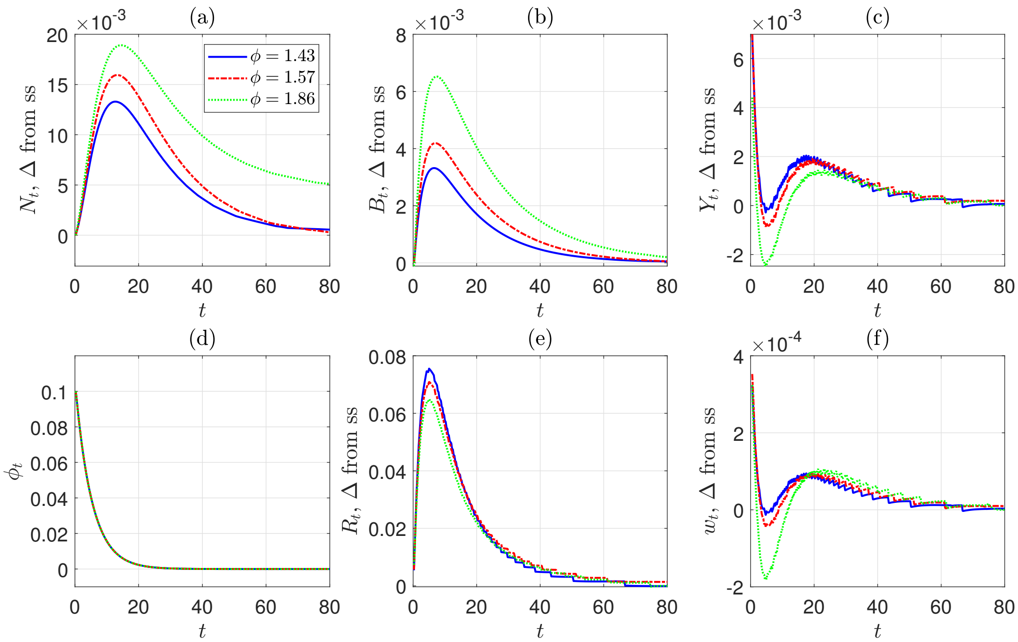

Figure 6: Impulse-responses of key macroeconomic aggregates following 10 percent shocks to the level of g for different levels of ϕ.

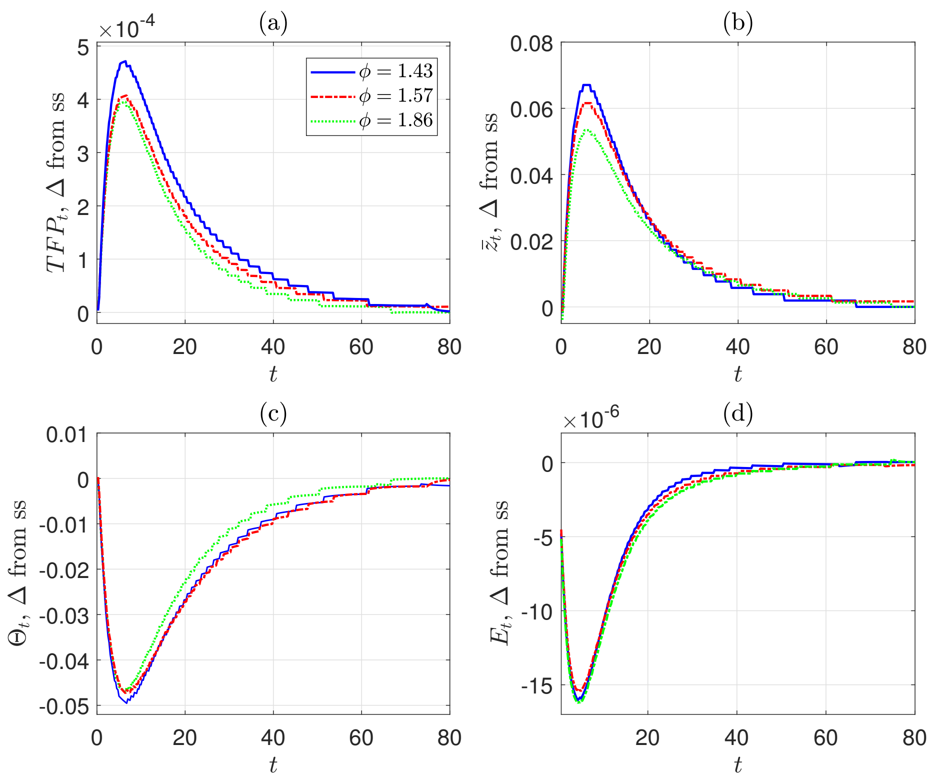

Figure 7: Impulse-responses of the endogenous total factor productivity, its components, and aggregate tax evasion following 10 percent shocks to the level of g for different levels of ϕ.

the market clearing condition in Eq. (15), $K_t = N_t - B_t$). Interestingly, when financial frictions are milder (portrayed by the red dashed lines vs solid blue ones), fluctuations in N_t and B_t are more significant. Still, the responses of aggregate productivity and tax evasion are milder. With tighter financial constraints, however, debt increases less. Consequently, the output's initial decrease is milder, and its recovery is relatively faster. Wages display an analogous behaviour (Figure 6, Panel (f)).

Another result is that while increments in public expenditure foster productivity and lower tax evasion in the aggregate, their effects are heterogeneous across entrepreneurs with different productivities. The magnitude of heterogeneity depends on how public goods affect the "shape" of the net worth distribution $\theta_t(z)$. This is the effect of the "net worth" channel, as also identified by González et al. (2022). We visualize this channel in Figure 8, which displays the responses of the distribution of entrepreneurs and their tax evasion policies conditional on z. The figure reveals that, while fostering productivity across all enterprises, public expenditure generates capital misallocation by reallocating capital from high- to low-productivity entrepreneurs (Panel (a)), thus generating the drop in the share of active entrepreneurs observed in Figure 7, Panel (c). The following proposition summarizes the result formally.

Proposition 3. ***(Public expenditure and productivity – I)*** *The impact of changes in public expenditure on the growth rate of aggregate productivity and, in turn, tax evasion depends on how these changes affect the average net worth dynamics across active entrepreneurs. Formally:*

$$\frac{\partial}{\partial g}\frac{d\ln TFP_t}{dt} = \epsilon\left\{\mathbb{E}^{\bar{\theta}}\left[\frac{\partial \mu_n(z)}{\partial g}\,|z_t \geq \bar{z}\right] - \mathbb{E}^{\tilde{\theta}}\left[\frac{\partial \mu_n(z)}{\partial g}\,|z_t \geq \bar{z}\right]\right\}. \tag{23}$$

in which $\bar{\theta}_t(z) := z\theta_t(z)\mathbb{I}_{z\geq\bar{z}_t}/\int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz$ *and* $\tilde{\theta}_t(z) := \theta_t(z)\mathbb{I}_{z\geq\bar{z}_t}/\int_{\bar{z}}^{z^{\max}} \theta_t(z)dz$.

Proof. See Appendix A.6. □

Concerning the cross-sectional response of tax evasion, e_t the second panel of Figure 4 shows that increments in g generate a reduction in tax evasion among low-productivity entrepreneurs and an increment among those whose productivity is at or above the threshold

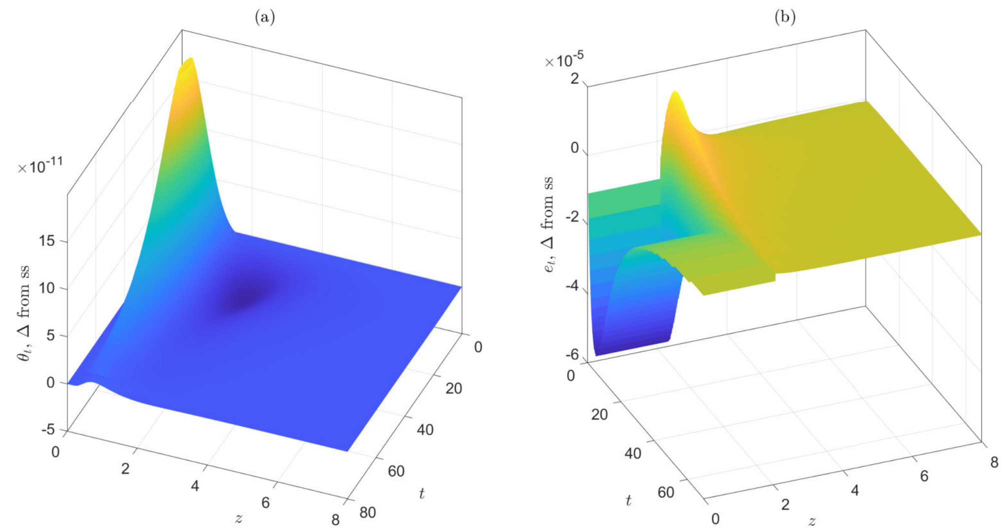

Figure 8: Impulse-response functions of the cross-sectional distribution of entrepreneurs and their tax evasion policies following a 10 percent MIT shock to the level of g.

level $\bar{z}_t$. Tax evasion among high-productivity agents, instead, remains almost unchanged.

The former outcome occurs because those entrepreneurs who were (and remain) inactive following the shocks earn a higher return from investing in bonds, which makes them mechanically less willing to expose themselves to auditing risk (see Eq. (12)).[23] The latter phenomenon arises because entrepreneurs who were active before the shock, thereby earning their firm's profits, become inactive in the aftermath of the shock.

To disentangle the forces behind this result, the following proposition clarifies the link between the change in TFP in response to changes in g and the productivity threshold $\bar{z}$.

Proposition 4. ***(Public expenditure and productivity – II)*** *The level of public expenditure g has an ambiguous effect on the economy's TFP; its sign and magnitude depend on the level of financial frictions ϕ. Formally:*

$$\frac{\partial TFP_t}{\partial g}\frac{g}{TFP_t} = \underbrace{\beta\left[1 - \theta_t(\bar{z})\left(\frac{\phi Z_t - \bar{z}}{Z_t}\right)\frac{R_t}{\pi_t}\right]}_{\gtreqless 0}. \tag{24}$$

Proof. See Appendix A.5. □

[23]Remember that, ceteris paribus, g fosters the returns of both capital and bonds.

According to Proposition 4, the total effect of increasing public expenditure on the economy's TFP is ambiguous and consists of two opposing forces. On the one hand, a higher public expenditure level fosters productivity across *all* enterprises, independently of their productivities; the effect is constant and proportional to the output elasticity of public capital β (see Eq. (3)). On the other hand, generalized productivity gains make it profitable for low$-z$ entrepreneurs to become active, lowering the threshold level $\bar{z}$. This result is akin to the "threshold" channel described in (González et al., 2022) in the context of optimal monetary policy. The channel exacerbates misallocation due to financial frictions and scales down the average productivity Z. Notably, the higher the leverage capacity ϕ (i.e., the lower the level of friction), the higher the sensitivity of the TFP response to misallocation. This mechanism rationalizes the observation of Figure 6, explaining the negative relation between financial frictions and the magnitude of the shadow economy's responses to public expenditure shocks.

6 Conclusions

We present a rich yet analytically tractable model exploring the dynamic interplay among tax evasion, financial frictions, and productivity in general equilibrium. Consistently with the empirical evidence, the model predicts a negative correlation between aggregate tax evasion and TFP and a positive association between shadow economy and financial frictions. Moreover, it shows that tax evasion entails heterogeneous and detrimental impacts across the distribution of entrepreneurs, exacerbating the capital misallocation resulting from financial frictions.

Investigating the role of the public sector, we highlight that public expenditure enhances the economy's TFP and reduces tax evasion. However, we also find that milder financial frictions constrain its effectiveness. This unexpected outcome arises because, while boosting productivity, public expenditure facilitates the market entry of low-productivity (and highly) evasive enterprises, an effect that is larger when frictions are weaker. This observation brings essential policy implications, suggesting that fiscal, tax evasion, and financial development policies have deeply interrelated effects.

To maintain analytical tractability and focus within the paper's scope, we have assumed a fixed aggregate labour supply and modelled households in a reduced form. Consequently, our model cannot tackle the exciting issue of analyzing the welfare effects of different fiscal policies when they interact with tax evasion and financial frictions. We leave this task for further research.

A Appendix: proofs and derivations

A.1 Firms' problem

This appendix discusses the solution to the firms' problem in Eq. (7). Firms' labour demand is unconstrained; therefore, its optimal level is given by the associated first-order condition, which yields

$$w = (1-\epsilon)\,\omega^{\beta}\,(zk)^{\epsilon}\,l^{-\epsilon}. \tag{25}$$

By substituting Eq. (25) in Eq. (7) and rearranging, one obtains the following relationship between firms' profits and their demand for capital:

$$\max_{k\leq\phi n} k\left[g^{\beta}z\left[\frac{(1-\epsilon)\,g^{\beta}}{w}\right]^{\frac{1-\epsilon}{\epsilon}} - R - z\omega^{\beta}\left[\frac{(1-\epsilon)\,g^{\beta}}{w}\right]^{\frac{1}{\epsilon}}\right]. \tag{26}$$

As this equation is linear in k, the optimal demand for capital is either $k=0$ or $k=\phi n$, depending on whether the term in square brackets is positive (i.e., on whether z is higher than some threshold $\bar{z}$). After rearranging, Eq. (8) follows suit.

A.2 Entrepreneurs' problem

Before solving the entrepreneurs' optimization problem, it is convenient to rewrite the dynamics in Eq. (9) by imposing the static balance sheet constraint $n_t = k_t + b_t$, considering the firm profit function in Eq. (9), and using the no-arbitrage condition that $R_t = r_t + \delta$, which yields

$$dn_t = [n_t(1-\tau_k+\tau_k e_t)(r_t+\varphi_t) - c_t]dt - e_t n_t \eta (r_t+\varphi_t)dJ_t. \tag{27}$$

Equipped with this equation, we can use standard stochastic control arguments (see for instance Pham, 2009, Chapter 2) to show that the value function V satisfies the following Hamilton-Jacobi-Bellman Equation (HJBE):

$$\lambda V = \max_{c,e} \left\{ \begin{array}{c} (1-\gamma)\rho V \left(\log c - \frac{1}{1-\gamma}\log((1-\gamma)V)\right) + \frac{\partial V}{\partial n}\left(n(1-\tau_k+\tau_k e)(r+\varphi) - c\right) + \\ +\frac{\partial V}{\partial z}\mu_z + \frac{1}{2}\frac{\partial^2 V}{\partial z^2}\sigma_z^2 + \lambda V\left(n\left(1-e\eta\left(r+\varphi\right)\right)\right) - \omega V \end{array} \right\}. \tag{28}$$

By taking first-order conditions, we get

$$c : \frac{(1-\gamma)\rho V}{c} = \frac{\partial V}{\partial n}, \tag{29}$$

$$e : \frac{\partial V}{\partial n} n\tau_k = \eta\lambda \frac{\partial V\left(n\left(1-e\eta\left(r+\varphi\right)\right)\right)}{\partial\left(n\left(1-e\eta\left(r+\varphi\right)\right)\right)}. \tag{30}$$

To characterize the problem's solution, we look for a candidate in the form $V(n,z) := v(z)n^{1-\gamma}/(1-\gamma)$, where v is an unknown function of z. By substituting this guess in the FOCs and rearranging, we obtain

$$c = n\rho, \text{ and } e = \frac{1-\left(\frac{\eta\lambda}{\tau_k}\right)^{\frac{1}{\gamma}}}{\eta\left(r+\varphi\right)}, \tag{31}$$

which also appear in the main text.

By substituting the optimal policies in Eq. (31) in the HJBE and rearranging, we obtain the following ODE:

$$v\left(\frac{\omega}{1-\delta} - \lambda\frac{\left(\frac{\eta\lambda}{\tau_k}\right)^{\frac{1-\gamma}{\gamma}}}{1-\delta} + \rho\left(1-\log\rho\right) - \frac{\tau_k}{\eta}\left(1-\left(\frac{\eta\lambda}{\tau_k}\right)^{\frac{1}{\gamma}}\right)\right) =$$

$$= v\left(\left(1-\tau_k\right)\left(r+\varphi\right) + \frac{\log v}{\gamma-1}\right) + \frac{\partial v}{\partial z}\mu_z + \frac{1}{2}\frac{\partial^2 v}{\partial z^2}\sigma_z^2,$$

whose solution yields the value of $v(z)$. This equation can be solved numerically by using an up-wind finite difference approximation scheme and imposing a "reflecting barrier" boundary conditions $\partial v(z^{\max})/\partial z = 0$. Further details appear in Dixit (2013) and Moll (2014).

A.3 Proof of Proposition 1

To derive the first point of the proposition, we integrate individual firms' labour and capital demand in Eq. (8) over $\int_0^\infty \int_{\mathbb{Z}} f_t(n,z)dndz$ and use Eq. (16) to obtain

$$L_t = \frac{\pi_t}{w_t}\frac{1-\epsilon}{\epsilon} N_t \phi Z_t, \tag{32}$$

$$K_t = \phi N_t \Theta_t, \tag{33}$$

where $\Theta_t \coloneqq 1 - \int_0^{\bar{z}} \theta_t(z)dz$. Then, we integrate firm-level output over the same support, which yields

$$Y_t = \int_0^\infty \int_{\mathbb{Z}} y_t f_t(n,z)dndz. \tag{34}$$

By using Eq. (3), Eq. (34) can be written as

$$Y_t = \phi g^\beta \left(\frac{\pi_t}{w_t}\frac{1-\epsilon}{\epsilon}\right)^{1-\epsilon} N_t Z_t,$$

where $Z_t \coloneqq \int_{\bar{z}}^\infty z_t \theta_t(z)dz$. By using Eqs. (32) and (33) and rearranging, Eq. (34) simplifies as

$$Y_t = g^\beta \left(\frac{\int_{\bar{z}}^{z^{\max}} z_t\theta_t(z)dz}{1-\int_0^{\bar{z}}\theta_t(z)dz}\right)^\epsilon K_t^\epsilon. \tag{35}$$

Setting $L_t = 1$, we obtain what appears in the main text.

To derive the second point of the proposition, we match the definition of π_t in Eq. (8) with Eqs. (32) and (33) to obtain

$$\pi_t = \epsilon g^\beta L_t^{1-\epsilon} K_t^{\epsilon-1} X_t^{\epsilon-1}. \tag{36}$$

Substituting Eq. (36) in $\bar{z} = r/\pi$, imposing a no-arbitrage condition, and rearranging yields the right-hand side of Eq. (19) as it appears in the text. The left-hand side is derived similarly starting from Eq. (32).

The transition dynamics of public debt in Point 3 of the proposition is obtained by substituting Eqs. (14) and (12) in Eq. (13) and using that $K_t + B_t = N_t$, which gives

$$\frac{dB_t}{dt} = r_t B_t + g\left(N_t - B_t\right) - \tau_l w L - \tau_k \int_{\mathbb{Z}} (r_t + \varphi_t) \underbrace{\int_0^\infty n f_t(n,z) dn}_{=N_t\theta_t(z)} dz - \\ + \left(\lambda - \frac{\tau_k}{\eta}\right)\left[1 - \left(\frac{\eta\lambda}{\tau_k}\right)^{\frac{1}{\gamma}}\right] \underbrace{\int_{\mathbb{Z}}\int_0^\infty n f_t(n,z) dn dz}_{=N_t} - N_t \omega \tau_\omega. \quad (37)$$

By matching Eq. (9) to Eq. (36) and rearranging, the forth term to the right-hand side of Eq. (37) can be rewritten as

$$-\tau_k \int_{\mathbb{Z}} \varphi_t \theta_t(z) dz - \tau_k r_t \underbrace{\int_{\mathbb{Z}} \theta_t(z) dz}_{=1},$$

and thus

$$\tau_k N_t \left(r_t + \int_0^\infty \varphi_t \theta_t(z) dz\right) = \tau_k N_t \left[r_t + \phi \Theta_t \left(\epsilon g^\beta K_t^{\epsilon-1} X_t^\epsilon - R_t\right)\right]. \quad (38)$$

By substituting this result back in Eq. (37), using $w = (1-\epsilon)Y$ and Eq. (35), and considering that $\frac{dB_t}{B_t}\frac{1}{dt} = \frac{d\ln B_t}{dt}$, we obtain the following dynamics appearing in the main text:

$$\frac{d\ln B_t}{dt} = r_t + g\left(\frac{N_t}{B_t} - 1\right) - \frac{\tau_l (1-\epsilon) Y_t}{B_t} + \\ - \frac{N_t}{B_t}\left[\tau_k \left(r_t + \phi\Theta_t \left(\epsilon Y_t K_t^{-1} - R_t\right)\right) - \left(\frac{\tau_k}{\eta} - \lambda\right)\left[1 - \left(\frac{\eta\lambda}{\tau_k}\right)^{\frac{1}{\gamma}}\right] + \omega\tau_\omega\right].$$

To obtain the dynamics of the aggregate entrepreneurs' net worth, we substitute the optimal policies in Eq. (10) into the dynamics balance sheet in Eq. (12), which gives

$$dN_t = \underbrace{N_t \left[(1-\tau_k)\left(r_t + \int_{\mathbb{Z}} \varphi_t \theta_t(z) dz\right) - \rho\right]}_{\text{Average return on capital plus consumption}} dt + \\ + \underbrace{N_t \frac{\tau_k}{\eta}\left[1 - \left(\frac{\eta\lambda}{\tau_k}\right)^{\frac{1}{\gamma}}\right]}_{\text{Average evasion}} dt - \underbrace{N_t \lambda \left[1 - \left(\frac{\eta\lambda}{\tau_k}\right)^{\frac{1}{\gamma}}\right]}_{\text{Average auditing fine}} dt - \underbrace{N_t \omega \tau_\omega dt}_{\text{Retirement}}, \quad (39)$$

where we have used that $\int_0^\infty dJ_{t,i}di = \lambda dt$ (i.e., auditing events are pair-wise independent Poisson processes) and $N_t\omega\tau_\omega$ denotes the mass of retired entrepreneurs whose net worth is collected by the government as a tax. Dividing both terms by N_t and using Eq. (38) to simplify the first term on the right-hand side, Eq. (21) in the main text follows suit.

A.4 Proof of Proposition 2

Following Moll (2014), we can derive the partial differential equation for the dynamics of $\theta_t(z)$ by using that the density function $f(z,n)$ satisfies the following Fokker-Plank equation (omitting functional dependence on time):

$$\begin{aligned}\frac{\partial f(z,n)}{\partial t} = &-\frac{\partial}{\partial n}\left[nf(z,n)\mu_n(z)\right] - \frac{\partial}{\partial z}\left[f(z,n)\mu_z(z)\right] + \frac{1}{2}\frac{\partial^2}{\partial z^2}\left[f(z,n)\sigma_z^2(z)\right] + \\ &- \lambda f(z,n) + \lambda f(z,n(1-(\eta\lambda/\tau_k)^{1/\gamma})) - \omega f(z,n) + \omega f(z,n(1-\tau_\omega)), \end{aligned} \tag{40}$$

where $n\mu_n$ denotes the drift of the stochastic differential in Eq. (10) after substituting the optimal controls in Eq. (12) (for a formal derivation, see Stokey, 2008). By differentiating the auxiliary function in Eq. (16), we obtain that (omitting functional dependence on z)

$$\frac{\partial \theta}{\partial t} = \frac{N\int_0^\infty n\frac{\partial f}{\partial t}dn - \dot{N}\int_0^\infty nfdn}{N^2}, \tag{41}$$

which can be substituted in the Fokker-Plank equation to obtain

$$\begin{aligned}\frac{\partial \theta}{\partial t} = &-\frac{\int_0^\infty n\left[\frac{\partial}{\partial n}\left(nf\mu_n\right) + \frac{\partial}{\partial z}\left(f\mu_z\right)\right]dn}{N} + \frac{1}{2}\frac{\int_0^\infty n\frac{\partial^2}{\partial z^2}\left(f\sigma_z^2\right)dn}{N} - \frac{\int_0^\infty n\left(\lambda+\omega\right)fdn}{N} + \\ &-\frac{d\ln N}{dt}\frac{\int_0^\infty nfdn}{N} + \frac{\int_0^\infty n\lambda f\left(n(1-(\eta\lambda/\tau_k)^{1/\gamma})\right)dn + \int_0^\infty n\omega f\left(n(1-\tau_\omega)\right)dn}{N}. \end{aligned} \tag{42}$$

Integrating by parts, the first three terms into the right-hand side simplify as

$$\int_0^\infty n\frac{\partial\left(nf\mu_n\right)}{\partial n}dn = -\mu_n\int_0^\infty nfdn, \tag{43}$$

$$\frac{\partial}{\partial n}\left(\int_0^\infty nfdn\mu_z\right) = N\partial_z\left(\theta\mu_z\right), \tag{44}$$

$$\int_0^\infty n \frac{\partial^2}{\partial z^2}\left(f\sigma_z^2\right) dn = N \frac{\partial^2}{\partial z^2}\left(\theta\sigma_z^2\right), \tag{45}$$

By using that entrepreneurs' evasion strategies are linear in n, the fourth and fifth terms become

$$\frac{\lambda \int_0^\infty nf\left(n\left(1-\left(\frac{\eta\lambda}{\tau_k}\right)^{\frac{1}{\gamma}}\right)\right) dn}{N} + \frac{\omega \int_0^\infty nf\left(n(1-\tau_\omega)\right) dn}{N} =$$
$$= \theta\left[\left(1-\left(\frac{\eta\lambda}{\tau_k}\right)^{\frac{1}{\gamma}}\right)\lambda + \omega(1-\tau_\omega)\right]. \tag{46}$$

Finally, we substitute Eqs. (43)-(46) in Eq. (42), use the identity in Eq. (16), and rearrange to rewrite Eq. (42) as it appears in the main text.

A.5 Proof of Proposition 4

By matching Eqs. (17) and (18), the economy's TFP equal

$$TFP_t = g^\beta \left(\frac{\int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz}{\int_{\bar{z}}^{z^{\max}} \theta_t(z)dz}\right)^\epsilon. \tag{47}$$

By using that $\bar{z} = R\omega^{-\frac{\beta}{\epsilon}}\epsilon^{-1}\left((1-\epsilon)/w\right)^{\frac{\epsilon-1}{\epsilon}}$, the partial derivative of Eq. (47) wrt g equals

$$\frac{\partial TFP}{\partial g} = \frac{\partial}{\partial g}\left[g^\beta\left(\frac{\int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz}{\int_{\bar{z}}^{z^{\max}} \theta_t(z)dz}\right)^\epsilon\right] = \frac{\partial g^\beta}{\partial g} \times \left(\frac{\int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz}{\int_{\bar{z}}^{z^{\max}} \theta_t(z)dz}\right)^\epsilon +$$
$$+ \epsilon g^\beta \left(\frac{\int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz}{\int_{\bar{z}}^{z^{\max}} \theta_t(z)dz}\right)^{\epsilon-1} \frac{\partial}{\partial \bar{z}}\left(\frac{\int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz}{\int_{\bar{z}}^{z^{\max}} \theta_t(z)dz}\right) \frac{\partial}{\partial g}\left(\frac{R_t}{g^{\frac{\beta}{\epsilon}}\epsilon\left(\frac{1-\epsilon}{w_t}\right)^{\frac{1-\epsilon}{\epsilon}}}\right). \tag{48}$$

By using that

$$\frac{\partial\left(\frac{\int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz}{1-\int_0^{\bar{z}}\theta_t(z)dz}\right)}{\partial \bar{z}} = \frac{\theta_t(\bar{z})\int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz - \bar{z}\theta_t(\bar{z})\left[1-\int_0^{\bar{z}}\theta_t(z)dz\right]}{\left(1-\int_0^{\bar{z}}\theta_t(z)dz\right)^2} = \phi\theta_t(\bar{z})\left(\phi Z_t - \bar{z}\right),$$

Eq. (48) simplifies as

$$\frac{\partial TFP}{\partial g} = TFP\left[\frac{\beta}{g} - \epsilon\theta_t(\bar{z})\frac{(\phi Z_t - \bar{z})}{Z_t}\frac{\beta}{\epsilon g}\left(\frac{R_t}{g^{\frac{\beta}{\epsilon}}\left(\frac{1-\epsilon}{w_t}\right)^{\frac{1-\epsilon}{\epsilon}}}\right)\right].$$

A.6 Proof of Proposition 3

Following González et al. (2022), the growth rate of aggregate TFP can be expressed as

$$\frac{dTFP_t}{dt}\frac{1}{TFP_t} = \frac{d\ln TFP_t}{dt} = \frac{d}{dt}\left[\beta\ln g + \epsilon\ln\left(\int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz\right) - \epsilon\ln\left(\int_{\bar{z}}^{z^{\max}}\theta_t(z)dz\right)\right],$$

which can be conveniently rearranged as

$$\frac{d\ln TFP_t}{dt} = \epsilon\left[\frac{\int_{\bar{z}}^{z^{\max}} z\dot{\theta}_t(z)dz}{\int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz} - \frac{\int_{\bar{z}}^{z^{\max}}\dot{\theta}_t(z)dz}{\int_{\bar{z}}^{z^{\max}}\theta_t(z)dz}\right]. \tag{49}$$

By taking the partial derivative of Eq. (49) wrt g while keeping $\dot{\omega}$ constant, we obtain

$$\frac{\partial}{\partial g}\frac{d\ln TFP_t}{dt} = \epsilon\left[\frac{\int_{\bar{z}}^{z^{\max}} z\frac{\partial\dot{\theta}_t(z)}{\partial g}dz}{\int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz} - \frac{\int_{\bar{z}}^{z^{\max}}\frac{\partial\dot{\theta}_t(z)}{\partial g}dz}{\int_{\bar{z}}^{z^{\max}}\theta_t(z)dz}\right],$$

which by using Eq. (22) and rearranging yields

$$\frac{\partial}{\partial g}\frac{d\ln TFP_t}{dt} = \epsilon\left[\frac{\int_{\bar{z}}^{z^{\max}} z\frac{\partial\mu_n}{\partial g}\theta_t(z)dz}{\int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz} - \frac{\int_{\bar{z}}^{z^{\max}}\frac{\partial\mu_n}{\partial g}\theta_t(z)dz}{\int_{\bar{z}}^{z^{\max}}\theta_t(z)dz}\right] +$$
$$-\frac{\partial\left(\frac{d\ln N_t}{dt} + \tau_\omega\omega - \left(\frac{\tau_k}{\eta} - \lambda\right)\left[1 - \left(\frac{\eta\lambda}{\tau_k}\right)^{\frac{1}{\gamma}}\right]\right)}{\partial g}\underbrace{\left(\frac{\int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz}{\int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz} - \frac{\int_{\bar{z}}^{z^{\max}}\theta_t(z)dz}{\int_{\bar{z}}^{z^{\max}}\theta_t(z)dz}\right)}_{=0}.$$

By using that the terms $\bar{\theta}_t(z) := \frac{z\theta_t(z)\mathbb{I}_{z\geq\bar{z}}}{\int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz}$ and $\tilde{\theta}_t(z) := \frac{\theta_t(z)\mathbb{I}_{z\geq\bar{z}}}{\int_{\bar{z}}^{z^{\max}}\theta_t(z)dz}$ can be interpreted as probability density functions, the above can be expresses as follows:

$$\frac{\partial}{\partial g}\frac{d\ln TFP_t}{dt} = \epsilon\left\{\mathbb{E}^{\bar{\theta}}\left[\frac{\partial\mu_n(z)}{\partial g}\,|z_t \geq \bar{z}\right] - \mathbb{E}^{\tilde{\theta}}\left[\frac{\partial\mu_n(z)}{\partial g}\,|z_t \geq \bar{z}\right]\right\}. \tag{50}$$

Lemma 1. ***(Aggregate TFP growth – sign)*** *By first-order stochastic dominance, the sign of Eq. (50) is uniquely determined by that of* $\mathbb{E}^{\bar{\theta}}\left[\frac{\partial \mu_n(z)}{\partial g}\,|z_t \geq \bar{z}\right]$.

Proof. The likelihood ratio

$$\frac{\bar{\theta}_t(z)}{\tilde{\theta}_t(z)} = z\frac{\int_{\bar{z}}^{z^{\max}} \theta_t(z)dz}{\int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz},$$

is non-decreasing in z, meaning that for each couple $z_1 \geq z_0$ it holds that

$$\frac{\bar{\theta}_t(z_1)}{\tilde{\theta}_t(z_1)} \geq \frac{\bar{\theta}_t(z_0)}{\tilde{\theta}_t(z_0)}. \tag{51}$$

By rearranging Eq. (51) and integrating with respect to z_0 over $[\bar{z},\, z^{max}]$, one gets that

$$\int_{\bar{z}}^{z_1} \bar{\theta}_t(z_1)\tilde{\theta}_t(z_0)dz_0 \geq \int_{\bar{z}}^{z_1} \bar{\theta}_t(z_0)\tilde{\theta}_t(z_1)dz_0;$$

that is,

$$\frac{\bar{\theta}_t(z_1)}{\tilde{\theta}_t(z_1)} \geq \frac{\bar{\Theta}_t(z_1)}{\tilde{\Theta}_t(z_1)}. \tag{52}$$

By integrating the same equation with respect to z_1 over the interval $[z_0, z^{max}]$, we obtain instead that

$$\int_{z_0}^{z^{\max}} \bar{\theta}_t(z_1)\tilde{\theta}_t(z_0)dz_1 \geq \int_{z_0}^{z^{\max}} \bar{\theta}_t(z_0)\tilde{\theta}_t(z_1)dz_1;$$

that is,

$$\frac{\bar{\theta}_t(z_0)}{\tilde{\theta}_t(z_0)} \leq \frac{1-\bar{\Theta}_t(z_0)}{1-\tilde{\Theta}_t(z_0)}. \tag{53}$$

By matching Eqs. (52) and (53) for $z_0 = z_1 = z$, we obtain the following first-order stochastic dominance relationship:

$$\bar{\Theta}_t(z) \leq \tilde{\Theta}_t(z)$$

□

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B Online appendix

B.1 Workers' problem

In the same spirit of Moll (2014), this appendix provides a micro-foundation of the "hand-to-mouth" behaviour of the representative worker in the model. This agent is endowed with one unit of labour and choose its labour supply l_t and consumption c_t to solve the following problem:

$$\max_{\{c_t, l_t \in [0,1]\}} \int_0^\infty e^{-\rho t} \log c_t dt, \tag{54}$$

subject to $\dot{n}_t = n_t r_t + w_t l_t - c_t$, in which w_t denotes the economy's competitive wage. Since the worker does not gain utility from leisure and her wage is positive, then her labour supply is always equal to one. Her optimal consumption choice, instead, satisfies the following Hamiltonian:

$$H = logc + \vartheta \left(nr + w - c \right),$$

in which ϑ denotes the co-state variable. Taking the first-order conditions yields

$$\frac{\partial H}{\partial c} = 0 \rightarrow c = \vartheta^{-1} \text{ and } \dot{\vartheta} = \rho \vartheta - \frac{\partial H}{\partial n} \rightarrow \dot{\vartheta} = \left(\rho - r \right) \vartheta, \tag{55}$$

which can be rearranged to obtain the Euler equation $\dot{c} = (r - \rho)c$. Equipped with this result, we can re-writing the balance sheet constraint of the worker as

$$\int_0^\infty c_t e^{-\int_0^t r_s ds} dt = n_0 + \int_0^\infty w_t e^{-\int_0^t r_s ds} dt.$$

By rearranging this equation and using that the worker has no initial endowment ($n_0 = 0$), one obtains that, in the steady state $\bar{c} = \bar{w}\rho/\bar{r}$. The hand-to-mouth behaviour emerges endogenously when $\rho \geq \bar{r}$, a condition ensuring that she is always willing to consume her entire wage.

Panel (a) – aggregate item						
Variable	Description	Source	Mean	Std. dev.	min	max
Shadow	Shadow economy share (%)	World bank	18.414	7.38	7.10	41.50
ln(GDP)	log GDP	Fred	10.39	0.478	8.76	11.79
FI	Fin. dev. index	IMF	0.61	0.22	0.10	1.00
D/GDP	Public debt-to-GDP	OECD	74.16	44.24	6.65	159.38
ΔTFP	Prod. growth (%)	OECD	0.265	1.85	-10.56	9.67
IG	public investment share (%)	OECD	23.87	7.59	2.96	51.2
ln(G)	log public expenditure	OECD	3.76	0.17	3.15	4.17
Panel (b) – firm avg TFP, by percentile						
Variable	Description	Source	Mean	Std. dev.	min	max
ΔTFP_{p_1}	Prod. growth (%), 1st pct	CompNet	-4.27	2.80	-23.28	-1.24
$\Delta TFP_{p_{25}}$	Prod. growth (%), 25th pct.	CompNet	2.00	2.80	-87.67	49.91
$\Delta TFP_{p_{50}}$	Prod. growth (%), 50th pct.	CompNet	2.22	8.28	-83.58	56.33
$\Delta TFP_{p_{75}}$	Prod. growth (%), 75th pct.	CompNet	3.93	13.78	-44.06	139.37
$\Delta TFP_{p_{90}}$	Prod. growth (%), 90th pct.	CompNet	3.46	13.78	-45.03	376.51
$\Delta TFP_{p_{95}}$	Prod. growth (%), 95th pct.	CompNet	4.02	28.94	-45.04	375.51
ΔTFP_{avg}	Prod. growth (%), average	CompNet	2.64	11.56	-53.62	162.93

Table 5: Variable description and summary statistics.

B.2 Data

This appendix provides supplementary material to our empirical analysis. Table 5 reports the summary statistics of the variables used in the empirical analysis of Section 3. Figure 9 reports the scatter plot of the (de-trended) variations in a country's TFP against contemporaneous variations in the share of its shadow economy. The red lines depict the best linear fit. Coherently with the aggregate-level estimates in Section 3, country-level data display a negative covariation between the two variables in 21 countries out of 23, with the only exceptions of Australia and South Korea. Figure 10 depicts the relation between the (de-trended) share of a county's shadow economy and the IMF financial development index. Coherently with the aggregate-level analysis, the data document a negative correlation between these variables across most OECD countries, with the exceptions of Austria, Ireland, Germany, and Portugal.

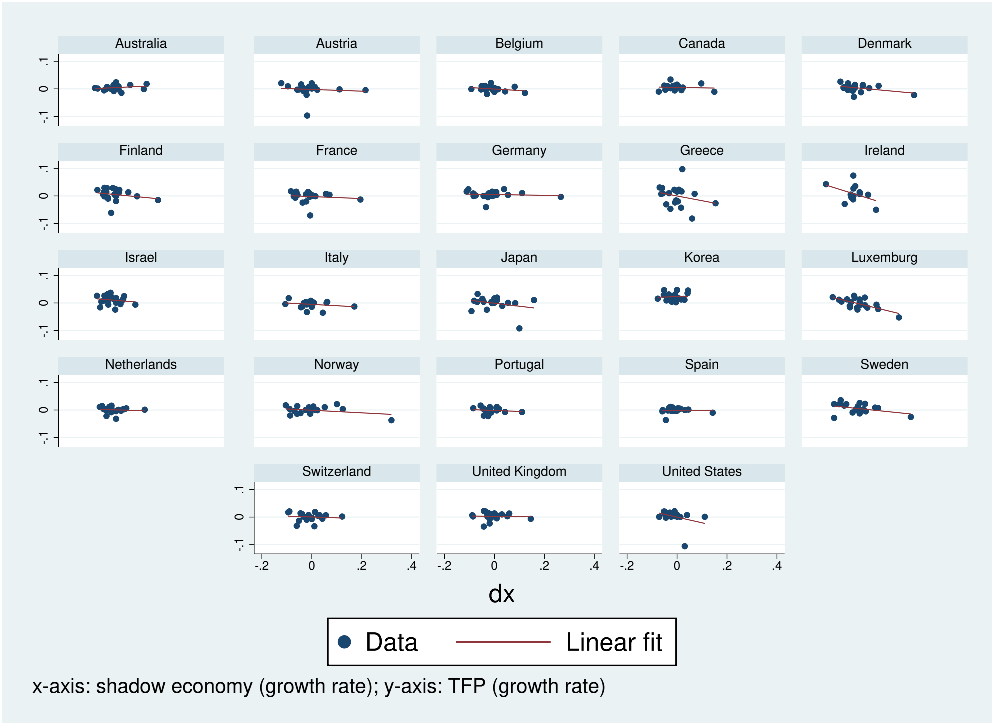

Figure 9: Correlation between variations in a country's TFP and the share of its shadow economy. Data sources: the World Bank dataset and Schneider et al. (2010) (shadow economy estimates); OECD statistics (TFP).

B.3 Solution algorithm and numerical approximation

B.3.1 Solution algorithm

Following Moll (2014), we numerically compute the model's steady-state equilibrium and its transition dynamics by recursively implementing the following steps.

1. For a given couple of aggregate net worth and public debt levels $\{N_t, B_t\}$ (two scalars) and a given net worth distribution across entrepreneurs $\theta_t(z)$ (a vector of size $1 \times I$), approximate the productivity threshold $\bar{z}$ and the average productivity Z_t by solving Eq. (17) over an equally-spaced grid $\mathbb{Z}^I = \begin{bmatrix} z_1 = 0 & \cdots & z_i & \cdots & z_I = z_{\max} \end{bmatrix}$.

2. Approximate the average productivity across active entrepreneurs by integrating numerically $Z_t \approx \sum_{i=1}^{I} \mathbb{I}_{z_i \geq \bar{z}} z_i \theta_t(i) \Delta z$ where $\Delta z = z_i - z_{i-1}$; compute R_t, r_t, and w_t by using Eq. (19).

3. Choose a discrete-time interval $dt \approx \Delta t$, obtain the density $\theta_{t+dt}(z)$ by approximating

Figure 10: Correlation between a country's share of its shadow economy and its IMF financial development index. Data sources: the World Bank and Schneider et al. (2010) (shadow economy estimates); IMF database (financial development index).

over $\mathbb{Z}^I$ the solution of Eq. (22), in which $d\ln N_t/dt \approx (\ln N_{t+\Delta t} - \ln N_t)/\Delta t$ is a suitable approximation of Eq. (21) over $\mathbb{Z}^I$ (details appear in Section B.3.2).

4. Compute $\{N_{t+\Delta t}, B_{t+\Delta t}\}$ by using a suitable approximation of Eqs. (20) and (21) over $\mathbb{Z}^I$ (details appear in Section B.3.2).

5. Check if $|N_{t+dt} - N_t| \wedge |B_{t+dt} - B_t| \wedge \|\theta_{t+dt}(z) - \theta_t(z)\|_\infty \leq \epsilon$. If yes, stop and define the steady-state equilibrium as $\{N_t, B_t, \theta_t(z)\} \coloneqq \{N_{ss}, B_{ss}, \theta_{ss}(z)\}$. Else, update $\{N_t, B_t, \theta_t(z)\} \to \{N_{t+\Delta t}, B_{t+\Delta t}, \theta_{t+\Delta t}(z)\}$ and repeat from Point 1.

B.3.2 Numerical approximation

To approximate the solution of the ODE in Eq. (22), $\theta_{t_j}(z_i) \approx \theta_{i,j}$, we follow Achdou et al. (2022) and adopt the following upwind finite difference scheme:

$$\frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta t} = l(z_i)\theta_{i,j+1} + \frac{\theta_{i+1,j+1} - \theta_{i,j+1}}{\Delta z} \min\{m(z_i), 0\} + \\ + \frac{\theta_{i,j+1} - \theta_{i-1,j+1}}{\Delta z} \max\{m(z_i), 0\} + n(z_i) \frac{\theta_{i+1,j+1} - 2\theta_{i,j+1} + \theta_{i-1,j+1}}{\Delta z^2}, \quad (56)$$

where

$$l(z) = \left[\mu_n(z) - \frac{N_{t+\Delta t} - N_t}{N_t} \frac{1}{\Delta t} - \left(\frac{\tau_k}{\eta} - \lambda\right)\left[1 - \left(\frac{\eta\lambda}{\tau_k}\right)^{\frac{1}{\gamma}}\right] - \tau_\omega \omega + \nu\left(\frac{3}{2} + \ln z\right)\right],$$

$$m(z) = \nu z (\ln z + 1.5), \quad \text{and} \quad n(z) = 0.5 z^2 \nu,$$

which allows us to rewrite Eq. (56) as the following linear system:

$$\begin{bmatrix} p(z_i) & q(z_i) & r(z_i) \end{bmatrix} \begin{bmatrix} \theta_{i-1,j+1} \\ \theta_{i,j+1} \\ \theta_{i+1,j+1} \end{bmatrix} = \theta_{i,j},$$

for $i = 2, ..., I$, where

$$p(z_i) = \Delta t \left[\frac{\max\{m(z_i), 0\}}{\Delta z} - \frac{n(z_i)}{\Delta z^2}\right],$$

$$q(z_i) = 1 - \Delta t \left(l(z_i) - \frac{\min\{m(z_i), 0\}}{\Delta z} + \frac{\max\{m(z_i), 0\}}{\Delta z} - \frac{2n(z_i)}{\Delta z^2}\right),$$

and

$$r(z_i) = -\Delta t \left[\frac{\min\{m(z_i), 0\}}{\Delta z} + \frac{n(z_i)}{\Delta z^2}\right].$$

The system can be solved iterating over $j = 1, 2, ..., J$ given an initial distribution $\theta_{i,0}$. We set the boundary conditions so that $\theta_{1,j} = 0$, and the mass preservation condition $\sum_{i=2}^{I} \theta_{i,j} dz = 1$

holds for all $j = 1, 2, ..., J$. Accordingly, the update function $\theta_{i,j+1}$ is given by

$$\underbrace{\begin{bmatrix} \theta_{2,j+1} \\ \theta_{3,j+1} \\ \vdots \\ \theta_{I-1,j+1} \\ \theta_{I,j+1} \end{bmatrix}}_{\theta^{j+1}_{(I-1)\times 1}} = \underbrace{\begin{bmatrix} q(z_2) & r(z_2) & 0 & \cdots & 0 \\ p(z_3) & q(z_3) & r(z_3) & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ dz & dz & dz & dz & dz \end{bmatrix}^{-1}}_{B^{-1}_{(I-1)\times(I-1)}} \underbrace{\begin{bmatrix} \theta_{2,j} \\ \theta_{3,j} \\ \vdots \\ \theta_{I-1,j} \\ 1 \end{bmatrix}}_{\theta^{j}_{(I-1)\times 1}}.$$

To implement the algorithm, we approximate the dynamics of aggregate net worth in Eq. (20) and the correspondent update by using the following Euler scheme:

$$\frac{N_{t+\Delta t} - N_t}{N_t}\frac{1}{\Delta t} \approx (1-\tau_k)\left[r_t + \phi\Theta_t\left(\epsilon\frac{Y_t}{N_t - B_t} - R_t\right)\right] + \left(\frac{\tau_k}{\eta} - \lambda\right)\left[1 - \left(\frac{\eta\lambda}{\tau_k}\right)^{\frac{1}{\gamma}}\right] - \rho,$$

where

$$Y_t \approx g^{\beta}\left(\frac{\sum_{i=1}^{I}\mathbb{I}_{z_i \geq \bar{z}} z_i \theta_t(i)\Delta z}{\sum_{i=1}^{I}\mathbb{I}_{z_i \geq \bar{z}}\theta_t(i)\Delta z}\right)^{\epsilon}(N_t - B_t)^{\epsilon},$$

$$\Theta_t \approx \sum_{i=1}^{I}\mathbb{I}_{z_i \geq \bar{z}}\theta_t(i)\Delta z,$$

and

$$R_t \approx \epsilon\bar{z}g^{\beta}\left(\phi\sum_{i=1}^{I} z_i\mathbb{I}_{z_i \geq \bar{z}}\theta_t(i)\Delta z\right)^{\epsilon-1}(N_t - B_t)^{\epsilon-1}.$$

Similarly, we approximate the update of the public debt level by using the following Euler approximation of Eq. (21):

$$\frac{B_{t+\Delta t} - B_t}{B_t}\frac{1}{\Delta t} \approx r_t + g\left(\frac{N_t}{B_t} - 1\right) - \frac{\tau_l(1-\epsilon)Y_t}{B_t} + \\ -\frac{N_t}{B_t}\left[\tau_k\left(r_t + \phi\Theta_t\left(\epsilon\frac{Y_t}{N_t - B_t} - R_t\right)\right) - \left(\frac{\tau_k}{\eta} - \lambda\right)\left[1 - \left(\frac{\eta\lambda}{\tau_k}\right)^{\frac{1}{\gamma}}\right] + \omega\tau_\omega\right]$$