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Non-Stationary Search and Assortative Matching

Nicolas Bonneton [1]
Christopher Sandmann [2]

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[1] University of Mannheim, Email: nicolas.bonneton@gmail.com
[2] London School of Economics, Email: c.sandmann@lse.ac.uk

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Nicolas Bonneton[†] and Christopher Sandmann[‡]

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Abstract

This paper studies assortative matching in a *non-stationary* search-and-matching model with non-transferable payoffs. Non-stationarity entails that the number and characteristics of agents searching evolve endogenously over time. Assortative matching can fail in non-stationary environments under conditions for which Morgan (1994) and Smith (2006) show that it occurs in the steady state. This is due to the risk of worsening match prospects inherent to non-stationary environments. The main contribution of this paper is to derive the weakest sufficient conditions on payoffs for which matching is assortative. In addition to known steady state conditions, more desirable individuals must be less risk-averse in the sense of Arrow-Pratt.

Keywords: non-stationary, assortative matching, random search, risk preferences, NTU

JEL classification: C73, C78, D81, E32

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[†]Universität Mannheim, nicolas.bonneton@gmail.com

[‡]London School of Economics, c.sandmann@lse.ac.uk

1 Introduction

Homer (Odyssey XVII, 218) claims that "Gods join like things with like things". This is arguably one of the oldest mentions of *positive assortative matching* (PAM) wherein individuals that share similar characteristics tend to be matched with one another. Interest in PAM is widespread, partly because it is so frequently observed.[1] To understand the determinants of PAM, it is essential to explore why economic agents choose to match assortatively. On a theoretical level, this amounts to identifying properties of match payoffs that lead to assortative matching. This research agenda was initiated by Becker (1973) in the context of a frictionless decentralized markets. We follow his line of inquiry in a model with *time-varying* search frictions that render finding a potential partner haphazard and time-consuming.

The theory of assortative matching amid search frictions is extensive (Smith (2006); Morgan (1994); Shimer and Smith (2000); Atakan (2006)).[2] However, and in line with the literature on heterogeneous agent models, formal results are confined to the steady state where match prospects do not evolve, and individual expectations over the future remain unchanged as time goes on. The assumption of stationarity makes complex models more tractable. But it also eclipses time-changing intertemporal trade-offs inherent in matching markets. For instance, rejecting a match today has different implications in an emptying search pool (e.g., the junior job market) with declining encounters over time compared to a cycling market (e.g., a seasonal housing market) with eventual match prospect rebounds. Recent applied articles highlight the relevance of understanding how these time-changing dynamics affect PAM.[3] Our approach maintains the more fundamental focus on individual choices as emphasized in the prior literature. In particular, we ask whether non-stationary match prospects can upset the ordering of individual match decisions, causing agents to match non-assortatively? To date, the theory offers no answer.

This paper is the first to derive sufficient conditions for PAM in a non-stationary search-and-matching model. Non-stationarity entails that the number and characteristics of unmatched agents evolve endogenously over time; individual match decisions adjust accordingly. The subsequent analysis reveals that the premise of stationarity not only simplifies but also changes equilibrium sorting. PAM fails in environments where it occurs

[1]Examples include the labor market, where skilled workers often join export-oriented firms employing sophisticated technology (Davidson et al. (2014), Felbermayr et al. (2014)), and family formation, where married individuals tend to share similar educational backgrounds, parental wealth, and fertility preferences (Mare (1991), Charles et al. (2012), Rasul (2008)).

[2]Chade et al. (2017) is a self-contained introduction to research on search and assortative matching.

[3]A recent literature studies the dynamics of labor market sorting within the quantitatively more tractable job ladder model. By assuming that firms hold all the bargaining power (Lise and Robin (2017)) or focusing on directed search (Baley et al. (2022)), these articles also simplify the individual decision problem—accept or wait for a better match—that lies at the heart of our analysis.

in the steady state. More generally, our results challenge the prevailing idea that the assumption of the steady state is always necessary for achieving tractability.

In line with the literature, we consider a continuous-time, infinite-horizon matching model in which pairs of vertically differentiated agents randomly meet one another. Upon meeting, agents observe each other's type. If both agents agree, they permanently exit the search pool and enjoy their respective match payoffs. Otherwise they continue waiting for a more suitable partner. An individual agent finds matching desirable whenever the payoff from matching exceeds the continuation value-of-search. PAM occurs if higher types match with sets of superior types.

We follow the non-transferable utility (NTU) paradigm, where match payoffs depend solely on the types of both partners. This paradigm applies in environments characterized by the absence of bilateral bargaining (e.g., rent-controlled housing, collective bargaining agreements in the labor market, see Felbermayr et al. (2014), or national wage setting, see Hazell et al. (2022)) or those where bilateral bargaining does not precede match formation (e.g., the classical hold-up problem in household bargaining or team production, see Mazzocco (2007), Rasul (2008), Doepke and Kindermann (2019)). As a special case, NTU search-and-matching also subsumes the one-sided search framework (e.g., consumers searching for products or entrepreneurs searching for a business case). In the NTU paradigm, PAM entails that higher types are relatively more selective about who they agree to match with.

To better grasp the dynamics of our model, we first characterize sorting in the simplest non-stationary matching market: a closed market with no new entrants, where agents have identical preferences over matched partners. However a special case of our general framework, it serves as a counterpart to the well-known steady-state analysis (see McNamara and Collins (1990), Burdett and Coles (1997), Smith (2006) and references therein) where block segregation emerges. Unlike in the steady state, as the meeting rate is decreasing over time there is only a single matching block at the top that is expanding as time goes on. Interestingly, less attractive types initially become choosier as they wait to be accepted by the most desirable agents.

i. Characterizing Assortative Matching in Non-Stationary Environments

To date, the literature has derived equilibrium sorting conditions by drawing on an explicit characterization of the value-of-search in the steady state. Non-stationary analysis forecloses this avenue, as the time-varying value-of-search is a complicated object to handle.[4] We circumvent the ensuing tractability issues by using a revealed preference

[4]The value-of-search is characterized by an integral over an infinite time horizon taking as its argument the population dynamics, which are themselves a solution to an infinite-dimensional system of integral equations.

argument: superior types, being more desirable, can exploit their superior match opportunities and replicate the expected match outcomes of any inferior type. These deviations must be weakly dominated by the actual value-of-search—establishing lower bounds on superior types' value-of-search. The lower bounds serve as the keystone of all of our equilibrium sorting results. In particular, we use these to derive novel and short proofs of two existing sorting results (Theorems 1 and 1′) that hold in the steady state: *if payoffs are log supermodular, then there is PAM when search is costly due to time discounting* as established by Smith (2006); *if payoffs are supermodular, then there is PAM when search entails an explicit time-invariant flow cost* as established by Morgan (1994).

In a non-stationary environment, steady state sufficient conditions are insufficient to guarantee PAM. Here, unlike in the steady state, the lowest type accepted today need not be the worst possible match outcome for all future times. As the search pool evolves over time, agents may face a less favorable selection of types to match with in the future. And an agent who initially rejects a given type, may accept an inferior type at a later stage. In effect, the agent's decision problem involves weighing a sure match payoff today against both the upside risk of matching with a superior type and the downside risk of ending up with an inferior type in the future. Supermodularity and log supermodularity do not resolve this trade-off. Log supermodularity implies that higher types gain relatively more from being matched with higher types. But it also implies that higher types lose out more from being matched with a lower type. We provide an example of a gradually clearing search pool in which the latter effect dominates: lower, not higher types are choosier. PAM does not occur despite log supermodular payoffs.

The main contribution of this article (Theorems 2 and 2′) is to derive an intuitive condition that guarantees PAM in non-stationary environments. We establish that if the respective steady state sufficient condition holds and payoffs satisfy *log supermodularity in differences*, then there is positive assortative matching across all equilibria. By log supermodularity in differences we mean that, for all $y_1 < y_2 < y_3$ and $x_1 < x_2$, we have

$$\frac{\pi(y_3|x_2) - \pi(y_2|x_2)}{\pi(y_2|x_2) - \pi(y_1|x_2)} \geq \frac{\pi(y_3|x_1) - \pi(y_2|x_1)}{\pi(y_2|x_1) - \pi(y_1|x_1)},$$

where $\pi(y|x)$ represents agent type x's payoff if matched with an agent of type y.[5] Log supermodularity in differences emerges as the missing condition because it ensures that the upside of matching with a higher type vis-à-vis the downside of matching with a lower type is always greater for higher types. Observe that this result holds irrespective of how

[5]Assuming differentiability, this condition is equivalent to log supermodularity of $d_y\pi(x|y)$. Log supermodularity in differences has been found to drive sorting results in areas such as matching with Nash bargaining, moral hazard, and menu pricing (e.g., Shimer and Smith (2000), Chade and Swinkels (2019) and Sandmann (2023)).

search cost is modeled. To ensure that PAM occurs in non-stationary environments, we require log supermodularity in differences under both discounting and explicit search cost.

We further prove that our conditions are the minimal ones under which matching away from the steady state is assortative: if one of the two is upset locally, then there exist environments for which PAM does not occur (Propositions 3 and 3′).

ii. Interpretation of Theorems 2 and 2′

Our analysis reveals a close link between the time-variant nature of search frictions and risk preferences. Interpreting type x's payoff over partners y as a utility function, log supermodularity in differences defines a ranking over risk preferences in the sense of Arrow (1965)-Pratt (1964). Accordingly, if the respective steady state sufficient condition holds, our main contribution states that *the weakest sufficient conditions for positive assortative matching is that more desirable individuals are less risk-averse.*

There is mounting empirical evidence that characteristics commonly attributed to desirability such as cognitive skills, education, health or income strongly correlate with risk preferences.[6] For instance, Dohmen et al. (2010) conduct a real-stakes experiment with a large, representative sample of the German population to test whether risk-aversion and patience are related with cognitive ability. The authors find that individuals with higher cognitive ability are both more willing to take financial risks and more patient. Moreover, Dohmen et al. (2011) find significant correlations between financial and non-financial measures of risk-aversion. This suggests that those individuals to which society attributes the greatest desirability are also the greatest risk-takers in matching markets. Provided that individual types represent cognitive skills, education, health or income, empirically elicited preferences correspond to utility functions that are log supermodular in differences. As such, our work provides a theoretical foundation as to why positive assortative matching may arise in decentralized matching markets that preclude bargaining prior to match formation.

Interpreting payoffs as a utility function is but one interpretation of our model. In many applied models, the curvature of match payoffs will be born out by the regulatory environment, production technology, or bilateral bargaining protocol that succeeds match formation, not differences in preferences. Section 5 illustrates this point by studying match formation in a marriage market where prospective partners differ in their fertility preferences. Non-stationarity in this market is due to declining match prospects as unmarried individuals age. We adopt Rasul (2008)'s framework in which efficient bilateral bargaining over fertility decisions is impeded by a hold-up problem; prospective fathers

[6]See Dohmen et al. (2010) and Dohmen et al. (2011), as well as Guiso and Paiella (2004), Frederick (2005), Benjamin et al. (2013) and Noussair et al. (2013) for evidence.

cannot pre-commit to transfers in exchange for achieving their desired fertility level. Our analysis takes the equilibrium payoffs from Rasul's model as a primitive of our search-and-matching model, establishing a connection between the two. This allows us to examine whether Rasul's payoffs satisfy our sufficient conditions for PAM. The predictions of our theory coincide with empirical observation on the strength of assortative matching.

iii. Scope of Theorems (2 and 2′)

Our main Theorems 2 and 2′ were derived in a framework where individual match prospects varied over time due to deterministic changes in the composition of the search pool. As discussed, these changes can introduce worsening match prospects over time that make individuals less selective in who they accept. Informed by hindsight, this leads to a feeling of regret: *some agents prefer a partner they had previously rejected over their current match.* Wherever there is regret, our condition, log supermodularity in differences, plays a key role in sustaining PAM.

To illustrate this point, we study two prominent model variations where regret occurs: In subsection 6.2 we consider environments where future match prospects are risky, e.g., due to random entry into the search pool. In subsection 6.3 we consider environments where individual attractiveness fluctuates over time, e.g., due to unemployment scarring or declining fertility with age. In the latter case, log supermodularity in difference is required to achieve PAM even if the search pool remains stationary.

In Section 6.4, we consider a model variation in which agents can strategically re-enter the search pool by breaking their match. Whether PAM occurs in this context depends on whether agents can commit to not break the match at a moment in time that is inopportune to their partner.

iv. Related Literature

Previous forays into non-stationary environments rely on two-types models or ad hoc payoff structures. Research shows that a sorting externality can give rise to endogenous cyclical equilibria (Burdett and Coles (1998)), render welfare-maximizing matching decisions non-stationary (Shimer and Smith (2001)), and sustain multiple equilibrium paths (Boldrin et al. (1993)).

"When is matching assortative?" is the central question in the theory of decentralized matching. Becker (1973) famously studied it in an idealized frictionless marriage market. His analysis emphasizes the role of pre-match negotiation in sorting. Under "complete rigidity" in the division of output at the moment of match creation (the NTU paradigm), e.g., due to a hold-up problem, PAM occurs when match payoffs are increasing in the partner's type.[7] Under "complete negotiability" at the moment of match creation (the TU paradigm), PAM occurs when match output satisfies increasing differences.[8,9]

Various authors have since extended Becker's initial analysis of frictionless matching markets.[10] Most related to ours is the strand of literature that takes into account search frictions, hitherto with an exclusive focus on the steady state.[11] A common finding is that Becker's conditions alone are insufficient to guarantee PAM, the exception being Atakan (2006). See Smith (2006) (time discounting) and Morgan (1994) (explicit search cost) for the NTU paradigm as well as Shimer and Smith (2000) (time discounting) and Atakan (2006) (explicit search cost) for the TU paradigm where payoffs are determined via Nash bargaining.[12,13]

The link between risk preferences and assortative matching has also been made in frictionless contexts in which the purpose of matching is to share risk that materializes after match creation (Serfes (2005), Chiappori and Reny (2016), Schulhofer-Wohl (2006) and Legros and Newman (2007)). These papers suggest that risk-loving individuals match with risk-averse ones to absorb the risk of the latter. Search frictions introduce risk that predates match creation.

2 The Model

We develop a continuous-time, infinite-horizon matching model in which heterogeneous agents engage in time-consuming and haphazard search for one another. When two agents

[7]More generally, Legros and Newman (2010) show that a co-ranking condition of types that requires local monotonicity of payoffs only is necessary and sufficient for PAM.

[8]This condition is commonly thought of as complementarity between assortative types. Increasing differences also plays a role for comparative statics: there is no less PAM with a more complementary production function Cambanis et al. (1976); more recently, Anderson and Smith (2023) impose additional structural assumptions under which they prove the stronger result that there is more PAM with a more complementary production function.

[9]Legros and Newman (2007) consider imperfect transfers that constitute a middle ground between the NTU and TU paradigm.

[10]The TU paradigm in particular has received great attention. Here the equilibrium matching coincides with the output-maximizing matching, allowing techniques from optimal transport to aid the analysis. See for instance Choo and Siow (2006), Chiappori et al. (2017) for the purpose of econometric analysis and Lindenlaub (2017) for studying PAM when agents' types are multidimensional.

[11]Following Postel–Vinay and Robin (2002), an applied literature incorporating search frictions in labor economics focuses on match-to-match transitions and simplifies the complexity of initial match creation by allowing firms to make take–it-or–leave–it wage offers conditional on worker characteristics. Lindenlaub and Postel-Vinay (2024) build on this framework to identify the dimensions in which matching is assortative when agent characteristics are multi-dimensional.

[12]Eeckhout and Kircher (2010) depart from random search to derive sufficient conditions for PAM in a model with directed search. One key difference is that the sellers cannot discriminate their prices based on the buyer's type. This may be attributed to additional information frictions that are not present in the canonical random search framework.

[13]In Bonneton and Sandmann (2021) we study assortative matching in the TU paradigm in non-stationary environments.

meet they observe each other's type. If both agents agree, they match and exit the search pool. Otherwise they continue waiting for a more suitable partner. Each agent maximizes her expected present value of payoffs, discounted at rate $\rho > 0$.

2.1 Individual Matching Problem

There are two distinct populations denoted X and Y, each containing a continuum of agents that seek to match with someone from the other population. Each agent is characterized by a type which belongs to the unit interval $[0, 1]$. Throughout, we denote by x a type of an agent from population X, and y a type of an agent from population Y.

Search. Over time agents randomly meet each other. Meetings follow an (inhomogeneous) Poisson point process. Such a process is characterized by the time-variant (Poisson) meeting rate $\lambda_t = (\lambda_t^X, \lambda_t^Y)$ so that $\lambda_t^X(y|x)$ is the rate at which type x meets type y agents at time t. We assume that higher types meet other agents at a weakly faster rate.

Assumption 1 (hierarchical search)**.** *Higher types meet other agents at a weakly faster rate; that is, $\lambda_t^X(y|x_2) \geq \lambda_t^X(y|x_1)$ for $x_2 > x_1$ and all y and $\lambda_t^Y(x|y_2) \geq \lambda_t^Y(x|y_1)$ for $y_2 > y_1$ and all x.*

Assumption 1 encompasses the commonly studied anonymous search where the meeting rate does not depend on one's own type.

Match payoffs. Agents derive a time-independent one-time payoff if matched with another agent and zero if unmatched: denote $\pi^X(y|x) > 0$ the lump-sum payoff of agent type x from population X when matched with agent type y from population Y. We assume that types are vertically differentiated:[14]

Assumption 2 (increasing match payoffs)**.** *Match payoffs $y \mapsto \pi^X(y|x)$ and $x \mapsto \pi^Y(x|y)$ are continuous and increasing in the partner's type, i.e., $\pi^X(y_2|x) > \pi^X(y_1|x)$ for $y_2 > y_1$ and all x, and $\pi^Y(x_2|y) > \pi^Y(x_1|y)$ for $x_2 > x_1$ and all y.*

Becker (1973) shows that in a frictionless matching market, Assumption 2 implies that the unique core allocation exhibits perfect positive assortative matching: agents match with a partner of equal rank. Targeting a single agent type is infeasible when there are search frictions, for the simple reason that such type may never be met.

Discussion of match payoffs. Lump-sum payoffs are natural if break-up costs are prohibitive (e.g., non-compete clauses), the purpose of the match serves a one-time goal, or

[14]Our results on PAM continue to hold if match payoffs are non-decreasing in the partner's type, none of our proofs require it. However, we prove that an equilibrium exists, see Sandmann and Bonneton (2023), only if match payoffs are increasing. This motivates the slightly stronger assumption here.

if agents enter a different search pool upon match destruction (e.g., as divorcees). When there are flow payoffs $f^X(y|x)$ (e.g., labor or housing markets), lump-sum payoffs capture the lifetime discounted value of the match, i.e., $\pi^X(y|x) = \int_t^\infty e^{-\rho(\tau-t)} f^X(y|x)d\tau = f^X(y|x)/\rho$. In effect, flow and lump-sum payoffs are indistinguishable as long as matches are expected to be permanent. When matches are not expected to be permanent, the costs associated with break-ups and the expected long duration of most matches may nonetheless outweigh any strategic considerations of re-entry at the time of matching.[15] In Section 6.4, we explore an alternative model where matches can be costlessly dissolved.

In what follows, we describe the economy from population X's point of view only. This is for ease of exposition. The construction for population Y follows immediately when reversing the role of X and Y, x and y.

Value-of-search. Upon meeting another unmatched agent, x weighs the immediate match payoff $\pi^X(y|x)$ against the value-of-search $V_t^X(x)$. Naturally, the (weakly dominant[16]) optimal matching decision is to accept to match with y whenever the payoff exceeds the option value-of-search:

$$\pi^X(y|x) \geq V_t^X(x). \tag{OS}$$

The optimal stopping rule determines the match indicator function:

$$m_t(x,y) = \begin{cases} 1 & \text{if } \pi^X(y|x) \geq V_t^X(x) \text{ and } \pi^Y(x|y) \geq V_t^Y(y), \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We denote $y_t(x)$ the infimum type with whom x is willing to match at time t so that $\pi^X(y|x) \geq V_t^X(x)$. As types are vertically differentiated, an agent type x is willing to match with any $y > y_t(x)$ at time t. A symmetric construction applies to $x_t(y)$.

The value-of-search is defined as the discounted expected future match payoff if currently unmatched:

$$V_t^X(x) = \int_t^\infty \int_0^1 e^{-\rho(\tau-t)} \pi^X(y|x)\, p_{t,\tau}^X(y|x) dy\, d\tau, \tag{2}$$

[15]In the context of marriage, a back-of-the-envelope computation suggests no less: assume that marriages break down at exponential rate $\beta = 1/30$ (as of 2010 the average marriage in England & Wales lasts for 30 years—cf. Office for National Statistics as computed by `https://www.rainscourt.com/average-length-marriage/`); (ii) the probability of re-entry into the search pool is α (in England & Wales $\alpha << \frac{3}{4}$ because the majority of marriages ends with the death of a spouse), and (iii) the annual discount rate is $\rho = 0.33$ (cf. Matousek et al. (2022)) Then the (steady state) match payoff is $\Pi = \frac{f}{\rho} + \beta\frac{\alpha V - f}{\rho+\beta} \approx 3.03f + 0.09\alpha V$. Or, match payoffs are largely explained by the flow payoff f only.

[16]By focusing on weakly dominant acceptance rules, we discard trivial equilibria in which agents mutually reject advantageous matches.

where $p^X_{t,\tau}(y|x)$ is the density of future matches with y at time τ conditional on x being unmatched at time t. This is a standard object and is characterized by the matching rate $\lambda^X_\tau(y|x)m_\tau(x,y)$.[17]

2.2 Population Dynamics

The rate at which agents meet each other depends on the size and composition of the population of unmatched agents. The distribution of types in the search pool at time t is characterized by a state $\mu_t = (\mu^X_t, \mu^Y_t)$ so that the mass of types $x \in U \subseteq [0,1]$ is $\int_U \mu^X_t(x)dx$. The initial distribution at time 0 is given by μ_0.[18]

Endogenous meetings. The individual meeting rate λ is a functional of the underlying state variable μ_t and time t. Then $\lambda^X_t(y|x) \equiv \lambda^X(t,\mu_t)(y|x)$ is agent type x's time t meeting rate with an agent type y given the prevailing time t and state μ_t. The meeting rates λ^X_t and λ^Y_t are not arbitrary but intricately linked. Coherence of the model demands that the number of meetings of agent types x with agent types y must be equal to the number of meetings of agent types y with agent types x:[19]

$$\lambda^X_t(y|x)\mu^X_t(x) = \lambda^Y_t(x|y)\mu^Y_t(y).$$

Evolution of the search pool. Population dynamics are governed by entry and exit. Any two agents x and y of opposite populations that meet and mutually consent to form a match permanently exit the search pool. The rate at which an individual agent type x matches and exits the market at time t—the hazard rate—is $\int_0^1 m_t(x,y)\lambda^X_t(y|x)dy$. Entry is characterized by a time- and state-variant rate $\eta = (\eta^X, \eta^Y)$ where $\eta^X_t(x) \equiv \eta^X(t,\mu_t)(x)$ is agent type x's time t entry rate. The economy can be non-stationary in that entry and exit need not be equal. We have:

$$\mu^X_{t+h}(x) = \mu^X_t(x) + \int_t^{t+h} \Big\{ -\mu^X_\tau(x) \int_0^1 \lambda^X_\tau(y|x)m_\tau(x,y)dy + \eta^X_\tau(x) \Big\} d\tau. \tag{3}$$

The economy is non-stationary whenever the integrand is non-zero.[20] For time-variant

[17] Formally, $p^X_{t,\tau}(y|x) = \lambda^X_\tau(y|x)m_\tau(x,y)\exp\{-\int_t^\tau \int_0^1 \lambda^X_r(z|x)m_r(x,z)dzdr\}$. Refer to Appendix A.1 in Sandmann and Bonneton (2023) for a formal derivation.

[18] Functions introduced are Lebesgue measurable throughout. This implies that the type distribution is atomless. Refer to Sandmann and Bonneton (2023) for an extensive discussion of the mathematical properties of the non-stationary search-and-matching model.

[19] Observe that when populations are symmetric and there is hierarchical search (Assumption 1), then coherence of the model implies that the meeting rate must be anonymous, i.e., $\lambda^X_t(y|x_1) = \lambda^X_t(y|x_2)$ for all x_1, x_2.

[20] Our formulation is that of an integral equation rather than a differential equation, because the time derivative of $\mu^X_t(x)$ does not exist everywhere.

entry (as in Figure 4) this will always be the case.

Equilibrium. An equilibrium of the search-and-matching economy of given initial search pool population μ_0 is a triple $(\boldsymbol{\mu}, \mathbf{V}, \mathbf{m})$, solution to (1), (2) and (3). In a companion paper, Sandmann and Bonneton (2023), we show that a non-stationary search equilibrium exists under minimal regularity conditions.[21] ,[22]

Note that our model relaxes common assumptions made in the literature, e.g. the economy is in the steady state, there are symmetric populations, search is anonymous and meeting and entry rates are given by specific functional forms. This level of generality helps identify the key assumptions to study equilibrium sorting: hierarchical search (Assumption 1) and vertically differentiated types (Assumption 2).[23]

3 One-block Block Segregation

To lay the groundwork for our study on assortative matching, we first characterize non-stationary sorting in a highly-stylized non-stationary matching market: a closed market with no new entrants, where potential meetings gradually decrease over time and agents have identical preferences over matched partners. This example allows us to understand better how non-stationary dynamics can affect individual match prospects and, thereby, optimal match decisions.

Formally, consider that agents have identical preferences over matched partners $\pi^X(y|x) = y$ and $\pi^Y(x|y) = x$. These payoffs are sometimes referred to as "pizzazz" (short for the quality of being attractive) and have been extensively studied in the steady state.[24] Then, we consider no entry, $\eta_t^X(y|x) = 0$, and quadratic search, $\lambda_t^X(y|x) = \mu_t^Y(y)$.

As no agents enter the search pool, meetings become scarcer as time goes on ($t \mapsto \lambda_t^X(y|x)$ is decreasing in time). Such a setting offers a simple model of the dynamics occurring during the junior academic job market, where the number of vacancies or candidates decreases as time advances. The decline in meeting rates can also arise from

[21] An equilibrium exists if: Meeting and entry rates are linearly bounded and Lipschitz continuous in the L^1 semi-norm over μ_t at all times t; there exists a constant $\Delta > 0$ so that $\pi^X(y_2|x) - \pi^X(y_1|x) \geq \Delta(y_2 - y_1)$ for all $y_2 > y_1$; total variation of $x \mapsto \pi^X(y|x)$ is finite.

[22] The proof of this result first appeared in an earlier version of this paper. We decided to write a stand-alone and more general existence paper, because for most dynamic heterogeneous agent models we do not know whether a time-dependent equilibrium exists (Achdou et al. (2014)). Sandmann and Bonneton (2023) is now written as a blueprint to establish equilibrium existence in a wide class of non-stationary models where there is a feedback loop between a time-moving aggregate state and individual decisions.

[23] The meeting technology λ encompasses the most commonly studied meeting rates found in the literature: linear (e.g. Mortensen and Pissarides (1994), Burdett and Coles (1997)) and quadratic search technologies (e.g. Shimer and Smith (2000) and Smith (2006)). The entry rate η encompasses several natural entry rates such as no entry and constant flows of entry (as in Burdett and Coles (1997)). In addition, entry may be generated by exogenous match destruction (as in Shimer and Smith (2000) and Smith (2006)).

[24] Subsequent results will apply for all multiplicatively separable payoffs, i.e., payoffs of the form $\pi^X(y|x) = \gamma_1^X(y)\gamma_2^X(x)$ and $\pi^Y(x|y) = \gamma_1^Y(x)\gamma_2^Y(y)$ that are increasing in the partner's type.

other factors like age, where socialization decreases as individuals grow older. Note that a steady state exists, but it is inherently uninteresting: nobody is left searching.

The following proposition characterizes equilibrium non-stationary sorting (for a slightly more general set-up):

Proposition 1 (one-block block segregation)**.** *Suppose that payoffs are multiplicatively separable and continuous in the partner's type*[25]*, and meeting rates* $t \mapsto \lambda_t^X(y|x)$ *and* $t \mapsto \lambda_t^Y(x|y)$ *are anonymous, decreasing and (for item 2.) tend to zero. Then there exist thresholds* $t \mapsto \underline{x}_t \in [0,1)$ *and* $t \mapsto \underline{y}_t \in [0,1)$*, decreasing if non-zero, so that:*

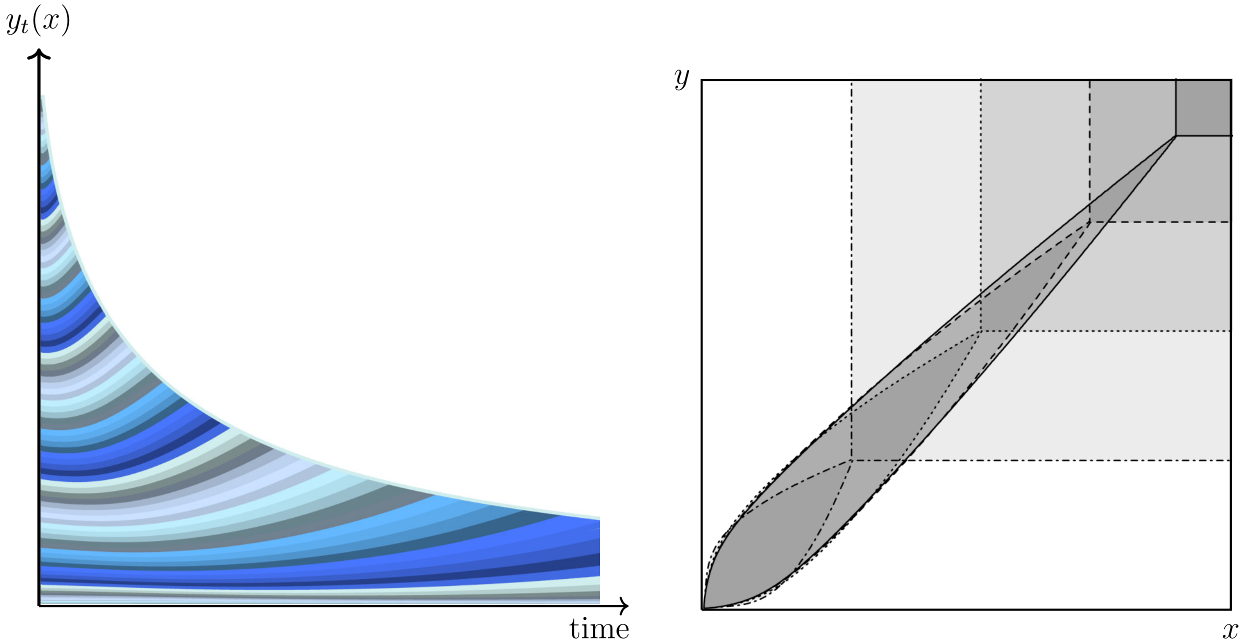
1. *agents with the most advantageous match opportunities match with the same set of agents:* $y_t(x) = \underline{y}_t$ *for all* $x \geq \underline{x}_t$*, and* $x_t(y) = \underline{x}_t$ *for all* $y \geq \underline{y}_t$*;*
2. *among agents with inferior match opportunities lower types are less selective:* $y_t(x_1) < y_t(x_2) < \underline{y}_t$ *for all* $x_1 < x_2 < \underline{x}_t$*, and* $x_t(y_1) < x_t(y_2) < \underline{x}_t$ *for all* $y_1 < y_2 < \underline{y}_t$*.*

Proposition 1, proven in Appendix A, shows that at any moment in time, an expanding set of the most desirable types forms a single matching class; agents within this class exclusively match with each other (see Figure 1b). This class of agents becomes less and less selective over time. The emergence of a perfectly pooled match acceptance threshold at the top derives from pizzazz preferences: individuals with the same preferences and identical match opportunities will make the same choices. In effect, all those who the most desirable agents accept will accept and reject matches in the same manner as the most desirable agents do.

Due to pooled match acceptance thresholds at the top, a decrease in the number of meetings over time does not uniformly deteriorate match opportunities. In fact, intermediate types experience an initial stark improvement in their match opportunities as they anticipate being accepted by the most desirable agents. Visibly (see Figure 1a), match acceptance thresholds rise for most types that have not yet joined the exclusive matching class. This indicates that certain types experience increasing selectivity despite a decrease in the meeting rate.

Comparison to the literature. Match outcomes under "pizzazz" payoffs have been extensively studied in the steady state. The emergence of block segregation (see McNamara and Collins (1990), Burdett and Coles (1997), Smith (2006) and references therein) is a prominent finding. Agents segregate into matching classes and exclusively form matches within their own class. The key difference is that in the steady state (and unlike in our non-stationary framework) block segregation is not limited to a single matching class at the top. Since the acceptance threshold of the most desirable agents

[25]Formally, payoffs are such that $\pi^X(y|x) = \gamma_1^X(x)\gamma_2^X(y)$ with γ_1^X strictly positive and γ_2^Y a continuous, increasing function.



(a) Match acceptance thresholds over time

(b) Match acceptance sets over time

Figure 1: Non-stationary sorting in a closed market with no new entrants

(a) Each color band corresponds to the range of acceptance thresholds chosen by a small interval of types prior to such types joining the matching block at the top. Once a band reaches the upper bound, individuals of said type form matches just like the most desirable individuals. Visibly, lesser ranked types become choosier amid the prospect of being accepted by the group of most desirable agents at the top. They eventually join the group of the choosiest agents. (b) Initially, there are few types accepted by the highest type, so the exclusive matching class at the top is small. Over time, as the number of desirable agents shrinks, this highest matching class expands to include ever more agents. Parameter values are $\lambda_t^X(y|x) = 5\mu_t^Y(y)$, $\eta_t^X(y|x) = 0$ and $\rho = 0.3$.

does not change over time, all agents accepted by the most desirable agent that was rejected by the first matching class also face identical match opportunities. Recursively, this process gives rise to distinct strata of matching classes encompassing all agents, where agents only match within their own class.

Our result shows that block segregation and perfect sorting can co-exist for different segments of the economy. At any moment in time, those agents that are universally accepted exclusively match with each other and form a (temporarily) impenetrable high society. By contrast, there is a natural and more fluid ordering of fortunes outside this exclusive matching class whereby more highly ranked individuals are choosier. This finding anticipates more general results on assortative matching presented in the next section.

4 Positive Assortative Matching

In this section, we derive the weakest sufficient conditions for positive assortative matching (PAM) across all equilibria, encompassing those embedded in non-stationary environments. This generalizes the insights from the preceding section in that we delineate conditions on payoffs for which higher types are choosier.

4.1 Definition of PAM

PAM means that agents of similar characteristics or rank tend to match with one another. When finding a partner entails search, the flow number of created matches depends on both the number of meetings that take place and individual match decisions. We use the definition of PAM by Shimer and Smith (2000) that disentangles physical search frictions from individual matching decisions. They look at hypothetical matches that would be formed if a meeting took place. Formally, define $U_t \equiv \{(x,y) : m_t(x,y) = 1\}$ the set of pairs who are willing to form a match at time t. Matching is assortative if, when any two agreeable matches in U_t are severed, both the greater two and the lesser two types can be agreeably rematched.

Definition 1 (PAM, (Shimer and Smith (2000))). *There is PAM at time t if $(x_1, y_2) \in U_t$ and $(x_2, y_1) \in U_t$ imply that $(x_1, y_1) \in U_t$ and $(x_2, y_2) \in U_t$ for all types $x_2 > x_1$ and $y_2 > y_1$.*

PAM can be recast in more intuitive terms: higher types match with sets of superior types; or, equivalently, higher types are relatively more selective about who they match with. The following proposition develops this idea formally. Recall that $y_t(x)$ is the infimum type with whom x is willing to match at time t so that $\pi^X(y|x) \geq V_t^X(x)$.

Proposition 2. *(i) If $x \mapsto y_t(x)$ and $y \mapsto x_t(y)$ are non-decreasing then there is PAM at time t. (ii) If there is PAM at time t then $x \mapsto y_t(x)$ and $y \mapsto x_t(y)$ are non-decreasing for all types whose individual matching sets $U_t^X(x) \equiv \{y : m_t(x,y) = 1\}$ and $U_t^Y(y) \equiv \{x : m_t(x,y) = 1\}$ are non-empty.*

Note that when the economy is non-stationary, agents may benefit from foregoing all current match opportunities provided they are sufficiently optimistic about the future. Thus individual matching sets may be empty, but not when there is PAM:

Remark 1. *With symmetric populations, PAM at time t implies that $x \in U_t(x)$ for all x.*

The proofs of Proposition 2 and Remark 1 are deferred to Appendix B.1. The first proof of the proposition, derived in the steady state for symmetric populations, is due to Shimer and Smith (2000).

4.2 The Mimicking Argument

To derive equilibrium sorting properties, we need to compare the value-of-search across types. Such a comparison is challenging, as the law of motion is intractable in non-stationary environments, making it impossible to characterize the value-of-search in closed form. To circumvent this problem, we apply a revealed preference argument, which we refer to as the *mimicking argument.*

We first note that the value-of-search, defined in equation (2), admits an integral representation over payoffs that subsumes the time dimension:

$$V_t^X(x) = \int_0^1 \pi^X(y|x) Q_t^X(y|x) dy \quad \text{where} \quad Q_t^X(y|x) \equiv \int_t^\infty e^{-\rho(\tau-t)} p_{t,\tau}^X(y|x) d\tau. \tag{4}$$

Here $Q_t^X(y|x)$ corresponds to a density that does not integrate to one: $\int_U Q_t^X(y|x)dy$ represents type x's discounted probability of forming a match with some other agent type $y \in U \subseteq [0,1]$ some time in the future.

Then observe that higher agent types have better match opportunities. The reasons are twofold. Since match payoffs are monotone (Assumption 2), an agent that is willing to match with a lower agent type x_1 is also willing to match with a higher agent type x_2. And since search is hierarchical (Assumption 1), x_2 meets other agents at a faster rate. Thus, agent type x_2 can in expectation match with all the agent types (and possibly even other, more attractive ones) that agent type x_1 is matching with. Both observations help establish the following lemma,[26] which is the keystone of our proofs for the sorting results in Theorems 1, 1′, 2 and 2′.

Lemma 1 (mimicking argument)**.** *The value-of-search admits the following lower bound:*

$$V_t^X(x_2) \geq \int_0^1 \pi^X(y|x_2) Q_t^X(y|x_1) dy \quad \textit{for all } x_2 > x_1 \in [0,1]. \tag{5}$$

To prove the lemma we define an auxiliary decision problem that allows more highly ranked agents x_2 to exactly replicate ("mimick") a lesser ranked agent x_1's matching rate. Such mimicking is feasible because higher types have better match opportunities. Then, by revealed preferences, mimicking leads to weakly smaller expected payoffs than following the optimal stopping rule (OS).

Proof. Fix $x \in [0,1]$ and $t \in \mathbb{R}_+$. And let $\mathscr{Q}_t^X(x)$ be the space of discounted probabilities $y \mapsto Q_t(y) \in \mathbb{R}_+$ generated by some matching rate $(\tau, y) \mapsto \nu_\tau(y)$ that is feasible, i.e., $\nu_\tau(y) \leq \lambda_\tau^X(y|x)$ and acceptable to y, i.e., $\nu_\tau(y) = 0$ if $\pi^Y(x|y) < V_\tau^Y(y)$. (Following standard arguments, the matching rate $(\tau, y) \mapsto \nu_\tau(y)$ defines the match density via $\tilde{p}_{t,\tau}^X(y|x) = \nu_\tau(y) \exp\big\{ - \int_t^\tau \int_0^1 \nu_r(z) dz dr \big\}$, whence the discounted match probability via

[26] Lemma 4, and thereby all subsequent results on PAM, readily extends to an environment where higher types are more patient as expressed by their discount factor, i.e., $\rho(x_2) < \rho(x_1)$ for all $x_2 > x_1$.

$\tilde{Q}_t(y) = \int_t^\infty e^{-\rho(\tau-t)}\tilde{p}^X_{t,\tau}(y)d\tau$.) By construction, $Q^X_t(\cdot|x) \in \mathcal{Q}^X_t(x)$ and

$$V^X_t(x) = \sup_{Q\in\mathcal{Q}_t(x)} \int_0^1 \pi^X(y|x)Q(y)dy.$$

Assumptions 1 and 2 imply that if $y \mapsto \nu_\tau(y)$ is feasible and acceptable for x_1 then it is feasible and acceptable for x_2. Hence, $\mathcal{Q}_t(x_1) \subseteq \mathcal{Q}_t(x_2)$ and

$$V^X_t(x_2) \geq \sup_{Q\in\mathcal{Q}_t(x_1)} \int_0^1 \pi^X(y|x_2)Q(y)dy.$$

The assertion of the lemma then follows because $Q^X_t(\cdot|x_1) \in \mathcal{Q}_t(x_1)$. □

4.3 Stationary Environment

We first use the mimicking argument to revisit the known steady state analysis. This allows us to make transparent how the assumption of stationarity facilitates PAM.

A condition on payoffs, log supermodularity, is sufficient for PAM in stationary environments:

Definition 2 (Log supermodularity). *Population X's payoffs are log supermodular if for all $y_2 > y_1$ and $x_2 > x_1$,*

$$\frac{\pi^X(y_2|x_2)}{\pi^X(y_1|x_2)} \geq \frac{\pi^X(y_2|x_1)}{\pi^X(y_1|x_1)}.$$

This condition means that higher types stand relatively more to gain from matching with higher types. If the inequality is reversed payoffs are log submodular. The following result is due to Smith (2006).

Theorem 1 (stationary PAM, Smith (2006)). *Suppose that both populations' payoffs are log supermodular. Then there is positive assortative matching (PAM) at all times in any stationary equilibrium.*

Smith's original proof, motivated by the analysis of block segregation, proceeds recursively from the highest type to the lowest type. We present here a shorter proof, based on Lemma 1, that makes explicit why this result is specific to stationary environments: in the steady state, agents always match with a weakly better type than the most desirable type rejected previously.

Proof. We prove the contrapositive. Let $x_2 > x_1$ be such that $y_t(x_2) < y_t(x_1)$ (the environment being stationary, this applies to all moments in time). This means that for any type $\underline{y} \in (y_t(x_2), y_t(x_1))$, agent type x_2 accepts $\underline{y}$ and x_1 rejects $\underline{y}$; whence, due to (OS), $\pi^X(\underline{y}|x_1) < V_t^X(x_1)$ and $\pi^X(\underline{y}|x_2) \geq V_t^X(x_2)$. Then recall the integral representation of the value-of-search (4) and apply the mimicking argument (Lemma 1):

$$\int_0^1 \pi^X(y|x_1)Q_t^X(y|x_1)dy > \pi^X(\underline{y}|x_1) \quad \text{and} \quad \int_0^1 \pi^X(y|x_2)Q_t^X(y|x_1)dy \leq \pi^X(\underline{y}|x_2). \tag{6}$$

In the steady state agents' matching decisions do not change over time. This implies that agents always match with a better type than those rejected previously. Formally, $Q_t^X(y|x_1) = 0$ for all $y < y_t(x_1)$ including $\underline{y}$, and we may adjust the bounds of integration in (6) accordingly. Finally, combining both inequalities yields

$$\int_{\underline{y}}^1 \frac{\pi^X(y|x_1)}{\pi^X(\underline{y}|x_1)}Q_t^X(y|x_1)dy > \int_{\underline{y}}^1 \frac{\pi^X(y|x_2)}{\pi^X(\underline{y}|x_2)}Q_t^X(y|x_1)dy, \tag{7}$$

which can only hold if match payoffs are not log supermodular. □

It is worthwhile to note that Theorem 1 extends beyond stationary environments. The proof only requires that agents unanimously perceive the economy as being on a weak upward trajectory. That is, at no time in the future do acceptance thresholds fall below the highest type currently rejected. This implies that *log supermodularity is sufficient to establish PAM whenever acceptance thresholds $x_t(y)$ and $y_t(x)$ are non-decreasing in time.* The steady state emerges as the knife-edge case where acceptance thresholds do not change.

4.4 Non-Stationary Environments

In a non-stationary environment, log supermodularity is insufficient to guarantee PAM. Here, unlike in the steady state, the lowest type accepted today need not be the worst possible match outcome for all future times. As the search pool evolves over time, agents may face a less favorable selection of types to match with in the future; an agent who rejects a given type initially may accept an inferior type at a later stage. This requires an agent to weigh the current acceptance decision against both the upside risk of matching with a superior type and the downside risk of ending up with an inferior type in the future.

Log supermodularity does not resolve this trade-off. On the one hand, payoff log supermodularity implies that higher types relatively better like to be matched with higher types. On the other hand, it stipulates that higher types stand more to lose from matching with a lower type. Depending on which effect dominates, higher or lower types are choosier. In particular, the higher type's fear of the worst outcome may upset PAM, even though payoffs are log supermodular. To build intuition, we first develop a simple three-type example that illustrates this point (see Figure 3 for an example with a continuum of types).

Example (PAM does not occur in a gradually clearing matching market)**.** *We construct a three-type example in which PAM is upset despite log supermodular payoffs. Populations are symmetric. The market gradually clears with no entrants joining the search pool* ($\eta_t(x) = 0$)*. Assuming quadratic search* ($\lambda_t(x'|x) = \mu_t(x')$)*, meetings are less and less likely to occur over time. Then consider payoffs that are increasing and log supermodular. In the appendix (see Corollary 1) we prove that this implies that agents of the lowest type* x_1 *accept everyone. The intermediate* x_2 *and high type* x_3 *payoffs are given as follows where* $\epsilon > 0$ *is small:*

	x_3	x_2	x_1
$\pi(\cdot\|x_3)$	$10+\epsilon$	1	ϵ
$\pi(\cdot\|x_2)$	10	1	$1-\epsilon$

In effect, the high type x_3 *is highly averse to matching with the lowest type* x_1*. The intermediate type, by contrast, is almost indifferent between the lesser two types.*

The example is solved numerically[27] *and illustrated in Figure 2, with time on the horizontal axis and the value-of-search on the vertical axis.*[28] *Owing to the gradually decreasing meeting rate, the high type's match opportunities deteriorate steadily. At the beginning of time she matches with high type agents* x_3 *only. But after time* t_1*, with only few agents left in the search pool, she also accepts to match with agents of intermediate type* x_2*. The intermediate type initially accepts fellow agents of type* x_2*. Yet, anticipating the possibility of matching with the highest type,* x_2 *experiences a surge in her value-of-search. This leaves her not only to reject the lowest, but also her own type between* t_0 *and* t_1*. (One could say that time interval* $[t_0, t_1]$ *is spent away from the search pool: Agents of type* x_2 *do not match with anyone!) Between time* t_1 *and* t_2 *type* x_2*, is the choosiest: the highest type finds the intermediate type acceptable, whereas the intermediate type does not. This upsets PAM.*

The main contribution of this paper is to establish sufficient conditions for which PAM occurs away from the steady state. First, a definition is in place.

[27] When the meeting rate is quadratic, solving the HJB differential equation characterizing the value-of-search in closed form is typically not possible. Closed-form solutions are reported in the examples on necessity and opportunistic match destruction; see Proposition 3 and Figure 5.

[28] The equilibrium is constructed backward in time, starting with an almost empty search pool far into the future. We further consider $\epsilon = 0.01$ and $\rho = 1$.

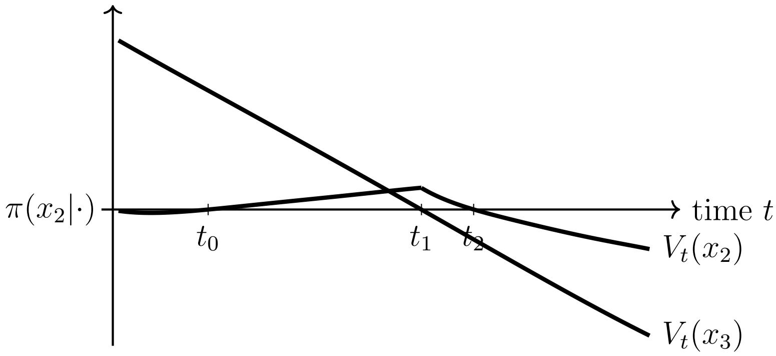


Figure 2: PAM is upset despite log supermodular payoffs —three type example

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Definition 3. *Population X's payoffs are log supermodular in differences if for all* $y_3 > y_2 > y_1$ *and* $x_2 > x_1$*,*

$$\frac{\pi^X(y_3|x_2) - \pi^X(y_2|x_2)}{\pi^X(y_2|x_2) - \pi^X(y_1|x_2)} \geq \frac{\pi^X(y_3|x_1) - \pi^X(y_2|x_1)}{\pi^X(y_2|x_1) - \pi^X(y_1|x_1)}.$$

If the inequality holds with the reverse sign, we say that payoffs satisfy log submodularity in differences. *Log supermodularity in differences*, a term that we introduce here, means that higher types stand relatively more to gain from matching with a high type than they stand to lose from matching with a low type. Log supermodularity in differences is equivalent to $d_y \pi^X(y|x)$ being log supermodular, insofar as such a derivative exists.[29,30,31]

We can interpret the payoff $\pi(\cdot|x) \equiv u_x(\cdot)$ as agent type x's utility function. This affords us an interpretation of log supermodularity in differences in terms of risk preferences. More specifically, Pratt (1964) shows that given arbitrary $x_2 > x_1$ the following statements are equivalent:

1. Agent type x_1 is weakly more risk-averse than agent type x_2; that is, x_1 does not accept a lottery that is rejected by x_2.[32]

2. For any $y_3 > y_2 > y_1$ we have

$$\frac{u_{x_2}(y_3) - u_{x_2}(y_2)}{u_{x_2}(y_2) - u_{x_2}(y_1)} \geq \frac{u_{x_1}(y_3) - u_{x_1}(y_2)}{u_{x_1}(y_2) - u_{x_1}(y_1)}.$$

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[29] See Proposition 7 in the textbook by Gollier (2004) for a proof.

[30] Log supermodularity is a condition that affects both the level and the curvature of a function. By contrast, log supermodularity in differences governs the curvature of a function only and is invariant to its level. In particular, if $\pi^X(y|x)$ is log supermodular in differences, then so is $\pi^X(y|x) - \pi^X(0|x)$. Moreover, $\pi^X(y|x) - \pi^X(0|x)$ is also log supermodular, whereas $\pi^X(y|x)$ need not be.

[31] The payoffs in the preceding example do not satisfy this condition, for the downside loss from matching with x_1 instead of x_2 is much larger for higher types: $\frac{\pi(x_3|x_3)-\pi(x_2|x_3)}{\pi(x_2|x_3)-\pi(x_1|x_3)} = \frac{9+\epsilon}{1-\epsilon} < \frac{9}{\epsilon} = \frac{\pi(x_3|x_2)-\pi(x_2|x_2)}{\pi(x_2|x_2)-\pi(x_1|x_2)}$.

[32] Formally, it holds that if $\int_0^1 u_{x_1}(y)dF(y) \geq (>)\, u_{x_1}(\overline{y})$, then also $\int_0^1 u_{x_2}(y)dF(y) \geq (>)\, u_{x_2}(\overline{y})$.

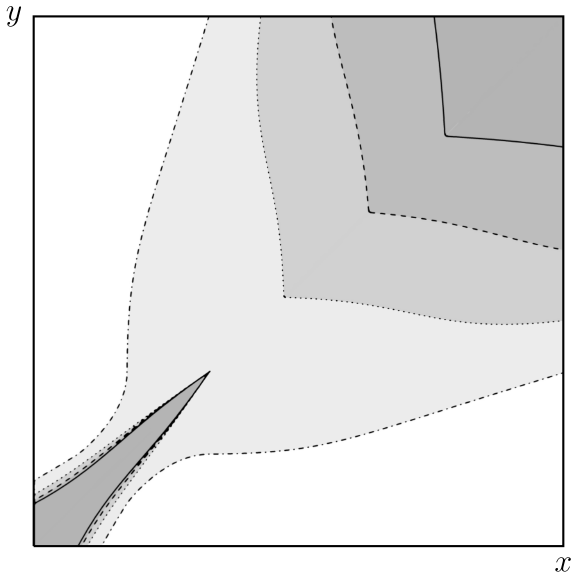


Figure 3: PAM is upset despite log supermodular payoffs

Note: Consider a rapidly clearing search pool with no entry. Symmetric payoffs are $\pi(y|x) = exp(1/16y - 2x^8(1-y)^8)$. These are log supermodular and log *sub*modular in differences. The figure depicts how match acceptance sets shrink over time: darker sets represent match acceptance sets at an earlier date. Initially, only the highest and the lowest types match. Intermediate types do not match up until they are accepted by the highest types. PAM fails initially because, prior to reaching an almost empty search pool, the most desirable agents are not the choosiest. Visually, at the top, the boundary of matching sets is decreasing.

The use of this result is twofold. First, it features prominently in the proof of Theorem 2. Second, it provides a simple interpretation of log supermodularity in differences: lesser ranked agent types are also more risk-averse. Here we are dealing with payoffs of course, not utilities. This is why we caution against viewing log supermodularity in differences solely in the guise of risk-aversion. The curvature of π is implied by the specific model in mind. It may consequently be derived from economic fundamentals rather than risk preferences.

Having established the terminology we can now state the main result:

Theorem 2 (non-stationary PAM). *Suppose that both populations' payoffs are log supermodular and log supermodular in differences. Then there is positive assortative matching (PAM) at all times in any (non-stationary) equilibrium.*

Proof. We prove, as in the stationary case, the contrapositive. Let $x_2 > x_1$ be such that $y_t(x_2) < y_t(x_1)$ at some time t. This means that for any $\underline{y} \in (y_t(x_2), y_t(x_1))$, agent type x_2 accepts $\underline{y}$ and x_1 rejects $\underline{y}$. Using identical arguments as in the proof of Theorem 1, i.e., representation (4) and Lemma 1, yields

$$\int_0^1 \pi^X(y|x_1)Q_t^X(y|x_1)dy > \pi^X(\underline{y}|x_1) \quad \text{and} \quad \int_0^1 \pi^X(y|x_2)Q_t^X(y|x_1)dy \leq \pi^X(\underline{y}|x_2). \quad (8)$$

Next, define $\overline{y} > \underline{y}$ such that $\pi^X(\overline{y}|x_1) \int_0^1 Q_t^X(y|x_1)dy = \pi^X(\underline{y}|x_1)$. (To see that such $\overline{y} \in [0,1]$ exists one must prove that $\pi^X(1|x_1) \int_0^1 Q_t^X(y|x_1)dy \geq \pi^X(\underline{y}|x_1) > \pi^X(\underline{y}|x_1) \int_0^1 Q_t^X(y|x_1)dy$ and apply the intermediate value theorem. The second inequality is trivially true. If the first inequality did not hold, then it must be that $\int_0^1 \left[\pi^X(y|x_1) - \pi^X(1|x_1)\right] Q_t^X(y|x_1)dy > 0$ due to (8) and in spite of non-decreasing match payoffs.) Log supermodularity implies that $1/\int_0^1 Q_t^X(y|x_1)dy = \frac{\pi^X(\overline{y}|x_1)}{\pi^X(\underline{y}|x_1)} \leq \frac{\pi^X(\overline{y}|x_2)}{\pi^X(\underline{y}|x_2)}$. Or, equivalently,

$$\pi^X(\underline{y}|x_2) \leq \pi^X(\overline{y}|x_2) \int_0^1 Q_t^X(y|x_1)dy. \tag{9}$$

Finally, normalize Q_t^X to recast the agents' decisions as a common choice in between a lottery F and the sure outcome $\overline{y}$. Formally, define $F(y) = \int_0^y Q_t^X(y'|x_1)dy' / \int_0^1 Q_t^X(y'|x_1)dy'$. It follows from (8) and (9) that

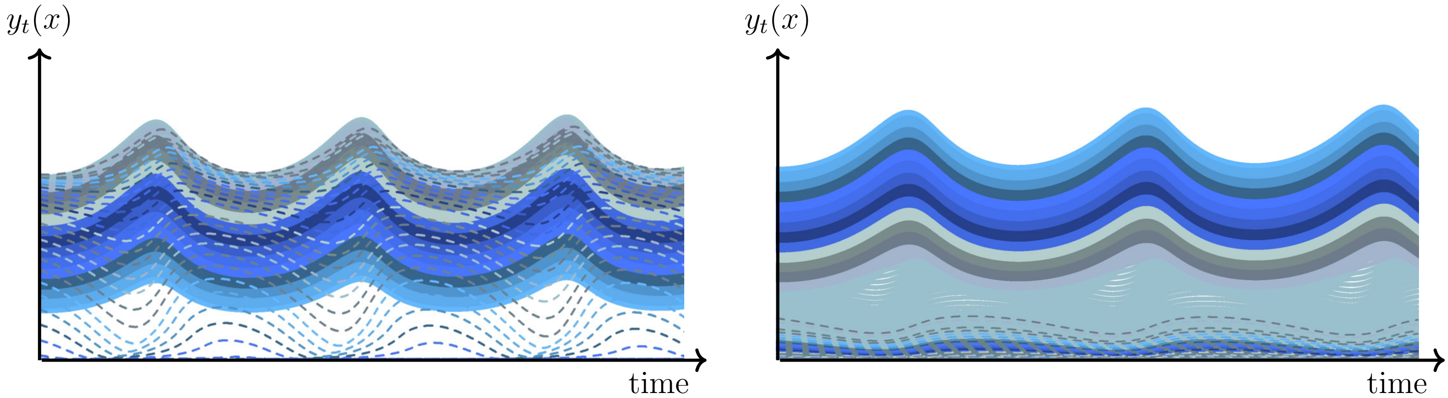
$$\int_0^1 \pi^X(y|x_1)dF(y) > \pi^X(\overline{y}|x_1) \quad \text{and} \quad \int_0^1 \pi^X(y|x_2)dF(y) \leq \pi^X(\overline{y}|x_2).$$

Or, type x_1 accepts the lottery that is rejected by type x_2. This runs counter to the characterization of log supermodularity in differences in terms of risk preferences and establishes a contradiction. □

Theorem 2 only applies if payoffs of both populations satisfy the sufficient conditions. Inspection of the proof of Theorem 2 reveals that if payoffs of population X (but not necessarily of population Y) are log supermodular and log supermodular in differences, then $x \mapsto y_t(x)$ is non-decreasing. The same comment pertains to Theorem 1.

To gain a visual understanding of the scope of Theorem 2, refer to Figure 4. In our simulations, we consider match acceptance thresholds with non-stationary cyclical entry, similar to the fluctuations in a dynamic seasonal housing market (cf. Ngai and Tenreyro (2014)). Despite the complex dynamics, when the conditions for PAM are met (as shown in Figure 4b), all acceptance thresholds remain in a specific order without any crossings. However, if these conditions are not satisfied, the sorting of thresholds may become intricate, leading to numerous crossings between agents' acceptance thresholds (as shown in Figure 4a). This is where PAM proves to be useful in imposing regularity on the dynamics of the matching problem.

Discussion. It may come as a surprise that risk preferences do not play as prominent a role in the steady state. After all, the decision to reject a certain match payoff today is a revealed preference for a risky, random match payoff sometime in the future—regardless of whether the environment is stationary or not. Our analysis shows that the randomness of search translates into less risk in the steady state. Indeed, in a stationary world, the lowest type accepted initially constitutes a bound on the worst possible match outcome for all future dates; the prospect of future matches below one's current acceptance threshold does not arise. This renders downside risk a feature of non-stationary environments only. In consequence, sorting in the steady state solely relies on a preference ranking over upside risk. Non-stationarity in contrast requires a preference ranking over any kind of lottery, entailing both upside and downside risk.



(a) Not PAM —Payoffs are Lsub and LsubD

(b) PAM—Payoffs are LS and LSD

Figure 4: Illustration of Theorem 2 with cyclical entry

Note: Populations are symmetric with payoffs given by $\pi(y|x) = y^{\frac{1}{2}+\frac{1}{2}x}$ (right) and $\pi(y|x) = y^{1-\frac{1}{2}x}$ (left). The former is LS and LSD, i.e., the conditions from Theorem 2, and the latter is neither. Entry is cyclical: $\eta_t(x) = 10\sin(8t)\phi(x)(\mu_t(x))^4$ where $\phi(x)$ is the lognormal density with logmean and logvar equal to 0.5. Further parameters are $\lambda_t(y|x) = \mu_t(y)/(\int_0^1 \mu_t(z)dz)^{\frac{1}{2}}$ and $\rho = 10$. Each color band corresponds to the range of acceptance thresholds chosen by a small interval of types. To highlight the crossing of acceptance thresholds, acceptance thresholds of types $x \in [0.1, 1]$ are depicted in plain color and acceptance thresholds of types $x \in [0, 0.1]$ are dashed. In the example where PAM fails, it is not the most desirable agents, but rather agents of a lower-ranked type with $x = 0.1$, who exhibit the highest level of selectivity.

4.5 Necessity

It is easy to provide examples in which PAM occurs, even when payoffs are neither log supermodular nor log supermodular in differences. As higher types are more likely to be accepted by others, higher types enjoy superior match opportunities and can therefore afford to be choosier, regardless of payoff curvature. Becker (1973) illustrates this point in a frictionless matching market. Adachi (2003) proves this to be the case more generally as search frictions vanish. This raises the question whether our conditions are needlessly strong.

In this section we show that log supermodularity and log supermodularity in differences are the minimal conditions under which PAM occurs in non-stationary environments. If either one condition reverses locally for some interval of types, then there exist primitives of the model under which PAM is upset. We show that this is particularly true when there is a gradually emptying search pool, arguably the simplest instance of a non-stationary environment. The additional requirement that there is zero entry and populations are symmetric merely disciplines the result.

Proposition 3 (weak sufficiency)**.** *Consider an economy with symmetric populations and zero entry and suppose that payoffs satisfy either of the following:*

1. *payoffs restricted to* $[\underline{x}, \overline{x}]^2 \subseteq [0,1]^2$ *are strictly log submodular, or*
2. *payoffs restricted to* $[\underline{x}, \overline{x}]^2 \subseteq [0,1]^2$ *are strictly log submodular in differences;*

then there exist meeting rates λ *and an initial search pool* μ_0 *such that PAM does not occur for some time preceding the (empty) steady state.*

The proof of Proposition 3 is deferred to the appendix. To prove this statement, we show that the set of model primitives for which PAM fails is non-empty, which entails choosing an appropriate meeting rate and type distribution that foster negative sorting for the entire class of payoffs considered. The proof thus revolves around two counterexamples.[33]

Counterexample 1. When there are only two types, both agent types face identical match opportunities once the superior type begins to indefinitely accept the mediocre type. If payoffs are strictly log submodular, the mediocre type relatively better likes matching with the higher type. Then PAM must be precisely upset the moment the high type changes her mind about the mediocre type.

Counterexample 2. We emphasize the role of risk as opposed to time by letting the expected time spent in the search pool become exceedingly small, all the while maintaining the downside risk of matching with the lowest type. To that end, we construct a meeting rate that allows for very frequent meetings initially, but renders meetings with high types extremely rare once its population has fallen below some threshold. As a result, a fraction of high types inevitably matches with the low type, while discounting plays virtually no role here. When payoffs are strictly log submodular in differences, the lesser, mediocre type is less risk-averse than the high type. In such a context, it is straightforward that log submodularity in differences makes the mediocre type relatively choosier, upsetting PAM.

5 Application

Our theory thus far focused on match creation and did not address the origin of payoffs. Many applied models on household bargaining or team production, by contrast, provide

[33]The proof of Proposition 3 relies on counterexamples involving finitely many types only. This is for analytical convenience only. Using bump functions, distributions over finitely many types can be approximated arbitrarily well by a continuous distribution over a continuum, so that one can construct analogous counterexamples with a continuum of types for which PAM is equally upset.

a rich description of the strategic interactions agents encounter once they are matched (see Chiappori and Mazzocco (2017) and references therein). We here illustrate how both approaches can be combined within an applied context to further our understanding of how contractual terms and cultural norms shape assortative matching.

5.1 Marriage and Fertility Choice

Rasul (2008) provides evidence for a hold-up problem in fertility choices: he highlights a commitment problem inferred from Malaysian data whereby men are unable to compensate women for the female cost of fertility choices once children have been born. As a result of this hold-up problem, household fertility choices fail to maximize marital surplus.

The key observation, made among ethnic Chinese marriages, is that male fertility preferences have no statistical power to explain realized fertility levels.[34] This is inconsistent with a model of bargaining at the moment of match creation (commonly referred to as the TU paradigm): If spouses could commit to transfers before marriage, the observed fertility outcomes should reflect a compromise between the preferences of both spouses. The multi-ethnic composition of Malaysia including both ethnic Malays and ethnic Chinese further provides an interesting case study into household bargaining. In Malay marriages, both spouses' fertility preferences have an equal, positive, and significant impact on fertility outcomes. Rasul attributes this difference to dramatically different attitudes towards divorce: divorce is rare among Chinese, whereas Malays had some of the highest divorce rates in the world. To reconcile both observations with the hold-up problem, Rasul formulates a game-theoretic model of household bargaining after fertility decisions have been made. In his model, the divergent fertility outcomes stem from differences in threat points during the post-fertility bargaining stage, namely divorce among Malays and non-cooperation within marriage among Chinese.[35]

Here we take Rasul's analysis a step further. If the male fertility preference does not affect fertility outcomes for Chinese couples, it seems intuitive that Chinese men tend to marry women whose desired fertility level is comparable to their own. To assess whether this hypothesis conforms with our theory, we take the equilibrium payoffs from Rasul's

[34] The discrepancy in the explanatory power of male and female preferences for realized fertility is a robust finding; it has recently been confirmed by Doepke and Kindermann (2019) by drawing on a longitudinal dataset including 19 countries across Eurasia and Australia.

[35] Historical divorce trends have since reversed, see `https://www.dosm.gov.my/portal-main/release-content/marriage-and-divorce-statistics-malaysia-2022`, suggesting that alternative explanations may play a role. Note for instance that Smith and Thomas (1998) document that Chinese are patrilocal, Malays are matrilocal. Hence it is conceivable that among Malays the husband's threat of moving the family away from the couple's first residence with the wive's parents explains the greater influence males exert on realized fertility within Malay couples. Irrespective of these alternative explanations, the original evidence for the hold-up problem among ethnic Chinese presented by Rasul (2008) remains highly pertinent to our analysis.

Marriage threat point:	Divorce (Malay's case)	No divorce (Chinese's case)
Payoff properties:		
Increasing	no	yes
Log supermodular	no	yes
Log supermodular in dif.	yes	yes
Empirical observation:	Less PAM	More PAM

Table 1: Do equilibrium payoffs from Rasul (2008) satisfy our condition for PAM? By "yes", we mean that the condition is satisfied for all parameter values, and "no" means that it is not always satisfied.

model as a primitive of our search-and-matching model, establishing a connection between the two. We then examine whether these payoffs satisfy our sufficient conditions for PAM.

Non-stationarity in the Malaysian marriage market emerges from time pressure associated with the age of marriage. In the data, women marry at a young age; the average female age at marriage is 16.51 (Malays) and 20.69 (Chinese). Analogously to a gradually clearing search pool, this suggests that female values-of-search decrease rapidly during the short window of time when women are of the conventional marital age.

As depicted in Table 1, Theorem 2 predicts PAM along fertility preferences only when divorce is not an admissible threat point. This prediction is consistent with empirical observations. Fertility preferences play a negligible role in explaining marriage patterns among Malays: 44% of couples have fertility preferences that differ by at least two children; 10% differ by more than four (see his Figure 3). The extent of these differences is much smaller among Chinese couples and not attributable to differences in the distribution of individual fertility preferences. This suggests that, remarkably, payoffs derived from a within-household decision model have predictive power for aggregate sorting in the marriage market.

Microfoundation of NTU payoffs: The married couple comprises a husband (y) and a wife (x). Types $x, y \in [0,1]$ encode preferences for greater fertility. Realized fertility q is at the sole discretion of the wife and subsumes both the quality and quantity of children born. Individual spouses wish to match their desired fertility but can be compensated via transfers. Match utility excluding transfers and sunk costs is

$$u^X(y|x) = v^X - \frac{1}{2}(q-x)^2,$$
$$u^Y(x|y) = v^Y - \frac{1}{2}(q-y)^2,$$

where v^X and v^Y are some private gains from marriage.

The timing of the game is as follow. In stage 1, the wife chooses fertility q and incurs sunk costs $cq^2/2$. In stage 2, the husband makes a transfer to his wife, determined via Nash bargaining with positive bargaining weights $\alpha^X, \alpha^Y : \alpha^X + \alpha^Y = 1$.[36] Bargaining outcomes hinge on the spouses' threat points. If bargaining breaks down, spouses lose the private benefits of marriage v^X, v^Y. In principle, spouses are then free to remarry (albeit from within the disadvantaged search pool of divorcees) and pursue their fertility goals with future marriage partners. If so, threat point utility is $\overline{u}_D^X(q;x) = \overline{u}_D^Y(q;y) = 0$. In the data, Chinese couples rarely divorce, rendering divorce implausible. Following Rasul, spouses enter into a non-cooperative marriage in which the mismatch in desired and realized fertility levels is irrevocable. Threat point utility is $\overline{u}_{NC}^X(q;x) = -\frac{1}{2}(q-x)^2$ and $\overline{u}_{NC}^Y(q;y) = -\frac{1}{2}(q-y)^2$.

We solve the game via backward induction. Details are deferred to Footnote 37.[37] By plugging equilibrium transfers and fertility decisions into utilities, we derive expected utilities of the bargaining game corresponding to the match payoffs of the search-and-matching model. We then check whether thus derived match payoffs satisfy our conditions for PAM. Results are summarized in Table 1. The theory, in line with empirical observations, predicts PAM only in the non-cooperative regime where divorce is inadmissible.

We begin with the non-cooperative regime. Female match payoffs are indiscriminate:

$$\pi_{NC}^X(y|x) = v^X - \frac{1}{2}\big(q_{NC}(x,y) - x\big)^2 + \alpha^X v^Y - \alpha^Y v^x.$$

[36]For comparison, in Rasul's paper the husband's bargaining power is θ, not α^X; α^Y becomes $1-\theta$. Fertility costs are non-parametric as given by $c(q)$. Finally, Rasul allows for a common fertility benefit, $\phi(q)$, that we normalize to be zero.

[37] Second stage transfers under either regime $R \in \{NC, D\}$ maximize the product of utilities:

$$\boldsymbol{t}_R(q;x,y) = \arg\max_{\boldsymbol{t}} \big(u^X(q;x) - \overline{u}_R^X(q;x) + \boldsymbol{t}\big)^{\alpha^X} \big(u^Y(q;y) - \overline{u}_R^Y(q;y) - \boldsymbol{t}\big)^{\alpha^Y}.$$

Under the threat of a non-cooperative marriage transfers notably do not depend on desired fertility levels:

$$\boldsymbol{t}_{NC}(q;x;y) = \alpha^X v^Y - \alpha^Y v^X \quad \text{and} \quad \boldsymbol{t}_D(q;x,y) = \alpha^X \big(v^Y - \frac{1}{2}(q-y)^2\big) - \alpha^Y \big(v^X - \frac{1}{2}(q-x)^2\big).$$

In the first stage, the wife anticipates the second stage payoffs and chooses fertility q accordingly. In either regime, $R \in \{NC, D\}$, realized fertility solves $q_R(x,y) = \arg\max_q u^X(q;x) + \boldsymbol{t}_R(q;x,y) - cq^2/2$.

$$q_{NC}(x,y) = \frac{x}{1+c} \qquad \text{and} \qquad q_D(x,y) = \frac{\alpha^X(x+y)}{c+2\alpha^X}.$$

Thus derived transfers and fertility levels correspond to Equations (14) and (15) in Rasul (2008)). Observe that the husband's fertility preference y is irrelevant in the non-cooperative regime. Under the threat of divorce, by contrast, the wife, despite being the sole decision-maker, symmetrically takes into account her husband's fertility preference y.

Since transfers and fertility decisions are independent of the spouse's fertility preference, only the wife's preference affects the fertility outcome. As a result, a single woman's optimal strategy is to not discriminate among potential husbands based on their fertility preferences and prioritize quick matching. Males, by contrast, can only realize their desired fertility level by marrying a woman whose preferences closely align. Their match payoffs are

$$\pi^Y_{NC}(x|y) = v^Y - \frac{1}{2}\big(q_{NC}(x,y) - y\big)^2 - \alpha^X v^Y + \alpha^Y v^x.$$

These are increasing in the partner's type x for all men with a sufficiently high fertility preference, more precisely for all $y \in [\frac{1}{1+c}, 1]$. Fortunately, this restriction is of little empirical relevance. In the data low fertility preferences among men are rare, as men typically desire more children than women. Monotonicity arises because women unilaterally bear the cost of child birth, yet are not compensated for it.

The weakest sufficient for assortative matching identified in this article is log supermodularity in differences. This holds because

$$d^2_{xy} \log d_x \pi^Y_{NC}(x|y) = \frac{d_x q_{NC}(x,y)}{(q_{NC}(x,y) - y)^2} > 0.$$

Our theory therefore predicts that men with a greater fertility preference are more willing to reject a women with a low fertility preference. Incidentally, the strength of this preference—and therefore the strength of sorting between men and women with high fertility preferences—rises as the female costs c associated with childbirth decrease. In the appendix we further derive the relevant expression that shows that payoffs are log supermodular.

In the regime where divorce is prevalent, attributed to Malays, both spouses' preferences carry equal weight in determining fertility outcomes. As a result, individual payoffs are single-peaked in the partner's type. The asymmetric distributions over fertility preferences between men and women then imply that men with the highest fertility preference will be the least desirable husbands. Conversely, women with the lowest fertility preference are the least desirable wives. As further developed in the appendix, these individuals then face the worst match opportunities in equilibrium, giving rise to negative assortative matching between the two groups.

6 Model Variations

In this section, we delve into four natural alternative specifications of the model. Each of these highlights the scope of our main sorting result.[38]

6.1 Explicit Search Cost

So far, we have embedded search cost through time discounting (as espoused by Shimer and Smith (2000) and Smith (2006)). In this section, we re-establish sufficient conditions for PAM adopting the other prominent representation of search cost: explicit search cost (see Morgan (1994), Chade (2001) and Atakan (2006)). Here, discounting plays no role ($\rho = 0$), and each agent in the search pool pays a flow cost c. Whereas time discounting captures the opportunity cost of time, explicit search cost elevates the act of search to be the critical cost.

As was the case under discounting, this framework has been exclusively studied in the steady state (see Morgan (1994)). In what follows, we broaden the scope of the analysis to consider all equilibria. We show that log supermodularity in differences is as essential to PAM under explicit search cost as it is under discounting.

Our analysis under explicit search cost is quasi-identical to the formal arguments presented under discounting, as such all proofs are deferred to the online appendix. We begin by re-stating an adapted version of the mimicking argument that incorporates explicit search cost. As under time-discounting, the value-of-search admits an integral representation over payoffs:

$$V_t^X(x) = \int_0^1 \pi^X(y|x) Q_t^X(y|x) dy - C_t^X(x) \quad \text{where} \quad Q_t^X(y|x) = \int_t^\infty p_{t,\tau}^X(y|x) d\tau. \tag{10}$$

Here $C_t^X(x)$ is the expected time that agent type x spends in the search pool from time t onward, multiplied by the explicit search cost c:

$$C_t^X(x) = c \int_t^\infty \int_0^1 (\tau - t)\, p_{t,\tau}^X(y|x) dy d\tau.$$

As under discounting, higher types have better match opportunities, and so can mimick lesser ranked agents' matching rates. Then an identical reasoning as in the proof of Lemma 1 establishes the following lower bound on the value-of-search:

[38] Existence of an equilibrium for the main model is due to Sandmann and Bonneton (2023). We are not aware of an existence result that applies under the variations of explicit search cost and negotiated match destruction, but conjecture that largely similar arguments as in Sandmann and Bonneton (2023) would establish the result. Existence in a model with aggregate risk is dealt with in Bonneton and Sandmann (2021).

$$V_t^X(x_2) \geq \int_0^1 \pi^X(y|x_2)Q_t^X(y|x_1)dy - C_t^X(x_1) \quad \text{for all } x_2 > x_1 \in [0,1]. \tag{11}$$

By adapting the proof of Theorem 1, Online Appendix .1 presents a short proof of the steady state result due to Morgan (1994): *Suppose that both populations' payoffs are supermodular. Then there is positive assortative matching (PAM) at all times in any stationary equilibrium.*[39]

Supermodularity is insufficient to guarantee positive assortative matching in non-stationary environments for the same reasons given in the analysis of search with discounting. Again, log supermodularity in differences turns out to be the missing sufficient condition that ensures PAM across all equilibria:

Theorem 2′ (non-stationary PAM with explicit search cost)**.** *Suppose that both populations' payoffs are supermodular and log supermodular in differences. Then there is positive assortative matching (PAM) at all times in any (non-stationary) equilibrium.*

This result (whose proof is deferred to Online Appendix .1) shows that, unlike steady state sufficient conditions, which differ between environments with discounting and explicit search cost, log supermodularity in differences ensures PAM in non-stationary equilibrium irrespective of how search cost is modeled. In Online Appendix .1, we further show that our conditions are weakly sufficient, i.e., the counterpart of Proposition 3:

Proposition 3′ (weak sufficiency with explicit search cost)**.** *Consider an economy with explicit search cost, symmetric populations and zero entry and suppose that payoffs restricted to* $[\underline{x}, \overline{x}]^2 \subseteq [0,1]^2$ *are strictly log submodular in differences. Then there exist meeting rates* λ *and an initial search pool* μ_0 *such that PAM does not occur for some time preceding the (empty) steady state.*

Quitting. One concern one may have is that in the preceding agents only exit the search pool if they match. The outside option of staying unmatched is normalized to zero under discounting. Under explicit search cost, by contrast, future expected search costs may accumulate and outweigh the expected benefits of matching; staying unmatched forever is infinitely costly. If agents lacking significant future match opportunities could quit the search pool, they would. We now consider the framework where agents can exit

[39] Recall that population X's payoffs are supermodular if for all $y_1 < y_2$ and $x_1 < x_2$

$$\pi^X(y_2|x_2) + \pi^X(y_1|x_1) \geq \pi^X(y_1|x_2) + \pi^X(y_2|x_1).$$

the search pool unmatched. In keeping with the discounting paradigm, we set the value of rejecting all match opportunities or, analogously, quitting the search altogether to be zero.

The option to exit the search pool is irrelevant in the steady state—those who are currently searching would never have entered if they then wanted to quit. However, it invalidates the conclusion of Theorem 2′ for non-stationary environments. Coercing unmatched agents to keep searching ensures that agents who are selective about who they match with must eventually match with someone. If, instead agents exit the search pool after an unsuccessful search, the probability that they match must be bounded away from one. In effect, inequalities (13) no longer amount to a comparison between a lottery and a certain outside option. Nonetheless, in the online appendix (see Theorem 2″), we prove that PAM can be recovered when in addition to the conditions from Theorem 2′, payoffs are *log supermodular*. Crucially, risk preferences play the same role as before. Adding log supermodularity to the sufficiency conditions is unsurprising in light of the analysis under time discounting. It allows us to normalize future match probabilities like in the proof of Theorem 2.

The following table summarizes the conditions on payoffs that ensure PAM for various environments in the NTU paradigm.

Frictionless	$\pi_2 > 0$ Becker (1973)	
Search frictions	**Stationary**	**Non-Stationary**
i) discounting	$\pi_2 > 0$, $(\log\ \pi)_{12} > 0$ Smith (2006)	$\pi_2 > 0$, $(\log\ \pi)_{12} > 0$, $(\log\ \pi_2)_{12} > 0$ This paper
ii) explicit search cost	$\pi_2 > 0$, $\pi_{12} > 0$ Morgan (1994)	$\pi_2 > 0$, $\pi_{12} > 0$, $(\log\ \pi_2)_{12} > 0$ This paper

Table 2: Sufficient conditions for PAM

Subscript 2 stands for the partial derivative in the partner's type and subscript 1 stands for the partial derivative in one's own type (assuming that these exist).

6.2 Aggregate Uncertainty

It is worthwhile to note that Theorem 2 straightforwardly extends to environments where aggregate fluctuations are stochastic and not deterministic.[40] Algebraically, aggregate

[40]Our focus on deterministic aggregate dynamics owes to the literature's initial focus on the steady state. In Bonneton and Sandmann (2021) (cf. Equation (2)), we explore a model with aggregate uncertainty, where uncertainty is driven by random entry into the search pool.

uncertainty merely compounds the individual idiosyncratic risk. Irrespective of the source of randomness—idiosyncratic or aggregate—future match prospects can be summarized by the discounted match probability $Q_t^X(y|x)$. Hence the integral representation of the value-of-search, see Equation (4), and the subsequent proofs of our main sorting results continue to apply without modification.

Our insights, therefore, carry over to environments where aggregate fluctuations in the state are unpredictable, such as unemployment rising following an (unexpected) economic crisis or sex imbalances being inflicted due to (an unexpected) war. Log supermodularity in differences plays a critical role whenever there is a positive probability that one's current match prospects deteriorate in the future.

6.3 Non-Stationary Types

It is also worthwhile to note that Theorem 2 extends to environments where time-variant match opportunities arise due to a change in individual characteristics rather than a change in the composition of the search pool. To ensure PAM in this context, we require log supermodularity in differences even in the steady state.

Formally, consider two-dimensional agent types (x, α_t) and (y, β_t) that vary across agent. α_t and β_t capture, depending on the application, time spent in the search pool or age. Then $\alpha_{t''} - \alpha_{t'} = t'' - t'$ and $\beta_{t''} - \beta_{t'} = t'' - t'$. We assume that age types α_t and β_t affect the agents' attractiveness to others, but not their preferences. Then y's match payoff when matching with an agent of type (x, α_t) is $\Pi^Y(x, \alpha_t|y)$.

The marriage market serves as an illustrative example where attractiveness plausibly evolves over time. The context of Malaysia as discussed in Section 5 is particularly striking: the average age at marriage for Malay females during the study period was 16.51. This observation is consistent with the idea that female desirability peaks and then declines at the earliest marriageable age, which is possibly driven by high fertility preferences and the advantages of cohabitation with the wife's younger parents.

To illustrate, consider non-stationary flow payoffs $f^Y_{\alpha_t}(x|y)$ that depend on the partner's age type α_t. For instance, flow payoffs may be given by $f^Y_{\alpha_t}(x|y) = e^{-\alpha_t}x$. Then the match payoff of matching with an agent of type (x, α_t) is given by

$$\Pi^Y(x, \alpha_t|y) = \int_{\alpha_t}^{\infty} e^{-\rho(\alpha_\tau - \alpha_t)} f^Y_{\alpha_\tau}(x|y) d\tau.$$

The following Theorem, corollary of Theorem 2, characterizes equilibrium sorting if individual types are non-stationary. The result states, as in our baseline model, that higher types of similar or more desirable age match with more desirable agents under identical conditions on payoffs as before.

Theorem 3 (PAM with non-stationary types)**.** *Suppose that both populations' payoffs are log supermodular and log supermodular in differences in x and y. Then for all α_t and $x_2 \geq x_1$, (x_1, α_t) accepts every (y, β_t) that (x_2, α_t) accepts. If moreover $\beta_t \mapsto \Pi^X(y, \beta_t|x)$ and $\alpha_t \mapsto \Pi^Y(x, \alpha_t|y)$ are non-increasing, then for all ages $\alpha_t'' \geq \alpha_t'$ and types $x_2 \geq x_1$, (x_1, α_t'') accepts every (y, β_t) that (x_2, α_t') accepts.*

If payoffs are determined via flow payoffs, payoff functions $(x, y) \mapsto \Pi^X(y, \beta_t|x)$ and $(x, y) \mapsto \Pi^Y(x, \alpha_t|y)$ satisfy LS and LSD for all α_t and β_t if $(x, y) \mapsto f^X_{\alpha_t}(y|x)$ and $(x, y) \mapsto f^Y_{\beta_t}(x|y)$ satisfy LS and LSD for all ages α_t, β_t. This is due to the fact that log supermodularity is preserved under integration (cf. Lemma 2 in Athey (2002) and Ahlswede and Daykin (1979) for the original result). Multiplicatively separable flow payoffs $g^Y(\beta_t)f^X(y|x)$ with LS and LSD $f^X(y|x)$ satisfy this.

The proof of this theorem is readily adapted from Theorem 2, since, just as in the baseline model, the value-of-search admits an integral representation over payoffs that subsumes the time dimension. For an analogously defined value-of-search that accounts for age, it holds that

$$V_t^X(x, \alpha_t) = \int_t^\infty \int_0^1 \Pi^X(y, \beta_\tau|x) Q_t^X(y, \beta_\tau|x, \alpha_t) dy d\tau.$$

Then the proof of Lemma 1 implies that

$$V_t^X(x_2, \alpha_t') \geq \int_t^\infty \int_0^1 \Pi^X(y, \beta_\tau|x_2) Q_t^X(y, \beta_\tau|x_1, \alpha_t'') dy d\tau.$$

for $x_2 > x_1$ and $\alpha_t' = \alpha_t''$ in general, and $\alpha_t' \leq \alpha_t''$ if match payoffs are non-increasing in age. If agents cease to be attractive to others as α_t grows, agents face downside risk even in the steady state. We therefore cannot simplify the problem as in the steady state proof of the baseline model. Instead, we need to proceed with the proof of Theorem 2 requiring log supermodularity in differences.

6.4 Strategic Match Destruction

Thus far, we have considered a model in which agents do not return to the initial search pool once a match is formed. This provides a natural setting if (i) break-up costs are prohibitive (e.g., non-compete clauses as studied by Shi (2023)), (ii) the purpose of the match serves a one-time goal, or (iii) if agents enter a different search pool upon match destruction (e.g., as divorcees). The literature, by contrast, has largely considered environments in which agents repeatedly enter and exit the search pool and derive flow payoffs while matched. In the steady state, both modeling specifications are indistinguishable because there is no reason for agents to match temporarily. In non-stationary environments, however, agents could be tempted to break their matches strategically once their match opportunities have improved. Whether PAM occurs in these environments depends on the terms under which existing matches are destroyed. In this section, we consider two frameworks for match destruction, one opportunistic and one negotiated, that lead to starkly different implications for PAM.

In this model variation, agents receive flow, not lump-sum payoffs. We denote $f^X(y|x)$ type x's flow match payoff when matched with type y. To evaluate this match, x must factor in that the match will only last until time $t' \in [t, \infty)$; after that x returns to the search pool. Type x's expected discounted match payoff of a match with y, created at time t and terminating at time t' is then given by the sum of discounted flow payoffs and type x's future value-of-search:

$$\Pi^X_{t,t'}(y|x) = \int_t^{t'} e^{-\rho(\tau - t)} f^X(y|x) d\tau + e^{-\rho(t'-t)} V^X_{t'}(x).$$

Agent type x accepts a match with y at time t lasting until time t' if the match payoff $\Pi^X_{t,t'}(y|x)$ exceeds his value-of-search $V^X_t(x)$. The time t' of match destruction will be governed by two very different paradigms.

Opportunistic Match Destruction

First, assume that a match is destroyed whenever at least one agent is better off leaving the match.[41] We call this paradigm opportunistic match destruction because agents pursue advantageous opportunities outside of the match without any regard for their partner's welfare or previous agreements that made the match viable in the first place.[42]

The high-type curse. In this alternative model, generating greater flow payoffs for their partners no longer guarantees superior match opportunities. The reason is simple: individuals may choose not to accept a match with a high-type agent because they anticipate being dumped in the future.[43]

[41]Formally, if x, y match at time t and separate at time $t' > t$, then for all $\delta > 0$ sufficiently small either $\Pi^X_{t,t+\delta}(y|x) < V^X_t(x)$ or $\Pi^Y_{t,t+\delta}(x|y) < V^Y_t(y)$; moreover, t' is the smallest time greater than t that satisifies this.

[42]Smith (1992) initially introduced this model in the context of a gradually clearing search pool (see his Part II, Section 7). However, the implications of opportunistic match destruction on high type match prospects are not discussed.

[43]In the context of labor economics, the high-type curse bears resemblance to the concept of *overqualification*. One reason for not hiring a worker deemed "too good" for a particular job is the concern of potential turnover and the associated costs of refilling the vacancy. Survey evidence indicates that overqualified workers are indeed more prone to leaving their jobs (e.g. Erdogan and Bauer (2011) Erdogan et al. (2011) Maynard et al. (2006)).

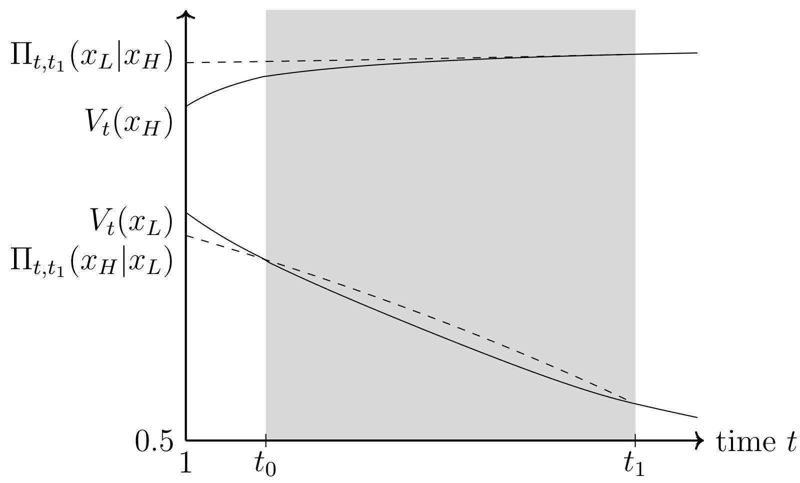


Figure 5: Opportunistic match destruction with two types

Note: agents' value-of-search are solid lines, their payoffs from matching with opposite types are dashed. The gray area is the time interval for temporary matching. Breakup occurs at t_1, and before t_0 the low types reject the high types.

Figure 5 illustrates this phenomenon (analytical resolution in Online Appendix .2.1); it depicts a market with two types, where high types gradually replace low types in meetings. Initially, from $t = 1$ to t_0, high types seek matches with low types, but all low types reject them, preferring to match permanently with lower types. From t_0 to t_1, as opportunities to meet with low types diminish, both low and high types are willing to engage in temporary matches (gray area). At time t_1, high types break up with low types to search for other high types. To grasp the market inefficiency fully, we observe that if high types could commit to permanent matches with low types during the time interval $[1, t_0)$, it would benefit both high and low types.

Consequently, the mimicking argument (Lemma 1) does not apply, and there are no conditions on payoffs that guarantee PAM in this context.

Proposition 4. *Posit conditions for PAM from Theorem 2. Under opportunistic match destruction, there exist meeting rates λ such that, for some time t, PAM does not occur.*

The proof is deferred to Online Appendix .2.1.

Whether these concerns go as far as translating into inferior match opportunities for highly qualified candidates remains to be seen. In the context of an online dating market, Bojd and Yoganarasimhan (2021) provides evidence that, if given a choice, lesser-ranked individuals do avoid meeting the most highly ranked individuals in favor of lesser ranked partners. However, conditional on an online match, the probability of sending a message increased in the receiver's ranked attractiveness.

Negotiated Match Destruction

Opportunistic match destruction points towards a commitment problem: prospective partners cannot commit to a match duration. We now consider a model variation where they can.

We distinguish between two possible match durations, one temporary, $\mathscr{T} \in (0,\infty)$, and one permanent, $\mathscr{P} = \infty$. Our analysis singles out the following negotiation protocol: if at time t in between the three alternative match durations, $\mathscr{T}, \mathscr{P}$ and no match, both x and y find $\mathscr{P}$ most desirable, then x and y match permanently; if instead both x and y find $\mathscr{T}$ more desirable than no match, yet they cannot agree to match permanently, then they match temporarily instead and exit the match at time $t + \mathscr{T}$. We focus on this protocol because it satisfies a property that we call envy-freeness: the higher-ranked type x_2 prefers his contractual terms and never envies those enjoyed by the lower-ranked type x_1.[44]

Lemma 2 (envy-freeness)**.** *If at time t types x_1 and y select match duration $\boldsymbol{t}_2 \in \{\mathscr{T}, \mathscr{P}\}$ and types x_2 and y select match duration $\boldsymbol{t}_1 \in \{\mathscr{T}, \mathscr{P}\}$ where $x_2 > x_1$, then x_2 prefers match duration $\boldsymbol{t}_2$ over match duration $\boldsymbol{t}_1$; that is, $\Pi^X_{t,t+\boldsymbol{t}_2}(y|x_2) \geq \Pi^X_{t,t+\boldsymbol{t}_1}(y|x_2)$.*

Envy-freeness guarantees that the mimicking argument (Lemma 1) holds, but it is not enough to ensure PAM among all pairs that match. When there are temporary matches, some partners can be a less-than-ideal placeholders that are only acceptable given current circumstances. Consequently, we expect the sorting patterns to depend on the nature of the match. To formalize this point, we divide the matching set: $U_t^{\mathscr{P}}$ is the set of pairs (x, y) which, if they meet at time t, match permanently. Similarly, $U_t^{\mathscr{T}}$ is the set of pairs (x, y) which, if they meet at time t, match temporarily. We say that there is PAM among temporary (permanent) matches if the associated matching set $U_t^{\mathscr{T}}$ $(U_t^{\mathscr{T}})$ satisfies the definition of PAM, i.e., Definition 1.[45]

The main result of this section is that the prevalence of envy-freeness guarantees PAM (only) among permanent matches.

Theorem 4 (PAM among permanent matches)**.** *Suppose both populations' flow payoffs are log supermodular and log supermodular in differences. Then there is positive assortative matching (PAM) among all permanent matches.*

[44] Our focus on binary match durations can further be justified by labor or rental market regulation prohibiting many conceivable contractual tenure agreements in many countries. Another reason for not extending our theory to a larger set of possible match durations is that envy-freeness is generally incompatible with Pareto-efficiency. Consequently, standard bargaining protocols such as Nash bargaining over all possible match duration are inadmissible for our result.

[45] Our result on opportunistic match destruction, Proposition 4, does not rely on the distinction between temporary and permanent matches. This is because, as shown in Online Appendix .2.1, there is no PAM even among permanent matches in this case.

We deduce from Theorem 4 that sorting need not occur among temporary matches, i.e., the set $U_t^{\mathcal{T}}$ need not satisfy PAM. To illustrate, consider an environment in which temporarily there are few meetings. Then every meeting will result in a match, be it permanent or temporary. Since the set of permanent matches satisfies PAM, the remaining temporary matching set is not assortative. Under opportunistic match destruction, by contrast, the theory does not predict any specific sorting patterns; without commitment almost all outcomes are possible.

The idea that there is higher assortative matching among permanent matches is consistent with anecdotal evidence from football (soccer). Buraimo et al. (2015) document that players who received higher grades from journalists tend to have longer remaining contract tenure. Tenure, in our model, is a proxy for being assortatively matched. And in light of widely perceived complementarities between players in football, assortatively matched players, on average receive better grades. Hence the correlation between tenure and grade.[46]

7 Conclusion

This article studies positive assortative matching in a general non-stationary matching model. We show that non-stationarity search dynamics introduce downside risk into the agents' optimal match acceptance problem not seen in the steady state. At the individual level, the most desirable accepting agent previously met may be more desirable than one's current partner. We develop a mimicking argument that allows us to deal with the ensuing tractability issues. Our analysis reveals a close link between the time-variant nature of search frictions and a ranking of the curvature of match payoffs. Where payoffs are literal descriptions of preferences, we find that the weakest sufficient conditions for positive assortative matching entail that more desirable individuals are less risk-averse in the sense of Arrow-Pratt. This result, combined with the empirical evidence, provides a theoretical foundation for why positive assortative matching arises in decentralized matching markets. Where preferences are derived from an explicit model of behavior on the match, our theory allows us to generate new predictions about how the parameters of this model affect assortative matching at the match formation stage. An application to fertility choices plagued by a hold-up problem in household bargaining illustrates this approach.

[46] We acknowledge that, at this stage, attributing a longer tenure to greater match complementarity rather than innate skill remains speculative. A comparative analysis with other team sports, where complementarities play a lesser role, could provide further insights.

A One-block Block Segregation

For expositional purposes, Proposition 1 has been presented first. However, its proof requires further results from the paper, particularly our main theorem, Theorem 2.

Proof of Proposition 1. Denote $\underline{y}_t = y_t(1)$. Step 1: We first show, as is to be expected, that $t \mapsto \underline{y}_t$ is decreasing. To see this, note from (2) and the fact that all other agents accept the highest agent type, that

$$V_t^X(1) = \sup_{(\hat{y}_\tau)_{\tau \geq t}} \int_t^\infty \int_0^1 e^{-\rho(\tau-t)} \pi^X(y|1) \lambda_\tau^X(y) 1\{y \geq \hat{y}_\tau\} \exp\Big\{-\int_t^\tau \int_0^1 \lambda_r^X(z) 1\{z \geq \hat{y}_r\} dz dr\Big\} dy dr.$$

Then fix arbitrary times $t_1 > t_0$. And consider the strategy where, from time t_1 onward, at any time t type 1 accepts type y agents as if it were time $t + t_0 - t_1$. This gives a lower bound for $V_{t_1}^X(1)$. In effect, $V_{t_1}^X(1) - V_{t_0}^X(1)$ is weakly greater than

$$\begin{aligned} V_{t_1}^X(1) - V_{t_0}^X(1) \geq \int_{t_0}^\infty \int_0^1 \pi^X(y|1) \Bigg\{ & \Big(\lambda_{\tau+t_1-t_0}^X(y) - \lambda_\tau^X(y)\Big) 1\{y \geq \underline{y}_\tau\} \cdot \\ & \exp\Big\{-\int_t^\tau \int_0^1 \rho + \lambda_{r+t_1-t_0}^X(z) 1\{z \geq \underline{y}_r\}\, dz dr\Big\} \;+\; \lambda_\tau^X(y) 1\{y \geq \underline{y}_\tau\} \cdot \\ & \Big(\exp\Big\{-\int_t^\tau \int_0^1 \rho + \lambda_{r+t_1-t_0}^X(z) 1\{z \geq \underline{y}_r\}\, dz dr\Big\} - \exp\Big\{-\int_t^\tau \int_0^1 \rho + \lambda_r^X(z) 1\{z \geq \underline{y}_r\}\, dz dr\Big\}\Big) \Bigg\} dy d\tau. \end{aligned}$$

The difference is strictly positive due to the fact that, having assumed decreasing meeting rates, both terms in round parentheses are strictly positive. This proves that $t \mapsto V_t^X(1)$ is decreasing in time, and since $y \mapsto \pi^X(y|1)$ is continuous, it follows that also $\underline{y}_t = \inf\big\{y : \pi^X(y|1) - V_t(1) \geq 0\big\}$ is decreasing.

Step 2: We prove item 1., i.e., that all agents $x \in [\underline{x}_t, 1]$ match with the same set of agents. To begin with, admit (as a corollary of Theorem 2) that $\underline{x}_\tau \geq x_\tau(y)$ for all $y \in [0, 1]$ and $\tau \geq t$. Since $\tau \mapsto \underline{x}_\tau$ is decreasing, we deduce that all agents $x \in [\underline{y}_t, 1]$ have identical future match opportunities. In effect,

$$V_t^X(x) = \sup_{(\hat{y}_\tau)_{\tau \geq t}} \int_t^\infty \int_0^1 e^{-\rho(\tau-t)} \pi^X(y|x) \lambda_\tau^X(y) 1\{y \geq \hat{y}_\tau\} \exp\Big\{-\int_t^\tau \int_0^1 \lambda_r^X(z) 1\{z \geq \hat{y}_r\} dz dr\Big\} dy dr$$

for all $x \in [\underline{x}_t, 1]$. Then recall that $\pi^X(y|x) = \gamma_1^X(x)\gamma_2^X(y)$ and compare with $V_t^X(1)$ as characterized above. It follows that $V_t^X(x) = \frac{\gamma_1^X(x)}{\gamma_1^X(1)} V_t^X(1)$ and

$$y_t(x) = \inf\left\{y : \gamma_2^X(y) - \frac{V_t(x)}{\gamma_1^X(x)} \geq 0\right\} = \inf\left\{y : \gamma_2^X(y) - \frac{V_t(1)}{\gamma_1^X(1)} \geq 0\right\} = y_t(1).$$

Step 3: We prove item 2., i.e., that among agents with inferior match opportunities, $x_1 < x_2 < \underline{x}_t$, higher agent types are more selective. That $y_t(x_1) \leq y_t(x_2)$ is an implication of Theorem 2. We now show that this inequality is strict when meeting rates tend to zero. First, observe that following standard arguments $t \mapsto V_t^X(x)$ is continuous (see Proposition 6 (i) in Sandmann and Bonneton (2023)) and tends to zero (because meeting rates tend to zero), and so the earliest times at which two agents with inferior match opportunities match with the most desirable agents are finite and favor the more desirable type: for any two $x_1 < x_2 < \underline{x}_t$, it holds that $\inf\{t : \underline{x}_t = x_1\} > \inf\{t : \underline{x}_t = x_2\}$. Then an identical construction as in step 1 implies that $\frac{V_t^X(x_2)}{\gamma_1^X(x_2)} > \frac{V_t^X(x_1)}{\gamma_1^X(x_1)}$. And since $y \mapsto \gamma_2^X(y)$ is continuous, it follows that

$$y_t(x_2) = \inf\left\{y : \gamma_2^X(y) - \frac{V_t(x_2)}{\gamma_1^X(x_2)} \geq 0\right\} > \inf\left\{y : \gamma_2^X(y) - \frac{V_t(x_1)}{\gamma_1^X(x_1)} \geq 0\right\} = y_t(x_1)$$

as was to be shown. □

B Positive Assortative Matching

B.1 Definition of PAM

Proof of Proposition 2. (i) Fix $x_1 < x_2$ and $y_1 < y_2$ so that $(x_1, y_2), (x_2, y_1)$ belong to the set of pairs that match upon meeting, U_t. Then $y_t(x_2) \leq y_1$ and $x_t(y_2) \leq x_1$, whence also $y_t(x_2) \leq y_2$ and $x_t(y_2) \leq x_2$ due to Assumption 2. It follows that $(x_2, y_2) \in U_t$. As to (x_1, y_1), note that since $y_t(x)$ and $x_t(y)$ are non-decreasing, it holds that $y_t(x_1) \leq y_t(x_2) \leq y_1$ and $x_t(y_1) \leq x_t(y_2) \leq x_1$, whence $(x_1, y_1) \in U_t$.

(ii) Suppose by contradiction that there is PAM, yet $y_t^X(x_2) < y_t^X(x_1)$ for some types $x_2 > x_1$ whose time t matching sets are non-empty.

Case 1: Suppose that there exists $\underline{y} \in [y_t(x_2), y_t(x_1)) \cap U_t^X(x_2)$. Then pick arbitrary $y_2 \in U_t^X(x_1)$. Clearly, $y_2 \geq y_t(x_1) > y_1$. And due to the lattice property, $(x_2, y_1), (x_1, y_2) \in U_t$ implies that $(x_1, y_1) \in U_t$. This contradicts the assertion that $y_1 < y_t(x_1)$.

Case 2: Suppose that $[y_t(x_2), y_t(x_1)) \cap U_t^X(x_2)$ is empty. Then pick arbitrary $y_2 \in U_t^X(x_2)$ and $y_1 \in [y_t(x_2), y_t(x_1))$. Clearly, $y_2 > y_1$ and $x_t(y_1) > x_2$. Whence, for any $x_3 \in U_t^Y(y_1)$ it must be that $x_3 > x_2$. In particular, $(x_2, y_2), (x_3, y_1) \in U_t$ implies $(x_2, y_1) \in U_t$ due to the lattice property. This contradicts the assertion that $x_t(y_1) > x_2$. □

Proof of Remark 1. Suppose there is PAM at time t. If $x \notin U_t(x)$ then $U_t(x)$ is empty. For otherwise there exists x' so that $(x, x'), (x', x) \in U_t$ but not $(x, x) \in U_t$ in spite of PAM. To conclude, denote (t_0, t_1) a maximal time interval during which x does not match with x. Then $V_{t_0}(x) \geq \pi(x|x)$ and $V_{t_0}(x) = e^{-(t_1-t_0)\rho} V_{t_1}(x)$. It follows that $V_{t_0}(x) = 0$ if t_1 is infinite and $V_{t_1}(x) \leq \pi(x|x)$ if t_1 is finite. Contradiction. □

B.2 Sufficiency

Corollary 1. *Suppose that payoffs are log supermodular and populations are symmetric. Then the lowest type will accept everyone, $0 \in U_t(0)$ for every t.*

Proof. We prove the contrapositive. Let (t_0, t_1) denote the maximal time interval during which $0 \notin U_t(0)$ for all $t \in (t_0, t_1)$. If $U_t(0)$ were empty throughout (t_0, t_1), $V_{t_0}(0) = e^{-(t_1-t_0)\rho} V_{t_1}(x) < \pi(0|0)$, yet $V_{t_0}(0) = \pi(0|0)$ which is absurd. Thus, there exists $t \in (t_0, t_1)$ and some non-zero type $x_2 \in U_t(0)$. Yet, due to identical arguments as in the proof of Theorem 1,

$$\int_0^1 \frac{\pi(x'|0)}{\pi(0|0)} Q_t(x'|0) dx' > \int_0^1 \frac{\pi(x'|x_2)}{\pi(0|x_2)} Q_t(x'|0) dx'.$$

As in the proof of Theorem 1, this can only hold if match payoffs are not log supermodular.

□

B.3 Necessity

Proof of Proposition 3. **Counterexample 1** There are two types, $x_2 > x_1$, whose payoffs are strictly log submodular. That is

$$\frac{\pi(x_2|x_2)}{\pi(x_1|x_2)} < \frac{\pi(x_2|x_1)}{\pi(x_1|x_1)}.$$

Search is quadratic, i.e. $\lambda(t, \mu_t) = \mu_t$ and there is no entry.

As match prospects are bleakening over time, there exists a time t^* beyond which the high type will always accept the low type and $V_{t^*}(x_2) = \pi(x_1|x_2)$. Drawing on the integral representation of the value-of-search we can express $V_{t^*}(x_2)$ as

$$V_{t^*}(x_2) = \sum_{j \in \{1,2\}} \pi(x_j|x_2) Q_{t^*}(x_j)$$

where $Q_{t^*}(x_j)$ is the probability of type x_2 matching with x_j - discounted by the time at which such event materializes. Now observe that if the low type found it desirable, she could always exactly replicate discounted match probabilities of the high type, that is

$$V_{t^*}(x_1) \geq \sum_{j \in \{1,2\}} \pi(x_j|x_1) Q_{t^*}(x_j).$$

Then $V_{t^*}(x_1) > \pi(x_1|x_1)$ and the low type rejects other low types at time t^*. For otherwise the integral representation of the value-of-search combined with the inequalities implies that

$$\sum_{j \in \{1,2\}} \frac{\pi(x_j|x_2)}{\pi(x_1|x_2)} Q_{t^*}(x_j) \geq \sum_{j \in \{1,2\}} \frac{\pi(x_j|x_1)}{\pi(x_1|x_1)} Q_{t^*}(x_j) \quad \Leftrightarrow \quad \frac{\pi(x_2|x_2)}{\pi(x_1|x_2)} \geq \frac{\pi(x_2|x_1)}{\pi(x_1|x_1)}$$

in spite of strict log submodularity.

Counterexample 2. Consider symmetric populations consisting of three types $x_1 < x_2 < x_3$. Omit superscripts. Suppose that $\frac{\pi(x_3|x_3)-\pi(x_2|x_3)}{\pi(x_2|x_3)-\pi(x_1|x_3)} < \frac{\pi(x_3|x_2)-\pi(x_2|x_2)}{\pi(x_2|x_2)-\pi(x_1|x_2)}$. Then x_3 is strictly more risk-averse than x_2.

We construct a sequence of equilibra indexed by n in which, for n sufficiently large, there exists a moment in time such that x_3 accepts x_2 whereas x_2 rejects a fellow x_2. Specifically, consider two distinct moments in time, t_0^n and 0 where t_0^n precedes 0: at time t_0^n the high type x_3 begins accepting the intermediate type x_2 and at time 0 the high type begins accepting the low type x_1; PAM will be upset because type x_2 will reject another type x_2 at time t_0^n.

The construction makes apparent that the failure of PAM at time t_0^n arises due to a reversal of risk preferences. As n grows large both (i) $t_0^n \to 0$ and (ii) the probability of matching after time 0 will go to zero. As a consequence agent type x_3's future match outcomes at time t_0^n converge towards a lottery assigning positive probability to both the event that x_3 match with another x_3 and to the event that x_3 match with an agent type x_1. Crucially, at time t_0^n agent types x_2 are accepted by agent types x_3. They thus face identical match opportunities. Like agent types x_3, they may either choose to play the lottery—or accept x_2. Note that since agent type x_3 is indifferent between playing the lottery, i.e., waiting, or accepting x_2, by virtue of being less risk-averse agent type x_2 must strictly prefer the lottery and therefore reject another type x_2.

To construct the failure of PAM analytically, we consider the simplest non-stationary matching environment conceivable. There is zero entry. Agent type x_2 is present in zero proportion and solely of hypothetical interest. Due to log supermodularity agent type x_1 will accept any agent he meets. Proceed then to define the (anonymous) meeting rate: it becomes stationary eventually and is piecewise constant over time. We set

$$\lambda_t(x_1) = n(1-h(n)) \quad \text{if } t \geq 0 \qquad \text{and} \qquad \lambda_t(x_3) = \begin{cases} nh(n) & \text{if } t \geq 0 \\ n & \text{if } t < 0. \end{cases}$$

$h(n)$ is determined as to ensure indifference of agent type x_3 between accepting and rejecting agent types x_1 for all $t \geq 0$. Then at time $t = 0$

$$\rho V_0^n(x_3) = n\big[h(n)\pi(x_3|x_3) + (1-h(n))\pi(x_1|x_3) - V_0^n(x_3)\big] \quad \text{and} \quad V_0^n(x_3) = \pi(x_1|x_3).$$

Here the equation on the left is the stationary HJB equation and the equation on the right is the indifference condition. The latter holds if

$$h(n) = \frac{\rho}{n} \frac{\pi(x_1|x_3)}{\pi(x_3|x_3) - \pi(x_1|x_3)}.$$

We assume that at time 0 agent types x_2 likewise accept agent types x_1 (log supermodular payoffs imply this). If they did not, PAM would be upset as we desire to show.

Finally, choose as time 0 'starting values' $(\mu_0(x_1), \mu_0(x_2), \mu_0(x_3))$ such that $\mu_0(x_2) = 0$ and $\frac{\mu_t(x_3)}{\mu_t(x_1)} = \frac{\lambda_t(x_3)}{\lambda_t(x_1)}$.[47]

Preceding time $t = 0$ the high type x_3's value-of-search is decreasing. Time $t_0^n < 0$, the moment in time at which agent type x_3 is indifferent between accepting and rejecting agent type x_2, likewise admits a closed-form representation: Recall that $V_0^n(x_3) = \pi(x_1|x_3)$ so that prior to time 0 the high type x_3 exclusively matches with other high types. Then an explicit characterization of x_3's value-of-search as defined in Equation (2) gives

$$\begin{aligned} V_{t_0^n}^n(x_3) &= \int_{t_0^n}^{0} e^{-\rho(\tau - t_0^n)} \pi(x_3|x_3) n\, e^{-n(\tau - t_0^n)} d\tau + e^{\rho t_0^n} e^{n t_0^n} \pi(x_1|x_3) \\ &= \frac{n}{\rho + n}\big[1 - e^{t_0^n(\rho+n)}\big]\pi(x_3|x_3) + e^{t_0^n(\rho+n)}\pi(x_1|x_3). \end{aligned}$$

And the indifference condition that characterizes t_0^n is $V_{t_0^n}^n(x_3) = \pi(x_2|x_3)$. The solution is given by

$$t_0^n = \frac{1}{\rho + n} \ln \frac{\frac{n}{\rho+n}\pi(x_3|x_3) - \pi(x_2|x_3)}{\frac{n}{\rho+n}\pi(x_3|x_3) - \pi(x_1|x_3)}$$

Clearly, $t_0^n < 0$ due to Assumption 2 and $t_0^n \to 0$ as n goes to infinity.

[47] Note that this construction does not run counter the requirement that the search technology be anonymous: following time 0 there is common acceptance of all types so that under any anonymous search technology the ratio $\frac{\mu_t(x_3)}{\mu_t(x_1)}$ remains constant for all $t \geq 0$. $\lambda_t(x_1)$ for $t < 0$ will be uniquely pinned down by $\lambda_t(x_3)$ and μ_t (in particular $\lambda_t(x_1) = \lambda_t(x_3)\frac{\mu_t(x_1)}{\mu_t(x_3)}$), but this is inconsequential as agent type x_3 rejects agent types x_1 at $t < 0$.

Agent type x_3's discounted match probabilities of matching with agent types x_1 and x_3, as defined in Equation (4), are denoted by $Q^n_{t^n_0}(x_1)$ and $Q^n_{t^n_0}(x_3)$ respectively. Following the above value-of-search, these are

$$\begin{aligned}Q^n_{t^n_0}(x_1) &= e^{t^n_0(\rho+n)}\int_0^\infty e^{-\rho\tau}n(1-h(n))e^{-n\tau}d\tau = e^{t^n_0(\rho+n)}\frac{n(1-h(n))}{\rho+n}\\ &= \frac{\frac{n}{\rho+n}\pi(x_3|x_3)-\pi(x_2|x_3)}{\frac{n}{\rho+n}\pi(x_3|x_3)-\pi(x_1|x_3)}\frac{n(1-h(n))}{\rho+n} = \frac{\pi(x_3|x_3)-\pi(x_2|x_3)}{\pi(x_3|x_3)-\pi(x_1|x_3)}+o(1)\equiv q+o(1)\end{aligned}$$

and

$$\begin{aligned}Q^n_{t^n_0}(x_3) &= \int_{t^n_0}^0 e^{-\rho(\tau-t^n_0)}ne^{-n(\tau-t^n_0)}d\tau + e^{t^n_0(\rho+n)}\int_0^\infty e^{-\rho\tau}nh(n)e^{-n\tau}d\tau\\ &= \frac{n}{\rho+n}\left[1-e^{(\rho+n)t^n_0}\right]+e^{(\rho+n)t^n_0}\frac{nh(n)}{\rho+n} = \frac{n}{\rho+n}-\frac{n(1-h(n))}{\rho+n}\frac{\frac{n}{\rho+n}\pi(x_3|x_3)-\pi(x_2|x_3)}{\frac{n}{\rho+n}\pi(x_3|x_3)-\pi(x_1|x_3)}\\ &= 1-\frac{\pi(x_3|x_3)-\pi(x_2|x_3)}{\pi(x_3|x_3)-\pi(x_1|x_3)}+o(1) = (1-q)+o(1).\end{aligned}$$

Here $o(1)$ denotes the Landau notation: $\lim_{n\to\infty} o(1) = 0$. In particular, note that $Q^n_{t^n_0}(x_1)+Q^n_{t^n_0}(x_3) = 1+o(1)$, meaning that the x_3's probability of matching instantaneously approaches 1 as n tends to infinity.

Now observe that, beginning from time t^n_0, agent type x_2 is accepted by agent type x_3, and thus faces identical match opportunities as an agent type x_3. Accordingly, x_2 can mimic the higher type x_3's match probabilities (see Lemma 1) so that

$$V^n_{t^n_0}(x_2) \geq \pi(x_1|x_2)q+\pi(x_3|x_2)(1-q)+o(1).$$

(Recall by construction that $\pi(x_2|x_3) = V^n_{t^n_0}(x_3) = \pi(x_1|x_3)q+\pi(x_3|x_3)(1-q)+o(1)$.) We then claim that $V^n_{t^n_0}(x_2) > \pi(x_2|x_2)$ for n sufficiently large, so that PAM does not occur at time t^n_0: the intermediate type x_2 rejects a fellow intermediate type x_2 that is accepted by high type agents x_3. Indeed, this follows from the characterization of risk preferences. Suppose by contradiction that $V^n_{t^n_0}(x_2) \leq \pi(x_2|x_2)$ for all $n\in\mathbb{N}$. Letting $n\to\infty$ gives

$$\pi(x_2|x_2) \geq \pi(x_1|x_2)q+\pi(x_3|x_2)(1-q) \quad \text{and} \quad \pi(x_2|x_3) = \pi(x_1|x_3)q+\pi(x_3|x_3)(1-q).$$

This means that (i) agent type x_3 is indifferent between the lottery assigning probability q to x_1 and $1-q$ to x_2 and the sure outcome x_2, whereas (ii) agent type x_2 weakly prefers the sure outcome x_2. This contradicts the assertion that agent type x_2 is strictly less risk-averse than agent type x_3. □

C Application

C.1 Marriage and Fertility Choice

In the non-cooperative regime, payoffs are log supermodular.

$$d^2_{xy} \log \pi^Y_{NC}(x|y) = \frac{2(1+c)(2\alpha^Y(1+c)^2 v^X + 2(1-\alpha^X)(1+c)^2 v^Y + (-x+y+cy)^2)}{\big(-2\alpha^Y(1+c)^2 v^X - 2(1-\alpha^X)(1+c)^2 v^Y + (-x+y+cy)^2\big)^2} > 0$$

Match payoffs in the divorce regime are

$$\begin{aligned}\pi^Y_D(x|y) &= (1-\alpha^X)v^Y + \alpha^Y v^X - \frac{1}{2}(1-\alpha^X)(q_D(x,y)-y)^2 - \frac{1}{2}\alpha^Y(q_D(x,y)-x)^2\\ \pi^X_D(y|x) &= (1-\alpha^Y)v^X + \alpha^X v^Y - \frac{1}{2}\alpha^X(q_D(x,y)-y)^2 - \frac{1}{2}(1-\alpha^Y)(q_D(x,y)-x)^2.\end{aligned}$$

These exhibit single-peakedness rather than monotonicity in the partner's type. Consider the case of equal bargaining weights, where $\alpha^X = \alpha^Y = \frac{1}{2}$, and assume $c = 1$. When we examine the derivatives, we find that $d_x\pi^Y_D(x|y) = \frac{1}{16}(3y-5x)$, which is positive if and only if $3y > 5x$. Similarly, $d_y\pi^X_D(y|x) = \frac{1}{16}(3x-5y)$, which is positive if and only if $3x \geq 5y$. These results indicate that the ideal husband for a female with fertility preference x (or the ideal wife for a male with fertility preference y) has a lower fertility preference than their own. The desirability of matching with low fertility individuals stems from several factors. Lower fertility individuals place less value on the threat of divorce, resulting in lower required transfers under Nash bargaining. However, matching with individuals with the lowest fertility also carries the cost of not achieving one's desired fertility level. Therefore, individuals with low to intermediate fertility preferences tend to have the best match opportunities. Aggregate matching patterns partially reflect the distribution of types: due to the asymmetry in fertility preferences between men and women, where men generally desire more children than women, men with very high fertility preferences and women with very low fertility preferences are particularly disadvantaged and less likely to match with intermediate types. As a result, negative assortative matching occurs between these groups.

Finally, note that payoffs are not always log supermodular. Under identical parameters

$$d^2_{xy} \log \pi^Y(x|y) = d^2_{xy} \log \pi^X(y|x) = \frac{2\left(48v^X + 48v^Y + 15x^2 - 50xy + 15y^2\right)}{(16v^X + 16v^Y - 5x^2 + 6xy - 5y^2)^2}$$

which is negative for non-assortative x, y whenever the benefits of marriage, v^X and v^Y, are small. Log supermodularity in differences does hold by contrast, as

$$d^2_{xy} \log d_x\pi^Y(x|y) = \frac{15}{(3x-5y)^2} > 0 \qquad \text{and} \qquad d^2_{xy} \log d_y\pi^X(y|x) = \frac{15}{(3y-5x)^2} > 0.$$

References

Achdou, Y., Buera, F. J., Lasry, J.-M., Lions, P.-L., and Moll, B. (2014). Partial differential equation models in macroeconomics. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 372(2028):20130397.

Adachi, H. (2003). A search model of two-sided matching under nontransferable utility. *Journal of Economic Theory*, 113(2):182–198.

Ahlswede, R. and Daykin, D. (1979). An inequality for weights of two families of sets, their unions and intersections. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 93:183–185.

Anderson, A. and Smith, L. (2023). The comparative statics of sorting.

Arrow, K. J. (1965). *Aspects of the theory of risk-bearing*. Yrjö Jahnssonin Säätiö.

Atakan, A. E. (2006). Assortative matching with explicit search costs. *Econometrica*, 74(3):667–680.

Athey, S. (2002). Monotone comparative statics under uncertainty. *The Quarterly Journal of Economics*, 117(1):187–223.

Baley, I., Figueiredo, A., and Ulbricht, R. (2022). Mismatch cycles. *Journal of Political Economy*, 130(11):2943–2984.

Becker, G. S. (1973). A theory of marriage: Part i. *Journal of Political Economy*, 81(4):813–846.

Benjamin, D. J., Brown, S. A., and Shapiro, J. M. (2013). Who is 'behavioral'? cognitive ability and anomalous preferences. *Journal of the European Economic Association*, 11(6):1231–1255.

Bojd, B. and Yoganarasimhan, H. (2021). Star-cursed lovers: Role of popularity information in online dating. *Marketing Science*, 41(1):73–92.

Boldrin, M., Kiyotaki, N., and Wright, R. (1993). A dynamic equilibrium model of search, production, and exchange. *Journal of Economic Dynamics and Control*, 17(5-6):723–758.

Bonneton, N. and Sandmann, C. (2021). Probabilistic assortative matching under nash bargaining. Toulouse School of Economics.

Buraimo, B., Frick, B., Hickfang, M., and Simmons, R. (2015). The economics of long-term contracts in the footballers' labour market. *Scottish Journal of Political Economy*, 62(1):8–24.

Burdett, K. and Coles, M. G. (1997). Marriage and class. *The Quarterly Journal of Economics*, 112(1):141–168.

Burdett, K. and Coles, M. G. (1998). Separation cycles. *Journal of Economic Dynamics and Control*, 22(7):1069–1090.

Cambanis, S., Simons, G., and Stout, W. (1976). Inequalities for ek(x,y) when the marginals are fixed. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 36:285–294.

Chade, H. (2001). Two-sided search and perfect segregation with fixed search costs. *Mathematical Social Sciences*, 42(1):31–51.

Chade, H., Eeckhout, J., and Smith, L. (2017). Sorting through search and matching models in economics. *Journal of Economic Literature*, 55(2):493–544.

Chade, H. and Swinkels, J. M. (2019). The no-upward-crossing condition, comparative statics, and the moral-hazard problem. *Theoretical Economics*.

Charles, K. K., Hurst, E., and Killewald, A. (2012). Marital Sorting and Parental Wealth. *Demography*, 50(1):51–70.

Chiappori, P.-A. and Mazzocco, M. (2017). Static and intertemporal household decisions. *Journal of Economic Literature*, 55(3):985–1045.

Chiappori, P.-A. and Reny, P. J. (2016). Matching to share risk. *Theoretical Economics*, 11(1):227–251.

Chiappori, P.-A., Salanié, B., and Weiss, Y. (2017). Partner choice, investment in children, and the marital college premium. *American Economic Review*, 107(8):2109–67.

Choo, E. and Siow, A. (2006). Who marries whom and why. *Journal of political Economy*, 114(1):175–201.

Davidson, C., Heyman, F., Matusz, S., Sjöholm, F., and Zhu, S. C. (2014). Globalization and imperfect labor market sorting. *Journal of International Economics*, 94(2):177–194.

Doepke, M. and Kindermann, F. (2019). Bargaining over babies: Theory, evidence, and policy implications. *American Economic Review*, 109(9):3264–3306.

Dohmen, T., Falk, A., Huffman, D., and Sunde, U. (2010). Are risk aversion and impatience related to cognitive ability? *American Economic Review*, 100(3):1238–60.

Dohmen, T., Falk, A., Huffman, D., Sunde, U., Schupp, J., and Wagner, G. G. (2011). Individual risk attitudes: Measurement, determinants, and behavioral consequences. *Journal of the European Economic Association*, 9(3):522–550.

Eeckhout, J. and Kircher, P. (2010). Sorting and decentralized price competition. *Econometrica*, 78(2):539–574.

Erdogan, B. and Bauer, T. N. (2011). The impact of underemployment on turnover and career trajectories. *Underemployment: Psychological, economic, and social challenges*, pages 215–232.

Erdogan, B., Bauer, T. N., Peiró, J. M., and Truxillo, D. M. (2011). Overqualified employees: Making the best of a potentially bad situation for individuals and organizations. *Industrial and organizational psychology*, 4(2):215–232.

Felbermayr, G., Hauptmann, A., and Schmerer, H.-J. (2014). International trade and collective bargaining outcomes: Evidence from german employer–employee data. *The Scandinavian Journal of Economics*, 116(3):820–837.

Frederick, S. (2005). Cognitive reflection and decision making. *Journal of Economic perspectives*, 19(4):25–42.

Gollier, C. (2004). *The Economics of Risk and Time*, volume 1. The MIT Press, 1 edition.

Guiso, L. and Paiella, M. (2004). The role of risk aversion in predicting individual behaviors.

Hazell, J., Patterson, C., Sasons, H., and Taska, B. (2022). National wage setting. Technical report.

Legros, P. and Newman, A. (2010). Co-ranking mates: Assortative matching in marriage markets. *Economics Letters*, 106(3):177–179.

Legros, P. and Newman, A. F. (2007). Beauty is a beast, frog is a prince: Assortative matching with nontransferabilities. *Econometrica*, 75(4):1073–1102.

Lindenlaub, I. (2017). Sorting Multidimensional Types: Theory and Application. *The Review of Economic Studies*, 84(2):718–789.

Lindenlaub, I. and Postel-Vinay, F. (2024). Multi-dimensional sorting under random search. *Journal of Political Economy*.

Lise, J. and Robin, J.-M. (2017). The macrodynamics of sorting between workers and firms. *American Economic Review*, 107(4):1104–35.

Mare, R. D. (1991). Five decades of educational assortative mating. *American Sociological Review*, 56(1):15–32.

Matousek, J., Havranek, T., and Irsova, Z. (2022). Individual discount rates: a meta-analysis of experimental evidence. *Experimental Economics*, 25:318–358.

Maynard, D. C., Joseph, T. A., and Maynard, A. M. (2006). Underemployment, job attitudes, and turnover intentions. *Journal of Organizational Behavior: The International Journal of Industrial, Occupational and Organizational Psychology and Behavior*, 27(4):509–536.

Mazzocco, M. (2007). Household intertemporal behaviour: A collective characterization and a test of commitment. *The Review of Economic Studies*, 74(3):857–895.

McNamara, J. M. and Collins, E. (1990). The job search problem as an employer–candidate game. *Journal of Applied Probability*, 27(4):815–827.

Morgan, P. B. (1994). *A model of search, coordination and market segmentation.* Department of Economics, State University of New York at Buffalo.

Mortensen, D. T. and Pissarides, C. A. (1994). Job creation and job destruction in the theory of unemployment. *The Review of Economic Studies*, 61(3):397–415.

Ngai, L. R. and Tenreyro, S. (2014). Hot and cold seasons in the housing market. *American Economic Review*, 104(12):3991–4026.

Noussair, C. N., Trautmann, S. T., and Van de Kuilen, G. (2013). Higher order risk attitudes, demographics, and financial decisions. *Review of Economic Studies*, 81(1):325–355.

Postel–Vinay, F. and Robin, J. (2002). Equilibrium wage dispersion with worker and employer heterogeneity. *Econometrica*, 70(6):2295–2350.

Pratt, J. W. (1964). Risk aversion in the small and in the large. *Econometrica*, 32(1-2):122–136.

Rasul, I. (2008). Household bargaining over fertility: Theory and evidence from malaysia. *Journal of Development Economics*, 86(2):215–241.

Sandmann, C. (2023). When are single-contract menus profit-maximizing? London School of Economics.

Sandmann, C. and Bonneton, N. (2023). Existence of a non-stationary equilibrium in search-and-matching models: TU and NTU.

Schulhofer-Wohl, S. (2006). Negative assortative matching of risk-averse agents with transferable expected utility. *Economics Letters*, 92(3):383–388.

Serfes, K. (2005). Risk sharing vs. incentives: Contract design under two-sided heterogeneity. *Economics Letters*, 88(3):343–349.

Shi, L. (2023). Optimal regulation of noncompete contracts. *Econometrica*, 91(2):425–463.

Shimer, R. and Smith, L. (2000). Assortative matching and search. *Econometrica*, 68(2):343–369.

Shimer, R. and Smith, L. (2001). Nonstationary search. *University of Chicago and University of Michigan mimeo*, 112:122.

Smith, J. P. and Thomas, D. (1998). On the road: Marriage and mobility in malaysia. *The Journal of Human Resources*, 33(4):805–832.

Smith, L. (2006). The marriage model with search frictions. *Journal of Political Economy*, 114(6):1124–1144.

Smith, L. A. (1992). Cross-sectional dynamics in a two-sided matching model. Technical report, M.I.T.

Online Appendix

The online appendix contains the missing proofs for the model variations presented in the main text.

.1 Explicit Search Cost

Theorem 1′ (stationary PAM with explicit search cost, Morgan (1994))**.** *Suppose that both populations' payoffs are supermodular. Then there is positive assortative matching (PAM) at all times in any stationary equilibrium.*

Proof. We prove the contrapositive. Let $x_2 > x_1$ be such that $y_t(x_2) < y_t(x_1)$ (the environment being stationary, this applies to all moments in time). Then for any type $\underline{y} \in (y_t(x_2), y_t(x_1))$ the optimal matching decision implies that $\pi^X(\underline{y}|x_1) < V_t^X(x_1)$, yet $\pi^X(\underline{y}|x_2) \geq V_t^X(x_2)$. Then apply the integral representation of the value-of-search and apply the mimicking argument:

$$\pi^X(\underline{y}|x_1) < \int_0^1 \pi^X(y|x_1)Q_t^X(y|x_1)dy - C_t^X(x_1) \text{ and } \int_0^1 \pi^X(y|x_2)Q_t^X(y|x_1)dy - C_t^X(x_1) \leq \pi^X(\underline{y}|x_2).$$

In the steady state agents always match with a weakly better type than the one rejected initially. Formally, $Q_t^X(y|x_1) = 0$ for all $y < y_t(x_1)$ including $\underline{y}$, and we may adjust the bounds of integration accordingly. Isolating $C_t^X(x_1)$, it follows that

$$\int_{\underline{y}}^1 \pi^X(y|x_1)Q_t^X(y|x_1)dy - \pi^X(\underline{y}|x_1) > \int_{\underline{y}}^1 \pi^X(y|x_2)Q_t^X(y|x_1)dy - \pi^X(\underline{y}|x_2).$$

Since $y_t(x_1) > 0$, agent type x_1's value-of-search exceeds the match payoff from matching with type 0. In effect, type x_1 must almost surely eventually exit the search pool so that $Q_t^X(\cdot|x_1)$ integrates to one. If not it must be that $V_t^X(x_1) = -\infty$, because there is a non-zero probability of incurring an infinite amount of search cost. The preceding inequality thus simplifies to

$$\int_{\underline{y}}^1 \left[\pi^X(y|x_1) + \pi^X(\underline{y}|x_2) - \pi^X(\underline{y}|x_1) - \pi^X(y|x_2)\right] Q_t^X(y|x_1)dy > 0,$$

which can impossibly hold if payoffs are not supermodular. $\square$

Proof of Theorem 2′. Suppose that matching is not assortative, i.e., there exist $x_2 > x_1$ such that $y_t(x_2) < y_t(x_1)$ at some time t. Then for any type $\underline{y} \in (y_t(x_2), y_t(x_1))$ the optimal matching decision implies that $\pi^X(\underline{y}|x_1) < V_t^X(x_1)$, yet $\pi^X(\underline{y}|x_2) \geq V_t^X(x_2)$. As before, an application of the mimicking argument implies that

$$\int_0^1 \pi^X(y|x_1)Q_t^X(y|x_1)dy - C_t^X(x_1) > \pi^X(\underline{y}|x_1) \text{ and } \int_0^1 \pi^X(y|x_2)Q_t^X(y|x_1)dy - C_t^X(x_1) \leq \pi^X(\underline{y}|x_2). \tag{12}$$

Next, define $\overline{y} > \underline{y}$ such that $\pi^X(\overline{y}|x_1) = \pi^X(\underline{y}|x_1) + C_t^X(x_1)$. Such $\overline{y} \in [0,1]$ does exist (for $\pi^X(\underline{y}|x_1) + C_t^X(x_1) \leq V_t^X(x_1) + C_t^X(x_1) \leq \pi^X(1|x_1)$; then conclude using the intermediate value theorem). Due to supermodularity,

$$\pi^X(\overline{y}|x_2) + \pi^X(\underline{y}|x_1) \geq \pi^X(\underline{y}|x_2) + \pi^X(\overline{y}|x_1) \quad \Leftrightarrow \quad \pi^X(\overline{y}|x_2) \geq \pi^X(\underline{y}|x_2) + C_t^X(x_1).$$

It follows that

$$\int_0^1 \pi^X(y|x_1)Q_t^X(y|x_1)dy > \pi^X(\overline{y}|x_1) \quad \text{and} \quad \int_0^1 \pi^X(y|x_2)Q_t^X(y|x_1)dy \leq \pi^X(\overline{y}|x_2). \tag{13}$$

It remains to observe that, as in the steady state, $Q_t^X(\cdot|x_1)$ is a density and integrates to one. Then type x_1 accepts a lottery that is rejected by type x_2. This runs counter to the characterization of log supermodularity in differences in terms of risk preferences and establishes a contradiction. □

Theorem 2″ (non-stationary PAM with explicit search cost and endogenous quits)**.** *Suppose that both populations' payoffs are supermodular, log supermodular and log supermodular in differences. Then there is positive assortative matching (PAM) at all times in any (non-stationary) equilibrium.*

Proof. When there are outside options, there are two stopping rules: match if $\pi^X(y|x) \geq V_t^X(x)$, exit if $0 \geq V_t^X(x)$. Now suppose that matching is not assortative. As before an application of the mimicking argument guarantees that there exist $x_2 > x_1$ and $\underline{y}$ such that (12) holds with the exception that $y \mapsto Q_t^X(y|x_1)$ need not integrate to one. Then consider two normalizations: let, as in the proof of Theorem 2, $\hat{y}$ be such that $\pi^X(\hat{y}|x_1)\int_0^1 Q_t^X(y|x_1)dy = \pi^X(\underline{y}|x_1)$. Clearly $\hat{y} > \underline{y}$. Then $\pi^X(\hat{y}|x_2)\int_0^1 Q_t^X(y|x_1)dy \geq \pi^X(\underline{y}|x_2)$ because payoffs are log supermodular. Next, let, as in the proof of Theorem 2′, $\overline{y}$ be such that $\pi^X(\overline{y}|x_1)\int_0^1 Q_t^X(y|x_1)dy = \pi^X(\hat{y})\int_0^1 Q_t^X(y|x_1)dy + C_t^X(x_1)$. Clearly $\overline{y} > \hat{y}$ because search cost are non-negative. Then $\pi^X(\overline{y}|x_2)\int_0^1 Q_t^X(y|x_1)dy \geq \pi^X(\hat{y}|x_2)\int_0^1 Q_t^X(y|x_1)dy + C_t^X(x_1)$ because payoffs are supermodular. Given both normalizations, inequalities (13) continues to hold which (as before) upsets the posited ranking of risk preferences. □

Proof of Proposition 3′. We follow the same steps as in Counterexample 2 in the proof of Proposition 3. For an identical set-up, type x_3's stationary HJB equation writes as $c = n\big[h(n)\pi(x_3|x_3) + (1-h(n))\pi(x_1|x_3) - V_0^n(x_3)\big]$. The indifference condition continues unchanged as $V_0^n(x_3) = \pi(x_1|x_3)$. One then deduces algebraically that $h(n)$ is well-defined (as given by $h(n)n = c/(\pi(x_3|x_3) - \pi(x_1|x_3)))$.

Next, consider the explicit characterization of the value-of-search preceding time 0 and succeeding the time t_0^n at which agent type x_3 is indifferent between accepting and rejecting agent type x_2:

$$\begin{aligned} V_{t_0^n}^n(x_3) &= \int_{t_0^n}^{0} ne^{-n(\tau - t_0^n)} d\tau\, \pi(x_3|x_3) + \Big(1 - \int_{t_0^n}^{0} ne^{-n(\tau-t_0^n)} d\tau\Big) \pi^X(x_1|x_3) \\ &\qquad - c\bigg\{ \int_{t_0^n}^{0} (\tau - t_0^n) ne^{-n(\tau-t_0^n)} d\tau + (-t_0^n)\Big(1 - \int_{t_0^n}^{0} ne^{-n(\tau-t_0^n)} d\tau\Big) \bigg\} \\ &= \underbrace{(1 - e^{nt_0^n})}_{\equiv Q_{t_0^n}^n(x_3)} \pi(x_3|x_3) + \underbrace{e^{nt_0^n}}_{\equiv Q_{t_0^n}^n(x_1)} \pi(x_1|x_3) - c\frac{1 - e^{nt_0^n}}{n}. \end{aligned}$$

Here, as before, we used that by construction $V_0^n(x_3) = \pi(x_1|x_3)$ and that during time interval $(t_0^n, 0)$ the high type only matches with fellow high type agents. The indifference condition is $V_{t_0^n}^n(x_3) = \pi(x_2|x_3)$ which implies that for all $n \in \mathbb{N}$

$$\pi(x_2|x_3) = Q_{t_0^n}^n(x_3)\pi(x_3|x_3) + Q_{t_0^n}^n(x_1)\pi(x_1|x_3) + o(1).$$

Beginning from time t_0^n, agent type x_2 is accepted by agent type x_3, and thus faces identical match opportunities as an agent type x_3. Accordingly, x_2 can mimic the type x_3's match probabilities so that

$$V_{t_0^n}^n(x_2) \geq Q_{t_0^n}^n(x_3)\pi(x_3|x_2) + Q_{t_0^n}^n(x_1)\pi(x_1|x_2) + o(1)$$

We then show that PAM does not occur at time t_0^n for n sufficiently large. Or, we show that $V_{t_0^n}^n(x_2) > \pi(x_2|x_2)$. If not, it must hold that

$$\pi(x_2|x_2) \geq Q_{t_0^n}^n(x_3)\pi(x_3|x_2) + Q_{t_0^n}^n(x_1)\pi(x_1|x_2) + o(1).$$

Since $Q_{t_0^n}^n(\cdot)$ is a probability measure, i.e. $Q_{t_0^n}^n(x_1) + Q_{t_0^n}^n(x_3) = 1$, this runs counter Arrow-Pratt's characterization of risk preferences whereby strict LsubD implies that x_3 is strictly more risk averse than x_2: $\pi(x_2|x_3) \leq Q_{t_0^n}^n(x_3)\pi(x_3|x_3) + Q_{t_0^n}^n(x_1)\pi(x_1|x_3)$ implies $\pi(x_2|x_2) < Q_{t_0^n}^n(x_3)\pi(x_3|x_2) + Q_{t_0^n}^n(x_1)\pi(x_1|x_2)$ Then taking the limit $n \to \infty$ establishes the desired contradiction. □

.2 Strategic Match Destruction

.2.1 Opportunistic Match Destruction

We first discuss the example depicted in Figure 5.

Set-up Consider symmetric populations where there are low types x_L and high types x_H. Both types share identical (pizzazz) preferences and value a match with a high (low) type according to flow payoff $f(x_H|x) = \rho\pi(x_H|x)$ ($f(x_L|x) = \rho\pi(x_L|x)$). Initially, there are only low types. These meet at rate $\lambda_t(x_L|x) = 1$. High types are never met, $\lambda_t(x_H|x) = 0$ for all $t < 1$. Over time $t > 1$, high types enter and crowd out meetings with low types: then the rate at which one meets the low type is $\frac{1}{t}$ and the rate at which one meets the high type is $1 - \frac{1}{t}$. Further parameter values are $\rho = 1/9$, $\pi(x_L|x) = 1$ and $\pi(x_H|x) = 1.17$.

Result The assortative pairs (high and high or low and low) always match upon meeting. Non-assortative pairs (high and low) sometimes match, but never permanently. Match outcomes upon meeting are characterized by three disjoint time intervals: During time interval $[1, t_0)$ the low type rejects a match with a high type agent. During time interval $[t_0, t_1)$ the low and the high type match upon meeting. At time t_1 high type agents destroy an existing match with a low type agent. We note that the inability of the high type agent to commit not to destroy the match at time t_1 hurts both the high and the low type: in the example, both types would be better off if, during time interval $[1, t_0)$, low and high type agents could match permanently.

Proof. Claim 1: As time tends to infinity, the high type is better off remaining unmatched than being matched with the low type. To see this, we compute the value-of-search for the high type when only matching with other high types. The Hamilton-Jacobi-Bellman (HJB) equation is expressed as $\rho V_t(x_H) = \big(1 - \frac{1}{t}\big)(\pi(x_H|x_H) - V_t(x_H)) + \dot{V}_t(x_H)$ with limiting condition $\lim_{t\to\infty} V_t(x_H) = \frac{\pi(x_H|x_H)}{1+\rho}$. The unique solution to this differential equation is

$$V_t(x_H) = \frac{\pi(x_H|x_H)}{1+\rho}\big(1 - \frac{\rho}{1+\rho}\frac{1}{t}\big).$$

For the given parameter values it then holds that $\lim_{t\to\infty} V_t(x_H) = \frac{1}{1+\rho}\frac{f(x_H|x_H)}{\rho} > \frac{f(x_L|x_H)}{\rho}$.

Claim 2: At time $t = 1$ the high type is better off matched with the low type than not matched. Suppose not. Then any arbitrarily short match duration δ is undesirable to the high type: $\Pi_{1,1+\delta}(x_L|x_H) < V_1(x_H)$. Plugging in the functional form of $\Pi_{t,t'}(x_L|x_H)$, dividing by δ and taking the limit $\delta \to 0$ then yields

$$\dot{V}_1(x_H) < \rho V_1(x_H) - f(x_L|x_H).$$

Moreover, note that if the high type rejects the low type at time $t = 1$, then he will do so at all times. Consequently, the value-of-search is as given in Claim 1. In particular, match opportunities with other high types are improving, and so $\dot{V}_1(x_H) > 0$. The inequality above then implies that $V_1(x_H) > f(x_L|x_H)/\rho$. Given the parameter values considered, this cannot hold:

$$V_1(x_H) = \frac{\pi(x_H|x_H)}{1+\rho}\big(1 - \frac{\rho}{1+\rho}\big) = 1.17\big(\frac{9}{10}\big)^2 = 0.9477 < 1 = f(x_L|x_H)/\rho.$$

We deduce that there exists a unique time $t_1 > 1$ at which point the high type no longer matches with the low type. Should the high type be matched with a low type, that match will be destroyed and both re-enter the search pool.

Claim 3: Time t_1 at which the high type destroys a match with a low type satisfies $\rho\pi(x_L|x_H) = \big(1 - \frac{1}{t_1}\big)\big(\pi(x_H|x_H) - V_{t_1}(x_H)\big)$. To derive this expression, note that t_1 satisfies

$$\Pi_{t_1,t_1+\delta}(x_L|x_H) < V_{t_1}(x_H) \qquad \text{and} \qquad \Pi_{t_1-\delta,t_1}(x_L|x_H) > V_{t_1-\delta}(x_H)$$

for $\delta > 0$ small. Then plug in the functional form of $\Pi_{t,t'}(x_L|x_H)$, divide by δ, take the limit $\delta \to 0$, recall the HJB equation and note that $\lambda_{t_1}(x_H) = 1 - \frac{1}{t_1}$.

The remainder of the example, notably the low type's optimal match acceptance strategy, is solved numerically via backward induction. As it turns out, for t close to t_1 the low type will accept a match with the high type, but not so for t close to 1. The earliest time t_0 at which the low type accepts the high type is the solution to $\Pi_{t_0,t_1}(x_H|x_L) = V_{t_0}(x_L)$. $\square$

Proposition 4 states that under opportunistic match destruction, there exist meeting rates and an initial search pool such that, for some time t, PAM does not occur. In light of the above example, proving this statement is straightforward when considering the entire set of (both temporary and permanent) matches created. In what follows, we prove a more stringent and less intuitive result: under opportunistic match destruction, PAM fails even among the set of permanent matches. We do so to not only highlight the extent to which opportunistic match destruction alters sorting, but also facilitate a comparison of our finding with the one concerning negotiated match destruction (Theorem 4).

The following proof shows that under opportunistic match destruction PAM can fail even among permanent matches.

Proof of Proposition 4. We show, by constructing a counterexample, that the set of model primitives for which PAM fails is non-empty.

Set up: Suppose that payoffs are strictly log supermodular. Consider a symmetric population. We choose types, $x_3 > x_1$, so that $x \mapsto f(y|x)$ is continuous at x_3,[48] and type $x_2 : x_1 < x_2 < x_3$ arbitrarily close to x_3. Similarly, let ϵ be an arbitrarily small, positive number. The meeting rate is given by:

- for $t_3 \leq t$, $\lambda_t(x_3) = \frac{\rho f(x_2|x_3)}{f(x_3|x_3)-f(x_2|x_3)}$ and $\lambda_t(x_1) = \lambda_t(x_2) = \epsilon$.
- for $t_2 \leq t < t_3$, $\lambda_t(x_1) = \lambda_t(x_3) = \lambda_t(x_2) = \epsilon$.
- for $t_1 \leq t < t_2$, $\lambda_t(x_1) = L$ with L an arbitrarily large number and $\lambda_t(x_3) = \lambda_t(x_2) = \epsilon$.
- for $t < t_1$, $\lambda_t(x_3) = \lambda$ with λ sufficiently large and $\lambda_t(x_1) = \lambda_t(x_2) = \epsilon$.

The timing is such that $t_3 - t_1 = -\frac{1}{\rho}\log(1 - \frac{f(x_1|x_1)}{f(x_3|x_1)}) - \epsilon$.

Intuition:

Before solving the model, we provide a heuristic description of equilibrium play to fix ideas. In the last phase of the game (after time t_3), the high type remains matched with the middle type while rejecting the low type. In contrast, the middle type matches with the low type. In anticipation of the high type destroying the match, the low type (for most of t_1 to t_3) is willing to match with both low and middle types but rejects high types. In the initial phase of the game (prior to t_1), there exists a time interval during which middle types reject other middle types while high types accept middle types; middle types are choosier because they can afford to match with low types in the subsequent phase, whereas high types cannot.

Resolution of the model: We solve the model by backward induction.

Starting from t_3 onwards, the economy enters a steady state, where high types exhibit indifference between remaining with middle types or departing (leading to their decision to stay). In effect, the stationary HJB equation and the parameter value of the meeting rate, $\lambda_{t_3}(x_3)$, imply that

$$V_{t_3}(x_3) = \frac{\lambda_{t_3}(x_3)f(x_3|x_3) + \epsilon f(x_2|x_3)}{\rho(\rho + \lambda_{t_3}(x_3) + \epsilon)} = \frac{f(x_2|x_3)}{\rho}.$$

Hence high types reject low types and destroy the match if previously matched to a low type. By contrast, the middle types do not quit when matched with a low type. Indeed,

[48] In light of our assumption that payoffs are of bounded variation in own type and therefore continuous almost everyhwere (cf. Footnote 21), almost any x_3 is an admissible choice.

they accept to stay with a low type if

$$V_{t_3}(x_2) = \frac{\lambda_{t_3}(x_3) f(x_3|x_2) + \epsilon f(x_2|x_2) + \epsilon f(x_1|x_2)}{\rho(\rho + \lambda_{t_3}(x_3) + 2\epsilon)} \leq \frac{f(x_1|x_2)}{\rho}.$$

To see that the latter inequality holds for ϵ sufficiently small, replace λ_{t_3} by its value. Then, letting epsilon go to zero, the inequality amounts to

$$\frac{f(x_3|x_2)}{f(x_1|x_2)} \leq \frac{f(x_3|x_3)}{f(x_2|x_3)}.$$

This holds due to the strict log supermodularity of payoffs.

From t_1 *to* t_2, any meeting that does not involve a low type agent occurs with vanishingly small probability. Devoid of better options, all types accept a temporary match with a low-type agent. However, anticipating that high types will terminate the match at time t_3, low types refuse to match with high types at time t_1 and for some time thereafter because the short-term benefit of being with a high type is outweighed by the long-term advantages of searching for a low type agent.

Formally, note that the low type payoff of being temporarily matched with a high type, $\Pi_{t_1,t_3}(x_3|x_1)$, satisfies

$$\Pi_{t_1,t_3}(x_3|x_1) = \int_{t_1}^{t_3} e^{-\rho(\tau - t_1)} f(x_3|x_1) + o(1)$$

where $o(1)$ tends to zero as ϵ tends to zero. And

$$\int_{t_1}^{t_3} e^{-\rho(\tau - t_1)} f(x_3|x_1) < \frac{f(x_1|x_1)}{\rho} \text{ is equivalent to } t_3 - t_1 < -\frac{1}{\rho} \log(1 - \frac{f(x_1|x_1)}{f(x_3|x_1)}).$$

Rejection of high types follows, because, for L sufficiently large, the low type's value-of-search at time t_1, $V_{t_1}(x_1)$, can be arbitrarily close to $\frac{f(x_1|x_1)}{\rho}$. And the second inequality holds due to the choice of parameter values. We deduce that there exists time $t_1' \in (t_1, t_2)$ before which the low type accepts matches with middle types, but not with high types.

Before t_1, by construction, the high type's value-of-search is decreasing. For λ sufficiently large, the high type exclusively matches with other high types so that $V_{-\infty}(x_3) \geq \frac{f(x_2|x_3)}{\rho}$. And due to the few meetings with non-low types during (t_1, t_3), $\frac{f(x_2|x_3)}{\rho} \geq V_{t_1}(x_3)$. Hence, there exists a time $t_0 < t_1$ from which point onward high type agents (permanently) accept a match with a middle type x_2.

We then write down the high-types' value-of-search. Note in particular that during time interval (t_1, t_1') high types match with vanishingly small probability. And during

$[t_1', t_2)$ they match temporarily until time t_3 with low types. Then it holds that

$$\frac{f(x_2|x_3)}{\rho} = V_{t_0}(x_3) = \int_{t_0}^{t_1} e^{-\rho(\tau-t_0)}\lambda e^{-\lambda(\tau-t_0)}\frac{f(x_3|x_3)}{\rho}d\tau$$
$$+ e^{-\lambda(t_1-t_0)}\int_{t_1'}^{t_3} e^{-\rho(\tau-t_0)}Le^{-L(\tau-t_1')}\Big(\int_{\tau}^{t_3} e^{-\rho(s-\tau)}f(x_1|x_3)ds\Big)\,d\tau$$
$$+ e^{-\lambda(t_1-t_0)}e^{-L(t_2-t_1')}e^{-\rho(t_3-t_0)}V_{t_3}(x_2) + o(1),$$

where, as before, $o(1)$ tends to zero as ϵ tends to zero. We then show that at time t_0 PAM among permanent matches is upset; intermediate type agents x_2 do not match permanently with one another. To see this, observe that the intermediate type x_2's value-of-search writes as

$$V_{t_0}(x_2) = \int_{t_0}^{t_1} e^{-\rho(\tau-t_0)}\lambda e^{-\lambda(\tau-t_0)}\frac{f(x_3|x_2)}{\rho}d\tau + e^{-\lambda(t_1-t_0)}\int_{t_1}^{t_2} e^{-\rho(\tau-t_0)}Le^{-L(\tau-t_1)}\frac{f(x_1|x_2)}{\rho}d\tau$$
$$+ e^{-\lambda(t_1-t_0)}e^{-L(t_2-t_1)}V_{t_3}(x_2) + o(1).$$

This reflects that type x_2 during time interval $[t_0, t_1)$ mostly matches with types x_3, during time interval $[t_1, t_2)$ mostly with types x_1, during time interval $[t_2, t_3)$ mostly with no-one, and during time interval $[t_3, \infty)$ mostly with types x_3. Then consider the following mimicking strategy: x_2 matches, as above, with x_1 during time interval $[t_1, t_2)$; but at t_3 the middle type x_2 breaks the match and returns to the search pool, thereby encountering the same situation as the unmatched high type x_3 following t_3:

$$V_{t_0}(x_2) \geq \int_{t_0}^{t_1} e^{-\rho(\tau-t_0)}\lambda e^{-\lambda(\tau-t_0)}\frac{f(x_3|x_2)}{\rho}d\tau$$
$$+ e^{-\lambda(t_1-t_0)}\int_{t_1}^{t_2} e^{-\rho(\tau-t_0)}Le^{-L(\tau-t_1)}\Big(\int_{\tau}^{t_3} e^{-\rho(s-\tau)}f(x_1|x_2)ds\Big)d\tau + e^{-\lambda(t_1-t_0)}e^{-\rho(t_3-t_0)}V_{t_3}(x_2) + o(1)$$
$$= \frac{\lambda f(x_3|x_2)}{\rho(\rho+\lambda)}(1-e^{-(\rho+\lambda)(t_1-t_0)}) + e^{-(\rho+\lambda)(t_1-t_0)}\frac{Lf(x_1|x_2)}{\rho}\int_{t_1}^{t_2}\big(e^{-\rho(\tau-t_1)} - e^{-\rho(t_3-t_1)}\big)e^{-L(\tau-t_1)}d\tau$$
$$+ e^{-\lambda(t_1-t_0)}e^{-\rho(t_3-t_0)}V_{t_3}(x_2) + o(1).$$

However, in comparison to high types, the middle type's advantage are matches with low types that, unlike for the high type, are also initiated with positive probability during time interval $[t_1, t_1')$. Then

$$V_{t_0}(x_3) - V_{t_0}(x_2) \leq \frac{\lambda}{\rho(\rho+\lambda)}(1-e^{-(\rho+\lambda)(t_1-t_0)})\big(f(x_3|x_3) - f(x_3|x_2)\big)$$

$$+ e^{-\lambda(t_1-t_0)}e^{-\rho(t_3-t_0)}\big(V_{t_3}(x_3)-V_{t_3}(x_2)\big)$$
$$- e^{-(\rho+\lambda)(t_1-t_0)}\frac{Lf(x_1|x_2)}{\rho}\Big(\frac{1-e^{-(\rho+L)(t_1'-t_1)}}{\rho+L} - e^{-\rho(t_3-t_1)}\frac{1-e^{-L(t_1'-t_1)}}{L}\Big) + o(1).$$

In the limit, as $x_2 \uparrow x_3$, both $f(x_3|x_2) \to f(x_3|x_3)$ and $V_{t_3}(x_2) \to V_{t_3}(x_3)$. We deduce that there exists a constant $c > 0$ (equal to the last term of the previous expression and corresponding to the middle type's advantage) so that $\lim_{x_2\uparrow x_3} V_{t_0}(x_3) - V_{t_0}(x_2) \leq -c < 0$. This proves that a middle type x_2 that is sufficiently close to x_3 can impossibly accept a fellow middle type x_2 at time t_0. For otherwise $V_{t_0}(x_2) \leq \frac{f(x_2|x_2)}{\rho}$. But then, noting that $V_t(x_3) = \frac{f(x_2|x_3)}{\rho}$, it holds that $\frac{f(x_2|x_3)-f(x_2|x_2)}{\rho} \leq V_{t_0}(x_3) - V_{t_0}(x_2)$ where the left-hand-side tends to zero as $x_2 \uparrow x_3$. However, in light of the above this would imply that $-c \geq 0$, which establishes the desired contradiction.

□

.2.2 Negotiated Match Destruction

Proof of Lemma 2. Suppose by contradiction that the higher-ranked x_2 prefers the lower-ranked type x_1's match duration with y. There are two possible cases: (i) x_1 and y match temporarily or (ii) x_1 and y match permanently. In both cases the following observation, owing to payoff monotonicity, will be helpful: if y prefers to match with x_1 for longer, then y also prefers to match with x_1 for longer:

$$\Pi^Y_{t,t+\boldsymbol{t}_2}(x_1|y) \geq \Pi^Y_{t,t+\boldsymbol{t}_1}(x_1|y) \quad \Rightarrow \quad \Pi^Y_{t,t+\boldsymbol{t}_2}(x_2|y) > \Pi^Y_{t,t+\boldsymbol{t}_1}(x_2|y) \qquad \text{for all } \boldsymbol{t}_2 > \boldsymbol{t}_1.$$

Then consider case (i): if x_1 and y match temporarily, then both must find matching temporarily more desirable than no match. And so y also finds matching temporarily with x_2 more desirable than no match. Following the suggested protocol, this means that x_2 can also match temporarily with y if so desired. Envy can impossibly prevail. Finally consider case (ii): if x_1 and y match permanently, then y must prefer matching permanently with x_1 over any other match duration. And so y must prefer matching permanently with x_2 over any other match duration. And so x_2 can also match permanently with y if so desired. Envy can impossibly prevail. □

Proof of Theorem 4. We first show that whenever x_2 enters into a permanent match with y, it cannot be the case that a lower ranked x_1 rejects to permanently match with y. To begin with, note that since the match between x_2 and y is permanent, it holds that $\Pi^X_t(y|x_2) = f^X(y|x_2)/\rho$; since x_2 was willing to match with y, it holds that $V^X_t(x_2) \geq \Pi^X_t(y|x_2)$. Next, a mimicking argument applies: by committing to identical match creation and destruction rates as x_1, x_2 can mimic the distribution over match flow payoffs enjoyed by x_1. Since, due to envy-freeness, this was initially in x_2's choice set, such mimicking must result in a lower value of search: write x_1's value-of-search as

$$V_t^X(x_1) = \int_0^1 f^X(y|x_1)Q_t(dy|x_1),$$

then mimicking ensures that the following is a lower bound on x_2's value-of-search:

$$V_t^X(x_2) \geq \int_0^1 f^X(y|x_2)Q_t(dy|x_1).$$

Then suppose by contradiction that x_1 rejects a permanent match with y, yet x_2 accepts such a match. If so, $V_t^X(x_1) > f^X(y|x_1)/\rho$ yet $V_t^X(x_2) \leq f^X(y|x_2)/\rho$ as noted above. This gives the familiar comparison over lotteries discussed in the proof of Theorem 2.

Why does this result imply that there is PAM among permanent matches? Clearly, if x_1, y_2 and x_2, y_1 agree to match permanently at time t, so do x_2 and y_2. And the preceding ensures that if x_2 matches permanently with y_1, then x_1 would agree to match permanently with y_1. Symmetrically, if y_2 matches permanently with x_1, then y_1 would agree to match permanently with x_1. In sum, if (x_1, y_2) and (x_1, y_2) are agreeable permanent matches, then so are (x_1, y_1) and (x_2, y_2). □