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Simple Allocation with Correlated Types

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Simple Allocation with Correlated Types*

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Abstract

An object is allocated among a number of agents. The optimal allocation depends on the agents' information about their peers, but each agent wants the object for themself. Monetary transfers are unavailable. We consider mechanisms where it is a dominant strategy to report truthfully. We show that deterministic mechanisms do not generally suffice for implementation and optimality, and that anonymous mechanisms cannot meaningfully elicit information. However, there are simple mechanisms—*jury mechanisms*—that are optimal when there are three or fewer agents, approximately optimal in symmetric environments with many agents, and the only deterministic mechanisms satisfying a relaxed anonymity notion. In a jury mechanism, each agent is either a *juror* or a *candidate*. The jurors decide which candidate wins the object, but jurors never win.

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1 Introduction

We consider environments where an object is allocated among a number of agents. The efficient allocation depends on how the agents evaluate their peers, but monetary transfers are not used to elicit this information. A number of environments fit this description:

(1) A group has to elect one of its members to a prestigious post. The group as whole benefits from selecting a qualified candidate, and each agent knows the qualities of their friends in the group. Monetary transfers would naturally be excluded in such an election.

(2) A community of households has to distribute a good among its members. Each member can vouch for the needs and valuations of their friends or neighbors. If some members are financially constrained, it may be infeasible or undesirable to have members compete for the good via bids.

(3) A funding agency splits a budget across researchers. Each researcher can evaluate others in their field. If all parties are risk neutral, the allocated share of the budget can be interpreted as the probability of being allocated the object. Additional monetary transfers would be self-defeating.

In these environments, asking the agents straightforwardly who "should" get the object does not guarantee satisfactory outcomes. In particular, if agents are primarily concerned with their own winning chances, they may exaggerate their individual qualities instead of impartially disclosing their peer information.

To better understand good allocation rules, we take a mechanism design approach and consider the following model. Each agent wants to win the object and is indifferent to which of the others wins. Allocating to an agent generates a social value. The agents have private information about these values—their *types*. We model peer information by allowing for an arbitrary joint distribution of types and values. Hence an agent's type may be informative about the types and values of all others.

We study mechanisms for maximizing the expected value of the allocation. In a mechanism, each agent is asked to report their type. We focus on mechanisms where truthfully reporting one's type is a dominant strategy; that is, we focus on dominant-strategy incentive-compatible (DIC) mechanisms. For the assumed preferences of the agents, DIC requires that one's report never influences one's own winning probability.

Let us highlight some of the differences to existing models (a detailed review follows later). Alon et al. (2011) and Holzman and Moulin (2013) consider DIC mechanisms (there called strategyproof or impartial) where the agents nominate one another to win the object. These nominations do not arise from some ground truth. By contrast, we fix a general joint distribution of types and values. This lets us study mechanisms where, say, two agents can share their private information and form a consensus about which of the others to nominate. Other work considers settings where non-monetary instruments for screening the agents are available, but where the agents have no peer information (for example, Ben-Porath et al., 2014, 2019).

We contribute two results demonstrating the difficulty of designing "simple" mechanisms for this problem: deterministic DIC mechanisms are not without loss, and anonymous DIC mechanism cannot meaningfully elicit information. We further contribute three positive results on so-called *jury mechanisms*. These mechanisms, described in detail below, solve the problem with three agents, are approximately optimal in symmetric environments with many agents, and are the only deterministic DIC mechanisms satisfying a relaxed notion of anonymity. Let us elaborate.

For each agent, there is a trade-off between allocating to the agent and using the agent's peer information. This trade-off arises since, on the one hand, DIC demands that a change in an agent's type does not affect that agent's own winning probability, but, on the other hand, the change in the type reveals information about the values from allocating to the others.

Optimally resolving this trade-off may require the use of stochastic mechanisms that cannot be implemented by randomizing over deterministic ones. That is, the set of DIC mechanisms may admit stochastic extreme points, and these can be uniquely optimal. Stochastic extreme points exist if and only if there are at least four agents and the type spaces are not "too small." The typical view in the literature is that one should use mechanisms that can be implemented by randomizing over deterministic ones (for example, Chen et al., 2019; Pycia and Ünver, 2015). We find that doing so is not generally without loss in the present problem.

Our next result is that all anonymous DIC mechanisms must ignore the reports of the agents. Here, anonymity means that all agents can make the same reports and that an agent's winning probability does not change when one permutes the reports of the others. We view anonymity as the familiar axiom from social choice theory that no agent play a special role in determining the chosen social alternative; that is, in determining who wins the object. As such, anonymity helps reduce the complexity of the mechanism, protects agents' privacy when evaluating their peers, and ensures that agents have the same rights as voters. Our negative result also sheds new light on a characterization due to Holzman and Moulin (2013) and Mackenzie (2015) of a slightly different notion of anonymity.

Our positive results concern the following class of mechanisms. In a *jury mechanism*, each agent is either a *juror* or a *candidate*. The allocation only depends on the reports of the jurors, and the object is always allocated to a candidate. Given that jurors cannot win, all jury mechanisms are DIC.

If there are three agents, then all DIC mechanisms are randomizations over deterministic jury mechanisms. In particular, a deterministic jury mechanism is optimal. This generalizes a known result for deterministic DIC mechanisms due to Holzman and Moulin (2013). Our key insight is that in the three-agent case all DIC mechanisms are actually randomizations over deterministic ones.

Next, we identify a condition on the environment under which deterministic jury mechanisms are approximately optimal with many agents. By "approximately optimal" we mean that the difference in expected values between an optimal deterministic jury mechanism and an optimal DIC mechanism vanishes as the number of agents diverges. The condition on the environment is that agents are exchangeable in terms of supplying information about the vector of values. Intuitively, when agents are exchangeable, increasing their number relaxes the aforementioned trade-off. In particular, there is essentially no loss from ignoring the reports of those agents who are sometimes allocated the object—this is the defining property of a jury mechanism.

For the last result, we consider a relaxed notion of anonymity—*partial anonymity*. Whereas the earlier notion of anonymity demands that an agent's winning probability be invariant with respect to *all* permutations of the others, partial anonymity only considers permutations of those agents that in the given mechanism *actually* influence the agent's winning probability. We show that all deterministic partially anonymous DIC mechanism are jury mechanisms.

The paper is organized as follows. We next discuss related work (Section 2) and present the model (Section 3). In Section 4, we introduce jury mechanisms and present the results for the three- and many-agent cases. In Section 5, we characterize when stochastic extreme points exist. In Section 6, we study anonymous mechanisms, presenting the two notions and the associated characterizations side-by-side. We conclude by discussing open questions (Section 7). All omitted proofs are in

Appendix A. Supplementary material is collected in Appendices B and C.

2 Related literature

Holzman and Moulin (2013) study axioms for peer nomination rules. In such a rule, agents nominate one another to receive a prize. Their central axiom—*impartiality*—is equivalent to DIC when each agent cares only about their own winning probability. As Holzman and Moulin note, many of their axioms have no obvious counterparts in a model with abstract types. Most relevant for us is their notion and characterization of anonymity, as well subsequent results due to Mackenzie (2015, 2020). We discuss the differences to our characterization in detail in Section 6.4.[1]

Alon et al. (2011) initiated a literature on optimal DIC mechanisms (there called *strategyproof* mechanisms) in a model where each agent nominates a subset of the others, and the aim is to select an agent nominated by many. Mechanisms are ranked according to approximation ratios[2] rather than according to expected values, and this leads to qualitatively different optimal mechanisms. For example, while jury mechanisms can be optimal in our model, the 2-*partition mechanism* of Alon et al. (2011), which is a natural analogue of jury mechanisms, is not optimal in their model.[3,4]

See Olckers and Walsh (2022) for a survey of the literature following Holzman and Moulin (2013) and Alon et al. (2011). Olckers and Walsh also report on some related empirical studies.

Other work in mechanism design focuses on non-monetary instruments for eliciting information For example, in the aforementioned paper of Ben-Porath et al. (2014),

[1]Further contributions to the literature following Holzman and Moulin (2013) include Edelman and Por (2021), Tamura (2016), and Tamura and Ohseto (2014). See also de Clippel et al. (2008).

[2]Given $\alpha \in [0, 1]$, a mechanism has an *approximation ratio* of α if it guarantees a fraction α of some benchmark value. The guarantee is computed across all realizations of the type profile; that is, across all possible approval sets. The benchmark value at a particular realization is the maximal number of approvals across agents.

[3]The 2-partition mechanism randomly splits the agents into two subsets, and then selects an agent from the first subset with the most approvals from agents in the second subset. Alon et al. (2011, Theorem 4.1) show that the 2-partition mechanism has an approximation ratio of $\frac{1}{4}$. Fischer and Klimm (2015) present a mechanism that achieves the strictly higher and optimal ratio of $\frac{1}{2}$.

[4]Further contributions to this literature include Aziz et al. (2016, 2019), Bjelde et al. (2017), Bousquet et al. (2014), Lev et al. (2021), and Mattei et al. (2020). See also Caragiannis et al. (2019, 2021), who consider additive approximations rather than approximation ratios.

the agents' types can be verified at a cost.[5] The typical assumption in this literature is that the agents do not have information about their peers. Most relevant for us are papers that study how a *Bayesian* incentive-compatible mechanism may use agents' peer information to incentivize truthtelling (Bloch et al., 2022; Kattwinkel, 2019; Kattwinkel and Knoepfle, 2021; Kattwinkel et al., 2022). The idea is that when agents have information about their peers, one can detect lies by cross-checking the agents' reports. We observe that the *dominant-strategy* incentive-compatible mechanisms that we consider do not use peer information in this manner. While DIC thus shuts down a screening channel, it leads to mechanisms that are far simpler for the agents to play. Relatedly, the fundamental insights of Crémer and McLean (1985, 1988) and McAfee and Reny (1992) on mechanisms *with transfers* do not apply here.

The papers of Baumann (2018) and Bloch and Olckers (2021, 2022) study related settings but focus on different questions. For instance, Bloch and Olckers (2022) study whether it is possible to reconstruct the ordinal ranking of agents from their reports when agents prefer a high rank.

We also contribute to the literature on the gap between stochastic and deterministic mechanisms[6] by fully characterizing when deterministic DIC mechanisms suffice for describing the set of DIC mechanisms in the present model. Methodologically, we show that here the existence of stochastic extreme points can be understood via a graph-theoretic result due to Chvátal (1975). We elaborate in Appendix B.

3 Model

A single indivisible object is to be allocated to one of n agents, where $n \geq 2$. For each agent i, let Ω_i be a finite set of reals representing the possible social values from allocating to agent i, and let Θ_i be a finite set representing agent i's possible private types. Let $\Omega = \times_{i=1}^{n} \Omega_i$ and $\Theta = \times_{i=1}^{n} \Theta_i$. Values and types are distributed according to a joint distribution μ over $\Omega \times \Theta$. At all type profiles, agent i strictly prefers winning the object to not winning it; agent i is indifferent to which of the others is

[5] See Epitropou and Vohra (2019), Erlanson and Kleiner (2019), and Li (2020) for further work with costly verification. Other examples of non-monetary instruments include promises of future allocations (Guo and Hörner, 2021), costly signaling (Chakravarty and Kaplan, 2013; Condorelli, 2012), allocative externalities (Bhaskar and Sadler, 2019; Goldlücke and Tröger, 2020), or ex-post punishments (Li, 2020; Mylovanov and Zapechelnyuk, 2017).

[6] See, for example, Budish et al. (2013), Chen et al. (2019), Jarman and Meisner (2017), Pycia and Ünver (2015), and Rivera Mora (2022).

allocated the object.

In a *(direct) mechanism*, each agent reports a type, and then the object is allocated to one of the agents according to some lottery. Formally, a mechanism is a function $\varphi\colon \Theta \to [0,1]^n$ satisfying $\sum_{i=1}^n \varphi_i = 1$. Here $\varphi_i\colon \Theta \to [0,1]$ denotes the winning probability of agent i. Since the object is allocated to one of the agents, these probabilities sum to 1. The requirement that the object is always allocated keeps with some earlier work (for example, Alon et al. (2011) and Holzman and Moulin (2013)). In Appendix B, we discuss mechanisms that do not always allocate.

A mechanism φ is *dominant-strategy incentive-compatible (DIC)* if truthfully reporting one's type is a dominant strategy. For the assumed preferences of the agents, a mechanism is DIC if and only if one's report never affects one's own winning probability.

To see the previous point in detail, let $u_i(\theta)$ denote the payoff to an agent i when i is allocated the object at a type profile θ. We normalize i's payoff when not allocated the object to 0, and we assume $u_i > 0$. DIC for a mechanism φ requires that all $i, \theta_i, \theta_i', \theta_{-i}$, and θ_{-i}' satisfy $u_i(\theta_i, \theta_{-i})\varphi_i(\theta_i, \theta_{-i}') \geq u_i(\theta_i, \theta_{-i})\varphi_i(\theta_i', \theta_{-i}')$. Since $u_i > 0$ and since θ_i and θ_i' are arbitrary, we must have $\varphi_i(\theta_i, \theta_{-i}') = \varphi_i(\theta_i', \theta_{-i}')$. That is, agent i's report never affects φ_i. Observe that nothing in this argument changes if $u_i < 0$. Hence we can equally model cases where some agents prefer not to be allocated the object.

We evaluate a DIC mechanism φ via the expected value of the allocation, which is given by $\mathbb{E}_{\omega,\theta}\left[\sum_{i=1}^n \varphi_i(\theta)\omega_i\right]$. When we say a DIC mechanism is *optimal*, we mean it maximizes the expected value among all DIC mechanisms. The Revelation Principle implies that DIC mechanisms are without loss: if a mechanism can be implemented in some dominant-strategy equilibrium of some game, then it is DIC.

Lastly, we define the following: A mechanism is *deterministic* if it maps to a subset of $\{0,1\}^n$. A mechanism is *stochastic* if it is not deterministic.

4 Jury mechanisms

In this section, we focus on the following class of mechanisms.

Definition 1. A mechanism φ is a *jury mechanism* if for all agents i we have the following: if the mechanism is non-constant in agent i's report, then agent i never wins, meaning $\varphi_i = 0$.

Given a jury mechanism, we refer to an agent as a *juror* if the mechanism is non-constant in their report. The set of jurors is called the *jury*, and the remaining agents are called *candidates*. All jury mechanisms are DIC since jurors never win.

The most natural jury mechanisms are those that allocate to the top candidate conditional on the jurors' reports. That is, when the set of jurors is J and jurors report types $(\theta_i)_{i \in J}$, the object is allocated to one of the candidates in

$$\arg\max_{k \in \{1,\dots,n\} \setminus J} \mathbb{E}_{\omega_k}[\omega_k | (\theta_i)_{i \in J}].$$

Assuming a common prior, this mechanism would be implemented by having the jurors share their private information via cheap-talk messages, update their beliefs about the candidates, and then award the object to the top candidate given their shared posterior belief. (For our proofs, however, it is convenient to allow the jurors to select a suboptimal candidate.)

A priori, all agents in the model are candidates for winning and suppliers of information. Jury mechanisms are special since the roles of candidates and jurors are assigned *before* the agents are consulted. There are more complicated mechanisms where an agent's "role" varies across type profiles, and we shall encounter such mechanisms later. As such, it is remarkable that there are situations where jury mechanisms are (approximately) optimal, as we discuss next.

4.1 Jury mechanisms solve the three-agent case

Theorem 4.1. *Let $n \leq 3$. A mechanism is DIC if and only if it is a convex combination of deterministic jury mechanisms. In particular, there is an optimal DIC mechanism that is a deterministic jury mechanism.*

With three agents, a jury mechanism admits at most one juror who deliberates between the other two. Therefore, all DIC mechanisms with three agents can be implemented by nominating a juror (according to some distribution over the set of agents), and then asking the juror to pick one of the others as a winner of the object. Optimally, the information of at least two of the agents is ignored. (With only two agents, all DIC mechanisms are constant.)

In the remainder of this subsection, we explain the steps in the proof of Theorem 4.1. We begin with a known result (Holzman and Moulin, 2013, Proposition 2.i).

Lemma 4.2. *If $n \leq 3$, then all deterministic DIC mechanisms are jury mechanisms.*

In the language of Section 5 of Holzman and Moulin (2013), a deterministic DIC mechanism is an *impartial award rule*. Their Proposition 2.i implies that, if $n \leq 3$, then in each impartial award rule there is at most one agent whose report influences the allocation, and this influential agent never wins. Such a rule is a jury mechanism.[7]

To the best of our knowledge, Lemma 4.2 has so far been limited to deterministic DIC mechanisms. We now close the gap to stochastic ones.

Lemma 4.3. *If $n \leq 3$, then all DIC mechanisms are convex combinations of deterministic DIC mechanisms.*

Lemma 4.3 completes the proof of Theorem 4.1. Indeed, Lemmata 4.2 and 4.3 immediately imply that all DIC mechanisms are convex combinations of deterministic jury mechanisms. Since the expected value is a linear function of the mechanism, at least one deterministic jury mechanism must be optimal.

To prove Lemma 4.3 we consider the extreme points of the set of DIC mechanisms. A routine argument shows that the set of DIC mechanisms is convex and compact (as a subset of Euclidean space). Hence, by the Krein-Milman theorem (Aliprantis and Border, 2006, Theorem 7.68), the set is given by the convex hull of its extreme points.

We show that all stochastic DIC mechanisms fail to be extreme points. Specifically, given an arbitrary stochastic DIC mechanism φ we construct a non-zero function f such that $\varphi + f$ and $\varphi - f$ are two other DIC mechanisms. To understand this construction, recall that a stochastic mechanism is one where, for at least one type profile, at least one agent enjoys an interior winning probability. Since the object is always allocated, some other agent must also enjoy an interior winning probability at the same profile. The function f represents a shift of a small probability mass between these two agents. This shift should be consistent with DIC (since we want $\varphi + f$ and $\varphi - f$ to be DIC), and hence we have to shift masses at multiple type profiles. What makes the construction of f difficult is that changing one agent's type may change which of the others enjoys an interior winning probability. Our argument

[7] Holzman and Moulin (2013) note that the result is essentially due to Kato and Ohseto (2002), who study pure exchange economics. For a discussion of this relationship, we refer to Section 1.4 of Holzman and Moulin (2013).

thus intuitively leans on there only being three agents. Indeed, we shall later see that the argument does not go through with four or more agents.

4.2 Approximate optimality of jury mechanisms

In this subsection, we identify environments in which jury mechanisms are approximately optimal if the number n of agents is large. As suggested in the introduction, DIC creates a tension between allocating to an agent and using the agent's peer information. This tension becomes easier to resolve with many agents. Indeed, we intuit that many DIC mechanisms become approximately optimal as $n \to \infty$. The insight of the upcoming result is that this includes the DIC mechanisms that resolve the tension in the most straightforward way—jury mechanisms.

The following example conveys the basic idea.

Example 1. For each agent i, the value ω_i of allocating to i depends on some common component s and some private component t_i. Specifically, for some function $\hat{\omega}_i$ we have $\omega_i = \hat{\omega}_i(s, t_i)$ with probability 1. The agents observe their private components, which are independently and identically distributed across agents and independent of s. All agents observe s. (So, agent i's type is $\theta_i = (s, t_i)$.) Let φ be an arbitrary DIC mechanism for these n agents. Now suppose a new agent $n+1$, who also observes the common component s, joins the group. Agent $n+1$ may observe some additional information, but this will not be relevant. We claim that there is a jury mechanism that only uses agent $n+1$ as a single juror and that does as well as φ. Note that, by ignoring the reports of agents 1 to n, the information contained in the public component s is not lost. The only information that is potentially lost is the first n agents' knowledge of their private components $t_1, \ldots, t_n$. Each agent i's private component t_i is informative only about i's own value (by independence). However, DIC of the original mechanism φ implies that t_i could not have been used to determine i's own allocation. Thus one does not actually lose any information when ignoring the reports of agents 1 to n.

The main result of this section generalizes the previous example as follows. Under an assumption on the distribution of types and values, an arbitrary DIC mechanism with n agents can be replicated by a jury mechanism when additional agents are around. If values remain bounded in n, an implication is that the loss from using an optimal jury mechanism vanishes as $n \to \infty$.

We introduce new notation to accommodate the growing number of agents. The agents share a common finite type space ($\Theta_1 = \Theta_i$ for all i). The prior distribution of values and types is now a Borel-probability measure μ on $\times_{i\in\mathbb{N}}(\Omega_i \times \Theta_i)$,where each Ω_i is a finite set of reals.[8]

The following assumption captures the idea that if i, j, and k are three distinct agents, then i and j have access to the same sources of information about ω_k.

Assumption 1. For all $n \in \mathbb{N}$, all $i \in \{1, \dots, n\}$, and all $\omega_i \in \Omega_i$, we have the following: Conditional on the value of agent i being equal to ω_i, the distribution of $(\theta_j)_{j\in\{1,\dots,n\}\setminus\{i\}}$ is invariant with respect to permutations of $\{1, \dots, n\} \setminus \{i\}$.

We are not assuming that i and j have the same information as k about ω_k. For example, in Example 1, the common component is the only information that i and j have about ω_k, but agent k actually observes ω_k.

When there are n agents (meaning that mechanisms only consult and allocate to the first n agents), let V_n denote the expected value from an optimal DIC mechanism. Let V_n^J denote the expected value from a jury mechanism with n agents that is optimal among jury mechanisms with n agents.

Theorem 4.4. *Let Assumption 1 hold. For all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $V_n \leq V_{n+m}^J$. If, additionally, the sequence $\{V_n\}_{n\in\mathbb{N}}$ is bounded,[9] then $\lim_{n\to\infty}(V_n - V_n^J) = 0$.*

In plain words, if m new agents are added to the group, a jury mechanism with $n+m$ agents does as well as an with an arbitrary DIC mechanism with n agents. The proof shows this claim for a jury mechanism that has the new m agents as jurors, and the old n agents as candidates, and where $m = n$. That is, a jury mechanism with the desired properties exists as soon as the number of agents is doubled. Depending on the exact distribution μ, a much smaller number of new agents may be needed; in Example 1, one new agent suffices.

Assumption 1 is stronger than what we really need. It suffices if, informally speaking, for all groups of agents $\{1, \dots, n\}$ there eventually comes a disjoint group of agents that is at least as well informed as $\{1, \dots, n\}$ about each other. Assumption 2 in Appendix A.1.2 formalizes this idea.

[8]Each of the finite sets Ω_i and Θ_i is equipped with the discrete metric. The product $\times_{i\in\mathbb{N}}(\Omega_i \times \Theta_i)$ is equipped with the product metric.

[9]A sufficient condition for boundedness of the sequence $\{V_n\}_{n\in\mathbb{N}}$ is that the values ω_i are bounded across agents. For example, suppose with μ-probability 1 we have $\omega_i \in [0, 1]$ for all $i \in \mathbb{N}$.

Remark 4.5. Theorem 4.4 does not assert that DIC mechanisms become approximately ex-post optimal conditional on the type profile. In Example 1, the only information that is used in the allocation is the common component. The common component need not pin down the entire profile of values.

5 Random allocations

In this section, we show that it typically does not suffice to consider deterministic mechanisms. This fact sheds light on the fundamental economic forces of the model and has practical implications for implementation, as we explain below.

5.1 Stochastic extreme points

One of way constructing a stochastic DIC mechanism is by randomizing over deterministic ones; that is, by taking a convex combination of deterministic DIC mechanisms. In this case, one of the deterministic mechanisms from the combination must generate a weakly higher expected value than the stochastic mechanism.

We therefore ask whether all stochastic DIC mechanisms can be represented as convex combinations of deterministic ones; that is, whether all extreme points of the set of DIC mechanisms are deterministic. In a nutshell, this is true if and only if there are at most three agents or the agents' type spaces are small.

Theorem 5.1. *All extreme points of the set of DIC mechanisms are deterministic if and only if at least one of the following is true:*

(1) There are at most three agents; that is, we have $n \leq 3$.

(2) All agents have at most two types; that is, for all i we have $|\Theta_i| \leq 2$.

(3) At least $(n-2)$-many agents have a degenerate type; that is, we have

$$|\{i \in \{1, \dots, n\} \colon |\Theta_i| = 1\}| \geq n - 2.$$

We already know from Lemma 4.3 that (1) is sufficient for all extreme points to be deterministic. Sufficiency of (2) is related to a generalization of the well-known Birkhoff-von Neumann theorem; sufficiency of (3) is economically and technically uninteresting, but must be included for completeness.[10] As for the other direction:

[10] The reader may wonder whether one can prove sufficiency of (1) to (3) by viewing the set of DIC

we momentarily give an example of a stochastic extreme point. The general claim that a stochastic extreme point exists when (1) to (3) all fail follows readily by extending this example.

An implication of Theorem 5.1 is that deterministic DIC mechanisms do not suffice for optimality. Indeed, for each extreme point there exists at least one distribution of types and values where the extreme point is the unique optimal DIC mechanisms.[11]

We do not expect stochastic extreme points to closely resemble mechanisms observed in practice. The literature discusses several issues. First, to reduce complexity and opaqueness, it is appealing to implement a mechanism by randomizing over deterministic mechanisms, announcing the selected mechanism, and only then collecting the agents' reports (see, for example, Pycia and Ünver (2015)). A stochastic extreme point is precisely a DIC mechanism that cannot be implemented in this way.[12] Second, to implement a stochastic extreme point, the designer must commit to honoring the outcome of a stochastic process (see, for example, Chen et al. (2019)). A commitment issue arises if the agents' collective information identifies a unique qualified agent but the mechanism nevertheless promises to flip a coin between this agent and a less qualified one.

Despite the above points, it may be acceptable to randomize if this happens "rarely" or is used to break ties between "similar" agents. As it happens, the optimality of stochastic extreme points is not limited to such cases. We next present an example where a stochastic extreme point is uniquely optimal. This stochastic

mechanisms as the set of solutions to a linear system of inequalities, checking for total unimodularity of the constraint matrix, and then invoking the Hoffman-Kruskal theorem (Korte and Vygen, 2018, Theorem 5.21). In the mechanism design literature, this approach is discussed in Pycia and Ünver (2015), for example. Here the approach works for the case where all type spaces are binary; our proof uses a result which can itself be derived from the Hoffman-Kruskal theorem. However, in the difficult case with three agents, the constraint matrix is not generally totally unimodular (see Appendix C.3).

[11]The argument is as follows. The set of DIC mechanisms is a polytope in Euclidean space that does not depend on the distribution. All extreme points of the polytope are exposed. Since all linear functionals on this polytope can be represented via some distribution, the claim follows. See Appendix C.1 for the formalities.

[12]In fact, in our model, stochastic extreme points cannot be implemented via any dominant-strategy equilibrium of any deterministic *indirect* mechanism. See Appendix C.2. We note, however, a result of Rivera Mora (2022) implying the following (for our model): *Given an arbitrary DIC direct mechanism, there is an* ex-post *equilibrium of a deterministic indirect mechanism that implements the given DIC direct mechanism.* In this ex-post equilibrium, the agents play mixed strategies that emulate the randomization on the part of the given DIC mechanism. These mixed strategies do not generally form a dominant-strategy equilibrium.

extreme point "frequently" randomizes between "dissimilar" agents.

5.2 An example of a stochastic extreme point

There are four agents, and their types are as follows:

$$\Theta_1 = \{\ell, r\}, \quad \Theta_2 = \{u, d\}, \quad \Theta_3 = \{f, c, b\}, \quad \Theta_4 = \{0\}. \tag{5.1}$$

Figure 1 shows (among other things that are not yet relevant) the type profiles of agents 1, 2, and 3; the degenerate type of agent 4 is omitted. The types of agents 1, 2, and 3 span a three-dimensional hyperrectangle. (Mnemonically, their types mean left, right, up, down, front, center, and back.) Each edge of the hyperrectangle represents a set of type profiles along which exactly one agent's type is changing. Hence DIC requires that the winning probability of this agent be constant along the edge. We identify such an edge by a pair (i, θ_{-i}), where i indicates the agent whose type is changing, and θ_{-i} indicates the fixed types of the others.

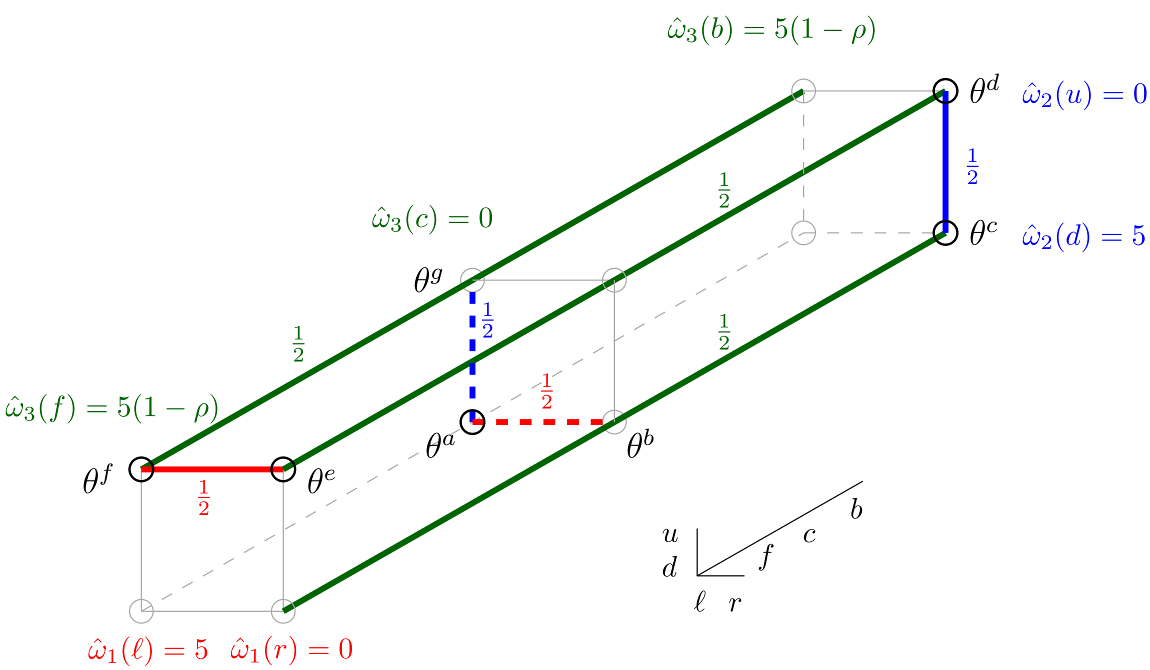


Figure 1: The set of types of agents 1, 2, and 3. The probabilities $\frac{1}{2}$ attached to the edges of the hyperrectangle represent the relevant values of the mechanism φ^*. The values from the allocation are as defined in (5.5). The distribution μ assigns probability $\frac{1}{5}$ to the profiles $\{\theta^a, \theta^c, \theta^d, \theta^e, \theta^f\}$. All other profiles have probability 0.

Let $\Theta^* = \{\theta^a, \theta^b, \theta^c, \theta^d, \theta^e, \theta^f, \theta^g\}$ be the set of labeled type profiles in Figure 1; these are the profiles

$$\begin{aligned} &\theta^a = (\ell, d, c, 0), \quad \theta^b = (r, d, c, 0), \quad \theta^c = (r, d, b, 0), \\ &\theta^d = (r, u, b, 0), \quad \theta^e = (r, u, f, 0), \quad \theta^f = (\ell, u, f, 0), \\ &\theta^g = (\ell, u, c, 0). \end{aligned} \tag{5.2}$$

Let V^* denote the set of bold edges in Figure 1 that connect the profiles in Θ^*; these are the edges

$$V^* = \{(1, \theta^a_{-1}), (3, \theta^c_{-3}), (2, \theta^c_{-2}), (3, \theta^e_{-3}), (1, \theta^e_{-1}), (3, \theta^f_{-3}), (2, \theta^a_{-2})\}.$$

Our candidate stochastic extreme point φ^* is defined as follows (see Figure 1): For all $i \in \{1, 2, 3\}$ and $\theta \in \Theta$, let

$$\varphi^*_i(\theta) = \begin{cases} \frac{1}{2}, & \text{if } (i, \theta_{-i}) \in V^*, \\ 0, & \text{otherwise.} \end{cases}$$

Further, for all $\theta \in \Theta$ let $\varphi^*_4(\theta) = 1 - \sum_{i \in \{1,2,3\}} \varphi^*_i(\theta)$. In plain words, at all profiles in Θ^*, exactly two bold edges of the hyperrectangle intersect at the profile; the mechanism φ^* randomizes evenly between the two agents of these edges. All remaining probability mass is assigned to agent 4. It is easy to verify from Figure 1 that φ^* is a well-defined DIC mechanism.

Further below we specify values Ω and a distribution μ such that φ^* is the unique optimal DIC mechanism. This implies that φ^* is an extreme point of the set of DIC mechanisms. Since the proof for uniqueness is somewhat involved, we next present a simple self-contained argument showing that φ^* is an extreme point.

Let φ be a DIC mechanism that receives non-zero weight in a convex combination that equals φ^*. We show $\varphi = \varphi^*$. For all profiles $\theta \in \Theta^*$, there are exactly two agents i and j such that (i, θ_{-i}) and (j, θ_{-j}) both belong to V^*; these are the two bold edges of the hyperrectangle that intersect at θ. Hence at θ the mechanism φ^* randomizes evenly between i and j. Since φ is part of a convex combination that equals φ^*, it follows that at θ the mechanism φ only randomizes between i and j, meaning $\varphi_i(\theta) = 1 - \varphi_j(\theta)$. Since φ is DIC, repeatedly applying this observation shows:

$$
\begin{aligned}
\varphi_1(\theta^a) = 1 - \varphi_3(\theta^c) = \varphi_2(\theta^c) &= 1 - \varphi_3(\theta^e) \\
&= \varphi_1(\theta^e) \\
&= 1 - \varphi_3(\theta^f) = \varphi_2(\theta^a) = 1 - \varphi_1(\theta^a).
\end{aligned} \tag{5.3}
$$

In particular, we have $\varphi_1(\theta^a) = 1 - \varphi_1(\theta^a)$, implying $\varphi_1(\theta^a) = \frac{1}{2}$. Hence all probabilities in (5.3) equal $\frac{1}{2}$. Hence φ agrees with φ^* at all profiles in Θ^*. By inspecting $\Theta \setminus \Theta^*$, we may easily convince ourselves that φ and φ^* also agree on $\Theta \setminus \Theta^*$. Thus φ^* is an extreme point.

We next construct an environment in which φ^* is uniquely optimal. We could do so by invoking a separating hyperplane theorem. However, this would be unsatisfying since we would gain no intuition for why randomization helps or for whether φ^* is uniquely optimal in a restricted class of environments. We shall gain both by considering environments in which values are *privately known*, in the following sense: for all agents i, the value of allocating to i is pinned down by a function $\hat{\omega}_i$ that depends only on θ_i.

We can describe an environment with privately known values by specifying a distribution μ over type profiles and, for all agents i, a function $\hat{\omega}_i \colon \Theta_i \to \mathbb{R}$ that governs the value of allocating to i. Our candidate distribution μ is given by (see Figure 1)

$$
\forall_{\theta \in \Theta}, \quad \mu(\theta) = \begin{cases} \frac{1}{5}, & \text{if } \theta \in \{\theta^a, \theta^c, \theta^d, \theta^e, \theta^f\} \\ 0, & \text{else.} \end{cases} \tag{5.4}
$$

Our candidates for $\hat{\omega}_1, \dots \hat{\omega}_4$ are parametrized by $\rho \in [0, \frac{1}{2}]$ and given by

$$
\begin{aligned}
\hat{\omega}_1(r) = \hat{\omega}_2(u) = \hat{\omega}_3(c) &= 0 \\
\hat{\omega}_1(\ell) = \hat{\omega}_2(d) &= 5 \\
\hat{\omega}_3(f) = \hat{\omega}_3(b) &= 5(1 - \rho) \\
\hat{\omega}_4 &= 0.
\end{aligned} \tag{5.5}
$$

Proposition 5.2. *The mechanism φ^* is an optimal DIC mechanism if and only if $\rho \in [0, \frac{1}{2}]$, and it is uniquely optimal if and only if $\rho \in (0, \frac{1}{2})$.*

In the introduction, we intuited that there is a trade-off between allocating to an agent and using that agent's information about others. In the present example, this trade-off involves agent 3 and depends on ρ.

To gain an intuition for the trade-off and the result, consider the case $\rho = 0$. Allocating to agent 3 is now ex-post optimal at *all except one* of the five profiles in the support of μ. Indeed, one optimal DIC mechanisms is the constant one that always allocates to agent 3. The mechanism φ^* is another optimal mechanism for $\rho = 0$, which is intuitively explained by agent 3's type being informative: if $\theta_3 = c$ realizes, the type profile must be θ^a, where θ^a is the unique type profile in the support of μ at which allocating to agents 1 or 2 is better than allocating to agent 3. The mechanism φ^* indeed allocates to agents 1 and 2 at θ^a.

Since ρ decreases the value from allocating to agent 3, it is now intuitive that φ^* does strictly better than always allocating to agent 3 for small but strictly positive values of ρ. In the formal proof, most of our effort goes towards showing that φ^* is in fact uniquely optimal for small but strictly positive values of ρ. The idea is that, among all DIC mechanisms that are optimal for $\rho = 0$, the mechanism φ^* is the unique one minimizing agent 3's overall winning probability.

If we increase ρ further, it eventually becomes optimal to use agent 3 as a source of information and never allocate to agent 3. The critical value turns out to be $\rho = \frac{1}{2}$. The intuition is confirmed by the fact that, if $\rho = \frac{1}{2}$, the following *jury* mechanism with agent 3 as a juror is optimal: if agent 3 reports f, agent 1 wins; if agent 3 reports c, a coin flip determines whether agent 1 or 2 wins; if agent 3 reports b, agent 2 wins.

Proposition 5.2 also helps illustrate the commitment issue discussed in the paragraphs following Theorem 5.1. At the profile θ^e, a coin flip determines whether agent 1 or 3 wins the object. Yet, at this profile, the value from allocating to agent 3 is strictly higher than the value from allocating to agent 1. In fact, a coin is flipped at all type profiles in the support of the distribution. For $\rho \in (0, \frac{1}{2})$, the mechanism designer is indifferent to the outcome of the coin flip at only one of these profiles.

Remark 5.3. Chen et al. (2019) show that, in certain mechanism design problems, given any stochastic mechanism there is a deterministic one that induces the same interim-expected allocations. Since the deterministic mechanism is not guaranteed to be DIC, their result does not contradict the suboptimality of deterministic DIC mechanisms in our model.

Remark 5.4. An alternative approach to showing the existence of a stochastic extreme point uses a graph-theoretic result due to Chvátal (1975), as we explain in Appendix B.2. For a certain graph G that we define in Appendix B.2, Chvátal's theorem implies that all extreme points are deterministic if and only if G is perfect. To be precise, the results of this appendix concern the related problem where the mechanism may dispose the object instead of allocating it to the agents. The associated characterization of extreme points is implied by Theorem 5.1, but not vice versa.

6 Anonymous juries

In this section, we study anonymous DIC mechanisms. Anonymity, formally defined below, is roughly the requirement that any two agents exert the same influence with their reports on the winning probability of any third agent. This is a desirable property as it helps protect the agents' privacy when they evaluate their peers, reduces the complexity of the mechanism, and ensures that the agents have the same voting rights.

We offer two insights. First, all anonymous DIC mechanisms ignore the reports of the agents. Second, we consider a relaxed notion of anonymity—*partial anonymity*—and show that all deterministic partially anonymous DIC mechanisms are jury mechanisms.

Throughout, we assume that the agents share a common type space, meaning $\Theta_1 = \ldots = \Theta_n$. In an equally valid interpretation, we can consider indirect mechanisms where all agents have the same message space and cannot influence their own winning probabilities.

6.1 Notions of anonymity

Anonymity and partial anonymity are defined next. Anonymity requires that, for all k, the winning probability of agent k does not change if one permutes the reports of the agents other than k. Partial anonymity relaxes anonymity as follows: When testing whether k's winning probability is affected by permutations, we only consider permutations of those agents who actually influence agent k. In particular, partial anonymity permits the set of agents who influence k to be a proper subset of

$\{1, \ldots, n\} \setminus \{k\}$.

Definition 2. Let the agents have a common type space. Let φ be a mechanism.

(1) Given i, j, and k that are all distinct, agents i and j are *exchangeable for* k if φ_k is invariant with respect to permutations of i's and j's reports; that is, for all profiles θ and θ' such that θ is obtained from θ' by permuting the types of i and j we have $\varphi_k(\theta) = \varphi_k(\theta')$.

(2) Given distinct i and k, agent i *influences* k if φ_k is non-constant in i's report; that is, there exist type profiles θ and θ' that differ only in i's type and satisfy $\varphi_k(\theta) \neq \varphi_k(\theta')$.

(3) The mechanism is *anonymous* if for all i, j, and k that are all distinct, agents i and j are exchangeable for k.

(4) The mechanism is *partially anonymous* if for all i, j, and k that are all distinct we have the following: if i and j both influence k, then i and j are exchangeable for k.

To state the upcoming characterization of partial anonymity, we also define what we mean by an anonymous jury.

Definition 3. Let the agents have a common type space. A jury mechanism has an *anonymous jury* if all jurors i and j are exchangeable for all agents k.

Remark 6.1. If Assumption 1 holds, then among jury mechanisms it is without loss to use one with an anonymous jury. Indeed, consider the jury mechanism that selects the candidate that is best conditional on the types of the jurors (breaking ties in some fixed order). Under Assumption 1, the identity of the favored candidate does not change when one permutes the jurors' types.

6.2 Anonymous DIC mechanisms ignore all reports

Theorem 6.2. *Let the agents have a common type space. All anonymous DIC mechanisms are constant.*

Note well that anonymity does *not* demand that i and j be exchangeable for i's own winning probability. If anonymity did demand this, the theorem would follow rather trivially from DIC.

The theorem is more subtly related to the requirement that the mechanism always allocates the object, as we explain next. This requirement lets us link the influence that two agents i and j exert on others to the influence that they exert on each other.

More concretely, assume towards a contradiction that at some profile θ agent i can increase φ_j by changing their report from θ_i to some θ_i'. By DIC and since the object is always allocated, this change in i's report decreases $\sum_{k\colon i\neq k\neq j} \varphi_k$. Now consider the profile that is obtained from θ by permuting the reports of i and j. By anonymity, agent j can change their report from θ_i to θ_i' to decrease $\sum_{k\colon i\neq k\neq j} \varphi_k$. Using again that the mechanism is DIC and that the object must be allocated, it follows that the change in agent j's report increases φ_i. In summary, if i can increase j's winning probability at some profile, then j must also be able to increase i's winning probability at a permuted profile. This observation suggests that i and j both win with "high" probability when both report θ_i'. In a deterministic mechanism, where winning probabilities are either 0 or 1, we thus arrive at a contradiction to there being only one object to allocate. We address stochastic mechanisms via a substantially more complex summation over winning probabilities across all pairs (i, j).

Remark 6.3. Theorem 6.2 implies that all DIC mechanisms satisfying the following stronger notion of anonymity are constant: Whenever the set of reports is permuted, then the same permutation is applied to the vector of winning probabilities. This stronger notion captures a sense in which agents are treated equally both as voters and winners.

Remark 6.4. An implication of Theorem 6.2 is that it is impossible to elicit information in environments where anonymity is without loss. Indeed, if the joint distribution of types and values is invariant with respect to all permutations of the agents, then it is without loss to use a DIC mechanism that satisfies the strong notion of anonymity from Remark 6.3. Hence in this case it is without loss to use a constant mechanism.

6.3 Partial anonymity

Theorem 6.2 implies that a non-constant DIC mechanism must admit some asymmetry in how it processes the reports of different agents. This brings us to partial anonymity. We offer the following characterization for deterministic mechanisms.

Theorem 6.5. *Let the agents have a common type space. A mechanism is deterministic, partially anonymous, and DIC if and only if it is a deterministic jury mechanism with an anonymous jury.*

To better understand the theorem, consider how a partially anonymous jury mechanism could fail to admit an anonymous jury. Given agents i and j, partial anonymity is silent on the winning probabilities of those agents k who are influenced by either i or j but not by both. By contrast, anonymity of the jury requires that all candidates are either influenced by all or none of the jurors. Accordingly, most of our effort goes towards proving that, in a deterministic partially anonymous DIC mechanism, if i and j influence *some* third agent k, then i and j influence exactly the same set of agents. Equipped with this fact, we show that the agents can be partitioned into equivalence classes with the following property: two agents in the same class do not influence one another, but influence the same (possibly empty) set of agents outside the class. Lastly, there cannot be multiple classes; indeed, else there is a profile where two distinct classes allocate the object to two distinct agents, which is impossible. The unique class defines an anonymous jury.

6.4 Discussion of Theorems 6.2 and 6.5

We conclude by discussing limitations of Theorems 6.2 and 6.5.

6.4.1 Disposal and randomization

The following definition will be useful: A *mechanism with disposal* is a function $\varphi\colon \Theta \to [0,1]^n$ satisfying $\sum_{i=1}^n \varphi_i \leq 1$. In plain words, this is a mechanism that does not necessarily always allocate the object to the agents. For a mechanism with disposal, DIC and anonymity are defined as above.

The next result shows via an example that Theorem 6.2 does not extend to mechanisms with disposal, and that Theorem 6.5 does not extend to stochastic mechanisms (without disposal).

Proposition 6.6. *Let the agents have a common type space T such that $|T| = 7$.*

(1) If $n = 3$, then the set of DIC mechanisms with disposal admits an extreme point that is stochastic and anonymous.

(2) If $n = 4$, then the set of DIC mechanisms (without disposal) admits an extreme point that is stochastic and partially anonymous.

The extreme point in (1) is non-constant (else it would be a convex combination of deterministic constant mechanisms). The extreme point in (2) is not a jury mechanism (else it would be a convex combination of deterministic jury mechanisms). The idea of the proof is to "symmetrize" the stochastic extreme point φ^* from Section 5.2. See Appendix A.3.3 for the proof and an informal sketch.

6.4.2 Anonymous ballots

Lastly, we discuss the assumption that all agents can make the same reports. Indeed, a third escape route from Theorem 6.2 (besides partial anonymity and disposal) entails message spaces with some inherent asymmetry across agents. This brings us to the results of Holzman and Moulin (2013) and Mackenzie (2015, 2020). They consider DIC mechanisms where agents nominate one another. Let us keep with the terminology of Holzman and Moulin by referring to these mechanisms as *impartial nomination rules*. This is the same mathematical object as a DIC mechanism when each agent i's type space is $\{1, \ldots, n\} \setminus \{i\}$. Their notion of anonymity—*anonymous ballots*—requires that the winning probabilities depend only on the number of nominations received by each agent.[13] Importantly, in a nomination rule agents cannot nominate themselves, and hence they all have distinct message spaces. By contrast, we have assumed that the agents have the same type space. Hence our notion of anonymity neither nests nor is nested by anonymous ballots.

Contrasting Theorem 6.2, there are *non-constant* impartial nomination rules with anonymous ballots. For one example, suppose one of the agents is selected uniformly at random as a juror, following which the juror's nomination determines a winner. See Mackenzie (2015, Theorem 1) for a full characterization of anonymous ballots. Mackenzie's result generalizes Theorem 3 of Holzman and Moulin (2013), who had previously shown that all *deterministic* impartial nomination rules with anonymous ballots are constant.

Mackenzie (2020) shows that impartiality and anonymous ballots are compatible

[13] Equivalently, the allocation is unchanged if one permutes the profile in a way that does not yield self-nominations (Mackenzie, 2015, Lemma 1.1). Mackenzie uses the name *voter anonymity* instead of anonymous ballots.

for deterministic nomination rules with disposal.[14] This parallels our discussion from Section 6.4 and contrasts the aforementioned Theorem 3 of Holzman and Moulin (2013). Mackenzie (2020, Theorem 1) also shows that when agents can nominate themselves, then deterministic impartial nomination rules with anonymous ballots must be constant. This is a special case of our Theorem 6.2 as anonymous ballots with self-nominations is stronger than anonymity.

7 Conclusion

We saw that jury mechanisms are optimal with three agents, and approximately-optimal when there are many exchangeable agents in the sense of Assumption 1. While DIC mechanisms cannot process all reports anonymously, jury mechanisms are the only deterministic partially anonymous DIC mechanisms. Lastly, outside of special cases of the model, the set of DIC mechanisms admits stochastic extreme points.

We conclude by discussing some interesting open problems.

The discussion on stochastic extreme points (Section 5.1) motivates restricting attention to deterministic mechanisms. We observe in Appendix C.4 that finding an optimal deterministic DIC mechanism can be cast as the problem of finding a maximum weight perfect matching in a certain hypergraph. If we relax the requirement that the object is always allocated, the problem can also be cast as finding a maximum weight independent set in another graph. Both of these problems are known to be NP-hard when general (hyper-)graphs and weights are considered. As such, it is interesting to investigate the hardness of the problem for the particular family of (hyper-)graphs that emerge from our model. (All weights can emerge via a suitable choice of the distribution of types and values.) If we include stochastic mechanisms in our search, finding an optimal DIC mechanism is a linear program and hence computationally tractable.

It is naturally interesting to extend the analysis to settings with multiple objects, allocated simultaneously or over many periods.[15] If the mechanism designer

[14] In fact, Mackenzie (2020, Theorem 2) shows that impartiality, anonymous ballots, and some other desirable axioms together characterize supermajority.

[15] See Guo and Hörner (2021) for recent work in this direction with a single agent. The literature following Alon et al. (2011) has also studied settings with multiple objects. Lipnowski and Ramos (2020) and de Clippel et al. (2021) study settings with limited or no commitment.

can commit to future allocations, this should lead to stronger foundations for jury mechanisms. Agents serving as jurors today can be promised a future spot as candidates, which may help justify excluding jurors as potential winners in the present. Alternatively, past winners may be expected to volunteer as jurors in the future.

The problem of finding an optimal composition of the jury is an interesting problem in itself. We expect interesting comparative statics when agents who are likely to have good information are also likely to yield a high value. In the example from the introduction where a group selects a president, say, an agent who is popular with others may be a suitable candidate (being well-liked for their pleasant qualities) but also have good information about others (being well-acquainted with everyone).

An important line of future research concerns optimal DIC mechanisms when agents care about the allocation to their peers. While DIC has different implications in such a model, our results provide insight in at least two cases. Firstly, in situations where agents evaluate their peers, it is seems inherently interesting to use a mechanism where agents cannot influence their individual chances of winning; that is, to impose the *impartiality* axiom of Holzman and Moulin (2013). Secondly, suppose agents have the following lexicographic preferences: each agent i strictly prefers one allocation to another if the former has i winning with strictly higher probability; if two allocations have the same winning probability for i, agent i ranks them according to some type-dependent preference. In some applications, this preference could reasonably capture i's opinion about who is the most deserving winner if it cannot be i themself. In particular, it could coincide with the preference of the mechanism designer. In this case, optimal jury mechanism are ex-post incentive compatible. However, an agent's preferences may also differ from those of the designer. This is plausibly the case when agents are biased in favor of friends or family, biased against minorities, or simply have a different notion of who deserves to win.[16] Fixing a jury of agents, the designer therefore also has to design a voting rule for eliciting the jurors' information.

[16]For example, Alatas et al. (2012), reporting on a field experiment on selecting beneficiaries of aid programs in Indonesian communities, find evidence of nepotism, though the welfare impact may be small relative to other upsides from involving the community in the decision. They also find evidence that community members have a poverty notion that differs from poverty as defined by per capita income. In this sense, if the central government wishes to select beneficiaries on the basis of per capita income, agents indeed hold a different notion of who deserves to win.

Appendices

In Appendices A.1 to A.3, respectively, we present the omitted proofs for Sections 4 to 6, respectively. Appendix B studies the model where the object does not have to be allocated. Appendix C contains results that were previously mentioned in passing.

Appendix A Omitted proofs

A.1 Jury mechanisms

A.1.1 Proof of Lemma 4.3

Proof of Lemma 4.3. If $n = 1$ or $n = 2$, it is easy to verify that all DIC mechanisms are constant. All constant mechanisms are convex combination of deterministic constant mechanisms, proving the claim. In what follows, let $n = 3$. Given an arbitrary stochastic DIC mechanism φ, we will find a non-zero function f such that $\varphi + f$ and $\varphi - f$ are two other DIC mechanisms. This shows that all extreme points of the set of DIC mechanisms are deterministic. Since this set is non-empty, convex and compact as a subset of Euclidean space, the claim follows from the Krein-Milman theorem.

In what follows, we fix a stochastic DIC mechanisms φ. Let us agree to the following terminology. In view of DIC, we drop i's type from φ_i. Given a profile θ, we refer to the equation $\sum_{i \in \{1,2,3\}} \varphi_i(\theta_{-i}) = 1$ as the *feasibility constraint* at profile θ. We refer to (i, θ_{-i}) as the *node of agent i with coordinates θ_{-i}*. Lastly, when we say $\varphi_i(\theta_{-i})$ is interior we naturally mean $\varphi_i(\theta_{-i}) \in (0, 1)$.

Most of the work will go towards proving the following auxiliary claim.

Claim A.1. *There are non-empty disjoint subsets R and B ("red" and "blue") of $\cup_{i \in \{1,2,3\}} (\{i\} \times \Theta_{-i})$ such that all of the following are true:*

(1) If $(i, \theta_{-i}) \in R \cup B$, then $\varphi_i(\theta_{-i})$ is interior.

(2) For all $\theta \in \Theta$, exactly one of the following is true:

(a) There does not exist $i \in \{1, 2, 3\}$ such that $(i, \theta_{-i}) \in R \cup B$.

(b) There exists exactly one $i \in \{1, 2, 3\}$ such that $(i, \theta_{-i}) \in R$, exactly one $j \in \{1, 2, 3\}$ such that $(j, \theta_{-j}) \in B$, and exactly one $k \in \{1, 2, 3\}$ such that $(k, \theta_{-k}) \notin R \cup B$.

Before proving Claim A.1, let us use it to complete the proof of Lemma 4.3. For a number ε to be chosen in a moment, let $f\colon \Theta \to \{-\varepsilon, 0, \varepsilon\}^3$ be defined as follows:

$$\forall_{\theta\in\Theta}, \quad f_i(\theta) = \begin{cases} -\varepsilon, & \text{if } (i, \theta_{-i}) \in R, \\ \varepsilon, & \text{if } (i, \theta_{-i}) \in B, \\ 0, & \text{if } (i, \theta_{-i}) \notin R \cup B. \end{cases}$$

By finiteness of Θ and Claim A.1, if we choose $\varepsilon > 0$ sufficiently close to 0, then $\varphi + f$ and $\varphi - f$ are two DIC mechanisms. Since f is non-zero, it follows that φ is not an extreme point. It remains to prove Claim A.1.

Proof of Claim A.1. Given candidate sets R and B, let us say a profile θ is *uncolored* if it falls into case (2.a) of Claim A.1. A profile *two-colored* if it falls into case (2.a) of Claim A.1. In this terminology, our goal is to construct sets R and B such that all $(i, \theta_{-i}) \in R \cup B$ satisfy $\varphi_i(\theta_{-i}) \in (0, 1)$, and such that all type profiles are either uncolored or two-colored.

Since φ is stochastic, we may assume (after possibly relabelling the agents and types) that there exists a profile θ^0 such that $\varphi_1(\theta_2^0, \theta_3^0)$ and $\varphi_2(\theta_1^0, \theta_3^0)$ are interior.

Let Θ_2° denote the set of types θ_2 for which $\varphi_1(\theta_2, \theta_3^0)$ is interior. Let $\Theta_2^\partial = \Theta_2 \setminus \Theta_2^\circ$. Similarly, let Θ_1° denote the set of types θ_1 such that $\varphi_2(\theta_1, \theta_3^0)$ is interior, and let $\Theta_1^\partial = \Theta_1 \setminus \Theta_1^\circ$. Notice that Θ_1° and Θ_2° are non-empty as, by assumption, agents 1 and 2 are enjoying interior winning probabilities at θ^0.

We consider two cases.

Case 1. Let $\Theta_1^\partial \neq \emptyset$ and $\Theta_2^\partial \neq \emptyset$.

We establish two auxiliary claims.

Claim A.2. *If $\theta_1 \in \Theta_1^\partial$, then $\varphi_2(\theta_1, \theta_3^0) = 0$. Similarly, if $\theta_2 \in \Theta_2^\partial$, then $\varphi_1(\theta_2, \theta_3^0) = 0$. If $(\theta_1, \theta_2) \in (\Theta_1^\circ \times \Theta_2^\partial) \cup (\Theta_1^\partial \times \Theta_1^\circ)$, then $\varphi_3(\theta_1, \theta_2)$ is interior.*

Proof of Claim A.2. Consider the first part of the claim. Let $\theta_1 \in \Theta_1^\partial$. Recalling that Θ_1° is non-empty, let us find a type $\theta_2 \in \Theta_1^\circ$. By definition, $\varphi_1(\theta_2, \theta_3^0)$ is interior. By definition of Θ_1^∂, we also know that $\varphi_2(\theta_1, \theta_3^0)$ must either equal 0 or 1. But it cannot equal 1 since $\varphi_2(\theta_1, \theta_3^0)$ and $\varphi_1(\theta_2, \theta_3^0)$ both appear in the feasibility constraint at the profile $(\theta_1, \theta_2, \theta_3^0)$, and since $\varphi_1(\theta_2, \theta_3^0)$ is interior. Thus $\varphi_2(\theta_1, \theta_3^0) = 0$, as desired.

A similar argument establishes the second claim.

As for the third claim, let $(\theta_1, \theta_2) \in \Theta_1^\circ \times \Theta_2^\partial$. The previous two paragraphs imply that at the profile $(\theta_1, \theta_2, \theta_3^0)$ the winning probability of agent 1 is 0. Moreover, by definition of Θ_1°, the winning probabiltiy of agent 2 is interior. Thus agent 3's winning probability at this profile must be interior, meaning $\varphi_3(\theta_1, \theta_2)$ is interior. A similar argument shows that $\varphi_3(\theta_1, \theta_2)$ is interior whenever (θ_1, θ_2) is in $\Theta_1^\partial \times \Theta_1^\circ$. □

The second auxiliary result is:

Claim A.3. *Let $\theta_3 \in \Theta_3$. If $\theta_2 \in \Theta_2^\circ$, then $\varphi_1(\theta_2, \theta_3)$ is interior. Similarly, if $\theta_1 \in \Theta_1^\circ$, then $\varphi_2(\theta_1, \theta_3)$ is interior.*

Proof of Claim A.3. We will prove the first part of the claim, the second being similar. Thus let $\theta_2 \in \Theta_2^\circ$. By assumption of Case 1, we may find $\theta_1^\partial \in \Theta_1^\partial$ and $\theta_2^\partial \in \Theta_2^\partial$. We make two auxiliary observations.

First, consider the profile $(\theta_1^\partial, \theta_2^\partial, \theta_3^0)$. According to Claim A.2, both agent 1's and agent 2's winning probabilities at this profile equal 0. Thus $\varphi_3(\theta_1^\partial, \theta_2^\partial) = 1$. But $\varphi_3(\theta_1^\partial, \theta_2^\partial)$ and $\varphi_2(\theta_1^\partial, \theta_3)$ both appear in the feasibility constraint at the profile $(\theta_1^\partial, \theta_2^\partial, \theta_3)$. Hence $\varphi_2(\theta_1^\partial, \theta_3) = 0$.

Second, since $\theta_1^\partial \in \Theta_1^\partial$ and $\theta_2 \in \Theta_2^\circ$, we infer from Claim A.2 that $\varphi_3(\theta_1^\partial, \theta_2)$ is interior.

The previous two observations imply that at the profile $(\theta_1^\partial, \theta_2, \theta_3)$ agent 2's winning probability is 0 and that agent 3's winning probability is interior. Hence $\varphi_1(\theta_2, \theta_3)$ is interior, as promised. □

We are ready to define the sets R and B. We assign the following colors (recall the terminology introduced in the paragraph before Claim A.1):

- red to all nodes of agent 1 with coordinates in $\Theta_2^\circ \times \Theta_3$,
- blue to all nodes of agent 3 with coordinates in $\Theta_1^\partial \times \Theta_2^\circ$,
- blue to all nodes of agent 2 with coordinates in $\Theta_1^\circ \times \Theta_3$,
- red to all nodes of agent 3 with coordinates in $\Theta_1^\circ \times \Theta_2^\partial$.

According to Claims A.2 and A.3, all of these nodes are interior. Moreover, all profiles are now either two-colored or uncolored: The profiles in $\Theta_1^\partial \times \Theta_2^\circ \times \Theta_3$ are two-colored via red nodes of agent 1 and blue nodes of agent 3; the profiles in $\Theta_1^\circ \times \Theta_2^\circ \times \Theta_3$ are two-colored via red nodes of agent 1 and blue nodes of agent 2; the profiles in $\Theta_1^\circ \times \Theta_2^\partial \times \Theta_3$ are two-colored via blue nodes of agent 2 and red nodes of 3; and the profiles in $\Theta_1^\partial \times \Theta_2^\partial \times \Theta_3$ are uncolored. ▲

Case 2. Suppose at least one of the sets Θ_1^∂ and Θ_2^∂ is empty. In what follows, we assume that Θ_2^∂ is empty, the other case being analogous (switch the roles of agents 1 and 2).

The assumption that Θ_2^∂ is empty means that $\varphi_1(\theta_2, \theta_3^0)$ is interior for all θ_2. Let Θ_1^* be the set of types θ_1 such that for all $\theta_2 \in \Theta_2$ the probability $\varphi_3(\theta_1, \theta_2)$ is interior. Notice that at this point Θ_1^* may or may not be empty; we will make a case distinction further below.

We first claim that if $\theta_1 \in (\Theta_1 \setminus \Theta_1^*)$, then $\varphi_2(\theta_1, \theta_3^0)$ is interior. Towards a contradiction, suppose this were false for some $\theta_1 \in (\Theta_1 \setminus \Theta_1^*)$. This means that we can find a type $\theta_2 \in \Theta_2$ such that $\varphi_2(\theta_1, \theta_3^0)$ and $\varphi_3(\theta_1, \theta_2)$ both fail to be interior. Recall from the previous paragraph that $\varphi_1(\theta_2, \theta_3^0)$ is interior for all θ_2. Hence at the profile $(\theta_1, \theta_2, \theta_3^0)$ only agent 1 is enjoying an interior winning probability; this is impossible.

Before proceeding further, let us assign the following colors:

- red to all nodes of agent 1 with coordinates in $\Theta_2 \times \{\theta_3^0\}$. These nodes are all interior since Θ_2^∂ is empty.
- blue to all nodes of agent 2 with coordinates in $(\Theta_1 \setminus \Theta_1^*) \times \{\theta_3^0\}$. The previous paragraph implies that these nodes are all interior.
- blue to all nodes of agent 3 with coordinates in $\Theta_1^* \times \Theta_2$. These nodes are all interior by definition of Θ_1^*.

Observe that all profiles in $\Theta_1 \times \Theta_2 \times \{\theta_3^0\}$ are now either two-colored or uncolored.

If Θ_1^* is empty, then the colors assigned above already define sets R and B with the desired properties, completing the proof. Thus suppose Θ_1^* is non-empty.

Let $\theta_3 \in \Theta_3 \setminus \{\theta_3^0\}$ be arbitrary. The fact that we have already assigned blue to the nodes of agent 3 with coordinates $\Theta_1^* \times \Theta_2$ requires us to assign some colors to the nodes of agents 1 or 2 whose 3'rd coordinate is θ_3. In this step, we will not color any further nodes of agent 3. We make a case distinction.

(1) Suppose that for all θ_1 in Θ_1^* the probability $\varphi_2(\theta_1, \theta_3)$ is interior. We assign red to all nodes of agent 2 with coordinates in $\Theta_1^* \times \{\theta_3\}$. This yields a coloring of the profiles in $\Theta_1 \times \Theta_2 \times \{\theta_3^0\}$ with the desired properties: The profiles in $\Theta_1^* \times \Theta_2 \times \{\theta_3\}$ are two-colored via red nodes of agent 2 and blue nodes of 3; the profiles in $(\Theta_1 \setminus \Theta_1^*) \times \Theta_2 \times \{\theta_3\}$ are uncolored.

(2) Suppose there exists $\tilde{\theta}_1 \in \Theta_1^*$ such that $\varphi_2(\theta_1, \theta_3)$ is interior. Given that $\varphi_3(\tilde{\theta}_1, \theta_2)$ is interior for all $\theta_2 \in \Theta_2$ (recall the definition of Θ_1^*), it must be

the case that, for all $\theta_2 \in \Theta_2$, the probability $\varphi_1(\theta_2, \theta_3)$ is interior.

We next claim that $\varphi_2(\theta_1, \theta_3)$ is interior for all $\theta_1 \in (\Theta_1 \setminus \Theta_1^*)$. Suppose this were false for some $\theta_1 \in (\Theta_1 \setminus \Theta_1^*)$. The previous paragraph tells us that $\varphi_1(\theta_2, \theta_3)$ is interior for all θ_2. Thus, if $\varphi_2(\theta_1, \theta_3)$ fails to be interior, then $\varphi_3(\theta_1, \theta_2)$ would have to be interior for all $\theta_2 \in \Theta_2$; this is a contradiction since $\theta_1 \in (\Theta_1 \setminus \Theta_1^*)$.

We now assign red to all nodes of agent 1 with coordinates in $\Theta_2 \times \{\theta_3\}$, and assign blue to all nodes of agent 2 with coordinates in $(\Theta_1 \setminus \Theta_1^*) \times \{\theta_3\}$. The previous two paragraphs imply that all of these nodes are interior. Moreover the profiles in $\Theta_1^* \times \Theta_2 \times \{\theta_3\}$ are two-colored via red nodes of agent 1 and blue nodes of agent 3, and the profiles in $(\Theta_1 \setminus \Theta_1^*) \times \Theta_2 \times \{\theta_3\}$ are two-colored via red nodes of agent 1 and blue nodes of agent 2.

If we apply this case distinction separately to all θ_3 in $\Theta_3 \setminus \{\theta_3^0\}$, this completes the construction of R and B in Case 2. ▲

Cases 1 and 2 together complete the proof of Claim A.1. □

□

A.1.2 Approximate optimality of jury mechanisms

In this part of the appendix, we prove Theorem 4.4. To distinguish a random variable from its realization, we denote the former using a tilde $\sim$. Given a set N of agents, we denote the profile of their types by θ_N, and the set of these profiles by Θ_N. For example, given $i \in N$, $\omega_i \in \Omega_i$, and $\theta_{N\setminus\{i\}} \in \Theta_{N\setminus\{i\}}$, we write $\mu\left(\tilde{\omega}_i = \omega_i, \tilde{\theta}_{N\setminus\{i\}} = \theta_{N\setminus\{i\}}\right)$ to mean the probability of the event that i's value is ω_i and the types of the other agents in N are $\theta_{N\setminus\{i\}}$.

Assumption 2. For all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ with the following property: Denoting $N = \{1, \ldots, n\}$ and $N' = \{n+1, \ldots, n+m\}$, there is a function $g\colon \Theta_{N'} \times \Theta_N \to \mathbb{R}_+$ with the following two properties:

(1) For all $i \in N$, all $\omega_i \in \Omega_i$ and $\theta_{N\setminus\{i\}} \in \Theta_{N\setminus\{i\}}$ we have

$$\begin{aligned} &\mu\left(\tilde{\omega}_i = \omega_i, \tilde{\theta}_{N\setminus\{i\}} = \theta_{N\setminus\{i\}}\right) \\ &= \sum_{\theta_{N'} \in \Theta_{N'}} \sum_{\theta_i \in \Theta_i} g(\theta_{N'}, \theta_{N\setminus\{i\}}, \theta_i)\mu\left(\tilde{\omega}_i = \omega_i, \tilde{\theta}_{N'} = \theta_{N'}\right). \end{aligned} \tag{A.1}$$

(2) For all $\theta_{N'} \in \Theta_{N'}$ we have

$$\sum_{\theta_N \in \Theta_N} g(\theta_{N'}, \theta_N) = 1. \tag{A.2}$$

Lemma A.4. *Assumption 1 implies Assumption 2.*

Proof of Lemma A.4. Let $m = n$. Let $N = \{1, \ldots, n\}$ and $N' = \{n+1, \ldots, 2n\}$, and let $\xi \colon N \to N'$ be a bijection. It is straightforward to verify that the function g defined as follows has the desired properties: For all $(\theta_N, \theta_{N'})$, let $g(\theta_N, \theta_{N'}) = 1$ if for all $i \in N$ the types of i and $\xi(i)$ agree; else, let $g(\theta_N, \theta_{N'}) = 0$. □

Proof of Theorem 4.4. The second part of the claim is immediate from the first. For the first part, let φ be an arbitrary DIC mechanism with n agents. Let $N = \{1, \ldots, n\}$. For this choice of N, we invoke Lemma A.4 to find m and g as in Assumption 2. Let $N' = \{n+1, \ldots, n+m\}$. We define our candidate jury mechanism as follows: For all $i \in N$, let $\psi_i \colon \Theta_{N'} \to \mathbb{R}^n$ be defined by

$$\forall_{\theta_{N'} \in \Theta_{N'}}, \quad \psi_i(\theta_{N'}) = \sum_{\theta_N \in \Theta_N} g(\theta_{N'}, \theta_N) \varphi_i^*(\theta_{N \setminus \{i\}}).$$

For all $i \in N'$, let $\psi_i = 0$. Let $\psi = (\psi_1, \ldots, \psi_m)$.

Notice that ψ only depends on the reports of agents in N'. Since N' is disjoint from N, we can show that ψ is a jury mechanism in the setting with $n + m$ agents by showing that ψ maps to probability distributions over N. It is clear that φ is non-negative (as g and ψ^* are non-negative). To verify that ψ almost surely allocates to an agent in N, we observe that for all profiles $\theta_{N'}$ we have the following (the first equality is by definition of ψ; the second is from the fact that φ^* is a well-defined mechanism when the set of agents is N; the third is from (A.2)):

$$\sum_{i \in N} \psi_i(\theta_{N'}) = \sum_{i \in N} \sum_{\theta_N \in \Theta_N} g(\theta_{N'}, \theta_N) \varphi_i^*(\theta_{N \setminus \{i\}}) = \sum_{\theta_N \in \Theta_N} g(\theta_{N'}, \theta_N) = 1,$$

as desired. We complete the proof by verifying that φ and ψ lead to the same expected value. We write the expected value from φ as follows (the first equality follows from

(A.1); the remaining equalities obtain by rearranging):

$$\begin{aligned}
&\sum_{i\in N}\sum_{\theta_{N\setminus\{i\}}}\sum_{\omega_i}\omega_i\mu\left(\tilde{\omega}_i=\omega_i,\tilde{\theta}_{N-i}=\theta_{N\setminus\{i\}}\right)\varphi_i^*(\theta_{N\setminus\{i\}})\\
=&\sum_{i\in N}\sum_{\theta_{N\setminus\{i\}}}\sum_{\omega_i}\omega_i\sum_{\theta_{N'}}\sum_{\theta_i}g(\theta_{N'},\theta_{N\setminus\{i\}},\theta_i)\mu\left(\tilde{\omega}_i=\omega_i,\tilde{\theta}_{N'}=\theta_{N'}\right)\varphi_i^*(\theta_{N\setminus\{i\}})\\
=&\sum_{i\in N}\sum_{\omega_i}\sum_{\theta_{N'}}\omega_i\mu\left(\tilde{\omega}_i=\omega_i,\tilde{\theta}_{N'}=\theta_{N'}\right)\sum_{\theta_{N\setminus\{i\}}}\sum_{\theta_i}g(\theta_{N'},\theta_{N\setminus\{i\}},\theta_i)\varphi_i^*(\theta_{N\setminus\{i\}})\\
=&\sum_{i\in N}\sum_{\omega_i}\sum_{\theta_{N'}}\omega_i\mu\left(\tilde{\omega}_i=\omega_i,\tilde{\theta}_{N'}=\theta_{N'}\right)\psi_i(\theta_{N'}).
\end{aligned}$$

This last expression is precisely the expected value from ψ. □

A.2 Random allocations

A.2.1 Proof of Proposition 5.2

Proof of Proposition 5.2. To keep calculations readable, it will be convient to adopt the following notation: When a DIC mechanism φ is given, we denote

$$\begin{aligned}
&\varphi_1(\theta^a)=p^{a|b},\quad \varphi_3(\theta^c)=p^{b|c},\quad \varphi_2(\theta^c)=p^{c|d},\quad \varphi_3(\theta^e)=p^{d|e},\\
&\varphi_1(\theta^e)=p^{e|f},\quad \varphi_3(\theta^f)=p^{f|g},\quad \varphi_2(\theta^a)=p^{g|a}.
\end{aligned}$$

The probabilities in the previous display do not fully describe the mechanism, but these are the only ones needed to evaluate the mechanism. For a given value of ρ, we denote the expected value from φ by $V_\rho(\varphi)$. Direct computation shows

$$V_\rho(\varphi)=p^{a|b}+p^{b|c}+p^{c|d}+2p^{d|e}+p^{e|f}+p^{f|g}+p^{g|a}-\rho\left(p^{b|c}+2p^{d|e}+p^{f|g}\right). \quad \text{(A.3)}$$

In particular, $V_\rho(\varphi^*)=4-2\rho$.

We first show that φ^* is uniquely optimal if $\rho\in(0,\frac{1}{2})$. The following auxiliary claim is central.

Claim A.5. *Let φ be a DIC mechanism distinct from φ^*. We have $V_{\frac{1}{2}}(\varphi)\leq V_{\frac{1}{2}}(\varphi^*)$. Further, there exists $\rho_\varphi\in(0,\frac{1}{2})$ such that $\rho\in(0,\rho_\varphi)$ implies $V_\rho(\varphi)<V_\rho(\varphi^*)$.*

Proof of Claim A.5. Inspection of Figure 1 shows that φ must satisfy the following system of inequalities:

$$\begin{aligned} &p^{a|b}+p^{g|a}\le 1, \quad p^{a|b}+p^{b|c}\le 1, \quad p^{c|d}+p^{b|c}\le 1, \quad p^{c|d}+p^{d|e}\le 1, \\ &p^{e|f}+p^{d|e}\le 1, \quad p^{e|f}+p^{f|g}\le 1, \quad p^{g|a}+p^{f|g}\le 1. \end{aligned} \tag{A.4}$$

Turning to the first part of the claim, we have to show $V_{\frac{1}{2}}(\varphi)\le V_{\frac{1}{2}}(\varphi^*)$. Direct computation shows $V_{\frac{1}{2}}(\varphi^*)=3$. Using (A.4), we can bound $V_{\frac{1}{2}}(\varphi)$ as follows.

$$\begin{aligned} V_{\frac{1}{2}}(\varphi) =& p^{a|b}+p^{b|c}+p^{c|d}+2p^{d|e}+p^{e|f}+p^{f|g}+p^{g|a}-\frac{1}{2}\left(p^{b|c}+2p^{d|e}+p^{f|g}\right) \\ =& p^{a|b}+\frac{p^{b|c}}{2}+p^{c|d}+p^{d|e}+p^{e|f}+\frac{p^{f|g}}{2}+p^{g|a} \\ =& \underbrace{p^{a|b}+p^{g|a}}_{\le 1}+\underbrace{\frac{p^{b|c}+p^{c|d}}{2}}_{\le\frac{1}{2}}+\underbrace{\frac{p^{c|d}+p^{d|e}}{2}}_{\le\frac{1}{2}}+\underbrace{\frac{p^{d|e}+p^{e|f}}{2}}_{\le\frac{1}{2}}+\underbrace{\frac{p^{e|f}+p^{f|g}}{2}}_{\le\frac{1}{2}} \\ \le& 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} \\ =& 3. \end{aligned}$$

Hence $V_{\frac{1}{2}}(\varphi)\le V_{\frac{1}{2}}(\varphi^*)$, as promised.

Now consider the second part of the claim. We show the contrapositive: If there exists a sequence $\{\rho_k\}_{k\in\mathbb{N}}$ in $(0,\frac{1}{2})$ that converges to 0 and such that $V_{\rho_k}(\varphi)\ge V_{\rho_k}(\varphi^*)$ holds for all k, then $\varphi=\varphi^*$. Let $\{\rho_k\}_{k\in\mathbb{N}}$ be such a sequence. For all ρ_k, the system (A.4) implies the following upper bound on $V_{\rho_k}(\varphi)$:

$$\begin{aligned} V_{\rho_k}(\varphi) =& \underbrace{p^{a|b}+p^{b|c}}_{\le 1}+\underbrace{p^{c|d}+p^{d|e}}_{\le 1}+\underbrace{p^{d|e}+p^{e|f}}_{\le 1}+\underbrace{p^{f|g}+p^{g|a}}_{\le 1} \\ &-\rho_k\left(p^{b|c}+2p^{d|e}+p^{f|g}\right) \\ \le& 4-\rho_k\left(p^{b|c}+2p^{d|e}+p^{f|g}\right). \end{aligned} \tag{A.5}$$

Since $V_{\rho_k}(\varphi)\ge V_{\rho_k}(\varphi^*)=4-2\rho_k$ and $\rho_k>0$, we find

$$p^{b|c}+2p^{d|e}+p^{f|g}\le 2. \tag{A.6}$$

Further, since $V_{\rho_k}(\varphi)\ge 4-2\rho_k$ holds for all k, taking limits implies $V_0(\varphi)\ge 4$.

Together with the bound in (A.5) we get $V_0(\varphi) = 4$; that is,

$$V_0(\varphi) = p^{a|b} + p^{b|c} + p^{c|d} + p^{d|e} + p^{d|e} + p^{e|f} + p^{f|g} + p^{g|a} = 4 \tag{A.7}$$

Hence (A.4) and (A.7) imply

$$p^{a|b} + p^{b|c} = p^{c|d} + p^{d|e} = p^{d|e} + p^{e|f} = p^{f|g} + p^{g|a} = 1. \tag{A.8}$$

We now bound $V_0(\varphi)$ a second time (the equality is by direct computation; the inequality follows from (A.4)):

$$V_0(\varphi) = p^{a|b} + p^{g|a} + p^{b|c} + p^{c|d} + 2p^{d|e} + p^{e|f} + p^{f|g} \leq 3 + 2p^{d|e}. \tag{A.9}$$

Hence $V_0(\varphi) = 4$ implies $p^{d|e} \geq \frac{1}{2}$. We next claim $p^{d|e} = \frac{1}{2}$. Towards a contradiction, suppose not, meaning $p^{d|e} > \frac{1}{2}$. Hence (A.8) implies $p^{c|d} = p^{e|f} < \frac{1}{2}$. Now, we also know from (A.6) and (A.7) that

$$p^{a|b} + p^{c|d} + p^{e|f} + p^{g|a} \geq 2$$

holds. However, in light of (A.4) we have $p^{a|b} + p^{g|a} \leq 1$, and hence the previous display requires $p^{c|d} + p^{e|f} \geq 1$. This contradicts $p^{c|d} = p^{e|f} < \frac{1}{2}$. Thus $p^{d|e} = \frac{1}{2}$.

Let us now return to the bound derived in (A.9). In view of $p^{d|e} = \frac{1}{2}$ and (A.4), we can infer from (A.9) that $p^{a|b} + p^{g|a} = p^{b|c} + p^{c|d} = p^{e|f} + p^{f|g} = 2p^{d|e} = 1$ holds. Together with (A.8), we find

$$p^{a|b} = 1 - p^{b|c} = p^{c|d} = 1 - p^{d|e} = p^{e|f} = 1 - p^{f|g} = p^{g|a}. \tag{A.10}$$

We already know that $p^{d|e} = \frac{1}{2}$ holds. Hence all probabilities (A.10) must equal $\frac{1}{2}$. This shows that φ agrees with φ^* at all profiles in $\Theta^* = \{\theta^a, \theta^b, \theta^c, \theta^d, \theta^e, \theta^f, \theta^g\}$. By inspecting $\Theta \setminus \Theta^*$, it is now easy to verify that φ and φ^* also agree on $\Theta \setminus \Theta^*$. □

We next use Claim A.5 to show that φ^* is uniquely optimal if $\rho \in (0, \frac{1}{2})$. Let φ be an arbitrary DIC mechanisms distinct from φ^*. Inspection of (A.3) shows that the difference $V_\rho(\varphi) - V_\rho(\varphi^*)$ is an affine function of ρ. That is, there exist reals a_φ and b_φ such that $V_\rho(\varphi) - V_\rho(\varphi^*) = a_\varphi + b_\varphi \rho$ holds for all $\rho \in [0, \frac{1}{2}]$. Let $\rho_\varphi \in (0, \frac{1}{2})$ be as in the conclusion of Claim A.5. If $\rho \in (0, \rho_\varphi)$, the choice of ρ_φ implies $V_\rho(\varphi) < V_\rho(\varphi^*)$, and so we are done. Hence in what follows we assume $\rho \in [\rho_\varphi, \frac{1}{2})$. We distinguish two cases.

(1) If $b_\varphi \leq 0$, then

$$V_\rho(\varphi) - V_\rho(\varphi^*) = a_\varphi + b_\varphi \rho \leq a_\varphi + b_\varphi \frac{\rho_\varphi}{2} = V_{\frac{\rho_\varphi}{2}}(\varphi) - V_{\frac{\rho_\varphi}{2}}(\varphi^*).$$

Now $\frac{\rho_\varphi}{2} \in (0, \rho_\varphi)$ and the choice of ρ_φ imply $V_{\frac{\rho_\varphi}{2}}(\varphi) - V_{\frac{\rho_\varphi}{2}}(\varphi^*) < 0$, and we are done.

(2) If $b_\varphi > 0$, then

$$V_\rho(\varphi) - V_\rho(\varphi^*) = a_\varphi + b_\varphi \rho < a_\varphi + b_\varphi \frac{1}{2} = V_{\frac{1}{2}}(\varphi) - V_{\frac{1}{2}}(\varphi^*).$$

Now Claim A.5 implies $V_{\frac{1}{2}}(\varphi) - V_{\frac{1}{2}}(\varphi^*) \leq 0$, and we are done.

Hence all $\rho \in (0, \frac{1}{2})$ and all DIC mechanisms φ distinct from φ^* satisfy $V_\rho(\varphi) < V_\rho(\varphi^*)$.

It remains to show that φ^* is not uniquely optimal if $\rho \in \{0, \frac{1}{2}\}$, and that φ^* is not optimal if $\rho \notin [0, \frac{1}{2}]$. To that end, recall the constant mechanism and the jury mechanism described in the paragraphs after Proposition 5.2. By direct computation one can show that the constant mechanism or the jury mechanism, respectively, generate the same expected value as φ^* if $\rho = 0$ or $\rho = \frac{1}{2}$, respectively. Thus φ^* is not uniquely optimal if $\rho \in \{0, \frac{1}{2}\}$. Since φ^* is uniquely optimal on $(0, \frac{1}{2})$, and since the expected value is affine in ρ, we conclude that φ^* is not optimal if $\rho \notin [0, \frac{1}{2}]$. □

A.2.2 Proof of Theorem 5.1

Lemma A.6. *If for all agents i we have $|\Theta_i| \leq 2$, then all extreme points of the set of DIC mechanisms are deterministic.*

For the proof, recall the following definitions for a given (simple undirected) graph G with node set V and edge set E. Given a node v, the set of edges which are incident to v is denoted $E(v)$. A *perfect matching* is a function $\psi\colon E \to \{0,1\}$ such that all $v \in V$ satisfy $\sum_{e \in E(v)} \psi(e) = 1$. The *perfect matching polytope* is the set $\{\psi\colon E \to [0,1]\colon \forall_{v \in V}, \sum_{e \in E(v)} \psi(e) = 1\}$.

Proof of Lemma A.6. Let us relabel types such that we have $\Theta_i \subseteq \{0,1\}$ for all i. First, suppose we have $\Theta_i = \{0,1\}$ for all i.

For all DIC mechanisms φ, all agents i and all profiles θ, we may drop i's report from i's winning probability, writing $\varphi_i(\theta_{-i})$ instead of $\varphi_i(\theta)$. Under this convention, we claim that the set of DIC mechanisms is the perfect matching polytope of the graph G that has node set $\{0,1\}^n$ and where two nodes are adjacent if and only if they differ in exactly one coordinate. (This graph is known as the *n-hypercube*.) Indeed, each node of the graph is a type profile θ, and each edge may be identified with a pair of the form (i, θ_{-i}). The set of edges incident to θ is the set $\{(i, \theta_{-i})\}_{i=1}^n$. Hence the constraint $\sum_{e \in E(v)} \psi(e) = 1$ is exactly the constraint that the object be allocated to one of the agents.

Now, the graph G described in the previous paragraph is bi-partite (partition the type profiles (that is, the nodes of G) according to whether the profile has an odd or even number of entries equal to 0). It follows from Theorem 11.4 of Korte and Vygen (2018) that all extreme points of the perfect matching polytope are perfect matchings. All perfect matchings represent deterministic DIC mechanisms. Hence all extreme points of the set of DIC mechanisms are deterministic.

The claim for the general case, where we have $\Theta_i \subseteq \{0,1\}$ for all i, follows from the previous paragraph by viewing a DIC mechanism on Θ as a mechanism on $\{0,1\}^n$ that ignores the reports of those whose type spaces are singletons. □

Lemma A.7. *If $|\{i \in \{1,\dots,n\} \colon |\Theta_i| \geq 2\}| \leq 2$, then all extreme points of the set of DIC mechanisms are deterministic.*

Proof of Lemma A.7. We may assume $n \geq 3$, as otherwise the claim follows from Lemma A.6. We will prove the claim for the case where $|\{i \in \{1,\dots,n\} \colon |\Theta_i| \geq 2\}| = 2$, the other cases being simpler. After possibly relabelling the agents, suppose we have $|\Theta_1| \geq 2$ and $|\Theta_2| \geq 2$. Let φ be a stochastic DIC mechanism. Notice that at all profiles θ where either agent 1 or agent 2 but not both is enjoying an interior winning probability, there must be an agent in $\{3,\dots,n\}$ who is also enjoying an interior winning probability; let i_θ denote one such agent. For a number $\varepsilon > 0$ to be chosen later, consider $f \colon \Theta \to \{-\varepsilon, 0, \varepsilon\}^n$ defined for all θ as follows:

(1) If $\varphi_1(\theta) \in (0,1)$ and $\varphi_2(\theta) \in (0,1)$, let $f_1(\theta) = \varepsilon$, let $f_2(\theta) = -\varepsilon$, and let $f_i(\theta) = 0$ for all $i \notin \{1,2\}$.

(2) If $\varphi_1(\theta) \in (0,1)$ and $\varphi_2(\theta) \notin (0,1)$, let $f_1(\theta) = \varepsilon$, let $f_{i_\theta}(\theta) = -\varepsilon$, and let $f_i(\theta) = 0$ for all $i \notin \{1, i_\theta\}$.

(3) If $\varphi_1(\theta) \notin (0,1)$ and $\varphi_2(\theta) \in (0,1)$, let $f_2(\theta) = -\varepsilon$, let $f_{i_\theta}(\theta) = \varepsilon$, and let

$f_i(\theta) = 0$ for all $i \notin \{2, i_\theta\}$.

Using that, for all θ, agent i_θ has a singleton type space, it is easy to see that $\varphi + f$ and $\varphi - f$ are two DIC mechanisms distinct from φ whenever ε is sufficiently small. Thus φ is not an extreme point. $\square$

Proof of Theorem 5.1. Lemmata 4.3, A.6 and A.7 imply that all extreme points are deterministic if one of the conditions (1) to (3) holds. Now let conditions (1) to (3) all fail. We know from Section 5.2 that a stochastic extreme point exists in the hypothetical situation where $n = 4$ and the set of type profiles is $\hat{\Theta} = \{\ell, r\} \times \{u, d\} \times \{f, c, b\} \times \{0\}$. Since (1) to (3) all fail, we can relabel the agents and types such that agents 1 to 4 have these sets as subsets of their respective sets of types. Let φ^* denote the stochastic extreme point Section 5.2. Using φ^*, it is straightforward to define a stochastic extreme point for the actual set of type profiles with n agents. To see this in detail, let us agree to the following notation: when $i \in \{1, 2, 3\}$, then $\hat{\Theta}_{-i}$ means the sets of type profiles of agents $\{1, 2, 3, 4\} \setminus \{i\}$ that belong to $\hat{\Theta}$. Now consider $\psi^* \colon \Theta \to \mathbb{R}^n$ defined as follows: For all $i \in \{1, \dots, n\} \setminus \{1, 2, 3, 4\}$, let $\psi_i^* = 0$; for all $i \in \{1, 2, 3\}$ and all $\theta \in \Theta$, let $\psi_i^*(\theta) = \varphi_i^*(\theta_1, \theta_2, \theta_3, \theta_4)$ if $(\theta_j)_{j \in \{1,2,3,4\} \setminus \{i\}} \in \hat{\Theta}_{-i}$, and let $\psi_i^*(\theta) = 0$ if $(\theta_j)_{j \in \{1,2,3,4\} \setminus \{i\}} \notin \hat{\Theta}_{-i}$; let $\psi_4^* = 1 - \sum_{i=1}^{3} \psi_i^*$. A moment's thought reveals that ψ^* is a well-defined DIC mechanism. To see that it is a stochastic extreme point, consider an arbitrary DIC mechanism ψ that appears in a convex combination that equals ψ^*. We know from Section 5.2 that ψ must agree with ψ^* whenever the types of agents 1 to 4 are in $\hat{\Theta}$. From here it is easy to see that ψ must agree with ψ^* at all other profiles, too. $\square$

A.3 Anonymous juries

A.3.1 Proof of Theorem 6.2

Proof of Theorem 6.2. Let φ be DIC and anonymous.

The following notation is useful. Let T denote the common type space. Let T^{n-1} with generic element θ^{n-1} denote the $(n-1)$-fold Cartesian product of T. We will frequently consider profiles obtained from a profile θ^{n-1} in T^{n-1} by replacing one entry of θ^{n-1}. For instance, we write (t, θ^{n-1}_{-j}) to denote the profile obtained by replacing the j'th entry of θ^{n-1} by t.

By DIC, for all i, we may drop i's type from i's winning probability. Thus we write $\varphi_i(\theta^{n-1})$ for i's winning probability when the types of the others are $\theta^{n-1} \in T^{n-1}$. Anonymity implies that $\varphi_i(\theta^{n-1})$ is invariant to permutations of θ^{n-1}.

We use the following auxiliary claim.

Claim A.8. *Let $i \in \{1, \ldots, n\}$, $t \in T$, $t' \in T$, and $\theta^{n-1} \in T^{n-1}$. Then*

$$\sum_{j=1}^{n-1} \left(\varphi_i(t, \theta^{n-1}_{-j}) - \varphi_i(t', \theta^{n-1}_{-j})\right) = 0. \tag{A.11}$$

Proof of Claim A.8. Let us arbitrarily label θ^{n-1} as $(\theta_j)_{j \in N \setminus \{i\}}$. Let us also fix an arbitrary type $\theta_i \in T$.

In an intermediate step, let j be distinct from i. For clarity, we spell out winning probabilities as follows: $\varphi_i(r_i = t, r_j = t', r_{-ij} = \theta_{-ij})$ means i's winning probability when i reports t, j reports t', and all remaining agents report θ_{-ij}. A permutation of i's and j's reports does not change the winning probabilities of the agents other than i and j. Since the object is allocated with probability one, we have

$$\begin{aligned} &\varphi_i(r_i = t, r_j = t', r_{-ij} = \theta_{-ij}) + \varphi_j(r_i = t, r_j = t', r_{-ij} = \theta_{-ij}) \\ =&\varphi_i(r_i = t', r_j = t, r_{-ij} = \theta_{-ij}) + \varphi_j(r_i = t', r_j = t, r_{-ij} = \theta_{-ij}). \end{aligned}$$

By rearranging the previous display, and by DIC, we obtain

$$\begin{aligned} &\varphi_i(r_i = t, r_j = t', r_{-ij} = \theta_{-ij}) - \varphi_i(r_i = t', r_j = t, r_{-ij} = \theta_{-ij}) \\ =&\varphi_j(r_i = t', r_j = \theta_j, r_{-ij} = \theta_{-ij}) - \varphi_j(r_i = t, r_j = \theta_j, r_{-ij} = \theta_{-ij}). \end{aligned} \tag{A.12}$$

Now consider summing (A.12) over all $j \in \{1, \ldots, n\} \setminus \{i\}$. This summation yields

$$\sum_{j\colon j \neq i} \left(\varphi_i(r_i = t, r_j = t', r_{-ij} = \theta_{-ij}) - \varphi_i(r_i = t', r_j = t, r_{-ij} = \theta_{-ij})\right) \tag{A.13}$$

$$= \sum_{j\colon j \neq i} \left(\varphi_j(r_i = t', r_j = \theta_j, r_{-ij} = \theta_{-ij}) - \varphi_j(r_i = t, r_j = \theta_j, r_{-ij} = \theta_{-ij})\right). \tag{A.14}$$

In (A.14), the profiles considered are all of the form $(r_i = t', r_{-i} = \theta_{-i})$ and $(r_i = t, r_{-i} = \theta_{-i})$, respectively. Note that by DIC we have $\varphi_i(r_i = t', r_{-i} = \theta_{-i}) - \varphi_i(r_i =$

$t, r_{-i} = \theta_{-i}) = 0$. Hence (A.14) equals

$$\sum_{j=1}^{n} \left(\varphi_j(r_i = t', r_{-i} = \theta_{-i}) - \varphi_j(r_i = t, r_{-i} = \theta_{-i})\right).$$

Since the object is always allocated, the term in the previous display equals 0. Hence the sum in (A.13) equals

$$\sum_{j\colon j \neq i} \left(\varphi_i(r_i = \theta_i, r_j = t', r_{-ij} = \theta_{-ij}) - \varphi_i(r_i = \theta_i, r_j = t, r_{-ij} = \theta_{-ij})\right) = 0.$$

We now revert to our usual notation. By DIC, we may drop i's report from φ_i. Since φ_i is permutation-invariant with respect to $N \setminus \{i\}$, we may also write

$$\varphi_i(r_i = \theta_i, r_j = t', r_{-ij} = \theta_{-ij}) = \varphi_i(t', \theta_{-j}^{n-1}) \qquad \text{and}$$
$$\varphi_i(r_i = \theta_i, r_j = t, r_{-ij} = \theta_{-ij}) = \varphi_i(t, \theta_{-j}^{n-1}).$$

Thus we obtain the desired equality $\sum_{j=1}^{n-1} \left(\varphi_i(t', \theta_{-j}^{n-1}) - \varphi_i(t, \theta_{-j}^{n-1})\right) = 0$. □

In what follows, let i be an arbitrary agent. We show i's winning probability is constant in the reports of others. To that end, let us fix an arbitrary type $t^* \in T$. For all $k \in \{0, \ldots, n-1\}$, let T_k^{n-1} denote the subset of profiles in T^{n-1} where exactly k-many entries are distinct from t^*. Let p_i denote i's winning probability when all other agents report t^*. We will show via induction over k that i's winning probability is equal to p_i whenever the others report a profile in T_k^{n-1}. This completes the proof since $T^{n-1} = \cup_{k=0}^{n-1} T_k^{n-1}$ holds.

Base case $k = 0$. Immediate from the definitions of p_i and T_0^{n-1}.

Induction step. Let $k \geq 1$. Let all $\hat{\theta}^{n-1} \in \cup_{\ell=0}^{k-1} T_\ell^{n-1}$ satisfy $\varphi_i(\hat{\theta}^{n-1}) = p_i$. Letting $\theta^{n-1} \in T_k^{n-1}$ be arbitrary, we show $\varphi_i(\theta^{n-1}) = p_i$.

By anonymity, we may assume that exactly the first k entries of θ^{n-1} are distinct from t^*. That is, there exist types $t_1, \ldots, t_k$ all distinct from t^* such that $\theta^{n-1} = (t_1, \ldots, t_k, t^*, \ldots, t^*)$.

Let $\tilde{\theta}^{n-1} = (t_1, \ldots, t_{k-1}, t^*, \ldots, t^*)$. This profile is obtained from θ^{n-1} by replacing

t_k by t^*. We now invoke Claim A.8 to infer

$$\sum_{j=1}^{n-1} \varphi_i(t_k, \tilde{\theta}^{n-1}_{-j}) = \sum_{j=1}^{n-1} \varphi_i(t^*, \tilde{\theta}^{n-1}_{-j}). \tag{A.15}$$

Consider the profiles appearing in the sum on the left of (A.15) as j varies from 1 to $n-1$.

(1) Let $j \leq k-1$. Since exactly the first $k-1$ entries of $\tilde{\theta}$ are distinct from t^*, it follows that $(t_k, \tilde{\theta}^{n-1}_{-j})$ is another profile where exactly $k-1$ entries differ from t^*. Hence the induction hypothesis implies $\varphi_i(t_k, \tilde{\theta}^{n-1}_{-j}) = p_i$.

(2) Let $j > k-1$. In the profile $(t_k, \tilde{\theta}^{n-1}_{-j})$, the first $k-1$ entries are $t_1, \ldots, t_{k-1}$, the j'th entry is t_k, and all remaining entries are t^*. Hence $(t_k, \tilde{\theta}^{n-1}_{-j})$ is a permutation of θ^{n-1}. Anonymity implies $\varphi_i(t_k, \tilde{\theta}^{n-1}_{-j}) = \varphi_i(\theta^{n-1})$.

Hence the sum on the left of (A.15) equals $\sum_{j=1}^{n-1} \varphi_i(t, \tilde{\theta}^{n-1}_{-j}) = (k-1)p_i + (n-k)\varphi_i(\theta^{n-1})$

Now consider the sum on the right of (A.15). For all j, a moment's thought reveals that the profile $(t^*, \tilde{\theta}^{n-1}_{-j})$ contains at most $(k-1)$-many entries different from t^*. By the induction hypothesis, therefore, the sum on the right of (A.15) equals $(n-1)p_i$.

The previous two paragraphs and (A.15) imply $(k-1)p_i + (n-k)\varphi_i(\theta^{n-1}) = (n-1)p_i$. Equivalently, $(n-k)(\varphi_i(\theta^{n-1}) - p_i) = 0$. Since $k \leq n-1$, we find $\varphi_i(\theta^{n-1}) = p_i$, as promised. □

A.3.2 Proof of Theorem 6.5

Proof of Theorem 6.5. We omit the straightforward verification that a jury mechanism with an anonymous jury is partially anonymous.

For the converse, let φ be deterministic, partially anonymous, and DIC. Let N denote the set of agents, and let T denote the common type space. For this proof, we write $\varphi(\theta)$ to mean the agent who wins at profile θ; this makes sense since φ is deterministic.

Let I_i denote the set of agents that influence agent i's winning probability. For all $j \in N$, let $A_j = \{i \in N : j \in I_i\}$ be the set of agents that are influenced by j. Let $I = \{i \in N : A_i \neq \emptyset\}$. We may assume that φ is non-constant, meaning $I \neq \emptyset$, as otherwise the proof is trivial.

Given two agents i and j, let $D_{i-j} = A_i \backslash A_j$, and $D_{j-i} = A_j \backslash A_i$, and $C_{ij} = A_j \cap A_i$, and $N_{ij} = N \setminus (A_i \cup A_j)$. Note that, by DIC, the set C_{ij} contains neither i nor j. Hence partial anonymity implies that for all $k \in C_{ij}$ the winning probability of k is invariant with respect to permutations of i and j.

When i, j, and k are given, we write $(t, t', t'', \theta_{-ijk})$ to mean the profile where i, j, and k, respectively, report t, t', and t'', respectively, and all others report θ_{-ijk}.

Claim A.9. *Let i and j be distinct. Let $\theta_{-ij} \in \Theta_{-ij}$. If there exists $\theta_i, \theta_j \in T$ such that $\varphi(\theta_i, \theta_j, \theta_{-ij}) \in D_{i-j}$, then all $\theta'_i, \theta'_j \in T$ satisfy $\varphi(\theta'_i, \theta'_j, \theta_{-ij}) \in D_{i-j}$.*

Proof of Claim A.9. We drop the fixed type profile θ_{-ij} of the others from the notation. To show $\varphi(\theta'_i, \theta'_j) \in D_{i-j}$, it suffices to show $\varphi(\theta'_i, \theta_j) \in D_{i-j}$ since if the latter is true then definition of D_{i-j} implies $\varphi(\theta'_i, \theta'_j) = \varphi(\theta'_i, \theta_j)$.

We first claim $\varphi(\theta_j, \theta_i) \in D_{i-j}$. If $\varphi(\theta_j, \theta_i) \in N_{ij}$, then $\varphi(\theta_j, \theta_i) = \varphi(\theta_i, \theta_j)$, and we have a contradiction to $\varphi(\theta_i, \theta_j) \in D_{i-j}$. If $\varphi(\theta_j, \theta_i) \in C_{ij}$, then partial anonymity implies $\varphi(\theta_i, \theta_j) \in C_{ij}$, and we have another contradiction to $\varphi(\theta_i, \theta_j) \in D_{i-j}$. If $\varphi(\theta_j, \theta_i) \in D_{j-i}$, then $\varphi(\theta_j, \theta_i) = \varphi(\theta_i, \theta_i) \in D_{j-i}$. However, from $\varphi(\theta_i, \theta_j) \in D_{i-j}$ we know $\varphi(\theta_i, \theta_j) = \varphi(\theta_i, \theta_i) \in D_{i-j}$; contradiction. Thus $\varphi(\theta_j, \theta_i) \in D_{i-j}$.

We next claim $\varphi(\theta'_i, \theta_j) \in (D_{i-j} \cup C_{ij})$. Towards a contradiction, suppose not. Then $\varphi(\theta'_i, \theta_j) \in (D_{j-i} \cup N_{ij})$, and hence $\varphi(\theta'_i, \theta_j) = \varphi(\theta_i, \theta_j) \notin D_{i-j}$. This contradicts the assumption $\varphi(\theta_i, \theta_j) \in D_{i-j}$.

In view of the previous paragraph, we can complete the proof by showing $\varphi(\theta'_i, \theta_j) \notin C_{ij}$. Towards a contradiction, let $\varphi(\theta'_i, \theta_j) \in C_{ij}$. Partial anonymity implies $\varphi(\theta_j, \theta'_i) \in C_{ij}$. We have shown earlier that $\varphi(\theta_j, \theta_i) \in D_{i-j}$ holds. Hence $\varphi(\theta_j, \theta'_i) \in D_{i-j}$, and this contradicts $\varphi(\theta_j, \theta'_i) \in C_{ij}$. Thus $\varphi(\theta'_i, \theta_j) \notin C_{ij}$ and the proof is complete. □

Claim A.10. *Let i, j, k be distinct. Let $\theta_k \in T$ and $\theta_{-ijk} \in \Theta_{-ijk}$ be such that all $\theta'_i, \theta'_j \in T$ satisfy $\varphi(\theta'_i, \theta'_j, \theta_k, \theta_{-ijk}) \in (C_{ij} \cup N_{ij})$. Then, all $\theta'_i, \theta'_j, \theta'_k \in T$ satisfy $\varphi(\theta'_i, \theta'_j, \theta'_k, \theta_{-ijk}) \in (C_{ij} \cup N_{ij})$.*

Proof of Claim A.10. Towards a contradiction, suppose $\varphi(\theta'_i, \theta'_j, \theta'_k, \theta_{-ijk}) \in (D_{i-j} \cup D_{j-i})$. Suppose $\varphi(\theta'_i, \theta'_j, \theta'_k, \theta_{-ijk}) \in D_{i-j}$, the other case being similar. The inclusions $\varphi(\theta'_i, \theta'_j, \theta_k, \theta_{-ijk}) \in (C_{ij} \cup N_{ij})$ and $\varphi(\theta'_i, \theta'_j, \theta'_k, \theta_{-ijk}) \in D_{i-j}$ together imply $\varphi(\theta'_i, \theta'_j, \theta'_k, \theta_{-ijk}) \in A_k$. Hence $\varphi(\theta'_i, \theta'_j, \theta'_k, \theta_{-ijk}) \in D_{k-j}$. We now invoke Claim A.9 to infer $\varphi(\theta'_i, \theta'_j, \theta_k, \theta_{-ijk}) \in D_{k-j}$. Since we also have $\varphi(\theta'_i, \theta'_j, \theta_k, \theta_{-ijk}) \in (C_{ij} \cup N_{ij})$, we infer $\varphi(\theta'_i, \theta'_j, \theta_k, \theta_{-ijk}) \in N_{ij}$. In particular, we have $\varphi(\theta'_i, \theta'_j, \theta_k, \theta_{-ijk}) \notin A_i$. Hence

$\varphi(\theta_i', \theta_j', \theta_k, \theta_{-ijk}) \in D_{k-i}$. We now invoke Claim A.9 to infer $\varphi(\theta_i', \theta_j', \theta_k', \theta_{-ijk}) \in D_{k-i}$. In particular, we have $\varphi(\theta_i', \theta_j', \theta_k', \theta_{-ijk}) \notin A_i$. This contradicts the assumption $\varphi(\theta_i', \theta_j', \theta_k', \theta_{-ijk}) \in D_{i-j}$. □

Claim A.11. *If $C_{ij} \neq \emptyset$, then $D_{i-j} \cup D_{j-i} = \emptyset$.*

Proof of Claim A.11. Let $k \in C_{ij}$. We may find a profile θ such that $\varphi(\theta) = k$ as else k's winning probability is constantly 0 (which would contradict $k \in C_{ij}$). Denoting by θ_{-ij} the types of agents other than i and j at θ, we appeal to Claim A.9 to infer that all $\theta_i', \theta_j' \in T$ satisfy $\varphi(\theta_i', \theta_j', \theta_{-ij}) \in (C_{ij} \cup N_{ij})$. Repeatedly applying Claim A.10 implies that all profiles θ' satisfy $\varphi(\theta') \in (C_{ij} \cup N_{ij})$. It follows that all agents in $D_{i-j} \cup D_{j-i}$ enjoy a winning probability that is constantly equal to 0. Recalling the definitions $D_{i-j} = A_i \setminus A_j$, and $D_{j-i} = A_j \setminus A_i$, it follows that $D_{i-j} \cup D_{j-i}$ is empty. □

Recall the definition $I = \{i \in N \colon A_i \neq \emptyset\}$. Consider the binary relation $\sim$ on I defined as follows: Given i and j in I, we let $i \sim j$ if and only if $C_{ij} \neq \emptyset$.

Claim A.12. *The relation $\sim$ is an equivalence relation. For all $i, j \in I$, if $i \sim j$, then $i \notin A_j$ and $A_i = A_j$.*

Proof of Claim A.12. It is clear that $\sim$ is symmetric. As for reflexivity, note that $i \in I$ implies $A_i = C_{ii} \neq \emptyset$. Turning to transitivity, suppose $i \sim j$ and $j \sim k$. Hence $C_{ij} \neq \emptyset$ and $C_{jk} \neq \emptyset$. Let $\ell \in C_{jk}$. Claim A.11 and $C_{ij} \neq \emptyset$ together imply $D_{j-i} = \emptyset$. Hence $\ell \in C_{jk}$ implies $\ell \in C_{ij}$. Hence $\ell \in C_{jk} \cap C_{ij}$, implying $\ell \in C_{ik}$. Hence $i \sim k$.

As for the second part of the claim, let $i \sim j$. Thus $C_{ij} \neq \emptyset$. Claim A.11 implies $D_{j-i} = D_{i-j} = \emptyset$. This immediately implies $A_i = A_j$. Together with DIC, we also infer $i \notin A_j$. □

Claim A.12 implies that we may partition I into finitely-many non-empty $\sim$-equivalence classes. (Recall that I is non-empty.) We now claim that there is exactly one $\sim$-equivalence class. Towards a contradiction, suppose not. In view of Claim A.12, this means that there are distinct i and j such that $A_i \cap A_j = \emptyset$ and $A_i \neq \emptyset \neq A_j$. Let J_i and J_j, respectively, denote the equivalence classes containing i and j, respectively. Let $k \in A_i$ and $\ell \in A_j$. Claim A.12 implies $k \notin J_i$ and $\ell \notin J_j$ and $k \neq \ell$. Since $k \in A_i$ and φ is deterministic, there is a type profile θ such that $\varphi(\theta) = k$; there must be another type profile θ' such that $\varphi(\theta') = \ell$. However, the definition of equivalence classes implies that k's winning probability depends only on the types of agents in J_i, and that ℓ's winning probability depends only on the types of agents in J_j. Hence there is a type profile where both k and ℓ are winning with probability 1 (such a type profile is obtained by changing at the profile θ the types of agents in J_j to their respective types at θ', and keeping all other types fixed). Contradiction.

Now, Claim A.12 implies that the members of the unique $\sim$-equivalence class do not influence one another, and that they influence the same set of others. By partial anonymity, it follows φ that is a deterministic jury mechanism with an anonymous jury. □

A.3.3 Proof of Proposition 6.6

We first give an informal sketch of the proof. The idea is to "symmetrize" the stochastic extreme point φ^* from Section 5.2.

In Section 5.2, there are four agents, the set of type profiles of agents 1 to 3 is a $2 \times 2 \times 3$ set $\hat{\Theta}$, and agent 4 has a singleton type space. Let us view allocating to agent 4 as disposing the object. Let us relabel the types of agents 1 to 3 so that they are all distinct. Across agents 1 to 3 we thus have a set T of seven distinct types. The 3-fold Cartesian product T^3 of T with itself contains six permutations of $\hat{\Theta}$ (one for each permutation of $\{1, 2, 3\}$). In Figure 2, the common type space is labelled $T = \{1, \ldots, 7\}$, and the six permutations of $\hat{\Theta}$ are depicted via six symbols (square, circle, etc.).

We can associate to each permutation of $\hat{\Theta}$ a permutation of the mechanism φ^*. The idea is to extend these permutations to a DIC mechanism with disposal on all of T^3. The difficulty is to verify that the resulting mechanism is well-defined. To see the issue, reconsider Figure 2. For each of the six subsets, imagine rays emanating from the subset and travelling parallel to the axes. Along the ray, exactly one agent's type changes. Hence DIC requires that this agent's winning probability remain constant along the ray. The rays emanating from distinct subset intersect, and we have verify that the sum of the associated winning probabilities does not exceed 1. We use two observations. The first is that, at most two such rays intersect simultaneously; this is a consequence of the fact that the types in $\hat{\Theta}$ are distinct across agents. The second is that the winning probabilities associated with the rays are at most $\frac{1}{2}$; this is a consequence of the construction of φ^* in Section 5.2.

Proof of Proposition 6.6. We first prove part (2) of the claim, assuming for a moment that part (1) is true. Let $\psi^*\colon T^3 \to [0,1]^3$ be a mechanism with disposal for three agents that meets the conclusion of part (1). We view ψ^* as a mechanism (without disposal) with four agents that ignores the report of agent 4, and where agent 4's winning probability equals the probability that ψ^* does not allocate the object to the first three agents. Using the assumed properties of ψ^*, we obtain a mechanism without disposal that is DIC, partially anonymous, and an extreme point of the set of DIC mechanisms without disposal for four agents.

It remains to prove part (1) of the claim. That is, we show that if $n = 3$, then there is an anonymous DIC mechanism with disposal that is an extreme point of the set of all DIC mechanisms with disposal.

Let us relabel the common type space as $T = \{1, 2, 3, 4, 5, 6, 7\}$. Let $T^3 = \times_{i=1}^3 T$ denote the 3-fold Cartesian product of T. Let $T_1 = \{1, 2\}$, $T_2 = \{3, 4\}$ and $T_3 = \{5, 6, 7\}$ and $\hat{\Theta} = T_1 \times T_2 \times T_3$. In Section 5.2, we constructed a stochastic DIC mechanism φ^* without disposal in a setting with 4 agents, where the types of agents 1, 2, and 3, respectively, are $\{\ell, r\}$, $\{u, d\}$, $\{f, c, b\}$, respectively, and where agent 4's type is degenerate. By relabeling types, we view φ^* as a mechanism with disposal with 3 agents on the set of type profiles $\hat{\Theta}$, and where allocating to agent 4 is identified with disposing the object. The arguments from Section 5.2 show that, if $n = 3$ and the set of type profiles is $\hat{\Theta}$, then φ^* is an extreme point of the set of DIC mechanisms with disposal.

For later reference, we note that, at all type profiles $\theta \in \hat{\Theta}$ and all $i \in \{1, 2, 3\}$, agent i's winning probability at θ under φ^* is either 0 or 1/2.

Our candidate mechanism will be denoted ψ^*. Let Ξ denote the set of permutations of $\{1, 2, 3\}$. Let $\Theta^* = \{\xi(\theta) \colon \theta \in \hat{\Theta}, \xi \in \Xi\}$ denote the set of type profiles obtained by permuting a type profile in $\hat{\Theta}$; see Figure 2. Fixing an arbitrary type profile in $\hat{\Theta}$, the types of the agents at this type profile are all distinct. Consequently, for all θ^* in Θ^* there is a unique profile θ in $\hat{\Theta}$ and ξ in Ξ such that $\theta^* = \xi(\theta)$.

For later reference, we also note that at an arbitrary type profile in Θ^*, the types of distinct agents must belong to distinct elements of the partition $\{T_1, T_2, T_3\}$.

We define ψ^* as follows: For all θ^* in Θ^*, we find the unique $(\theta, \xi) \in T \times \Xi$ such that $\theta^* = \xi(\theta)$, and then let

$$(\psi_i^*(\theta^*))_{i=1}^n = \left(\varphi^*_{\xi(i)}(\xi(\theta))\right)_{i=1}^n . \tag{A.16}$$

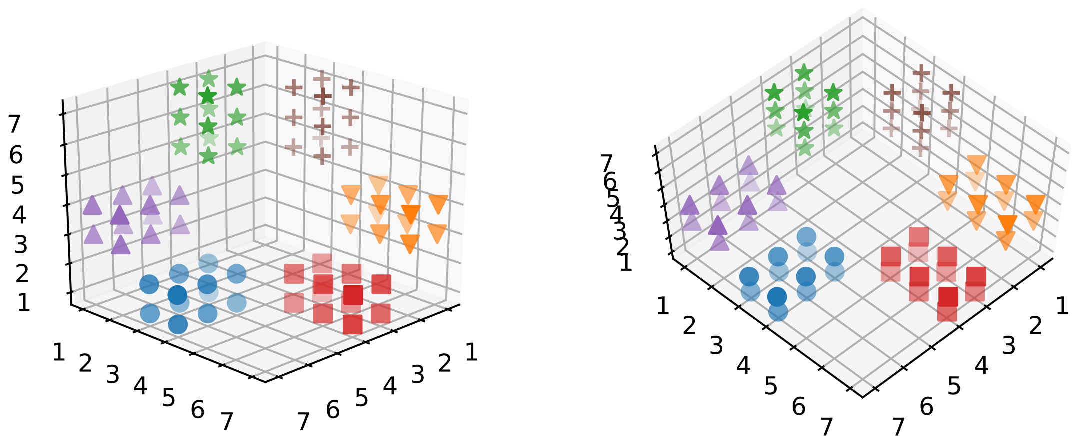


Figure 2: The set Θ^* viewed from two different angles. Each agent is associated with a distinct axis. Each symbol (square, circle, upward-pointing triangle, etc.) identifies a particular permutation of $\{1, 2, 3\}$. For instance, the upward-pointing triangles are obtained from the downward-pointing triangles by permuting the two agents on the horizontal axes.

For the remaining profiles, we proceed as follows: For all agents i and profiles θ, if θ differs from at least one profile θ^* in Θ^* in agent i's type and no other agent's type, then i's winning probability at θ equals i's winning probability at θ^* (which makes sense since the latter probability has already been defined in (A.16)); else, if no such profile θ^* in Θ^* exists, then agent i's winning probability is set equal to 0.

To complete the argument, we have to show that ψ^* is well-defined, DIC, stochastic, anonymous, and an extreme point of the set of DIC mechanisms with disposal. Assuming for a moment that the mechanism is well-defined, it is clear that the mechanism is stochastic, and one can easily verify from the definition that it is DIC and anonymous. To show that it is an extreme point of the set of DIC mechanisms, we proceed via the arguments from Section 5.2. Indeed, we know from Section 5.2 that all DIC mechanisms ψ with disposal that appear in a candidate convex combination must agree with ψ^* on $\hat{\Theta}$, and hence on Θ^*. It is then straightforward to verify that such a mechanism ψ must also agree with ψ^* on $\Theta \setminus \Theta^*$.

It remains to show that ψ^* is well-defined. Towards a contradiction, suppose there is a profile $\theta = (\theta_1, \theta_2, \theta_3)$ in Θ such that $\sum_{i=1}^{3} \psi_i^*(\theta) > 1$. By construction, all $i \in \{1, 2, 3\}$ satisfy $\psi_i^* \in \{0, \frac{1}{2}\}$. Hence all three agents enjoy non-zero winning probabilities at θ. By definition of ψ^*, we can infer the following: Since agent 1's winning probability at θ is non-zero, there exists t_1 such that $(t_1, \theta_2, \theta_3) \in \Theta^*$. Similarly, there are t_2 and t_3 such that $(\theta_1, t_2, \theta_3) \in \Theta^*$ and $(\theta_1, \theta_2, t_3) \in \Theta^*$. Recall that $\{T_1, T_2, T_3\}$ is a partition of T. Hence, for all agents i, there is a unique integer $\xi(i)$ in $\{1, 2, 3\}$ such that $\theta_i \in T_{\xi(i)}$. We now recall that if a profile is in Θ^*, then the types of distinct agents belong to distinct elements of the partition $\{T_1, T_2, T_3\}$. Hence we infer from $(t_1, \theta_2, \theta_3) \in \Theta^*$ that $\xi(2) \neq \xi(3)$ holds. Similarly, from $(\theta_1, t_2, \theta_3) \in \Theta^*$ and $(\theta_1, \theta_2, t_3) \in \Theta^*$ we infer $\xi(1) \neq \xi(2)$ and $\xi(1) \neq \xi(3)$. Taken together, we infer $\theta \in \Theta^*$. Hence the vector of winning probabilities at θ is a permutation of the vector of winning probabilities at a profile θ' in $\hat{\Theta}$. At the profile θ', the winning probabilities under ψ^* agree with φ^*. Thus there is a profile where the winning probabilities under φ^* sum to a number strictly greater than 1. This contradicts the fact that φ^* is a well-defined mechanism on $\hat{\Theta}$. □

Appendix B Supplementary material: Disposal

In this part of the appendix, we relax the requirement that the object always be allocated. An intepretation is that the mechanism designer can dispose or privately consume the object. Accordingly, we refer to such mechanisms as mechanisms with disposal. We discuss how this affects our results from the main text (Appendix B.1). Further, we show how the existence of stochastic extreme points of the set of DIC mechanisms with disposal can be related to a certain graph (Appendix B.2).

Beginning with the definitions, a *mechanism with disposal* is a function $\varphi\colon \Theta \to [0, 1]^n$ satisfying

$$\forall_{\theta \in \Theta}, \quad \sum_{i=1}^{n} \varphi_i(\theta) \leq 1.$$

A mechanism from the main text will be referred to as a mechanism with no disposal. If there is no risk of confusion, we will drop the qualifiers "with disposal" or "with no disposal".

A mechanism with disposal is DIC if and only if for arbitrary i the winning probability φ_i is constant in i's report. We will sometimes drop i's report θ_i from $\varphi_i(\theta_i, \theta_{-i})$.

A jury mechanism with disposal is defined as in the basic model: For all i, if agent i influences the allocation, then i never wins the object.

We normalize the value from not allocating the object to 0.

A mechanism with n agents and disposal can be viewed as a mechanism with no disposal and with $n+1$ agents where agent $n+1$ has a singleton type space; the value from allocating to $n+1$ is always 0. Likewise, if there are other agents with singleton type spaces, we can always renormalize values and view allocating to one of these agents as disposing the object. In what follows, whenever considering mechanisms with disposal, let us thus simplify by assuming that no agent has a singleton type space; that is, for all agents i we have $|\Theta_i| \geq 2$.

B.1 Results from the main text

Here we discuss how our results change when the mechanism can dispose the object.

To begin with, we have the following analogue of Theorem 5.1.

Theorem B.1. *Fix n and $\Theta_1, \ldots, \Theta_n$. For all agents i, let $|\Theta_i| \geq 2$. All extreme points of the set of DIC mechanisms with disposal are deterministic if and only if at least one of the following is true:*

(1) We have $n \leq 2$.

(2) For all agents i we have $|\Theta_i| = 2$.

Proof of Theorem B.1. As discussed above, a DIC mechanism with n agents and disposal is a DIC mechanism with $n+1$ agents and no disposal. The claim follows from Theorem 5.1. □

Further below, we provide an alternative proof of Theorem B.1 that does not invoke Theorem 5.1 but relies on graph-theoretic results. We emphasize that Theorem B.1 does not imply Theorem 5.1. Namely, we cannot conclude from Theorem B.1 that if $n = 3$ all extreme points of the set of DIC mechanisms with no disposal are deterministic.

It follows from Theorem B.1 that Theorem 4.1 (jury mechanisms with 3 agents) carries over to mechanisms with disposal in the sense that all mechanisms with disposal and 2 agents are convex combinations of deterministic jury mechanisms with disposal. Note that, according to Theorem B.1, this result does not extend to $n = 3$. With $n = 2$, a jury mechanism with disposal admits a single juror whose report determines whether or not the object is disposed or allocated to the other agent.

Proposition 5.2 (on the suboptimality of deterministic DIC mechanisms) analogizes straightforwardly to mechanisms with disposal. Indeed, note that in our proof of Proposition 5.2 agent 4 was simply a dummy agent with value normalized to 0.

Theorem 4.4 (approximate optimality of jury mechanisms under Assumption 1 and large n) extends to mechanisms with disposal in a straightforward way, with no changes to the proof.

We already showed via Proposition 6.6 that Theorem 6.2 does not extend to mechanisms with disposal. In fact, the non-constant mechanism constructed in the proof of Proposition 6.6 actually satisfies an even stronger notion of anonymity. Namely, whenever one permutes the type profiles, the vector of winning probabilities is permuted in the same manner.

We next turn to partial anonymity for mechanisms with disposal. In particular, we show that Theorem 6.5 extends under a slight strengthening of partial anonymity. Given a mechanism φ, let $\varphi_0 = 1 - \sum_{i=1}^{n} \varphi_i$ denote the probability that the object is not allocated.

Definition 4. Let φ be a mechanism with disposal. Let $N = \{1, \ldots, n\}$ and $N_0 = N \cup \{0\}$.

(1) Given distinct $i \in N$ and $k \in N_0$, agent i *influences* k if φ_k is non-constant in i's report.

(2) The mechanism is *partially $*$-anonymous* if for all $i \in N$, $j \in N$, and $k \in N_0$ that are all distinct and are such that i and j influence k, agents i and j are exchangeable for k.

In words, partial anonymity is strengthened by demanding that the disposal probability φ_0 is permutation-invariant with respect to those agents who influence φ_0.

It follows from Theorem 6.5 that a deterministic partially $*$-anonymous DIC mechanism with disposal is a deterministic jury mechanism with an anonymous jury. To see this, let us view disposing the object as allocating to agent 0. Now, agent 0 does not have the same type space as the other agents. Since this was a maintained assumption of Section 6, we cannot yet appeal to Theorem 6.5. But, we can simply view the mechanism as a mechanism where agent 0's type space is same as the type spaces of the others, and where agent 0's report is always ignored. By now appealing to Theorem 6.5, the claim follows.

B.2 Stochastic extreme points and perfect graphs

In this section, we relate the existence of stochastic extreme points with disposal to a graph-theoretic property called perfection.

B.2.1 Preliminaries

We first recall several definitions for a simple undirected graph G with nodes V and edges E.

An *induced cycle of length* k is a subset $\{v_1, \dots, v_k\}$ of V such that, denoting $v_{k+1} = v_1$, two nodes v_ℓ and $v_{\ell'}$ in the subset are adjacent if and only if $|\ell - \ell'| = 1$.

The *line graph* of G is the graph that has as node set the edge set of G; two nodes of the line graph are adjacent if and only if the two associated edges of G share a node in G.

A *clique* of G is a set of nodes such that every pair in the set are adjacent. A clique is *maximal* if it is not a strict subset of another clique. A *stable set* of G is a subset of nodes of which no two are adjacent. The *incidence vector* of a subset of nodes $\hat{V}$ is the function $x\colon V \to \{0, 1\}$ that equals one on $\hat{V}$ and equals zero otherwise. Let $S(G)$ denote the set of incidence vectors belonging to some stable set of G.

The upcoming result uses another property of graphs called *perfection*. For our purposes, it will be enough to know the following facts, all of which may be found in Korte and Vygen (2018).

Lemma B.2. *All bi-partite graphs and line graphs of bi-partite graphs are perfect. If a graph admits an induced cycle of odd length greater than five, then it is not perfect.*

Our interest in perfect graphs is due to the following theorem of Chvátal (1975, Theorem 3.1); one may also find it in Korte and Vygen (2018, Theorem 16.21).

Theorem B.3. *A graph G with node set V and edge set E is perfect if and only if the convex hull $\operatorname{co} S(G)$ is equal to the set*

$$\left\{ x\colon V \to [0, 1] \colon \text{all maximal cliques } X \text{ of } G \text{ satisfy} \sum_{v \in X} x(v) \le 1 \right\}. \tag{B.1}$$

The set $\operatorname{co} S(G)$ is the *stable set polytope* of G. The set in (B.1) is the *clique-constrained stable set polytope* of G.

B.2.2 The feasibility graph

We next define a graph G such that the set of deterministic DIC mechanisms with disposal corresponds to $S(G)$, and such that the set of all DIC mechanisms with disposal coincides with the clique-constrained stable set polytope of G. In view of Theorem B.3, the question of whether all extreme points are deterministic thus reduces to checking whether G is a perfect graph.

Consider the following graph G with node set V and edge set E. Let

$$V = \cup_{i=1}^{n} \left(\{i\} \times \Theta_{-i}\right),$$

and let two nodes (i, θ_{-i}) and (j, θ'_{-j}) be adjacent if and only if $i \neq j$ and there is a type profile $\hat{\theta}$ satisfying $\hat{\theta}_{-i} = \theta_{-i}$ and $\hat{\theta}_{-j} = \theta'_{-j}$. We refer to G as the *feasibility graph*.

Informally, a node (i, θ_{-i}) is the index for agent i's winning probability when the type profile of the others is θ_{-i}. Two nodes are adjacent if and only if there is a profile $\hat{\theta}$ such that the associated winning probabilities simultaneously appear in the feasibility constraint

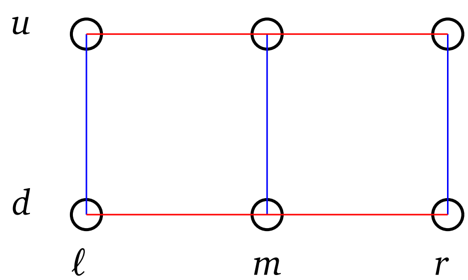
$$\sum_{i=1}^{n} \varphi_i(\hat{\theta}_{-i}) \leq 1 \tag{B.2}$$

of the profile $\hat{\theta}$.

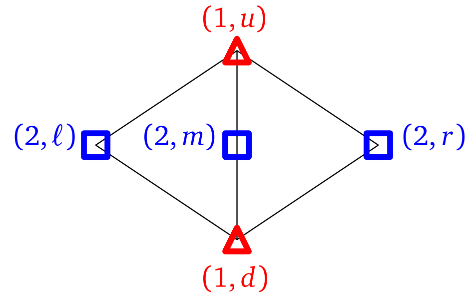
Figure 3 shows the feasibility graph in an example with two agents; Figure 4 shows it in an example with three agents.

Given a node $v = (i, \theta_{-i})$ of G, let us write $\varphi(v) = \varphi_i(\theta_{-i})$. Note that a clique in the feasibility graph is a subset of nodes of V such that the winning probabilities associated with these nodes all appear in the same feasibility constraint (B.2). It follows that there is a one-to-one mapping between maximal cliques of G and type profiles. For a DIC mechanism with disposal, the feasibility constraint (B.2) may thus be equivalently stated as follows: For all maximal cliques of X of G, we have $\sum_{v \in X} \varphi(v) \leq 1$. Thus the set of DIC mechanisms with disposal coincides with the set (B.1). One may similarly verify that the set of deterministic DIC mechanisms with disposal coincides with $S(G)$. In view of Theorem B.3, we deduce:

Lemma B.4. *All extreme points of the set of DIC mechanisms with disposal are*



(a) The set of type profiles Θ. Circles represent type profiles.



(b) The graph G. Red triangles represent nodes of G that are associated with agent 1. Blue squares represent nodes associated with agent 2.

Figure 3: There are two agents with types $\Theta_1 = \{\ell, m, r\}$ and $\Theta_2 = \{u, d\}$.

deterministic if and only if G is perfect.

This leads us to the following alternative proof of Theorem B.1.

Alternative proof of Theorem B.1. Let $n = 2$. Observe that the node set of G may be partitioned into the sets $\{1\} \times \Theta_2$ and $\{2\} \times \Theta_1$. By definition, two nodes (i, θ_{-i}) and (j, θ_{-j}) are adjacent only if $i \neq j$. Thus G is bi-partite. Since every bi-partite graph is perfect (Lemma B.2), the claim follows from Theorem B.3.

Suppose $|\Theta_i| = 2$ holds for all i. We may relabel the types so that $\Theta_i = \{0, 1\}$ holds for all i. In this case G is the line graph of a bi-partite graph; namely the bi-partite graph with node set $\{0, 1\}^n$ and where two nodes are adjacent if and only if they differ in exactly one entry. The line graph of a bi-partite graph is perfect (Lemma B.2), and so the claim again follows from Theorem B.3.

Lastly, suppose $n \geq 3$ and $|\Theta_i| > 2$ for at least one i. We will show that G admits an odd induced cycle of length seven. In view of Lemma B.2 and Theorem B.3, this proves that there exists a stochastic extreme point. Let us relabel the agents and types such that the type spaces contain the following subsets of types:

$$\tilde{\Theta}_1 = \{\ell, r\} \quad \text{and} \quad \tilde{\Theta}_2 = \{u, d\} \quad \text{and} \quad \tilde{\Theta}_3 = \{f, c, b\}$$

all hold. Let θ_{-123} be an arbitrary type profile of agents other than 1, 2 and 3 (assuming such agents exist). One may verify that the following is an induced cycle

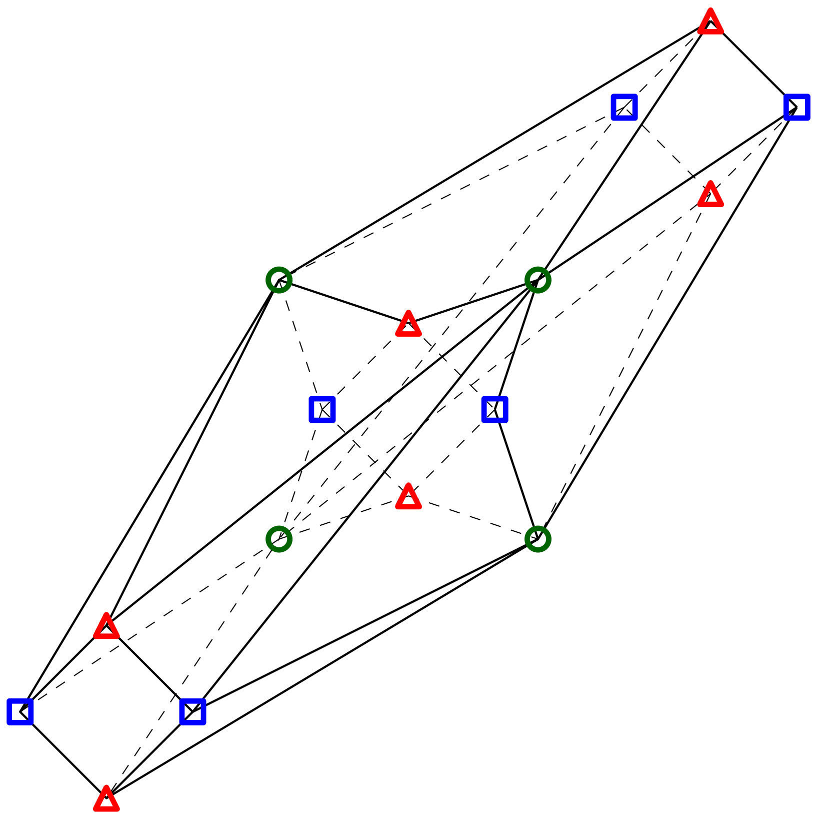


Figure 4: The feasibility graph G in an example with three agents. Agents 1 and 2 each have two possible types. The nodes of G associated with agents 1 and 2, respectively, are depicted by red triangles and blue squares, respectively. Agent 3 has three possible types; the associated nodes are depicted by green circles. One may view this as the graph G associated with the four-agent environment of Section 5.2, except that all nodes of the dummy agent 4 are omitted.

of length seven:

$$\begin{aligned}(2,(\ell,c,\theta_{-123})) &\leftrightarrow (1,(d,c,\theta_{-123}))\\ &\leftrightarrow (3,(r,d,\theta_{-123}))\\ &\leftrightarrow (2,(r,b,\theta_{-123}))\\ &\leftrightarrow (3,(r,u,\theta_{-123}))\\ &\leftrightarrow (1,(u,f,\theta_{-123}))\\ &\leftrightarrow (3,(\ell,u,\theta_{-123}))\\ \leftrightarrow (2,(\ell,c,\theta_{-123}))\,.\end{aligned}$$

□

The proof in the main text for the existence of a stochastic extreme point is slightly more elaborate than the one given above since in the former we explicitly spell out the extreme point. (The proof in the main text uses one of the agents as a dummy, and therefore also works for mechanisms with disposal.) In our view, the advantage of the more elaborate argument is that it facilitates the construction of environments where all deterministic DIC mechanisms fail to be optimal. This lets us give an interpretation as to why it may be optimal to use a lottery. That said, it is clear how the induced cycle defined in the proof of Theorem B.1 relates to the construction from the main text. The nodes of the cycle correspond to the bold edges of the hyperrectangle in Figure 1.

Appendix C Supplementary material: Additional results

C.1 All extreme points are candidates for optimality

For the following lemma, observe that the set of DIC mechanisms depends only on the number of agents and their type spaces.

Lemma C.1. *Let $n \in \mathbb{N}$. Let $\Theta_1, \dots, \Theta_n$ be finite sets, and let $\Theta = \times_{i=1}^{n} \Theta_i$. If φ is an extreme point of the set of DIC mechanisms when there are n agents and the set of type profiles is Θ, then there exists a set Ω of value profiles and a distribution μ over $\Omega \times \Theta$ such that in the environment (n, Ω, Θ, μ) the mechanism φ is the unique optimal DIC mechanism.*

Proof of Lemma C.1. The set of DIC mechanisms is a polytope in Euclidean space (being the set of solutions to a finite system of linear inequalities). Hence all its extreme points are exposed (Aliprantis and Border, 2006, Corollary 7.90). Hence there is a function $p\colon \{1, \dots, n\} \times \Theta \to \mathbb{R}$ such that for all DIC mechanisms ψ different from φ we have $\sum_{i,\theta} p_i(\theta)(\varphi_i(\theta) - \psi_i(\theta)) > 0$. By suitably choosing Ω and μ, the function p represents the objective function of our model. For example, one possible choice of Ω and μ is as follows: Let the marginal of μ on Θ be uniform; for all agents i, let Ω_i be the image of p_i; for all θ, conditional on the type profile realizing as θ, let the value of allocating to agent i be $|\Theta| p_i(\theta)$. □

C.2 Implementation with deterministic outcome functions

An indirect mechanism specifies a tuple $M = (M_1, \dots, M_n)$ of finite message sets, and an outcome function $g\colon \times_i M_i \to \Delta\{0, \dots, n\}$. (Given a finite set X, we denote by ΔX the set of distributions over X.) The outcome function is *deterministic* if for all m the distribution $g(m)$ is degenerate. A *strategy* of agent i in (M, g) is a function $\sigma_i\colon \Theta \to \Delta M_i$; let Σ_i denote the set of strategies of agent i in (M, g). A DIC mechanism φ is *implementable (in dominant strategies)* via (M, g) if there is a dominant-strategy equilibrium $(\sigma_1, \dots, \sigma_n)$ of (M, g) such that all profiles θ satisfy $\varphi(\theta) = \sum_m g(m) \prod_i \sigma_i(m_i|\theta_i)$.

Lemma C.2. *If a stochastic DIC mechanism φ is implementable via an indirect mechanism with a deterministic outcome function, then φ is not an extreme point of the set of DIC mechanisms.*

Proof of Lemma C.2. Towards a contradiction, suppose φ is an extreme point. As in the proof of Lemma C.1, we may find $p\colon \{1, \dots, n\} \times \Theta \to \mathbb{R}$ such that all DIC mechanisms ψ distinct from φ satisfy $\sum_{i,\theta} p_i(\theta)(\varphi_i(\theta) - \psi_i(\theta)) > 0$. However, since φ is implementable via an indirect mechanism with a deterministic outcome function, Proposition 1 of Jarman and Meisner (2017) implies that there is a deterministic DIC mechanism ψ such that

$$\forall_{\theta \in \Theta}, \quad \sum_i p_i(\theta)(\varphi_i(\theta) - \psi_i(\theta)) \leq 0.$$

Hence $\sum_{i,\theta} p_i(\theta)(\varphi_i(\theta) - \psi_i(\theta)) \leq 0$. Since φ is stochastic, we have $\psi \neq \varphi$; contradiction. □

C.3 Total unimodularity

This section of the appendix discusses another potential approach for showing that all extreme points are deterministic. Our aim is to explain why this approach does not help us for the proof of Theorem 5.1 in the difficult case with three agents.

For a function $\varphi\colon \Theta \to [0, 1]^n$ to be a DIC mechanism, the function should satisfy the following:

$$\begin{aligned} \forall_{i,\theta}, \quad & 1 \geq \varphi_i(\theta) \\ \forall_{i,\theta_i,\theta_i',\theta_{-i}}, \quad & 0 \geq \varphi_i(\theta_i, \theta_{-i}) - \varphi_i(\theta_i', \theta_{-i}) \geq 0 \\ \forall_\theta, \quad & 1 \geq \sum_i \varphi_i(\theta) \geq 1 \end{aligned} \tag{C.1}$$

For a suitable matrix A and vector b, the set of DIC mechanisms is then the polytope $\{\varphi\colon A\varphi \geq b, \varphi \geq 0\}$. Here, the matrix A has one row for every constraint in (C.1) (after splitting the constraints into one-sided inequalities). Each column of A identifies a pair of the form (i, θ).

A matrix or a vector is *integral* if its entries are all in $\mathbb{Z}$. A polytope is *integral* if all its extreme points are integral. In this language, all extreme points of the set of DIC mechanisms are deterministic if and only if the polytope $\{\varphi\colon A\varphi \geq b, \varphi \geq 0\}$ is integral.

Recall that a matrix is *totally unimodular* if all its square submatrices have a determinant equal to -1, 0, or 1. A submatrix of a totally unimodular matrix is itself totally unimodular.

Our interest in total unimodularity is due the Hoffman-Kruskal theorem (Korte and Vygen, 2018, Theorem 5.21).

Theorem C.3. *An integral matrix A is totally unimodular if and only if for all integral vectors b all extreme points of the set $\{\varphi\colon A\varphi \geq b, \varphi \geq 0\}$ are integral.*

Thus a sufficient condition for all extreme points of the set of DIC mechanisms to be deterministic is that the constraint matrix A be totally unimodular. Unfortunately:

Lemma C.4. *For all agents i, let $|\Theta_i| \geq 2$. Let $n = 3$. If there exists i such that $|\Theta_i| \geq 3$, then A is not totally unimodular.*

Proof of Lemma C.4. Towards a contradiction, suppose A is totally unimodular. Consider the constraint matrix $\tilde{A}$ and vector $\tilde{b}$ that define the set of DIC mechanisms with disposal (where such mechanisms are defined in Appendix B). That is, φ is a DIC mechanism with disposal if and only if $\tilde{A}\varphi \geq \tilde{b}$ and $\varphi \geq 0$. Notice that $\tilde{A}$ is obtained from A by dropping all rows corresponding to constraints of the form $\sum_i \varphi_i(\theta) \geq 1$; the vector $\tilde{b}$ is obtained from b by dropping the corresponding entries. In particular, the matrix $\tilde{A}$ is a submatrix of A. Since A is totally unimodular, we conclude that $\tilde{A}$ is totally unimodular. We infer from Theorem C.3 that all extreme points of the set of DIC mechanism with disposal are deterministic. Since $n = 3$, since all agents have at least binary types, and since at least one agent has non-binary types, we have a contradiction to Theorem B.1. □

We can give an alternative proof of Lemma C.4 that does not require Theorem B.1. Consider the following characterization of total unimodularity due to Ghouila-Houri (1962) (Korte and Vygen, 2018, Theorem 5.25).

Theorem C.5. *A matrix A with entries in $\{-1, 0, 1\}$ is totally unimodular if and only if all subsets C of columns of A satisfy the following: There exists a partition of C into subsets C^+ and C^- such that for all rows r of A we have*

$$\left(\sum_{c \in C^+} A(r, c) - \sum_{c \in C^-} A(r, c) \right) \in \{-1, 0, 1\}. \tag{C.2}$$

Alternative proof of Lemma C.4. Let us relabel the agents and types such that the type spaces contain the following subsets:

$$\tilde{\Theta}_1 = \{\ell, r\} \quad \text{and} \quad \tilde{\Theta}_2 = \{u, d\} \quad \text{and} \quad \tilde{\Theta}_3 = \{f, c, b\}$$

Fixing an arbitrary type profile θ_{-123} of agents other than 1, 2, and 3, let us define the type profiles $\{\theta^a, \theta^b, \theta^c, \theta^e, \theta^f, \theta^g\}$ as in (5.2) in Section 5.2. That is, let

$$\begin{aligned} \theta^a = (\ell, d, c, \theta_{-123}), \quad \theta^b &= (r, d, c, \theta_{-123}), \quad \theta^c = (r, d, b, \theta_{-123}), \\ \theta^d &= (r, u, b, \theta_{-123}), \quad \theta^e = (r, u, f, \theta_{-123}), \\ \theta^f &= (\ell, u, f, \theta_{-123}), \quad \theta^g = (\ell, u, c, \theta_{-123}). \end{aligned}$$

Recall that each column of A corresponds to an entry of the form (i, θ). We will argue that the following set C of columns does not admit a partition in the sense of

Theorem C.5.

$$\begin{aligned} C = \{&(1,\theta^a),(1,\theta^b),(3,\theta^b),(3,\theta^c),\\ &(2,\theta^c),(2,\theta^d),(3,\theta^d),(3,\theta^e),\\ &(1,\theta^e),(1,\theta^f),(3,\theta^f),(3,\theta^g),\\ &(2,\theta^g),(2,\theta^a)\} \end{aligned}$$

Towards a contradiction, suppose C does admit a partition into sets C^+ and C^- in the sense of Theorem C.5. Let us assume $(1,\theta^a) \in C^+$, the other case being similar. Note that θ^a and θ^b differ only in the type of agent 1. Consider the row of A corresponding to the DIC constraint for agent 1 at these type profiles. By referring to (C.2) for this row, we deduce $(1,\theta^b) \in C^+$. Next, via a similar argument, the constraint that the object is allocated at θ^b requires $(3,\theta^b) \in C^-$. Continuing in this manner, it is easy to see that $(1,\theta^a)$ must be in C^-. Since $(1,\theta^a)$ is assumed to be in C^+, we have a contradiction to the assumption that C^+ and C^- are a partition of C. □

C.4 Maximum weight perfect hypergraph matching

In this section, we explain that the problem of finding an optimal deterministic DIC mechanism corresponds to finding a maximum weight perfect matching on a certain hypergraph.

The hypergraph has as vertices the set of type profiles. Its hyperedges are those type profiles along which the type of exactly one agent i varies across Θ_i. That is, a set of type profiles e is a hyperedge if and only if there exist $i \in \{1,\dots,n\}$ and $\theta_{-i} \in \Theta_{-i}$ such that $e = \{(\theta_i,\theta_{-i})\colon \theta_i \in \Theta_i\}$. We index this hyperedge by (i,θ_{-i}). The weight attached to hyperedge (i,θ_{-i}) is $\mathbb{E}_{\omega_i}[\omega_i|\theta_{-i}]$.

In a matching of this hypergraph, including edge (i,θ_{-i}) in the matching corresponds to allocating to agent i at all type profiles incident to (i,θ_{-i}); this respects DIC for agent i. In a perfect matching, each type profile is covered by some edge; this respects the requirement that the object is always allocated.

If we relax the requirement that the object is always allocated (Appendix B), we instead consider the larger set of all matchings on the hypergraph. Such a matching can also be interpreted as a stable set of the feasibility graph introduced in Appendix B.2.2.

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