Discussion Paper Series – CRC TR 224

Discussion Paper No. 519
Project B 04

Auctions with Frictions: Recruitment, Entry, and Limited Commitment

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March 2024

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Support by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through CRC TR 224 is gratefully acknowledged.

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Rheinische Friedrich-Wilhelms-Universität Bonn - Universität Mannheim

Auctions with Frictions: Recruitment, Entry, and Limited Commitment*

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August 18, 2023

Abstract

Auction models are convenient abstractions of informal price-formation processes that arise in markets for assets or services. These processes involve frictions such as bidder recruitment costs for sellers, participation costs for bidders, and limitations on sellers' commitment abilities. This paper develops an auction model that captures such frictions. We derive several novel predictions; in particular, we find that outcomes are often inefficient, and the market sometimes unravels.

To organize a successful auction, it is essential for the seller to recruit bidders and motivate them to participate; indeed, this can have a greater impact on revenue than the details of the bidding mechanism.[1] Recruitment is often difficult, because bidders may need to incur substantial costs to evaluate an item, line up financing, and prepare bids. Sellers' recruitment costs and bidders' participation costs are particularly likely to be significant in the sale of idiosyncratic assets.

This paper investigates how sellers' recruitment efforts and bidders' entry decisions jointly determine auction participation and outcomes. We conduct our analysis under the assumption of limited seller's commitment ability, which is pervasive in auctions for idiosyncratic assets.

*We gratefully acknowledge financial support from the German Research Foundation (DFG) through Germany's Excellence Strategy—EXC 2126/1390838866 and through the CRC TR 224 (Project B04). We thank Yi-Chun Chen and Audra Boone for helpful comments and Simon Block for research assistance.

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[1] See Bulow and Klemperer (1996) for the revenue effects of attracting an additional bidder.

The main novelty of the model is that it combines these elements—costly recruitment, costly participation, and limited commitment. It captures the fundamental tension between the seller's interest in recruiting more bidders—to intensify competition and draw high-value bidders—and the resultant bidders' concern about costly entry into an overly competitive auction. When the seller cannot commit to participation levels, this tension may lead to excessive recruitment effort and, in some cases, a complete shutdown of trade. These inherent inefficiencies are the subject of our first set of insights.

A related set of insights concerns the effects of the bidders' ability to observe the level of participation (or the seller's ability to disclose it credibly). We identify conditions under which observability can promote or suppress trade. These insights are shown to translate into results comparing the first-price auction (FPA) and the second-price auction (SPA): these are not equivalent in the presence of recruitment and entry costs, and their ranking depends on these costs.

The tensions and insights described above have not been studied before, since the existing literature has studied entry and recruitment separately.

Our insights may help explain the viability of costly intermediary services that recruit bidders, enable seller commitment, and reduce bidders' costs. Sellers' willingness to pay $20-30\%$ of their revenues to auction houses such as Christie's and Sotheby's[2] indicates the magnitude of the inefficiencies that these services reduce.

Our model features a seller who offers a single item for sale. In the recruitment stage, the seller makes a costly effort to attract bidders. The random number of bidders contacted follows a Poisson distribution whose mean is determined by the seller's effort. A contacted bidder decides whether to incur a cost that enables him to discover his private value and participate in the auction. The bidding stage features a first-price auction (FPA). The seller cannot commit to the level of the recruitment effort (which is unobservable to the bidders) or to a reserve price. We consider two variants of the auction stage: the *PO scenario* ("participation-observable") and the *PU scenario* ("participation-unobservable").

In the PO scenario, the bidders observe the number of auction participants before bidding. This can happen, for example, if the bidding is in person at the auction site. In this case, our analysis identifies the inefficiencies mentioned above (excessive recruitment and potential market shutdown).

In the PU scenario, the bidders do not observe the number of participants and the

[2] Ashenfelter and Graddy (2005) describe the fees and other institutional details for such auction houses.

seller cannot credibly disclose the number of bidders.[3] Unobservability generates a new consideration: an incentive for the seller to secretly reduce recruitment. This may give rise to multiple equilibria sustained by different levels of fulfilled expectations. In particular, an equilibrium with no trade always exists.

The two scenarios cannot be uniformly ranked: trade volume and profits are higher in the PO scenario for some recruitment and participation cost configurations and in the PU scenario for others. In particular, trade is supported only by the PO scenario in some circumstances, and only by the PU scenario in others. This reversal creates incentives for the seller to either conceal or disclose participation information (in settings where this can credibly be done). Which of these scenarios is more profitable depends on the nature of the seller's commitment problem—whether she would like to commit to a recruitment effort greater or smaller than the equilibrium effort.

The same results can be viewed from another angle: they apply verbatim to a comparison of the FPA and SPA, both with unobservable participation. This is because, in terms of payoffs and participation, the PO equilibrium outcome is equivalent to the outcome of the dominant-strategy equilibrium of the SPA (for which observability does not matter). In the absence of the frictions considered here—costly recruitment, costly participation, and lack of commitment—these two auction formats yield equal profit and surplus. With such frictions, they are not equivalent, because they affect the seller's recruitment incentives; their ranking depends on recruitment and entry costs.

In Section 5, we examine the effect of bidders' uncertainty about the seller's recruitment effort, modeled by introducing uncertainty about the seller's recruitment cost. We show that the market may unravel almost completely: almost all seller types may stay out of the market, even though each type would be active in equilibrium if it were commonly known. This result is driven by a "sampling curse": conditional on being recruited, each bidder believes that the seller is likely to have recruited many other bidders, rendering participation unprofitable.

The core of our model is the interaction between costly recruitment and costly participation when the seller cannot commit. Other features, such as the Poisson arrivals or whether bidders learn their values before or after entry, are not essential for the main insights. Several extensions illustrate the robustness of our qualitative findings. In particular, in the online appendix we consider the case in which bidders know their values before entry, as well as the cases in which the seller can set an entry fee/subsidy or a reserve price. In an earlier version of the paper, we established the same main insights

[3] For example, even with in-person bidding, other bidders may be "shills" or may be bidding via agents.

with deterministic (rather than Poissonian) recruitment.

Anecdotal evidence. Ample anecdotal evidence demonstrates the relevance of the key elements in our auction model: limited seller commitment, recruitment costs, and entry costs.

Limited commitment. Subramanian (2010) and Boone and Mulherin (2004, 2009) study merger and takeover proceedings, which often involve auctions of some form. Despite the high stakes involved, such auctions are often conducted in a way that suggests limited seller commitment. First, many of these auctions (the majority, according to Boone and Mulherin, 2009, p. 31) are "informal," in the sense that they are a mixture of auctions and negotiations rather than "a structured process where the rules are laid out in advance." Second, sellers seem unable to credibly commit ex-ante to a level of participation or its disclosure.[4] Sellers' commitment ability is sometimes further limited by confidentiality agreements with certain bidders (see also Gentry and Stroub, 2018), or other legal considerations, such as the reluctance of courts to enforce certain contract clauses.

Recruitment costs. Subramanian (2010) describes the critical role of bidder recruitment in merger and acquisition auctions. Milgrom (2003) states that, based on his consulting experience, the marketing of an auction is often more critical for its success than clever design. Fees paid by sellers to intermediaries go partly towards recruitment efforts. Recruitment costs may reflect also implicit costs, such as the costly disclosure of sensitive information to motivate potential buyers.[5]

Entry costs. An extensive empirical literature documents the importance of bidders' entry and participation costs; see, for example, Gentry and Stroub (2018) and the work discussed there.

The process of obtaining bids for home repair provides an example of an informal auction that will be familiar to many readers, in which both recruitment and entry costs play a major role. A homeowner may wish to suggest to prospective contractors that they have some competition, but not so much as to scare them away.

[4] Subramanian (2010) provides examples of sellers trying to increase competitive pressure by using fictitious bidders.

[5] Bulow and Klempere (1996, p 190) mention the implicit costs of revealing information as an additional (unmodeled) reason for restricting bidder numbers. To give an idea of recruitment in practice, the first case discussed in Boone and Mulherin (2009) is the sale of a firm, Blount Inc., where 65 potential buyers where contacted, of which 28 signed confidentiality agreements, and 2 submitted a bid (Lehman Brothers won).

Related literature. Our model's main novelty is the combination of costly recruitment and costly entry with limited commitment. There are strands of the literature discussing each of these frictions in isolation; we are not aware of any references that discuss all three jointly, or that derive insights similar to ours.

An extensive literature on auctions with costly entry has found that when bidders enter before learning their values, their entry decisions are efficient, and the seller's incentives align with the social planner's as she obtains the full surplus (McAfee and McMillan, 1987; Levin and Smith, 1994; Crémer, Spiegel, and Zheng, 2007).[6] These models correspond to versions of our model with positive entry costs and an exogenously given expected number of potential bidders.

Szech (2011) examines costly recruitment in an FPA where all contacted bidders enter. Her model corresponds to our PO scenario with costless entry. She shows that the seller's profit-maximizing choice of recruitment effort generally exceeds the efficient one. Lauermann and Wolinsky (2017, 2021) also feature costly recruitment and costless entry, but in a common value setting. They focus on different questions related to information aggregation with a privately informed seller.

Milgrom (1987), McAfee and Vincent (1997), and Liu et al. (2019) study limited commitment to a reserve price in auctions with a fixed set of bidders.

Our model can be viewed as a simultaneous search model in which the seller is the searcher. Renaming the actors turns our model into a stochastic version of the simultaneous search model of Burdett and Judd (1983), with the added features of heterogeneous production costs and price-quoting costs.

1 The PO auction: Observable participation

1.1 The model

One seller owns an indivisible item that has value 0 to her. She makes recruitment effort $\gamma \geq 0$, resulting in a random number of prospective bidders that is Poisson-distributed with mean γ; i.e., the probability of her contacting t bidders is $\frac{\gamma^t}{t!}e^{-\gamma}$. The cost of effort γ is γs, for some $s > 0$.

The prospective bidders are ex-ante symmetric. A prospective bidder i who decides to participate incurs a cost $c > 0$. He then observes his own value v_i for the item and the

[6] However, this is not the case when bidders have private information at entry (Samuelson, 1984; Ye, 2007).

total number n of bidders who chose to enter the auction (including i himself). The v_i are private values, independently and identically distributed with a cumulative distribution function (c.d.f.) G, with support $[0,1]$, a continuous density g, and increasing virtual values, $v - \frac{1-G(v)}{g(v)}$. The bidders do not observe γ. Finally, the participating bidders submit bids. The highest bidder wins and pays his bid.

When an auction ends with winning bid p, the payoff is $p-\gamma s$ for the seller, v_i-p-c for the winning bidder i, $-c$ for each participating bidder who lost, and 0 for each contacted bidder who declined entry.

1.2 Interaction: Strategies and equilibrium

The seller's strategy is her recruitment effort $\gamma \geq 0$. Bidder i's strategy is (q_i, β_i), where $q_i \in [0,1]$ is the entry probability and $\beta_i : [0,1] \times \{1,2,...\} \to [0,1]$ describes i's bid as a function of his information (v_i, n)—that is, his private value and the number of participating bidders. Bidder i's belief concerning the seller's effort, conditional on being contacted—but before observing (v_i, n)—is a probability measure μ_i on $[0,\infty)$.

We study symmetric behavior in which all bidders employ the same strategy (q, β) and hold the same belief μ. An **equilibrium** consists of γ^*, q^*, and β^* such that the following hold:

(E1) The effort γ^* maximizes the seller's expected payoff given q^* and β^*.

(E2) There exists a belief μ such that

- (i) q^* and β^* maximize each bidder's payoff, given μ and the other bidders' strategy (q^*, β^*);
- (ii) if $\gamma^* > 0$, then $\mu(\gamma^*) = 1$, i.e., the belief is confirmed on the path;
- (iii) if $\gamma^* = 0$, then the seller's payoff is not negative for any $\hat{\gamma}$ in the support of μ, given (q^*, β^*).

Thus, the equilibrium allows only pure recruitment and bidding strategies; mixing is allowed only in the bidders' entry decisions, $q \in [0,1]$.

Off-path beliefs arise only when $\gamma^* = 0$, but their role is not negligible since this is an important case of extreme market failure. The last condition in the equilibrium definition imposes a refinement on the off-path beliefs, which allows us to rule out no-trade equilibria that rely on unfounded beliefs. This will be discussed in Section 6.3, where we present alternative ways to obtain the needed refinement.

The random number of actual participants in the auction is Poisson distributed with mean

$$\lambda := q\gamma.$$

Given the Poisson distribution, λ is not just the expected number of participants from an outsider's perspective, but also the expected number of competitors of a participating bidder; see Myerson (1998).

For convenience, we will mostly use λ (instead of γ). Thus, the bidders' belief μ will be over λ, and the equilibrium will be expressed in terms of $\lambda^* := q^*\gamma^*$.

2 Equilibrium analysis for the PO scenario

2.1 Solving backward

The interaction in the PO scenario unfolds in three stages: recruitment, entry, and bidding. We can solve for the equilibrium backward.

Stage 3: Bidding. Once the number of participants n is realized, the ensuing auction is a standard symmetric FPA with independent private values drawn from the c.d.f. G. Such an auction has a unique symmetric equilibrium (see, e.g., Krishna 2010),

$$\beta_{FPA}(v,n) = v - \int_0^v \left[\frac{G(y)}{G(v)}\right]^{n-1} dy, \tag{1}$$

and so $\beta^* = \beta_{FPA}$ is the bidding strategy in every equilibrium. The main properties used below are that $\beta_{FPA}(v,n)$ is increasing in v and n, with $\beta_{FPA}(v,n) = 0$ if $n = 1$ and $\beta_{FPA}(v,n) \to v$ as n becomes large.

Stage 2: Entry. Let $U(\lambda)$ be the bidders' ex-ante expected payoff (gross of the cost of entry), given a Poisson-distributed number of participating bidders with mean λ who use β_{FPA}.

The main properties of U used in the equilibrium analysis are illustrated in Figure 1: U is continuous and decreasing in λ, with $U(0) = E[v]$ and $\lim_{\lambda\to\infty} U(\lambda) = 0$. These properties follow immediately from the properties of β_{FPA} discussed above, as verified in the appendix in Claim 11.

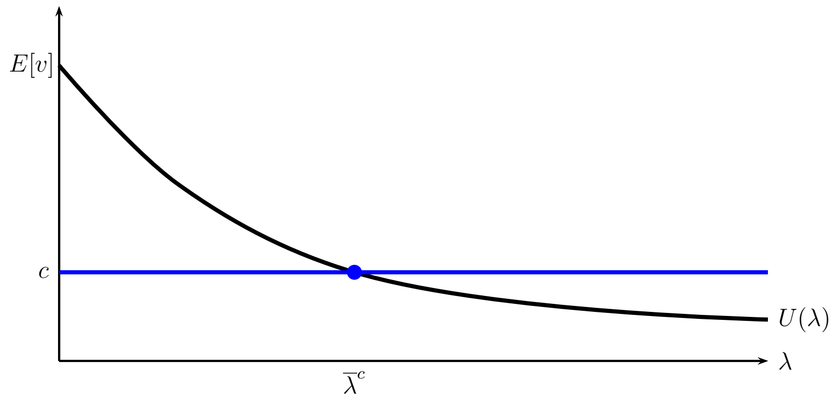

Figure 1: The bidder's payoffs $U(\lambda)$.

Given the bidders' belief μ concerning λ, their optimal entry decision q satisfies

$$\begin{aligned} E_\mu[U(\hat{\lambda})] > c &\quad\Rightarrow\quad q = 1, \\ E_\mu[U(\hat{\lambda})] < c &\quad\Rightarrow\quad q = 0. \end{aligned} \tag{2}$$

The case of $c \geq U(0)$ is uninteresting, since it means that no bidder enters. We therefore assume from now on that $c < U(0)$ (we already assumed $0 < c$). Since U is continuous and strictly decreasing to 0, for every c there is a unique $\overline{\lambda}^c$ such that

$$U(\overline{\lambda}^c) = c. \tag{3}$$

This is the bidders' break-even participation level: given λ, a bidder's expected payoff from entering is nonnegative if and only if $\lambda \leq \overline{\lambda}^c$. The upper bar in $\overline{\lambda}^c$ will serve as a reminder that this is the maximal scale acceptable to bidders. In any equilibrium,

$$\lambda^* \leq \overline{\lambda}^c \text{ and } q^* = 1 \text{ if } \lambda^* \in (0, \overline{\lambda}^c). \tag{4}$$

Stage 1: Recruitment. Given q and β_{FPA}, the seller's problem is to choose recruitment effort γ to maximize profit. The choice of effort γ at cost s is equivalent to the choice of $\lambda = q\gamma$ at cost s/q. Let $R_o(\lambda)$ be the seller's expected revenue given the participation level λ and β_{FPA}. (The subscript o here and later indicates that participation is observable.[7])

[7] There is no subscript on U since payoff equivalence implies it is independent of observability as we will see later.

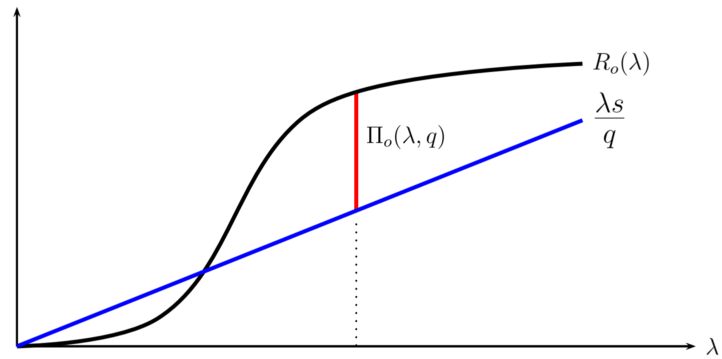

Figure 2: Revenue, cost, and profit.

The profit as a function of λ and $q > 0$ is

$$\Pi_o(\lambda, q) = R_o(\lambda) - \lambda \frac{s}{q},$$

with $\Pi_o(0, 0) = 0$ and $\Pi_o(\lambda, 0) = -\infty$ for $\lambda > 0$.

In any equilibrium, $\lambda^* \in \arg\max \Pi_o(\lambda, q^*)$. Below, we describe the solution to this maximization problem and relate it to the properties of the revenue, R_o.

Figure 2 shows the essential properties of R_o, verified in Claim 12: it is increasing and continuously differentiable, $R_o(0) = 0$, and $\lim_{\lambda \to \infty} R_o(\lambda) = 1$. Thus, an interior λ maximizes $\Pi_o(\lambda, q)$ only if

$$R_o'(\lambda) = \frac{s}{q}, \tag{5}$$

that is, marginal revenue equals marginal cost. (The first-order condition is necessary but not sufficient because R_o—and hence Π_o—is not concave.)

Figure 3 depicts the marginal revenue R_o' (blue) and the average revenue $\frac{R_o}{\lambda}$ (red): R_o' is single-peaked, $R_o'(0) = 0$, and $\lim_{\lambda \to \infty} R_o'(\lambda) = 0$. Therefore, the average revenue has the same basic shape, and its maximum is at its intersection with R_o', denoted by

$$\bar{s}_o := \max_{\lambda} \frac{R_o(\lambda)}{\lambda}.$$

Thus, when $\frac{s}{q} > \bar{s}_o$, the profit-maximizing λ is 0 (the seller would incur a loss at all positive λ). When $\frac{s}{q} < \bar{s}_o$, the profit-maximizing λ is the larger of the two solutions to the first-order condition (5). Let $\lambda_o\left(\frac{s}{q}\right)$ denote this solution. When $\frac{s}{q} = \bar{s}_o$, both $\lambda = 0$ and $\lambda = \lambda_o(\bar{s}_o)$ are profit-maximizing.

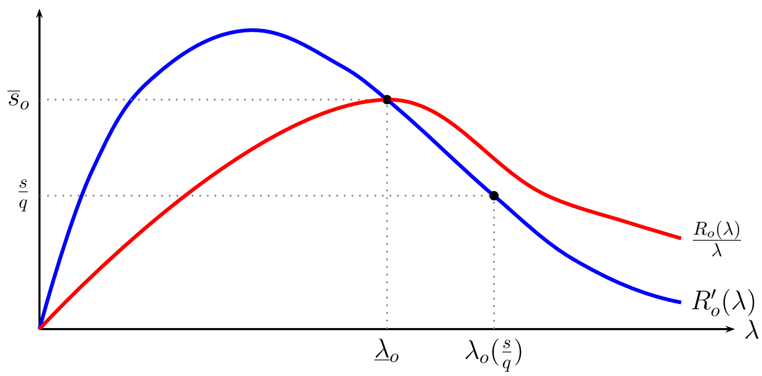

Figure 3: Marginal revenue, average revenue, and marginal cost.

Let

$$\underline{\lambda}_o := \lambda_o\left(\bar{s}_o\right);$$

that is, $R'_o(\underline{\lambda}_o) = \bar{s}_o$. Then, in every equilibrium,

$$\begin{aligned} \frac{s}{q^*} &> \bar{s}_o \Rightarrow \lambda^* = 0, \\ \frac{s}{q^*} &< \bar{s}_o \Rightarrow \lambda^* = \lambda_o(\frac{s}{q^*}), \\ \frac{s}{q^*} &= \bar{s}_o \Rightarrow \lambda^* \in \{0, \underline{\lambda}_o\}. \end{aligned} \tag{6}$$

In particular, in equilibrium, the seller will never choose a λ between 0 and $\underline{\lambda}_o$, her minimal scale. (The lower bar in $\underline{\lambda}_o$ serves as a reminder of that.) Roughly speaking, the first few bidders are complementary to each other because they stimulate competitive bidding, leading to an initially increasing marginal revenue. Therefore, the seller contacts either no bidders, or some minimal number of bidders above $\underline{\lambda}_o$. The complementarity of bidders results from our assumption that the seller cannot commit to an optimal reserve price.[8]

Solving backward through the three stages above, we conclude that an equilibrium is a profile (λ^*, q^*, μ^*) such that λ^* satisfies (6), q^* satisfies (2), and either $\lambda^* > 0$ and $\mu^*(\lambda^*) = 1$, or $\lambda^* = 0$ and $\Pi_o(\hat{\lambda}, q^*) = 0$ for all $\hat{\lambda}$ in the support of μ^*.

[8]With no reserve price, the marginal revenue from the first bidder at $\lambda = 0$ is 0. With a positive reserve price, the marginal revenue is positive at $\lambda = 0$, but it is still initially increasing and the complementarity is still present as long as the reserve price is below the optimal one. The online appendix contains an extensive discussion of (optimal) reserve prices.

2.2 The equilibrium outcome

We now use the characterization above to describe the unique equilibrium outcome. We start with the case of large c, such that $\overline{\lambda}^c < \underline{\lambda}_o$.

Proposition 1 *If $\overline{\lambda}^c < \underline{\lambda}_o$, then $\lambda^* = 0$ (no trade) is the unique equilibrium outcome for every $s > 0$.*

Proof. For $\lambda^* > 0$ to be an equilibrium, it must be that

$$\underline{\lambda}_o \leq \lambda^* \leq \overline{\lambda}^c. \tag{7}$$

The right inequality follows from bidder optimality, (4), and the left inequality from seller optimality, (6). If $\overline{\lambda}^c < \underline{\lambda}_o$, these inequalities cannot hold simultaneously, so there is no equilibrium with $\lambda^* > 0$.

We now show that an equilibrium with $\lambda^* = 0$ exists. If $s > \bar{s}_o$, then by (6), $\lambda = 0$ is the seller's unique optimal choice for any q^*. Therefore, $\lambda^* = 0$ with $q^* = 1$ and $\mu^*(0) = 1$ is an equilibrium in this case.

If $s \leq \bar{s}_o$, then the following is an equilibrium: $\lambda^* = 0$, q^* satisfies $\bar{s}_o = \frac{s}{q^*}$, and μ^* has support on $\{0, \underline{\lambda}_o\}$ with $\mu^*(0)U(0) + \mu^*(\underline{\lambda}_o)U(\underline{\lambda}_o) = c$. Since $\overline{\lambda}^c < \underline{\lambda}_o$, such a μ^* exists because $U(0) > c > U(\underline{\lambda}_o)$. The choice of μ^* implies $E_{\mu^*}(U(\lambda)) = c$, so q^* is bidder-optimal. The choice of q^* also implies $\max_\lambda \Pi_o(\lambda, q^*) = 0$ and $\arg\max_\lambda \Pi_o(\lambda, q^*) = \{0, \underline{\lambda}_o\}$. Hence, $\lambda^* = 0$ is seller-optimal, and μ^* satisfies equilibrium condition E2 (iii). ■

Thus, when $\overline{\lambda}^c < \underline{\lambda}_o$, even if s is small and contacting bidders is arbitrarily easy, there is no trade. This starkly illustrates the potential inefficiencies arising from limited commitment. For example, if effort were observable, the seller could choose an effort just below $\overline{\lambda}^c$ so as to give bidders strict incentives to participate and choose $q = 1$, yielding revenue close to $R_o\left(\overline{\lambda}^c\right)$. This would be profitable for small s. However, because effort is unobservable, the bidders understand that the seller will target a participation level above $\underline{\lambda}_o$, and are therefore unwilling to participate. Note that the initial complementarity of bidders means it is never optimal for the seller to recruit just a few bidders.

Given $\underline{\lambda}_o$, the no-trade condition $\overline{\lambda}^c < \underline{\lambda}_o$ obtains when c is too large. If c is small enough, then $\overline{\lambda}^c > \underline{\lambda}_o$, and trade is possible for certain values of s. Figure 4 gives the essential information about the equilibria in this case.

There are three cases, corresponding to three configurations of s, which are illustrated in the figure by $\{s_L, s_M, s_H\}$. At $s_H > \bar{s}_o$, the seller cannot profitably recruit bidders and

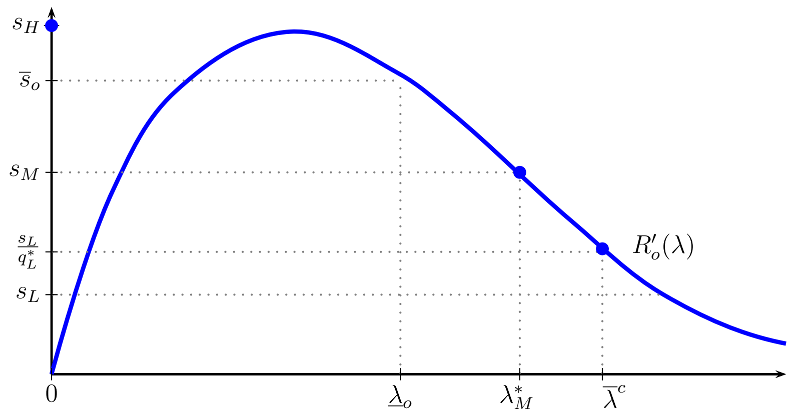

Figure 4: Equilibria of the PO scenario when c is not too large.

so $\lambda_H^* = 0$, even though c is small. For the intermediate s_M, the seller will choose the larger solution to the first-order condition, $s = R_o'(\lambda_M^*)$ and $q^* = 1$. For even smaller s, $s_L = R_o'(\lambda)$ would imply $\lambda > \overline{\lambda}^c$, which is above the bidders' maximal acceptable scale. In this case, q^* will adjust so that $R_o'(\lambda_L^*) = \frac{s_L}{q^*}$ implies $\lambda_L^* = \overline{\lambda}^c$.

Proposition 2 *If $\overline{\lambda}^c > \underline{\lambda}_o$, then the equilibrium outcome is as follows:*

(i) If $s > \bar{s}_o$, then $\lambda^* = 0$ and $q^* = 1$.

(ii) If $\bar{s}_o > s > R_o'(\overline{\lambda}^c)$, then $\lambda^* = \lambda_o(s)$ and $q^* = 1$.

(iii) If $R_o'(\overline{\lambda}^c) > s$, then $\lambda^* = \overline{\lambda}^c$ and $\frac{s}{q^*} = R_o'(\overline{\lambda}^c)$.

Finally, if either $s = \bar{s}_o$ or $\overline{\lambda}^c = \underline{\lambda}_o$, then $\lambda^* = 0$ and $\lambda^* = \underline{\lambda}_o$ are both equilibrium outcomes. Thus, except in these special cases, the equilibrium is unique. The main step in the proof of Proposition 2 in the appendix is to show that the equilibrium refinement in condition E2 (iii) rules out no-trade equilibria when $s < \bar{s}_o$.

To be willing to bear the cost of entry, bidders must believe that the seller is not recruiting too aggressively. This is immediate for large s. For small s, this is achieved in equilibrium when bidders are sufficiently reluctant to enter (q^* is sufficiently small) so that the marginal recruitment cost is high enough to induce the seller to stop at $\overline{\lambda}^c$.

The welfare implication of Proposition 2 for small s is that

$$\lambda^* \frac{s}{q^*} = \overline{\lambda}^c R_o'(\overline{\lambda}^c) = \text{const} > 0. \tag{8}$$

So even if s is vanishingly small, the total recruitment cost does not vanish, because it is determined by incentives rather than by s.[9] For small s, the total recruitment effort is wasteful. It could be avoided if the seller could commit to some effort level below $\overline{\lambda}^c$, as discussed after Proposition 1.

The magnitude of the inefficiency depends on c. When c is small, the total recruitment cost for small s is also small:[10]

$$\lim_{c\to 0} \overline{\lambda}^c R_o'(\overline{\lambda}^c) = 0.$$

The occurrence of wasteful recruitment even for small s is due to the seller's inability to commit and costly bidder entry.

2.3 Other auction formats and bargaining

The characterization of equilibrium relies only on the properties of the reduced-form payoffs U and R_o. Thus, we could have started with some reduced-form bargaining model that maps participation into a split of the available surplus between the seller and bidders. If the participating bidders' payoffs are decreasing to 0 in the level of participation and if the seller's marginal revenue as a function of participation is single-peaked, then the characterization results from Propositions 1 and 2 hold.

In particular, our results extend, of course, to all standard auctions in which the bidder with the highest value wins, such as the SPA or the all-pay auction. This follows immediately from payoff and revenue equivalence, which imply that U and R_o are the same across these auction formats.

3 The PU auction: Unobservable participation

After bidders have sunk their entry costs, the seller wants to convince them they have many competitors. However, she cannot do this unless she can credibly disclose participation because of her incentive to exaggerate the number of bidders. Subramanian (2010) gives numerous examples of such exaggeration: realtors pretend to get calls from other interested parties; companies suggest the existence of additional bidders during takeover

[9]Equation (8) shows that total recruitment costs are proportional to $R_0'(\overline{\lambda}^c)$. A subtle implication of this is that the seller may be better off in a different trading regime that gives her a lower marginal revenue—as we will see when we discuss the PU scenario.

[10]When $c \to 0$, we have $\bar{\lambda}^c \to \infty$. However, $\lim_{\lambda\to\infty} \lambda R_o'(\lambda) = 0$, since R_o is bounded on $\mathbb{R}_+$ and R_o' is positive and monotonically decreasing for $\lambda \geq \underline{\lambda}_o$.

negotiations; at Sotheby's and other auction houses, auctioneers make up "chandelier bids" to stimulate bidding.[11]

The PU scenario captures the effects of limited disclosure ability. It is the same model as in the PO scenario, except that the bidders *cannot* observe the number of other participants at any stage (before or after entry). The equilibrium definition from Section 1.2 remains the same. To simplify the exposition, we further restrict attention to equilibria in which the bidders' beliefs have point support, that is, $\mu^*(\hat{\lambda}) = 1$ for some $\hat{\lambda}$.[12] We index the magnitudes for this scenario with the subscript u (for "unobservable").

3.1 Solving backward

As before, the interaction unfolds in three stages—recruitment, entry, and bidding—and the equilibrium can be solved for backward.

Stage 3: Bidding. Since bidders do not observe the actual participation, this stage corresponds to an FPA with an uncertain number of bidders. A straightforward application of revenue equivalence characterizes its equilibrium.[13]

Claim 1 *An FPA with a Poisson-distributed number of bidders with mean λ has a unique symmetric bidding equilibrium,*

$$\beta_\lambda(v) = v - \int_0^v e^{-\lambda(G(v)-G(x))} dx. \tag{9}$$

In every equilibrium, the bidders have the belief $\mu^*(\hat{\lambda}) = 1$ for some $\hat{\lambda}$, and so their equilibrium strategy is $\beta^* = \beta_{\hat{\lambda}}$.

Stage 2: Entry. The bidders' ex-ante expected payoff given λ and β_λ is $U(\lambda)$, just as in the PO scenario. Again, this is immediate from payoff equivalence and β_λ being increasing; see the proof of Claim 1. Thus, as before, q^* satisfies (2). Hence, in every equilibrium, $\lambda^* \leq \overline{\lambda}^c$, and for all $\lambda^* \in (0, \overline{\lambda}^c)$, $q^* = 1$.

[11]The use of fictitious competitors or bids is limited by law, although US courts permit mild versions of it as "sales talk." On the other hand, non-disclosure requirements may prevent sellers from revealing the presence of actual competitors.

[12]Since we assume pure strategies by the seller, this is only a restriction in the case of $\lambda^* = 0$, and, as will become clear later, it entails no further loss of generality.

[13]For a textbook analysis of an auction with an uncertain number of bidders, see, e.g., Krishna (2009, Section 3.2.2).

Stage 1: Recruitment. Let $R_u(\lambda, \beta)$ be the seller's expected revenue given λ and β. Given $q > 0$, her expected payoff is

$$\Pi_u(\lambda, \beta, q) = R_u(\lambda, \beta) - \lambda \frac{s}{q};$$

it is 0 for $\lambda = q = 0$, and it is $-\infty$ for $q = 0$ and $\lambda > 0$. The functions R_u and hence Π_u are concave in λ (Claim 13 in the appendix).[14] Therefore, for any $\hat{\lambda}$ and q, a necessary and sufficient condition for maximization of Π_u with respect to λ is

$$\frac{\partial}{\partial \lambda} R_u(\lambda, \beta_{\hat{\lambda}}) \leq \frac{s}{q}, \tag{10}$$

with equality holding for $\lambda > 0$.

Let

$$\xi(\lambda) := \frac{\partial}{\partial \lambda} R_u(\lambda, \beta_{\hat{\lambda}})_{\hat{\lambda} = \lambda},$$

which is the marginal revenue with respect to λ where it coincides with the bidders' expectation $\hat{\lambda}$. Our previous discussion implies that $\lambda^* > 0$ is part of an equilibrium if

$$\xi(\lambda^*) = \frac{s}{q^*}$$

and either $\lambda^* < \overline{\lambda}^c$ with $q^* = 1$ or $\lambda^* = \overline{\lambda}^c$ with $q^* \in (0, 1]$.

3.2 The equilibrium outcomes

Claim 2 *For all s and c, the PU scenario always has a no-trade equilibrium with $\lambda^* = 0$.*

In the no-trade equilibrium, $q^* = 1$, $\beta^*(v) \equiv 0$, and $\mu^*(0) = 0$. Thus, if off the equilibrium path a bidder is contacted, he believes himself to be the only bidder and bids 0. This means it is indeed optimal for the seller to recruit no bidders. Being unable to commit to the recruitment effort or disclose the level of participation, the seller cannot break out of this equilibrium (even if s and c are small).

Turning to equilibria with trade, we consider the function ξ as illustrated in Figure 5. As shown in the figure and verified in the appendix (Claim 13), ξ is continuous, $\xi(\lambda) > 0$

[14] This follows from the monotonicity of the given $\beta_{\hat{\lambda}}$ and the concavity in λ of the first order statistic of a random sample distributed as Poisson(λ).

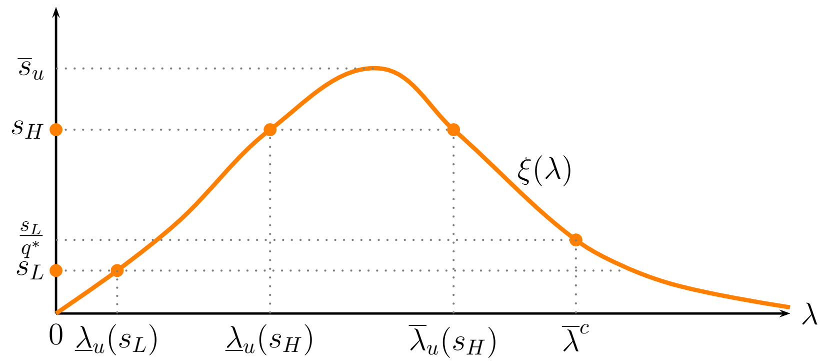

Figure 5: The function ξ and the equilibria of the PU scenario for s_H and s_L.

except at $\lambda = 0$, and $\xi(\lambda) \to 0$ for $\lambda \to \infty$.[15] We let

$$\overline{s}_u := \max_{\lambda} \xi(\lambda).$$

For $s \leq \overline{s}_u$, $\underline{\lambda}_u(s)$ and $\overline{\lambda}_u(s)$ denote the minimal and maximal values of λ such that $\xi(\lambda) = s$. The figure shows three equilibria (marked with dots) for each of two s values, $s_H > s_L$:

- For s_H, the equilibria are at $\lambda^* = 0$, $\lambda^* = \underline{\lambda}_u(s_H)$, and $\lambda^* = \overline{\lambda}_u(s_H)$.
- For s_L, the equilibria are at $\lambda^* = 0$, $\lambda^* = \underline{\lambda}_u(s_L)$, and $\lambda^* = \overline{\lambda}^c$.

Figure 5 is for the case where c is small enough so that $\overline{\lambda}^c$ is above the maximizer of ξ. In this case, when s is small, the equilibrium with the largest participation is at $\overline{\lambda}^c$. As in the PO scenario, the total recruitment effort in this equilibrium is independent of s. For small s, q^* adjusts and the total effort $(s/q^*)\overline{\lambda}^c$ remains constant.

In the PU scenario, there may be equilibria with trade even if c is large so that $\overline{\lambda}^c$ is small. Figure 6 depicts such a case. Since $\xi(0) = 0$ and ξ is continuous, $\underline{\lambda}_u(s)$ is below $\overline{\lambda}^c$ for s small enough, such as s_L in the figure.

Note that equilibria like the one with $\lambda^* = \underline{\lambda}_u(s_L)$ in the case of s_L are pseudo-unstable, since ξ crosses s from below.[16]

[15] When G is uniform, ξ is single-peaked, as depicted. However, we do not know whether ξ is single-peaked for general G.

[16] This is in the sense that if the actual and expected λ are for some reason displaced upwards (downwards) from the equilibrium point, the seller has an incentive to recruit more (less) aggressively.

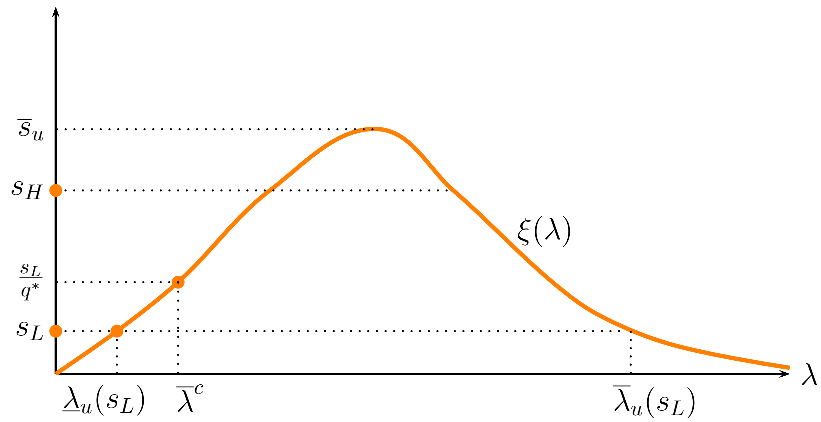

Figure 6: The equilibria of the PU scenario with large c.

4 Comparison of the PO and PU scenarios

4.1 Ranking reversals

In the PO scenario, the incentive to recruit is driven by two considerations: increasing the likelihood that high-value bidders will appear and inducing more aggressive bidding. In the PU scenario, only the former consideration is present. This difference is reflected by the stronger marginal incentive to recruit in the PO scenario. As shown below, it leads to ranking reversals: for small c and not-too-small s, the PO scenario generates higher participation and profit than the PU scenario; the opposite is true for large c or small s.

Figure 7, depicting R_o' and ξ, shows equilibria of the two scenarios for two levels of s. Recall that $\bar{s}_o$ and $\bar{s}_u$ are the maximal values of s for which an equilibrium with positive λ exists in the PO and the PU scenario, respectively. The following claim states the essential features of R_o' and ξ. It formalizes the idea that when participation is observable, the incentive to increase competition leads to strictly higher marginal revenue.

Claim 3 *The functions R_o' and ξ relate as follows:*

$$\begin{aligned} R_o'(\lambda) &> \xi(\lambda) \text{ for all } \lambda > 0; \\ \bar{s}_o &> \bar{s}_u. \end{aligned}$$

Of course, when the bidders' beliefs are correct, revenue equivalence holds, i.e.

$$R_u(\lambda, \beta_{\hat{\lambda}}) = R_o(\lambda) \text{ for } \hat{\lambda} = \lambda, \tag{11}$$

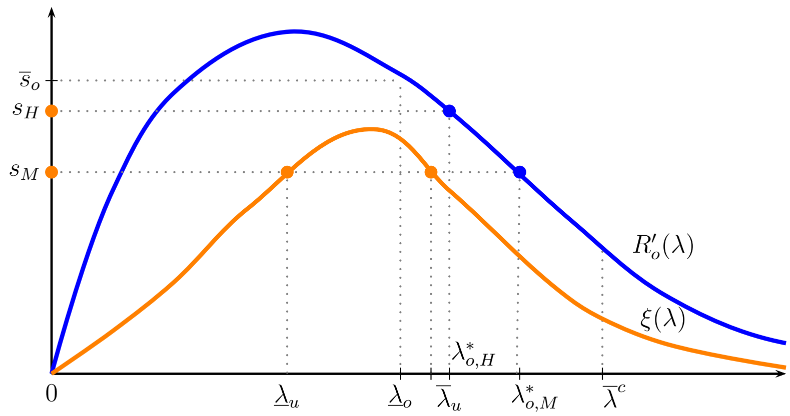

Figure 7: Comparison of the PO and PU scenarios, showing R_0' and ξ and the equilibria for s_H and s_L.

and so $\Pi_o(\lambda, q) = \Pi_u(\lambda, \beta_\lambda, q)$ for all λ and q. The proof of Claim 3 and our subsequent discussion make extensive use of this fact. In particular, the first part of the claim is immediate upon taking the total derivative in (11), which yields

$$R_o'(\lambda) = \underbrace{\frac{\partial}{\partial \lambda} R_u(\lambda, \beta_{\hat{\lambda}})|_{\hat{\lambda}=\lambda}}_{=\xi(\lambda)} + \frac{d}{d\hat{\lambda}} R_u(\lambda, \beta_{\hat{\lambda}})|_{\hat{\lambda}=\lambda}. \tag{12}$$

The claim follows from $\frac{d}{d\hat{\lambda}} R_u(\lambda, \beta_{\hat{\lambda}}) > 0$, which holds because $\beta_{\hat{\lambda}}$ is strictly increasing in $\hat{\lambda}$. Thus, as claimed, there is a stronger recruitment incentive in the PO scenario, precisely because greater participation makes bidders behave more aggressively.

We make four observations about the seller's relative profits in the two scenarios.

Observation 1. There always exists a robust no-trade equilibrium in the PU scenario. In the PO scenario, such an equilibrium exists only if either $s > \bar{s}_o$ or $\bar{\lambda}^c < \underline{\lambda}_o$. Without the ability to disclose the level of participation, the seller may be trapped in an equilibrium without competition.

Observation 2. For intermediate s and small c, the seller's profit is higher in the PO than in the PU scenario. Thus, the ability to disclose participation helps the seller.

Claim 4 *If $\bar{\lambda}^c > \underline{\lambda}_o$ and s satisfies $\bar{s}_o > s > R_o'\left(\bar{\lambda}^c\right)$, the seller's equilibrium profit is strictly higher in the PO scenario than in any equilibrium of the PU scenario.*

The claim is illustrated in Figure 7. If the seller in the PU scenario could commit to λ_o^*, she would get the same profit as in the PO scenario, by revenue equivalence (11). However, without commitment, this is not sustainable, because in the PU scenario she would have an incentive to secretly reduce λ. The bidders, anticipating this, would plan to bid less aggressively than if they expected λ_o^*, thus augmenting the seller's incentive to reduce λ. When $\overline{s}_o > s > \overline{s}_u$, these self-reinforcing considerations drive the maximal PU equilibrium participation to 0—complete "unraveling" of the market—even though $s < \overline{s}_o$ implies a positive λ_o^*. If $s < \overline{s}_u$, then λ_u^* may settle at some positive level, albeit lower than λ_o^*. In either case, this implies lower profit in the PU scenario.

We can also state this insight in terms of the entry costs c: for any $s < \overline{s}_o$, if c is small enough, then $s > R_o'\left(\bar{\lambda}^c\right)$. Hence, when c is small, the seller's profit is higher in the PO scenario. Roughly speaking, for small c, the seller's main concern is to stimulate competition (rather than entry), and for this the ability to disclose participation is helpful.

Observation 3. For small s and small c, the seller's profit is higher in the PU scenario than in the PO scenario, because the former reduces wasteful recruitment. Thus, the ability to disclose participation may hurt the seller.

Claim 5 *If $\bar{\lambda}^c > \underline{\lambda}_o$ and $s < \xi\left(\bar{\lambda}^c\right)$, then the seller's equilibrium profit in the PO scenario is strictly smaller than in her optimal equilibrium in the PU scenario.*

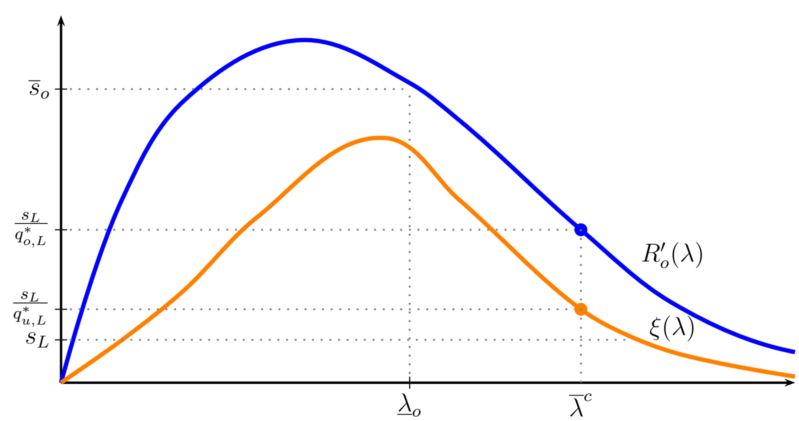

Figure 8: The seller's profit is lower in the PO scenario than the PU scenario for small s and c.

The claim is illustrated by s_L in Figure 8. The key idea is that the total recruitment costs are proportional to the marginal revenue in each scenario; therefore, $R_o'\left(\bar{\lambda}^c\right) > \xi\left(\bar{\lambda}^c\right)$

implies lower recruitment costs in the PU scenario. Thus, when s is sufficiently small so that the seller would like to commit to limiting her recruitment, the lower recruitment incentives of the PU scenario are helpful, because they reduce wasteful recruitment.

Observation 4. When c is large enough so that $\bar{\lambda}^c < \underline{\lambda}_o$, there is no trade in the PO scenario. By contrast, as illustrated in Figure 6, in the PU scenario there is an equilibrium with trade for all c provided s is small enough, because here the lower recruitment incentives help the seller commit to recruiting less than $\bar{\lambda}^c$.

4.2 Comparison of first- and second-price auctions

As noted in Section 2.3, in terms of payoffs and costs, the PO equilibrium is equivalent to the dominant-strategy equilibrium of an SPA. Since for the latter equilibrium it does not matter whether participation is observable, we may think of the format as an SPA with unobservable participation. Hence, the insights obtained in comparing the PO and PU scenarios in Section 4.1 also arise in comparing the SPA and FPA with unobservable participation.

Thus, while the FPA and SPA are equivalent in terms of equilibrium profit and welfare when participation is observable, they are not equivalent with unobservable participation because they give different recruitment incentives.

4.3 Disclosure

Suppose that the seller could credibly commit in advance either to always disclose or to never disclose the number of participants prior to the bidding. This is equivalent to the seller choosing between the PO and PU scenarios. Thus, if such commitment is possible, the comparison in Sections 4.1–4.2 applies also to the disclosure question. In particular, our discussion implies that the seller may prefer to commit in advance to disclosure or non-disclosure depending on s and c.

5 Uncertainty about recruitment costs

It is natural to suppose that the bidders may be uncertain about the seller's recruitment effort (even in equilibrium). Here, we model this by assuming that the bidders are uncertain about s and study the implications.

First, in Sections 5.1 and 5.2, we consider a binary setting in which the seller's recruitment cost is either high or low, $s_H > s_L > 0$. Naturally, optimal recruitment is larger for s_L than for s_H. This implies a "sampling curse": even if both costs are equally likely ex ante, conditional on being recruited, a bidder will believe s_L is more likely, because he is more likely to have been recruited in this case. A small s_L can imply very cautious bidder entry (small q). This makes entry by s_H unprofitable, pushing it out of the market.

In Section 5.3, we ask how general this inefficiency is and study a setting with recruitment costs uniformly distributed on $[0, \bar{s}_o]$.[17] We find that in equilibrium, the seller actively recruits bidders whenever her recruitment cost is below some cutoff. We show that this cutoff is never interior: either the seller is active for all cost levels, or the market unravels completely, with no trade happening. Roughly speaking, because of the sampling curse, if the cutoff were interior, the bidders would believe that the seller had an even lower recruitment cost.

5.1 Binary setup

We minimally modify the PO model of Section 1.1 to capture the uncertainty about costs (and hence recruitment). The privately known seller's type ω has marginal recruitment cost s_ω and occurs with prior probability ρ_ω, for $\omega \in \{L, H\}$. Type L is more efficient: $s_H > s_L > 0$. A seller of type ω selects recruitment effort γ_ω.

Contacted bidders decide whether to enter, then observe their own values and the number of participants, and finally submit bids in an FPA. The bidders' symmetric entry and bidding strategy (q,β) and the state-dependent participation rates $\boldsymbol{\lambda}{:=}(\lambda_L, \lambda_H)$, where $\lambda_\omega = q\gamma_\omega$, are just as in the PO scenario.

In any symmetric equilibrium, β must be the unique symmetric equilibrium strategy $\beta_{FPA}(v, n)$ of the FPA (see (1)). Therefore, for any given participation rate λ, the seller's revenue and the bidders' ex-ante expected payoff are the same as in the PO scenario. Hence, the seller of type ω has profit

$$\Pi_\omega(\lambda_\omega, q) = R_o(\lambda_\omega) - \lambda_\omega \frac{s_\omega}{q}.$$

Given the bidders' belief μ (the distribution over λ conditional on being contacted), their expected payoff is $E_\mu(U(\lambda))$ and their optimal entry decision q satisfies (2).

[17] Formally, we consider a uniform distribution on $[\underline{s}, \bar{s}_o]$ for some $\underline{s} > 0$ and then consider $\underline{s} \to 0$. This circumvents the problem that a seller with $s = 0$ would choose $\gamma = \infty$.

For $\boldsymbol{\lambda} \neq 0$, let

$$\phi_\omega(\boldsymbol{\lambda}) = \frac{\rho_\omega \lambda_\omega}{\Sigma \rho_\omega \lambda_\omega}.$$

Since $\boldsymbol{\lambda} \neq 0$ implies $\boldsymbol{\gamma}:=(\gamma_L, \gamma_H) \neq 0$, it follows that $\phi_\omega(\boldsymbol{\lambda})$ is the probability of ω conditional on a bidder being contacted by the seller. An **equilibrium** consists of $\boldsymbol{\lambda}^* = (\lambda_L^*, \lambda_H^*)$ and q^* such that the following hold:

(E'1) $\lambda = \lambda_\omega^*$ maximizes $\Pi_\omega(\lambda, q^*)$.

(E'2) There exists a belief μ such that

- (i) q^* is optimal given μ, i.e., it satisfies (2);
- (ii) if $\boldsymbol{\lambda}^* \neq (0,0)$, then $\mu(\lambda_\omega^*) = \phi_\omega(\boldsymbol{\lambda}^*)$ (confirmation on path);
- (iii) if $\boldsymbol{\lambda}^* = (0,0)$, the seller's payoff is not negative for any λ in the support of μ.

Claim 6 *There exists an equilibrium.*

For the equilibrium analysis, we can simply import what we know from the PO scenario to the current setting. The following discussion and Figure 9 prove Claim 6 above and Claim 7 below.

For the following discussion, recall from the PO scenario that $\overline{s}_o$ is the maximal s that sustains equilibrium with trade; that $\lambda_o(z)$ is the profit-maximizing λ for a given $z \leq \overline{s}_o$ (i.e., the larger solution of $R_o'(\lambda) = z$); that $\underline{\lambda}_o$ is the seller's minimum profitable scale ($\underline{\lambda}_o = \lambda_o(\overline{s}_o)$); and that $\overline{\lambda}^c$ is the maximal λ for which bidder entry is beneficial ($U(\overline{\lambda}^c) = c$).

Let

$$\widehat{\lambda}_\omega(q) = \begin{cases} \lambda_o(\frac{s_\omega}{q}) & \text{if} \quad \frac{s_\omega}{q} \leq \overline{s}_o, \\ 0 & \text{if} \quad \frac{s_\omega}{q} > \overline{s}_o, \end{cases}$$

and $\widehat{\boldsymbol{\lambda}}(q) = \Big(\widehat{\lambda}_L(q), \widehat{\lambda}_H(q)\Big)$. The PO analysis immediately implies that $\lambda_\omega^* = \widehat{\lambda}_\omega(q^*)$. Therefore, an equilibrium with $\boldsymbol{\lambda}^* = (0,0)$ exists if and only if $\frac{s_L}{q^*} \geq \overline{s}_o$, which can occur if and only if $s_L \geq \overline{s}_o$ or $\overline{\lambda}^c \leq \underline{\lambda}_o$, and it is unique if one of these inequalities is strict.

To consider equilibria with trade, $\boldsymbol{\lambda}^* \neq (0,0)$, let $V(\boldsymbol{\lambda})$ denote the bidders' expected payoff at $\boldsymbol{\lambda} = (\lambda_L, \lambda_H)$,

$$V(\boldsymbol{\lambda}) = \Sigma \phi_\omega(\boldsymbol{\lambda}) U(\lambda_w). \tag{13}$$

In an equilibrium with trade, q^* has to satisfy

$$\begin{aligned} q^* \in (0,1) \quad &\Rightarrow \quad V(\widehat{\boldsymbol{\lambda}}(q^*)) = c, \\ q^* = 1 \quad &\Rightarrow \quad V(\boldsymbol{\lambda}^*) \geq c. \end{aligned} \tag{14}$$

Obviously, $s_H > \bar{s}_o$ implies $\lambda_H^* = 0$ in any equilibrium, and we are back in the PO scenario with commonly known $s = s_L$, for which existence and characterization are already established. Therefore, the only interesting case to consider is $\bar{s}_o > s_H > s_L > 0$.

Figure 9 depicts $V(\widehat{\boldsymbol{\lambda}}(q))$ as a function of q. The intersection points between $V(\widehat{\boldsymbol{\lambda}}(\cdot))$ and c correspond to (14), and therefore capture the possible equilibria with trade. The minimal q that enables a profitable positive scale for seller type ω is $\bar{q}_\omega$ such that $\bar{s}_o = \frac{s_\omega}{\bar{q}_\omega}$. At $\bar{q}_L$, type L becomes active with the minimal positive scale $\underline{\lambda}_o$; at $\bar{q}_H$, type H also joins with the minimal scale $\underline{\lambda}_o$. This explains the discontinuities of $V(\widehat{\boldsymbol{\lambda}}(q))$ at $\bar{q}_L$ and $\bar{q}_H$.[18]

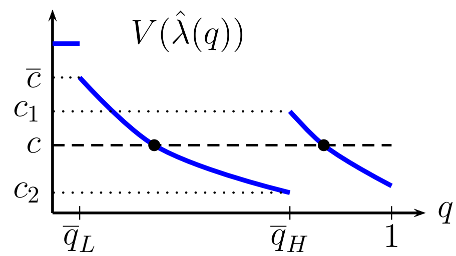

Figure 9: Equilibrium with two seller types.

The following claim summarizes what the above discussion and the diagram have established.

Claim 7 *For $\bar{s}_o > s_H > s_L > 0$, the equilibrium set is characterized by three cutoffs $\bar{c} > c_1 > c_2$:*

(i) For $c > \bar{c}$, the unique equilibrium has $\lambda_L^* = \lambda_H^* = 0$.

(ii) For $c \in (c_1, \bar{c})$, the unique equilibrium with trade has $\lambda_L^* > 0 = \lambda_H^*$.

(iii) For $c < c_2$, the unique equilibrium with trade has $\lambda_L^* > \lambda_H^* > 0$.

(iv) For $c \in (c_2, c_1)$, there are two equilibria with trade, one with $\lambda_L^* > 0 = \lambda_H^*$ and one with $\lambda_L^* > \lambda_H^* > 0$.

Remark. We have restricted our attention to pure strategies for the seller. However, if we admit randomized strategies, then for $c \in (c_2, c_1)$ there is also a third equilibrium in which $\lambda_L^* > 0$ and λ_H^* is randomized between a positive level and 0 that ensures $V(\widehat{\boldsymbol{\lambda}}(\bar{q}_H)) = c$.

[18] In more detail, $\widehat{\boldsymbol{\lambda}}(q) = (0,0)$ for $q < \bar{q}_L$; it jumps to $(\underline{\lambda}_o, 0)$ at $\bar{q}_L$ and increases continuously on $[\bar{q}_L, \bar{q}_H)$ according to $(\lambda_o(\frac{s_L}{q}), 0)$; it jumps again at $\bar{q}_H$ to $(\lambda_o(\frac{s_L}{\bar{q}_H}), \underline{\lambda}_o)$, and thereafter continues according to $(\lambda_o(\frac{s_L}{q}), \lambda_o(\frac{s_H}{q}))$. As for V, note that it is decreasing until $\bar{q}_H$ given that $(\lambda_o(\frac{s_L}{q}), 0)$ is increasing, and in this range, $V(\lambda_o(\frac{s_L}{q}), 0) = U\left(\lambda_o(\frac{s_L}{q})\right)$. As noted, at $\bar{q}_H$, H becomes active. Moreover, at this point, $\lambda_o(\frac{s_L}{\bar{q}_H}) > \underline{\lambda}_o$, which implies that $V(\underline{\lambda}_o, 0) > V(\lambda_o(\frac{s_L}{\bar{q}_H}), \underline{\lambda}_o)$. This is seen in the diagram as $V(\widehat{\boldsymbol{\lambda}}(\cdot))$ being higher at $\bar{q}_L$ than at $\bar{q}_H$.

5.2 Unraveling

As noted above, when s_L is sufficiently small relative to s_H, only type L is active in equilibrium (i.e., $\lambda_H^* = 0$). This is so even when s_H itself is small enough so that, if it were commonly known, the equilibrium would involve active recruiting.

Claim 8 *Suppose $c > 0$. For any $s_H > 0$ and $\rho_H > 0$, there exists a threshold $S(s_H, \rho_H)$ such that $s_L < S(s_H, \rho_H)$ implies $\lambda_L^* > 0$ and $\lambda_H^* = 0$.*

When s_L is small, q^* must be small as well, for otherwise λ_L^* would be very large and entry would be unprofitable for bidders. However, a small q^* means high marginal recruitment cost s_H/q^* for seller type H, making participation unprofitable for this type. More formally, given s_H and ρ_H, for sufficiently small values of s_L, $V(\lambda_o(s_L/\overline{q}_H), \underline{\lambda}_o) < c$. Hence, for any $q \geq \overline{q}_H$ (that accommodates the participation of H), $V(\lambda_o(s_L/q), \lambda_o(s_H/q)) \leq V(\lambda_o(s_L/\overline{q}_H), \underline{\lambda}_o) < c$. Thus, it must be that $q^* < \overline{q}_H$, and the unique equilibrium is with $\lambda_L^* = \overline{\lambda}^c$ and $\lambda_H^* = 0$. This outcome is inefficient: a seller of type H might fail to trade even when s_H is quite low and would result in active trade if it were known.

This insight does not depend on the two-type assumption: if there are more than two seller types, then if the lowest s is low enough, all types with higher s will still be shut out of the market. However, it does depend on the discreteness of the set of seller types, since the argument relies on making the ratio of the lowest to the next lowest cost small while keeping their probabilities constant. We now ask whether a similar unraveling occurs in an environment where very low costs are associated with very low probabilities.

5.3 Continuum of seller types

We now consider a continuum of possible seller types. The marginal recruitment cost s is distributed uniformly on $[\underline{s}, \overline{s}_o]$, where $\underline{s} > 0$ and $\overline{s}_o$ is as defined above (the maximal s compatible with active recruitment in the commonly-known-type case). Our previous model extends immediately to this environment. Identifying ω with s itself, we write λ_s and $\boldsymbol{\lambda} = (\lambda_s)_{s \in [\underline{s}, \overline{s}_o]}$.

The definition of equilibrium also extends almost directly. For each s, λ_s^* and q^* satisfy the conditions (E'1) and (E'2) of Section 5.1, with s and λ_s replacing s_ω and λ_ω, respectively. The equilibrium belief density μ also satisfies the analogous conditions. In particular, let

$$\phi_s(\boldsymbol{\lambda}) := \frac{\lambda_s}{\int_{\underline{s}}^{\overline{s}_o} \lambda_s ds}.$$

If $\lambda_s^* \neq 0$ for some $s \in (\underline{s}, \overline{s}_o]$, then $\mu(\lambda_s^*) = \phi_s(\boldsymbol{\lambda}^*)$ for all s. Let

$$V(\boldsymbol{\lambda}) := \int_{\underline{s}}^{\overline{s}_o} \phi_s(\boldsymbol{\lambda}) U(\lambda_s) ds = \frac{\int_{\underline{s}}^{\overline{s}_o} \lambda_s U(\lambda_s) ds}{\int_{\underline{s}}^{\overline{s}_o} \lambda_s ds}. \tag{15}$$

In an equilibrium with trade, q^* satisfies (14).

We already know from the discrete-types case that partial unraveling is possible, in the sense that trade might be shut down for some type, even though trade would be sustainable if that type were commonly known. The question is whether it is possible to have complete or nearly complete unraveling in equilibrium, even when c is low enough to allow trade when s is commonly known.

Obviously, if some seller type in $[\underline{s}, \overline{s}_o]$ is active in equilibrium, so is every lower type. Hence the equilibrium has a cutoff structure, and moreover, the cutoff must be $q^*\overline{s}_o$. It follows from the previous discussion that for $s < q^*\overline{s}_o$, $\lambda_s^* = \lambda_o(s/q^*) > 0$, and for $s > q^*\overline{s}_o$, we have $\lambda_s^* = 0$, where $\lambda_o(z)$ is the profit-maximizing λ in the PO scenario when the marginal recruitment cost is z. Let $\boldsymbol{\lambda_o}$=$(\lambda_o(s))_{s\in[\underline{s},\overline{s}_o]}$, and recall that $\overline{c}$ is the maximal cost level compatible with trade in the PO scenario (i.e., $\overline{\lambda}^{\overline{c}} = \lambda_o(\overline{s}_o) = \underline{\lambda}_o$).

Claim 9 *The unique equilibrium outcomes are as follows:*

(i) $c \geq \overline{c}$*: there is no trade,* $\boldsymbol{\lambda}^* = 0$*;* $q^* = \underline{s}/\overline{s}_o$*;*

(ii) $c \leq V(\boldsymbol{\lambda_o})$*: all types are active,* $\boldsymbol{\lambda}^* = \boldsymbol{\lambda_o}$*;* $q^* = 1$*;*

(iii) $V(\boldsymbol{\lambda_o}) < c < \overline{c}$*: only* $s \in [\underline{s}, \overline{s}_o q^*]$ *are active, with*

$$\lambda_s^* = \begin{cases} \lambda_o(s/q^*) > 0 & \textit{for} \quad s \in [\underline{s}, \overline{s}_o q^*], \\ 0 & \textit{for} \quad s > \overline{s}_o q^*, \end{cases} \tag{16}$$

and $q^* \in (0,1)$ *is such that* $V(\boldsymbol{\lambda}^*) = c$*.*

The proof of the claim is in the appendix in Section 7.4.1. Since by definition $\overline{c} = U(\underline{\lambda}_o)$, for any $c < \overline{c}$ and commonly known $s < \overline{s}_o$, the equilibrium in the PO scenario involves trade. In contrast, Part (iii) of Claim 9 identifies a range of $c < \overline{c}$ and $s < \overline{s}_o$ for which there is no trade.

The extent of such unraveling depends on c and $\underline{s}$. Proposition 3 identifies a threshold $\underline{c} < \overline{c}$ such that if $c > \underline{c}$, then the unraveling is nearly complete when $\underline{s}$ is small; if $c < \underline{c}$, trade always takes place regardless of how small $\underline{s}$ is.

Given c and $\underline{s}$, the probability of no recruitment in equilibrium is $\Pr(\{s : \lambda_s^* = 0\} | c, \underline{s})$.

Proposition 3 *There is some* $\underline{c}$ *such that* $0 < \underline{c} < \overline{c}$ *and*

(i) for any $c \in (\underline{c}, \overline{c})$, $\lim_{\underline{s} \to 0} \Pr(\{s : \lambda_s^* = 0\} | c, \underline{s}) = 1$;

(ii) for any $c < \underline{c}$ *and any* $\underline{s} < \overline{s}_o$, $\Pr(\{s : \lambda_s^* = 0\} | c, \underline{s}) = 0$.

Note again that nearly complete unraveling occurs for a range of $c < \overline{c}$ for which trade would take place at any commonly known $s \in (0, \overline{s}_o]$. The proof of the Proposition is in the appendix in Section 7.4.2.

6 Discussion and extensions

This section discusses welfare, seller commitment, and the refinement. The online appendix discusses reserve prices, fees, bidder heterogeneity (known values at the time of entry), and a numerical analysis for a uniform value distribution.

6.1 Welfare

Welfare $W(\lambda, q)$ is identified with the total surplus,

$$W(\lambda, q) := T(\lambda) - \lambda \frac{s}{q} - \lambda c,$$

where $T(\lambda) = \int_0^1 v \lambda e^{-\lambda(1-G(v))} g(v) dv = \int_0^1 [1 - e^{-\lambda(1-G(v))}] dv$ is the expected value of the first order statistic given Poisson(λ)-distributed participation. Let λ^w and q^w denote the welfare-maximizing magnitudes.

Proposition 4 *(i) We have* $q^w = 1$*. (ii) If* $U(0) > s + c$*, then* λ^w *is the unique value of* λ *satisfying*

$$U(\lambda) = c + s. \tag{17}$$

If $U(0) < s + c$*, then* $\lambda^w = 0$.

Proof. Part (i) is obvious. For Part (ii), note that

$$T'(\lambda) = \int_0^1 (1 - G(v)) [1 - e^{-\lambda(1-G(v))}] dv = U(\lambda),$$

where the second equality uses the characterization of U in (19) from Claim 11. Since U is strictly decreasing, T is strictly concave. It follows that (17) is the first-order condition for welfare maximization, and the condition is sufficient, proving the claim. ■

The critical equality is

$$T'(\lambda) = U(\lambda). \tag{18}$$

For intuition, recall the equivalence of the expected payoffs to those of the SPA, where each bidder's payoff is equal to his marginal contribution to the total surplus.

There are two types of inefficiency in equilibrium in the PO scenario. First, as we already know, we can have $q^* < 1$ in equilibrium, which immediately means wasted recruitment effort. Second, as shown below, for almost all pairs (s, c) in the PO scenario, $\lambda^* \neq \lambda^w$, and both excessive participation, $\lambda^* > \lambda^w$, and deficient participation, $\lambda^* < \lambda^w$, may arise in equilibrium.

For the equilibrium of the PO scenario to coincide with the welfare maximum, we must have $R_o'(\lambda^*) = s$ and $U(\lambda^*) = s + c$. Since both U and R_o' are independent of s and c, these equalities cannot be expected to hold simultaneously (they fail for almost all c and s). Thus, in general, the equilibrium does not maximize welfare.

Figure 10 depicts a possible relationship between $U(\lambda)$ and $R_o'(\lambda)$. Its relevant features are consistent with a uniform value distribution, that is, $G(v) = v$.

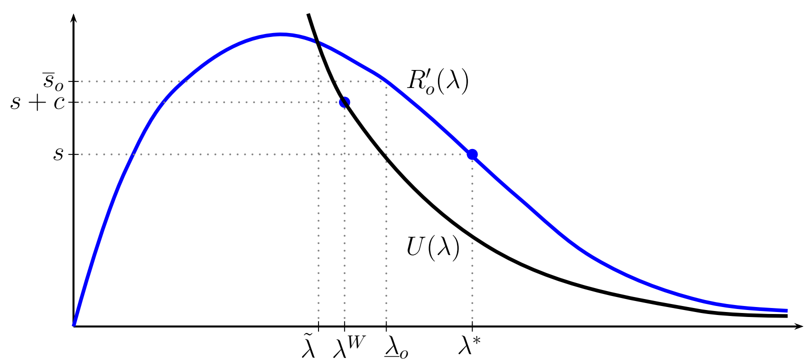

Figure 10: Welfare.

In this case, since $U(\lambda) < R_o'(\lambda)$ for any $\lambda \geq \underline{\lambda}_o$, it follows that $\lambda^* > \lambda^w$ in any equilibrium with trade. If $\lambda^* < \overline{\lambda}^c$, then $s + c > s = R_o'(\lambda^*) > U(\lambda^*)$; if $\lambda^* = \overline{\lambda}^c$, then $s + c > c = U(\lambda^*)$. In the case of $\lambda^* = \overline{\lambda}^c$, there is also the inefficiency of $q^* < 1$, except when $s = R_o'(\overline{\lambda}^c)$. On the other hand, there is a range of (s, c) combinations such that $s + c < U(0)$ requires trade, $\lambda^w > 0$, but either $s > \overline{s}_o$ or $\overline{\lambda}^c < \underline{\lambda}_o$ precludes trade in equilibrium, meaning $\lambda^w > \lambda^* = 0$.

We have not examined in detail the relationship between equilibrium and welfare in the PU scenario, but we expect the equilibria to be generally inefficient in that scenario as well. Since the maximal equilibrium in the PU scenario involves lower participation than that of the PO scenario, there will be less inefficiency due to excessive recruiting.[19]

For a general G satisfying our regularity assumptions, we have already established that U is decreasing and R'_o is single-peaked, as shown in Figure 10. The fact that U intersects R'_o for the first time at some point $\widetilde{\lambda}$ to the right of the maximum of R'_o also holds for general G (see Claim 14 in the appendix). Some other details in the figure have not been established analytically for general G,[20] but these details do not affect the overall conclusion that the equilibria are suboptimal.

The excessive recruitment noted above for the uniform distribution recalls the result in Szech (2011) that when G exhibits an increasing hazard rate, the equilibrium participation in an FPA with linear recruitment cost will exceed the welfare-maximizing level. Our model is somewhat different because of the stochastic arrival and entry costs, but the insight is similar.

The condition (18) implies that, for a given λ, the individual bidders' entry decisions are efficient. This is the counterpart in our model of a central finding in the literature on costly entry; see Levin and Smith (1994).

Note that λ^w maximizes welfare only within the constraints of the original Poisson contacting process. For example, welfare would be higher if the planner could coordinate entry among the contacted bidders, avoiding excessive participation numbers when the realized number of contacted bidders is too high.[21]

6.2 Commitment

Consider the PO scenario under the assumption that the bidders can observe the seller's choice of recruitment effort γ.

Claim 10 *Suppose γ is observable and $s \leq \bar{s}_o$. Then the seller's profit-maximizing effort $\hat{\gamma}$ is as follows:*

(i) if $\bar{\lambda}^c \geq \underline{\lambda}_o$, then $\hat{\gamma} = \min\{\bar{\lambda}^c, \lambda_o(s)\}$;

(ii) if $\bar{\lambda}^c < \underline{\lambda}_o$, then $\hat{\gamma} = \bar{\lambda}^c$ if $R_o(\bar{\lambda}^c) \geq \bar{\lambda}^c s$ and $\hat{\gamma} = 0$ otherwise.

[19] When the value distribution G is uniform, then numerically $\xi < U$, meaning, the seller may often recruit too few bidders.

[20] If G is uniform, we have shown that $U_o(\lambda)$ and $R'_o(\lambda)$ intersect only once and that $\widetilde{\lambda}$ is below $\underline{\lambda}_o$. For general G, we have not established these properties. However, loosely speaking, we expect $U_o(\lambda)$ to be mostly below $R'_o(\lambda)$ since $R_o(\lambda)$ is below $T(\lambda)$ and converging to it.

[21] This is similar to the observation of Levin and Smith (1994) that the randomness over participation numbers in symmetric mixed equilibria reduces welfare relative to the deterministic participation numbers in asymmetric pure equilibria.

The claim is immediate from our previous discussions. Of course, $q = 1$ for all choices of $\hat{\gamma}$. Whenever $\hat{\gamma} = \bar{\lambda}^c$, commitment to $\hat{\gamma} = \bar{\lambda}^c$ strictly improves the seller's profit (except for non-generic parameters): it enables positive trade when $\bar{\lambda}^c < \underline{\lambda}_o$ and $R_o(\bar{\lambda}^c) > \bar{\lambda}^c s$, and it saves on recruitment costs when $\bar{\lambda}^c > \underline{\lambda}_o$ and $R_o'(\bar{\lambda}^c) > s$. If $\hat{\gamma} \neq \bar{\lambda}^c$, then commitment does not change the outcome.

Regarding welfare, our observations from Section 6.1 imply that when $\hat{\gamma} = \bar{\lambda}^c$, the recruitment choice with commitment is always inefficiently large, since $\bar{\lambda}^c > \lambda^w$ if $s > 0$.

6.3 Uniqueness of equilibrium in the PO scenario

The equilibrium outcome of the PO scenario is unique for almost all values of s and c (except when $s = \bar{s}_o$ or $\overline{\lambda}^c = \underline{\lambda}_o$), given the refinement imposed by the last condition of the equilibrium definition in Section 1.2.[22] Without the refinement, the no-trade outcome is always an equilibrium; more precisely,

- if $s > \bar{s}_o$ or $\overline{\lambda}^c < \underline{\lambda}_o$, then no-trade is still the unique equilibrium outcome;
- if $s < \bar{s}_o$ and $\overline{\lambda}^c > \underline{\lambda}_o$, there are now two equilibrium outcomes: one with $\lambda^* > 0$ and one with $\lambda^* = 0$.

In the second case, the additional no-trade equilibrium $\lambda^* = 0$ is supported by the off-path belief $\mu(\overline{\lambda}^c) = 1$ and $q^* \in (0, \frac{s}{\bar{s}_o}]$. That is, bidders contacted off-path conjecture that $\lambda = \overline{\lambda}^c$, which makes them indifferent among all choices of q, including q^*. Such an equilibrium violates the refinement, since $\overline{\lambda}^c$ implies strictly negative profits given $\overline{\lambda}^c > \underline{\lambda}_o$ and $q^* \leq \frac{s}{\bar{s}_o}$.[23]

Observe that this no-trade equilibrium is unconvincing on other grounds as well. First, when $s < \bar{s}_o$ and $\overline{\lambda}^c > \underline{\lambda}_o$, it is Pareto dominated by the equilibrium with trade. Second, it is not robust to perturbations. Consider a perturbation in which the seller is required to choose at least an effort $\gamma \geq \varepsilon > 0$, for some small $\varepsilon > 0$. As $\varepsilon \to 0$, this perturbed game has a unique limit outcome that corresponds to the equilibrium with trade. This is because for any $q \in (0, 1)$ such that $\frac{s}{q} \geq \bar{s}_o$, the seller's best response is either $\lambda = \varepsilon$ or $\underline{\lambda}_o$ (or mixing between them). However, in all these cases, $\overline{\lambda}^c > \underline{\lambda}_o$ implies that the bidders have a strict incentive to enter, so that $q = 1$.

[22] If $\gamma^* = 0$, then no $\hat{\gamma}$ in the support of μ yields negative profits. A slightly more general formulation would require every $\hat{\gamma}$ in the support of μ to be a best response by the seller to q^* and β^*.

[23] If $q^* = \frac{s}{\bar{s}_o}$, then $\lambda = \underline{\lambda}_o$ is also a best response, but still $\underline{\lambda}_o \neq \bar{\lambda}^c$.

Formally, since this game is not finite (it has a continuum of actions and an unbounded number of players), we cannot directly apply the concept of stability in the sense of Kohlberg and Mertens (1986). However, for a discretized version in which the seller chooses λ from a finite grid (that contains 0, $\overline{\lambda}^c$, and $\underline{\lambda}_o$), we can define a refinement in the spirit of stability, requiring that the equilibrium be immune to all vanishing fully mixed perturbations. It is fairly immediate that the no-trade equilibrium will fail such refinement, while the unique equilibrium with trade will survive it.[24]

We can also confirm the instability of the no-trade equilibrium indirectly by observing that it fails the invariance property of stable equilibrium. To see this, consider the equivalent extensive form in which the seller first chooses between $\lambda = 0$, which terminates the game, and another action, "$\lambda > 0$", which stands for all positive recruitment efforts. After taking the action "$\lambda > 0$", the seller chooses the specific λ and the bidders make their entry decisions. The unique subgame-perfect equilibrium here is the equilibrium with trade, by the same argument as used above for the variation with $\gamma \geq \varepsilon$.

6.4 Concluding remarks

This paper contributes to the market approach in auction theory, treating auctions as abstractions of less formal price formation.[25] We examine the roles and interactions of three ubiquitous frictions in such scenarios: costly recruitment, costly bidder entry, and the seller's inability to commit. Our findings demonstrate that their interaction can lead to significant inefficiencies and, in some cases, the complete unraveling of trade.

Many open questions remain. For instance, it might be interesting to study the comparative statics of the outcome with respect to the value distribution—how does the latter affect the seller's recruitment effort and the inefficiency? Relatedly, one could explicitly model a setting where the bidders acquire information at some cost or where the seller provides information to bidders at some cost, implicitly subsidizing entry. As highlighted in the introduction, the fundamental inefficiencies of informal auctions may induce demand for intermediaries; it may be worthwhile to study the role of such intermediaries in our framework.

[24] Note, however, that the no-trade equilibrium will survive an analogously defined refinement in the spirit of perfect equilibrium, since we can focus on a sequence of perturbations for which the expectation conditional on $\lambda > 0$ is $\overline{\lambda}^c$.

[25] This perspective has its roots in the early literature (Milgrom, 1979; Wilson, 1977).

7 Appendix

7.1 Proofs for the PO scenario

7.1.1 Bidders' ex-ante expected payoff

Claim 11 *The bidders' payoff is*

$$U(\lambda) = \int_0^1 e^{-(1-G(v))\lambda}[1 - G(v)]dv. \tag{19}$$

In particular, U is strictly decreasing and continuous, $U(0) = E[v]$, and $\lim_{\lambda\to\infty} U(\lambda) = 0$.

Proof. When a bidder with value v is in an auction with a total of n bidders, he wins with probability $G(v)^{n-1}$ given that β_{FPA} is strictly increasing. Also, the lowest type obtains no payoff, $U(n, 0) = 0$. Therefore, the usual envelope argument implies that the expected equilibrium payoffs must be $U(n, v) = \int_0^v G(x)^{n-1} dx$; see Krishna (2000).[26]

Hence, when n is drawn from a Poisson distribution with mean λ, the expected payoff of type v is

$$\sum_{n=1}^{\infty} e^{-\lambda} \frac{\lambda^{n-1}}{(n-1)!} U(n, v) = \int_0^v \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n G(x)^n}{n!} dx = \int_0^v e^{-\lambda(1-G(x))} dx.$$

Therefore, the ex-ante expected payoff is

$$U(\lambda) = \int_0^1 \left(\int_0^v e^{-\lambda(1-G(x))} dx \right) g(v)\, dv.$$

Changing the order of integration yields (19). Inspection of the right-hand side of (19) immediately implies the claimed properties of U. ■

Payoff equivalence. The above characterization of the bidders' payoffs applies to any standard auction format in which type v in an auction with n bidders wins with probability $G(v)^{n-1}$ and $U(n, 0) = 0$. In particular, the bidders' expected payoff in an SPA is also $U(\lambda)$.

[26] We start from the characterization of β_{FPA}, since we have already stated it. Of course, the standard proof works in the opposite direction.

7.1.2 The seller's revenue

Claim 12 *(i)* $R_o(\lambda)$ *is strictly increasing,* $R_o(0) = 0$*, and* $\lim_{\lambda\to\infty} R_o(\lambda) = 1$*.*

(ii) $R_o(\lambda)$ *is continuously differentiable,* $R_o'(0) = 0$*,* $R_o'(\lambda) \to 0$ *as* $\lambda \to \infty$*, and* R_o' *is single-peaked.*

(iii) $\frac{R_o(\lambda)}{\lambda}$ *is single-peaked; at its peak,* $\frac{R_o(\lambda)}{\lambda} = R_o'(\lambda)$*.*

Proof. The total surplus (gross of the recruitment costs) is the expectation of the first order statistic of v given Poisson(λ) distributed participation,

$$\text{Total Surplus}(\lambda) = \int_0^1 \left[1 - e^{-(1-G(v))\lambda}\right] dv.$$

It is equal to the sum of the revenue, $R_o(\lambda)$, and the total bidders' expected payoff, $\lambda U(\lambda)$. Hence, revenue is simply the difference of Total Surplus(λ) and $\lambda U(\lambda)$, that is,

$$R_o(\lambda) = \int_0^1 \left[1 - e^{-(1-G(v))\lambda} - e^{-(1-G(v))\lambda}(1 - G(v))\lambda\right] dv. \tag{20}$$

Therefore, simple rewriting shows

$$\frac{d}{d\lambda}R_o(\lambda) = \int_0^1 \lambda(1 - G(v))^2 e^{-(1-G(v))\lambda} dv. \tag{21}$$

Parts (i) and (ii). Positivity, continuity, and values at $\lambda = 0$ and $\lambda \to \infty$ are obvious from (20) and (21). To establish that R_o' is single-peaked, consider the second derivative,

$$\begin{aligned} \frac{d^2}{d\lambda^2}R_o(\lambda) &= \int_0^1 (1 - G(v))^2 e^{-(1-G(v))\lambda} dv - \int_0^1 \lambda(1 - G(v))^3 e^{-(1-G(v))\lambda} dv \\ &= e^{-\lambda}\left(\frac{1}{g(0)} - \int_0^1 (1 - G(v))^2 e^{G(v)\lambda}\left[v - \frac{1 - G(v)}{g(v)}\right]_v' dv\right), \end{aligned} \tag{22}$$

using integration by parts.

Recall that by assumption, $\left[v - \frac{1-G(v)}{g(v)}\right]_v' > 0$. Thus, the integral on the last line of (22) is positive and increasing in λ, while the first term is positive and independent of λ. Therefore, $\frac{d^2}{d\lambda^2}R_o(\lambda) < 0$ for large λ, and once it turns negative, it stays negative. Inspection of the first line of (22) reveals that $\frac{d^2}{d\lambda^2}R_o(\lambda) > 0$ for $\lambda \in [0, \varepsilon]$ for some $\varepsilon > 0$. The two observations imply that $\frac{d}{d\lambda}R_o(\lambda)$ is single-peaked.

Part (iii). Immediate from Parts (i)–(ii) and $d(R_o(\lambda)/\lambda)d\lambda = \left[R_o'(\lambda) - \frac{R_o(\lambda)}{\lambda}\right]/\lambda$. ■

Revenue equivalence. The above argument applies again to any standard auction format in which type v in an auction with n bidders wins with probability $G(v)^{n-1}$. In particular, $R_o(\lambda)$ is also the seller's revenue in the SPA.

7.1.3 Proof of Proposition 2

Suppose $s > \bar{s}_o$. We already noted in the proof of Proposition 1 that there is a unique equilibrium. In that equilibrium, $\lambda^* = 0$, and $\mu^*(0) = 1$ (since all λ in the support must be profit-maximizing). Hence, $q^* = 1$ from bidder optimality.

For the cases with $s < \bar{s}_o$, the profiles described satisfy the bidders' and the seller's optimality conditions. The refinement is vacuous when $\lambda^* > 0$, so these are equilibria.

We show that there are no other equilibria with $\lambda^* > 0$. We first show that when $\lambda^* > 0$,

$$\lambda^* = \min\{\overline{\lambda}^c, \lambda_o(s)\}. \tag{23}$$

From seller optimality (6), it follows that if $\lambda^* > 0$, then $\lambda^* \geq \underline{\lambda}_o$ and $R_o'(\lambda^*) = \frac{s}{q^*}$. From bidder optimality (2), it follows that q^* may differ from 1 only if $\lambda^* = \overline{\lambda}^c$. Therefore, the only possibilities are $\lambda^* = \overline{\lambda}^c$ or $\lambda^* = \lambda_o(s)$. If $\overline{\lambda}^c > \lambda_o(s)$, then for any q, the fact that R_0' is decreasing means that $R_o'\left(\overline{\lambda}^c\right) < \frac{s}{q}$, so $\overline{\lambda}^c$ cannot be an equilibrium outcome. If $\overline{\lambda}^c < \lambda_o(s)$, then $U(\lambda_o(s)) < c$, so $\lambda_o(s)$ cannot be an equilibrium outcome. This proves (23).

In the case $\bar{s}_o > s > R_o'(\overline{\lambda}^c)$, since R_o' is decreasing, $\lambda_o(s) < \overline{\lambda}^c$. So (23) requires $\lambda^* = \lambda_o(s)$, and $\lambda_o(s) < \overline{\lambda}^c$ requires $q^* = 1$, which establishes the uniqueness of the equilibrium with $\lambda^* > 0$ in this case.

In the case $s < R_o'(\overline{\lambda}^c)$, since R_o' is decreasing, $\lambda_o(s) > \overline{\lambda}^c$. So (23) requires $\lambda^* = \overline{\lambda}^c$. Hence, (6) requires that q^* satisfy $R_o'(\overline{\lambda}^c) = \frac{s}{q^*}$, establishing the uniqueness of the equilibrium with $\lambda^* > 0$ in this case, too.

It remains to show that there is no equilibrium with $\lambda^* = 0$.

Seller optimality requires that $\max_\lambda \Pi_o(\cdot, q^*) = 0$ when $\lambda^* = 0$. So, by the equilibrium refinement, $\Pi_o(\lambda, q^*) = 0$ for any λ in the support of μ^*. Thus, by (6), the support of μ^* is contained in $\{0, \underline{\lambda}_o\}$. Since $\underline{\lambda}_o < \overline{\lambda}^c$ by the hypothesis of the proposition, and since U is decreasing, $E_{\mu^*}[U(\lambda)] > c$, and so $q^* = 1$. However, when $s < \bar{s}_o$, we have $\Pi_o(\underline{\lambda}_o, 1) > 0$, which contradicts the requirement that $\max_\lambda \Pi_o(\cdot, q^*) = 0$; thus, there is no equilibrium with $\lambda^* = 0$.

7.2 Proofs for the PU scenario

7.2.1 Proof of Claim 1: The bidding strategy

A standard overbidding argument implies that the equilibrium bidding strategy $\beta_{\hat{\lambda}}$ must be strictly increasing whenever $\hat{\lambda} > 0$. Hence, given expected participation $\hat{\lambda}$, a bidder with value v wins with probability $e^{-\hat{\lambda}(1-G(v))}$. Hence, the standard envelope argument implies that the equilibrium payoffs of a bidder with value v are given by

$$\int_0^v e^{-\hat{\lambda}(1-G(x))} dx,$$

and the ex-ante expected payoffs are $U(\hat{\lambda})$, as in the PO scenario.

Given a winning probability $e^{-\hat{\lambda}(1-G(v))}$, this implies that the bid $\beta_{\hat{\lambda}}(v)$ must solve

$$\int_0^v e^{-\hat{\lambda}(1-G(x))} dx = e^{-\hat{\lambda}(1-G(v))} \left(v - \beta_{\hat{\lambda}}(v)\right),$$

and so

$$\beta_{\hat{\lambda}}(v) = v - \int_0^v e^{-\hat{\lambda}(G(v)-G(x))} dx.$$

Revenue equivalence. Given any increasing bidding strategy, the allocation is the same in the PO and the PU scenario, which implies that the total realized surplus is the same. When $\hat{\lambda} = \lambda$, the envelope argument implies that the bidders' ex-ante expected payoffs are the same in the two scenarios. Therefore the revenues are also the same:

$$R_o(\lambda) = R_u(\lambda, \beta_\lambda). \tag{24}$$

7.2.2 Claim 13

Claim 13 *(i) $R_u(\lambda, \beta_{\hat{\lambda}})$ is twice differentiable (in λ and $\hat{\lambda}$), and for $\hat{\lambda} > 0$ it is strictly concave in λ.*

(ii) The function $\xi(\lambda)$ is continuous, $\xi(0) = 0$, and $\lim_{\lambda\to\infty} \xi(\lambda) = 0$.

Let $F_u(\cdot|\lambda, \beta_{\hat{\lambda}})$ be the distribution of the price received by the seller, given that actual participation is Poisson(λ)-distributed and all bidders bid according to $\beta_{\hat{\lambda}}$, where the no-trade event is identified with price 0. Let $\widetilde{\beta}_\lambda^{-1}$ denote the "generalized inverse" of β_λ, defined as follows: $\widetilde{\beta}_\lambda^{-1} = \beta_\lambda^{-1}$ over $[0, \beta_\lambda(1))$ and $\widetilde{\beta}_\lambda^{-1} \equiv 1$ over $[\beta_\lambda(1), 1]$. Note that this implies that $\widetilde{\beta}_0^{-1} \equiv 1$. Therefore,

$$F_u\left(b|\lambda,\beta_{\hat{\lambda}}\right)=e^{-\lambda\left(1-G\left(\widetilde{\beta}_{\hat{\lambda}}^{-1}(b)\right)\right)}. \tag{25}$$

Observe that F_u is indeed a cumulative distribution function and is well defined for $\hat{\lambda}=0$ as well: since $\beta_{\hat{\lambda}}$ is non-decreasing for any $\hat{\lambda}\geq 0$, $\widetilde{\beta}_{\hat{\lambda}}^{-1}$ is non-decreasing and so is F; since $\widetilde{\beta}_{\hat{\lambda}}^{-1}(1)=1$, $F_u\left(1|\lambda,\beta_{\hat{\lambda}}\right)=1$, and $F_u\left(0|\lambda,\beta_{\hat{\lambda}}\right)=e^{-\lambda}<1$. Then,

$$R_u(\lambda,\beta_{\hat{\lambda}})=\int_0^1[1-F_u\left(b|\lambda,\beta_{\hat{\lambda}}\right)]db=\int_0^1[1-e^{-\lambda\left(1-G\left(\widetilde{\beta}_{\hat{\lambda}}^{-1}(b)\right)\right)}]db, \tag{26}$$

where the last equality is obtained by substitution from (25). This and the characterization of $\beta_{\hat{\lambda}}$ in (9) imply that R_u is twice continuously differentiable in λ and $\hat{\lambda}$:

$$\frac{\partial}{\partial\lambda}R_u(\lambda,\beta_{\hat{\lambda}})=\int_0^1\left(1-G\left(\widetilde{\beta}_{\hat{\lambda}}^{-1}(b)\right)\right)e^{-\lambda\left(1-G\left(\widetilde{\beta}_{\hat{\lambda}}^{-1}(b)\right)\right)}db. \tag{27}$$

If $\hat{\lambda}>0$, then $\widetilde{\beta}_{\hat{\lambda}}^{-1}(b)<1$ for all b small enough. Therefore,

$$\frac{\partial^2}{\partial\lambda^2}R_u(\lambda,\beta_{\hat{\lambda}})<0, \tag{28}$$

so that $R_u\left(\lambda,\beta_{\hat{\lambda}}\right)$ and $\Pi_u(\lambda,\beta_{\hat{\lambda}},q)$ are strictly concave in λ. By definition,

$$\xi\left(\lambda\right)=\int_0^1\left(1-G\left(\widetilde{\beta}_{\lambda}^{-1}(b)\right)\right)e^{-\lambda\left(1-G\left(\widetilde{\beta}_{\lambda}^{-1}(b)\right)\right)}db.$$

The continuity of $\xi\left(\lambda\right)$ and its other properties follow directly from this functional form and the properties of $\widetilde{\beta}_{\lambda}^{-1}$. This proves the claim.

7.3 Comparison of PO and PU scenarios

7.3.1 Proof of Claim 3

Using (26),

$$\begin{aligned}\frac{\partial}{\partial\hat{\lambda}}R_u(\lambda,\beta_{\hat{\lambda}})|_{\hat{\lambda}=\lambda} &= \left(\frac{\partial}{\partial\hat{\lambda}}\int_0^1[1-e^{-\lambda\left(1-G\left(\widetilde{\beta}_{\hat{\lambda}}^{-1}(b)\right)\right)}]db\right)|_{\hat{\lambda}=\lambda}\\ &= -\int_0^1\lambda g\left(\widetilde{\beta}_{\lambda}^{-1}(b)\right)\frac{\partial}{\partial\lambda}\widetilde{\beta}_{\lambda}^{-1}(b)e^{-\lambda\left(1-G\left(\widetilde{\beta}_{\lambda}^{-1}(b)\right)\right)}db,\end{aligned}$$

and from (9),

$$\frac{\partial}{\partial\lambda}\widetilde{\beta}_{\lambda}^{-1}(b)=-\frac{\frac{\partial}{\partial\lambda}\beta_{\lambda}(v)}{\frac{\partial}{\partial v}\beta_{\lambda}(v)}=-\frac{\int_0^v\left(G(v)-G(x)\right)e^{-\lambda(G(v)-G(x))}dx}{\lambda g(v)\int_0^v e^{-\lambda(G(v)-G(x))}dx}<0,$$

where $v=\widetilde{\beta}_{\lambda}^{-1}(b)$. Therefore, we have $\frac{\partial}{\partial\hat{\lambda}}R_u(\lambda,\beta_{\hat{\lambda}})_{\hat{\lambda}=\lambda}>0$ for all λ, which implies Part (i) of the claim given (12).

Part (ii): We have

$$R_u(\lambda,\beta_{\lambda})=\int_0^{\lambda}\frac{\partial}{\partial t}R_u(t,\beta_{\lambda})dt>\lambda\frac{\partial}{\partial\lambda}R_u(\lambda,\beta_{\hat{\lambda}})_{\hat{\lambda}=\lambda},\tag{29}$$

since (28) implies that $\frac{\partial}{\partial t}R_u(t,\beta_{\lambda})$ is strictly decreasing in t. Since by revenue equivalence $R_u(\lambda,\beta_{\lambda})=R_o(\lambda)$ for all λ, it follows from (29) that

$$\frac{R_o(\lambda)}{\lambda}>\frac{\partial}{\partial\lambda}R_u(\lambda,\beta_{\hat{\lambda}})_{\hat{\lambda}=\lambda}.$$

The claim then follows from $\bar{s}_o=\max\frac{R_o(\lambda)}{\lambda}$ and $\bar{s}_u=\max\frac{\partial}{\partial\lambda}R_u(\lambda,\beta_{\hat{\lambda}})_{\hat{\lambda}=\lambda}$.

7.3.2 Proof of Claim 4

The claim is immediate for s such that $\bar{s}_o>s>\bar{s}_u$, since in this case there is positive trade in the PO scenario but no trade in the PU scenario. (The case corresponds to s_H in the figure.)

For values of s in the range $\bar{s}_u\geq s>R_o'\left(\bar{\lambda}^c\right)$, the equilibrium of the PO scenario has $0<\lambda_o^*<\bar{\lambda}^c$ and $q^*=1$. (This case corresponds to s_M in the figure.) By Claim 3, the λ_u^* in every equilibrium of the PU scenario must be different. For both scenarios and all equilibria, $q^*=1$, given that participation is below $\bar{\lambda}^c$. Now, a revealed-preference argument implies that the seller's profit is strictly higher in the PO scenario: given $q^*=1$, the seller could have chosen to recruit just λ_u^* bidders. By revenue equivalence (11), the seller's payoff at λ_u^* is

$$\Pi_o(\lambda_u^*,q^*)=R_u(\lambda_u^*,\beta_{\lambda_u^*})-s\lambda_u^*=\Pi_u(\lambda_u^*,\beta_{\lambda_u^*},q^*).$$

However, $\lambda_o^*\neq\lambda_u^*$ is the unique profit-maximizing choice, and so $\Pi_o(\lambda_o^*,q^*)>\Pi_u(\lambda_u^*,\beta_{\lambda_u^*},q^*)$.

7.3.3 Proof of Claim 5

Given the hypothesis, in the PO scenario, the seller's equilibrium profit is

$$R_o\left(\bar{\lambda}^c\right) - \bar{\lambda}^c R_o'\left(\bar{\lambda}^c\right). \tag{30}$$

In the PU scenario, there is an equilibrium with the same participation, $\lambda_u^* = \bar{\lambda}^c$, and q_u^* satisfying $\frac{s}{q_u^*} = \xi\left(\bar{\lambda}^c\right)$. Hence, the seller's profit in this equilibrium is

$$R_u(\bar{\lambda}^c, \beta_{\bar{\lambda}^c}) - \bar{\lambda}^c \xi\left(\bar{\lambda}^c\right). \tag{31}$$

From Claim 3, $R_o'\left(\bar{\lambda}^c\right) > \xi\left(\bar{\lambda}^c\right)$, meaning that total recruitment costs are higher in the PO scenario. Thus, given that $R_o\left(\bar{\lambda}^c\right) = R_u(\bar{\lambda}^c, \beta_{\bar{\lambda}^c})$ by revenue equivalence, the PO profit in (30) is strictly smaller than the PU profit in (31).

7.4 Uncertain recruitment costs—Proofs

7.4.1 Proof of Claim 9

The equilibrium has a cutoff structure with cutoff $q^*\overline{s}_o$, and with λ_s^* as in (16). It is also immediate that the configurations described in Parts (i)–(iii) are equilibria.

Part (i). For $c > \overline{c}$, the equilibrium is just the same as the $\lambda = 0$ equilibrium of the PO scenario with $s = \underline{s}$. That is, the support of the off-path beliefs is $\{0, \lambda_o(\overline{s}_o)\}$, and these values are optimal for type $\underline{s}$ given $q^* = \underline{s}/\overline{s}_o$. The probabilities μ satisfy $\mu(0)U(0) + \mu(\lambda_o(\overline{s}_o))U(\lambda_o(\overline{s}_o)) = c$. The uniqueness is also the same as in the corresponding PO scenario. For $c = \overline{c}$, apart from the above equilibrium, there is also an equilibrium in which only type $\underline{s}$ can be active. Since type $\underline{s}$ is of zero measure, we think of this as a no-trade outcome as well.

Parts (ii) and (iii). The λ_s^* and q^* are optimal, and there are no off-path moves. To see that the equilibria in Parts (ii) and (iii) are unique among those with $\lambda_s > 0$ for some s, suppose that, in either scenario, there are two equilibria with $q_1^* < 1$ and $q_2^* > q_1^*$. The corresponding equilibrium values of λ, namely $\lambda_s^*(q_1^*)$ and $\lambda_s^*(q_2^*)$, are given by (16). Hence, $V(\boldsymbol{\lambda}^*(q_2^*)) < V(\boldsymbol{\lambda}^*(q_1^*)) = c$, in contradiction to $q_2^* > 0$. Therefore, to establish uniqueness, we only have to rule out the no-trade equilibrium. Such an equilibrium can be supported only by the beliefs μ described in the proof of Part (i). But $c < \overline{c} = U(\lambda_o(\overline{s}_o))$ implies that $\mu(0)U(0) + \mu(\lambda_o(\overline{s}_o))U(\lambda_o(\overline{s}_o)) = c$ cannot hold.

7.4.2 Proof of Proposition 3

Let $V(\boldsymbol{\lambda_o}, \underline{s})$ from (15) with $\underline{s}$ as explicit argument. Its limit for $\underline{s} \to 0$ is

$$\underline{c} := \frac{\int_0^{\overline{s}_o} \lambda_o(s) U(\lambda_o(s))ds}{\int_0^{\overline{s}_o} \lambda_o(s)\, ds} = \lim_{\underline{s}\to 0} \frac{\int_{\underline{s}}^{\overline{s}_o} \lambda_o(s) U(\lambda_o(s))ds}{\int_{\underline{s}}^{\overline{s}_o} \lambda_o(s)\, ds}.$$

The limit exists since its argument is increasing in $\underline{s}$. Now, $\underline{c} > 0$ follows from $\int_{\underline{s}}^{\overline{s}_o} \lambda_o(s)\, ds$ being uniformly bounded, which is established by the following calculation:

$$\begin{aligned} \int_{\underline{s}}^{\overline{s}_o} \lambda_o(s)\, ds &= [s\lambda_o(s)]_{\underline{s}}^{\overline{s}_o} - \int_{\underline{s}}^{\overline{s}_o} s\lambda_o'(s)ds = [s\lambda_o(s)]_{\underline{s}}^{\overline{s}_o} - \int_{\underline{s}}^{\overline{s}_o} R_o'(\lambda_o(s))\lambda_o'(s)ds \\ &= [s\lambda_o(s)]_{\underline{s}}^{\overline{s}_o} - \int_{\lambda_o(\underline{s})}^{\lambda_o(\overline{s}_o)} R_o'(\lambda)d\lambda = R_o(\lambda_o(\underline{s})) - \underline{s}\lambda_o(\underline{s}) - [R_o(\lambda_o(\overline{s}_o)) - \overline{s}_o\lambda_o(\overline{s}_o)] \\ &= \Pi_o(\lambda_o(\underline{s})) - \Pi_o(\lambda_o(\overline{s}_o)) = \Pi_o(\lambda_o(\underline{s})) \leq 1 \end{aligned}$$

The first equality is from integration by parts, the second from the first-order condition of profit maximization, $s = R_o'(\lambda_o(s))$, the third from changing the integration variable, and the last from $\Pi_o(\lambda_o(\overline{s}_o)) = 0$ (by definition of $\overline{s}_o$) and that revenue and profit are bounded by the maximal posssible valuation, 1.

Hence, $\int_0^{\overline{s}_o} \lambda_o(s)\, ds = \lim_{\underline{s}\to 0} \int_{\underline{s}}^{\overline{s}_o} \lambda_o(s)\, ds \leq 1$ implying $\underline{c} > 0$. Since $U(\lambda_o(\overline{s}_o)) > U(\lambda_o(s))$ for $s < \overline{s}_o$, we have $\underline{c} < \overline{c} = U(\lambda_o(\overline{s}_o))$.[27]

If $c \in (\underline{c}, \overline{c})$, then for small enough $\underline{s}$, $V(\boldsymbol{\lambda_o}, \underline{s}) < c$, the equilibrium is given by Part (iii) of Claim 9 and q^* satisfies

$$c = \frac{\int_{\underline{s}}^{\overline{s}_o q^*} \lambda_o\left(\frac{s}{q^*}\right) U(\lambda_o\left(\frac{s}{q^*}\right))ds}{\int_{\underline{s}}^{\overline{s}_o q^*} \lambda_o\left(\frac{s}{q^*}\right) ds} = \frac{\int_{\underline{s}/q^*}^{\overline{s}_o} \lambda_o(s) U(\lambda_o(s))ds}{\int_{\underline{s}/q^*}^{\overline{s}_o} \lambda_o(s)\, ds}, \tag{32}$$

where the second equality follows from change of the integration variable.[28]

When $\underline{s} \to 0$, it must be that $q^* \to 0$, for otherwise the RHS of (32) converges to $\underline{c} < c$, contradicting the equation. Therefore,

$$\lim_{\underline{s}\to 0} \Pr(\{s : \lambda_s^* = 0\}|c, \underline{s}) = \lim_{\underline{s}\to 0} \frac{\overline{s}_o - q^*\overline{s}_o}{\overline{s}_o - \underline{s}} = 1.$$

[27]In fact, $\lim_{\underline{s}\to 0} \Pi_o(\lambda_o(\underline{s})) = 1$ from $\underline{s}\lambda_o(\underline{s}) \to 0$ (see Footnote 10), and so $\int_0^{\overline{s}_o} \lambda_o(s)\, ds = 1$, implying that $\mathbb{E}[\lambda(s)] = \int_0^{\overline{s}_o} \lambda_o(s) \frac{ds}{\overline{s}_0} = \overline{s}_0$.

[28]This step is where we make use of the uniform distribution.

If $c < \underline{c}$, then for any $\underline{s}$, $V(\boldsymbol{\lambda_o}, \underline{s}) > c$, the equilibrium is given by Part (ii) of Claim 9 and $q^* = 1$. Therefore, $\Pr(\{s : \lambda_s^* = 0\}|c, \underline{s}) = 0$.

7.5 Welfare–Proofs

Claim 14 *(i) For any Λ, there is $\lambda > \Lambda$ such that $U(\lambda) < R_o'(\lambda)$. (ii) There is $\widetilde{\lambda} > \overline{\lambda}$ such that $U(\lambda) \geq R_o'(\lambda)$ for $\lambda \leq \widetilde{\lambda}$ and $U(\lambda) < R_o'(\lambda)$ at least over some interval just above $\widetilde{\lambda}$.*

Proof. Obviously, $R_o(\lambda)$ is also the residual surplus not received by the bidders,

$$R_o(\lambda) = T(\lambda) - \lambda U(\lambda),$$

and $R_o(\lambda) \to T(\lambda)$ as $\lambda \to \infty$.

Part (i). If there is Λ such that $U(\lambda) > R_o'(\lambda)$ for all $\lambda \geq \Lambda$, then, by (18), for all such λ, $T(\lambda) - R_o(\lambda) > T(\Lambda) - R_o(\Lambda) > 0$, which contradicts the fact that $R_o(\lambda) \to T(\lambda)$ as $\lambda \to \infty$.

Part (ii). By (18),

$$R_o'(\lambda) = -\lambda U'(\lambda) = \lambda \int_0^1 e^{-(1-G(v))\lambda}[1 - G(v)]^2 dv \tag{33}$$

and

$$U(\lambda) - R_o'(\lambda) = U(\lambda) + \lambda U'(\lambda) = \int_0^1 e^{-(1-G(v))\lambda}[1 - G(v)]\left[1 - (1 - G(v))\lambda\right] dv. \tag{34}$$

Therefore,

$$\begin{aligned} R_o''(\lambda) &= -U'(\lambda) - \lambda U''(\lambda) = \int_0^1 e^{-(1-G(v))\lambda}[1 - G(v)]^2 dv \\ &\quad -\lambda \int_0^1 e^{-(1-G(v))\lambda}[1 - G(v)]^3 dv \\ &= \int_0^1 e^{-(1-G(v))\lambda}[1 - G(v)]^2 \left[1 - (1 - G(v))\lambda\right] dv. \end{aligned} \tag{35}$$

Recall that $R_o'(\lambda)$ is single-peaked and let $\overline{\lambda}$ denote the argument of the peak. Thus, $R_o''(\overline{\lambda}) = 0$, and it follows from (35) that there must be x such that $(1 - G(x))\overline{\lambda} = 1$, so the integrand on the right-hand side of (35) is positive for $v > x$ and negative for $v < x$.

Therefore,

$$\begin{aligned}
0 &= R_o''(\overline{\lambda}) < \int_0^x e^{-(1-G(v))\overline{\lambda}}[1-G(x)][1-G(v)]\left[1-\left(1-G(v)\right)\overline{\lambda}\right]dv \\
&\quad + \int_x^1 e^{-(1-G(v))\overline{\lambda}}[1-G(x)][1-G(v)]\left[1-\left(1-G(v)\right)\overline{\lambda}\right]dv \\
&= [1-G(x)]\int_0^1 e^{-(1-G(v))\overline{\lambda}}[1-G(v)]\left[1-\left(1-G(v)\right)\overline{\lambda}\right]dv \\
&= [1-G(x)][U(\overline{\lambda})-R_o'(\overline{\lambda})].
\end{aligned}$$

The first inequality follows from $1-G(x) < 1-G(v)$ for the range $v < x$ where the integrand is negative, and from $1-G(x) > 1-G(v)$ for the range $v > x$ where the integrand is positive; the last equality follows from (34). Therefore, $U(\overline{\lambda}) > R_o'(\overline{\lambda})$. Since U is decreasing and R_o' is increasing for $\lambda < \overline{\lambda}$, it follows that $U(\lambda) > R_o'(\lambda)$ for all $\lambda \leq \overline{\lambda}$. This and Part (i) imply that U and R_o' first intersect at some $\widetilde{\lambda} > \overline{\lambda}$. ■

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A Online appendix

This appendix is not intended for publication. It includes the discussion of some variations of the model, namely, optimal fees, (optimal) reserve prices, and bidder entry with known values. Finally, we illustrate our findings with a numerical example in which the value distribution is uniform.

A.1 Fees to influence participation

The question of optimal entry fees (or subsidies, when they are negative) is of secondary importance for this paper. First, it belongs more to the "design" paradigm of auction theory, which assumes significant seller commitment power and which we, therefore, de-emphasize in this paper. Second, non-serious bidders and sellers may abuse entry fees, so their credible implementation may require commitment and enforcement capabilities.

Here, we put aside those issues and consider a flat fee that is collected from, or offered to, each auction entrant in the PO scenario. Let D denote this fee ($D < 0$ means it is a subsidy). The subsequent interaction is formally equivalent to the PO scenario with bidders' cost $c + D$ and seller's marginal cost $\frac{s}{q} - D$. Let $\lambda^*(D)$ and $q^*(D)$ be the unique equilibrium magnitudes given D, and let $\overline{\lambda}^{c+D}$ be the solution to $U(\overline{\lambda}^{c+D}) = c + D$.

Claim 15 *(i) If the seller can commit to recruitment effort γ, then profit is maximized at $\gamma = \lambda^w$ with $D = s$.*
(ii) Suppose that the seller cannot commit to γ. If there exists a D that enables trade (i.e., $s - D \leq \overline{s}_o$ and $\overline{\lambda}^{c+D} \geq \underline{\lambda}_o$), then profit is maximized with D^* that satisfies $s - D^* = R_o'(\overline{\lambda}^{c+D^*})$, with

$$\lambda^*(D^*) = \overline{\lambda}^{c+D^*} \text{ and } q^*(D^*) = 1.$$

Part (ii) implies that the profit-maximizing fee is related to the equilibrium configuration that prevails when fees cannot be imposed (i.e., the case of $D = 0$). If $\lambda^*(0) < \overline{\lambda}^c$ (i.e., recruiting is unconstrained when fees are not allowed), then $D^* > 0$—a fee. If $\lambda^*(0) = \overline{\lambda}^c$, then $D^* < 0$—a subsidy.

Proof of Claim 15: Part (i). If the seller commits to $\gamma = \lambda^w$ and imposes an entry fee D such that $U(\lambda^w) = c + D$, then all contacted bidders will choose to enter: $q = 1$. Therefore, the surplus is maximal and the seller fully appropriates it since the bidders' payoff is 0. Since $U(\lambda)$ is decreasing and $\overline{\lambda}^c > \lambda^w$, it follows that $D = s > 0$.

Part (ii). Given D, we noted that this is the PO scenario with seller cost $\frac{s}{q} - D$ and bidders' cost $c + D$. Thus, in equilibrium given D, either $\lambda^*(D) \leq \overline{\lambda}^{c+D}$ and $R'_o(\lambda^*(D)) = s - D$, or $\lambda^*(D) = \overline{\lambda}^{c+D}$ and $R'_o(\lambda^*(D)) = \frac{s}{q^*} - D$.

If $\lambda^*(D) < \overline{\lambda}^{c+D}$, then any fee $D' > D$ such that the inequality still holds yields $\lambda^*(D') > \lambda^*(D)$ and higher profit.

If $\lambda^*(D) = \overline{\lambda}^{c+D}$ and $R'_o(\lambda^*(D)) > s - D$, then $q^*(D) < 1$. In this case, a fee of $D' < D$ defined by

$$s - D' = \frac{s}{q^*(D)} - D$$

results in $q^*(D') = 1$, $\overline{\lambda}^{c+D'} > \overline{\lambda}^{c+D}$, and $\lambda^*(D) = \lambda^*(D')$. This and the equality of the marginal recruitment costs imply that the profits for D and D' are equal as well. But then, by the argument of the previous paragraph, a fee slightly higher than D' would be even more profitable.

Thus, by elimination, D^* satisfies $\lambda^*(D^*) = \overline{\lambda}^{c+D^*}$ and $R'_o(\overline{\lambda}^{c+D^*}) = s - D^*$. ■

The welfare effects of fees depend on whether the seller can commit. With commitment, the optimal fee leads to an efficient equilibrium outcome: the seller chooses the welfare-maximizing effort γ^w and all bidders enter when contacted. Without fees, the outcome is generally inefficient, as discussed in Section 6.1. In contrast, without commitment, fees might actually decrease welfare. For instance, in the PO scenario without fees, if the parameters are such that $\lambda^w < \lambda^* < \overline{\lambda}^c$, then the profit-maximizing fee is strictly positive and pushes the equilibrium λ farther away from λ^w.

The version of our model with $s = 0$ and seller commitment is related to the model of Levin and Smith (1994). In this case, Claim 15(i) implies $D = 0$, which is consistent with their finding that an auction without fees maximizes the seller's profit.[29]

A.2 Reserve price

Here we discuss the effects of a reserve price r—a minimum bid below which the item is not sold. Before turning to the details, we note that the imposition of a reserve price requires commitment power that might not be available in the less formal settings we have in mind; see the discussion in the introduction. However, it is still interesting to understand the role of reserve prices even if their use is limited or imperfect.

[29] Furthermore, since the seller captures the full surplus, even if she could set a positive reserve price, doing so would only lower profits.

Assume that the auctions in both scenarios are subject to a reserve price $r > 0$ (not necessarily the optimal one). The equilibrium then differs in some details, but not in the main qualitative features, from that of the $r = 0$ case analyzed above. Graphically, the marginal revenue curves in the diagrams change somewhat: for small λ they lie above the $r = 0$ curve (in particular, the intercept at $\lambda = 0$ is $r(1 - G(r))$ rather than 0), and for large λ they lie below the $r = 0$ curve. However, their general properties (such as the single-peakedness of $dR_o/d\lambda$ and the relationship between the PO and PU curves) remain the same, as does the relationship between the curves and the nature of the equilibria. One immediate implication of the intercept at $\lambda = 0$ being $r(1 - G(r))$ is that, in the PU scenario, the no-trade equilibrium $\lambda = 0$ continues to exist only for $s \geq r(1 - G(r))$. For smaller s, the equilibrium necessarily involves trade. The reserve price also affects the bidders' entry decisions, since it lowers the benefit of entry for any level of anticipated participation.

Recall from the literature that, under the assumptions maintained on G, the revenue-maximizing $r_{\max}$ for a standard auction solves $r = \frac{1-G(r)}{g(r)}$. It follows immediately that this is also true for the FPA with stochastic participation in the PU scenario. Therefore, if the seller commits to r only after bidders enter, then the profit-maximizing r is $r_{\max}$.[30]

Let us add the argument r to our functions, writing $U(\lambda; r)$, $R_o(\lambda; r)$, $\Pi_o(\lambda, q; r)$, etc.

Claim 16 *(i) For a given λ, $R_o(\lambda; r)$ (and hence[31] $R_u(\lambda, \beta_\lambda(r))$) is maximized at $r_{\max}$.*
(ii) If the seller commits to r only after bidders enter, the reserve price is $r_{\max}$ in any equilibrium.

If the seller can commit to a reserve price before bidders enter, then it affects entry; hence the profit-maximizing r may differ from $r_{\max}$. Suppose the seller commits to a reserve price r, and then the interaction proceeds as in the PO scenario. Essentially the same arguments as in the $r = 0$ case establish that in the subgame following the selection of r, there is a unique equilibrium. Let $\lambda^*(r)$, $q^*(r)$, and $\overline{\lambda}^c(r)$ denote the equilibrium magnitudes in that subgame, and let r^* denote the seller's profit-maximizing r, i.e., $r^* = \arg\max_r \Pi_o(\lambda^*(r), q^*(r); r)$.

Claim 17 *In the PO scenario, the following hold:*
(i) If $\lambda^*(r^*) > 0$ and the bidders' entry cost does not constrain the equilibrium, i.e.,

[30] Of course, since $r_{\max}$ maximizes the revenue in any realized auction, it also maximizes the expected revenue in both scenarios, given any fixed participation rate λ.

[31] By revenue equivalence, $R_u(\lambda, \beta_\lambda(r)) = R_o(\lambda; r)$.

$\lambda^*(r^*) < \overline{\lambda}^c(r^*)$, *then* $r^* = r_{\max}$.
(ii) If the bidders' entry constrains the equilibrium, i.e., $\lambda^*(r^*) = \overline{\lambda}^c(r^*)$, *then* $r^* \neq r_{\max}$.

The proof is at the end of this section. However, both parts of this claim are almost immediate. In Part (i), the bidders' entry does not constrain the equilibrium, so the seller has no reason to deviate from $r_{\max}$. In Part (ii), the bidders' entry does constrain the equilibrium, so the first-order effect of a change in r at $r = r_{\max}$ is its effect on entry, which does not vanish.

The introduction of $r > 0$ affects both the seller's profit and the bidders' expected benefit. First, it makes the auction more profitable, which increases the range of s for which an equilibrium with trade can be sustained; i.e., $\overline{s}_o(r) > \overline{s}_o(0)$. Second, it lowers the bidders' benefit from entry for any expected level of participation, which decreases the maximal level of participation for which entry is profitable; i.e., $\overline{\lambda}^c(r) < \overline{\lambda}^c(0)$.

Intuitively, it seems that r^* should be lower than $r_{\max}$, because decreasing r slightly when it is above $r_{\max}$ makes the auction more profitable and relaxes the bidders' entry constraint. However, this intuition is incomplete, because changing r would change q^* and increase the total recruitment cost. For this reason, although $r^* < r_{\max}$ might hold in general, we have been able to establish it only under additional conditions that guarantee that the $\overline{\lambda}^c(r)$ values corresponding to the r values in the relevant range are not too small. This will be the case if c is not too large.[32]

Analogous results most likely hold for the equilibria with trade in the PU scenario, but we have not proved this. However, it is immediate that if $s \leq r[1-G(r)]$ and c is not prohibitive, then the no-trade outcome is not an equilibrium in the PU scenario. Since $r[1-G(r)]$ is maximized at $r_{\max}$, it follows that if $s < r_{\max}[1-G(r_{\max})]$, the seller can avoid the no-trade outcome by selecting an appropriate reserve price.

We now prove Claim 17. Obviously, r^* satisfies $\frac{d\Pi_o(\lambda_o^*(r),q^*(r);r)}{dr}|_{r=r^*} = 0$. Observe that

$$\begin{aligned}
\frac{d\Pi_o(\lambda_o^*(r),q^*(r);r)}{dr} &= \frac{d}{dr}\left[R_o(\lambda_o^*(r);r) - \frac{s}{q^*(r)}\lambda_o^*(r)\right] \\
&= \left(\frac{\partial R_o(\lambda_o^*(r);r)}{\partial\lambda} - \frac{s}{q^*(r)}\right)\frac{d\lambda_o^*(r)}{dr} + \frac{\lambda_o^*(r)s}{(q^*(r))^2}\frac{dq^*(r)}{dr} + \frac{\partial R_o(\lambda_o^*(r);r)}{\partial r} \\
&= \frac{\lambda_o^*(r)s}{(q^*(r))^2}\frac{dq^*(r)}{dr} + \frac{\partial R_o(\lambda_o^*(r);r)}{\partial r},
\end{aligned}$$

[32] The precise condition is $\overline{\lambda}^c(r)\,[2-G(r)] > 1$.

where the first term on the second line vanishes because it is the first-order condition with respect to λ. Also observe that, using integration by parts,

$$R_o(\lambda; r) = 1 - e^{-\lambda(1-G(r))}\left[r - \frac{1-G(r)}{g(r)}\right] - \int_r^1 e^{-(1-G(b))\lambda}\left[b - \frac{1-G(b)}{g(b)}\right]_b' db,$$

and therefore

$$\frac{\partial}{\partial r} R_o(\lambda; r) = -g(r)\lambda e^{-(1-G(r))\lambda}\left[r - \frac{1-G(r)}{g(r)}\right].$$

Hence, $\frac{\partial}{\partial r} R_o(\lambda; r) = 0$ if and only if $r = r_{\max}$.

Now if $\lambda_o^*(r^*) < \overline{\lambda}^c(r^*)$, then $q^*(r) = 1$ in a neighborhood of r^*. Hence $\frac{dq^*(r)}{dr}|_{r=r^*} = 0$ and

$$\frac{d\Pi_o(\lambda_o^*(r), q^*(r); r)}{dr} = \frac{\partial R_o(\lambda_o^*(r); r)}{\partial r}.$$

Therefore, $\frac{d\Pi_o(\lambda_o^*(r),q^*(r);r)}{dr} = 0$ if and only if $r = r_{\max}$, implying $r^* = r_{\max}$.

If $\lambda_o^*(r) = \overline{\lambda}^c(r)$, then $\frac{dq^*(r)}{dr}$ is obtained from total differentiation of the first-order condition with respect to λ, $\frac{\partial R_o(\lambda_o^*(r);r)}{\partial \lambda} - \frac{s}{q^*(r)} = 0$. Thus,

$$\frac{dq^*(r)}{dr} = -\frac{\frac{\partial^2 R_o(\lambda_o^*(r);r)}{\partial \lambda^2}\frac{d\lambda_o^*(r)}{dr} + \frac{\partial^2 R_o(\lambda_o^*(r);r)}{\partial \lambda \partial r}}{\frac{s}{(q^*(r))^2}}.$$

Now, $\frac{d\lambda_o^*(r)}{dr} = \frac{d\overline{\lambda}^c(r)}{dr} = -\frac{\frac{\partial U(\lambda_o^*(r);r)}{\partial r}}{\frac{\partial U(\lambda_o^*(r);r)}{\partial \lambda}} < 0$ and $\frac{\partial^2 R_o(\lambda_o^*(r);r)}{\partial \lambda^2} < 0$ from the second-order condition of profit maximization with respect to λ. Furthermore, at $r = r_{\max}$ both $\frac{\partial^2}{\partial \lambda \partial r} R_o(\lambda; r) = 0$ and $\frac{\partial R_o(\lambda_o^*(r);r)}{\partial r} = 0$. Therefore, at $r = r_{\max}$,

$$\frac{d\Pi_o(\lambda_o^*(r), q^*(r); r)}{dr} = \lambda_o^*(r)\frac{\partial^2 R_o(\lambda_o^*(r); r)}{\partial \lambda^2}\frac{\frac{\partial U(\lambda_o^*(r);r)}{\partial r}}{\frac{\partial U(\lambda_o^*(r);r)}{\partial \lambda}} < 0,$$

implying that $r^* \neq r_{\max}$. This finishes the proof.

A.3 Bidder entry with known values

The models discussed so far feature costly information acquisition: bidders learn their private values only after incurring the cost c. If, however, their values are readily known and their main costs lie in bid preparation or other aspects of bidding, then it would be more suitable to assume that their costly entry decisions take place with knowledge of their values. We now outline how to expand our analysis to cover this case. A full analysis would take too much space, but we believe that it would be doable and that the main qualitative insights would be the same as for the models discussed earlier. In particular, we show below for the case of small s that the recruitment cost is higher in the PO scenario than in the PU scenario.

Consider the PO scenario in this case. If entry is profitable for a bidder with value v, then it is profitable for all bidders with higher values. Therefore, a prospective bidder will enter if and only if his value v exceeds a certain cutoff $\underline{v} \in (0,1)$, at which he is indifferent about entry.

As before, let γ denote the Poisson rate of contacts made by the seller. The probability that a contacted bidder enters (the counterpart of q above) is $1 - G(\underline{v})$, and the effective Poisson rate of entry into the auction is $\lambda = \gamma(1 - G(\underline{v}))$. For a given $\underline{v}$, the seller's problem of choosing γ at marginal cost s is equivalent to choosing λ at marginal cost $s/(1-G(\underline{v}))$. As before, it will be convenient to express the relevant magnitudes in terms of λ rather than γ.

The bidding game among entrants is an FPA with observable participation and private values independently drawn from $[\underline{v}, 1]$. In equilibrium, if there is only one entrant, the winning bid is 0; if there are two or more entrants, the bids lie in $[\underline{v}, 1]$ and are monotone in values. Therefore, the seller's revenue is 0 if fewer than two bidders enter, and otherwise it is the appropriate equilibrium winning bid which lies in $[\underline{v}, 1]$. Given λ and $\underline{v} < 1$, the seller's payoff $\Pi_o(\lambda, \underline{v})$ is

$$\Pi_o(\lambda, \underline{v}) = R_o(\lambda, \underline{v}) - \lambda s/(1 - G(\underline{v})). \tag{36}$$

Since the equilibrium bids are monotone in values when there are two or more entrants, the marginal entrant with value $\underline{v}$ will win only if he is the sole entrant, in which case he will pay 0. The probability that he is the sole entrant is $e^{-\lambda}$. Therefore, his payoff from entering is $\underline{v}e^{-\lambda}$, and his indifference with respect to entry implies

$$\underline{v}e^{-\lambda} = c. \tag{37}$$

An equilibrium with trade is characterized by some $\lambda > 0$ and $\underline{v} < 1$ such that λ maximizes $\Pi_o(\lambda, \underline{v})$ and $\underline{v}$ satisfies (37).

Consider next the PU scenario. Here, too, a bidder enters if his value v exceeds a threshold $\underline{v}$. Given the Poisson rate γ of contacts made by the seller, the effective Poisson rate of entry into the auction is $\lambda = \gamma(1 - G(\underline{v}))$. As before, we express all magnitudes in terms of λ rather than γ. The bidding game among entrants is an FPA with unobservable participation and independent private values drawn from $[\underline{v}, 1]$. Given that bidders expect an effective Poisson rate $\widehat{\lambda}$ of entry, the entrants' equilibrium bidding strategy, $\beta(v; \underline{v}, \widehat{\lambda})$, is strictly increasing in $v \in [\underline{v}, 1]$.

With probability $e^{-\lambda}$, no bidders enter, in which case the seller's revenue is 0; otherwise it is the winning bid. Let $R_u(\lambda, \underline{v}, \widehat{\lambda})$ denote the expected winning bid given λ, $\widehat{\lambda}$, and $\underline{v} < 1$. The seller's payoff $\Pi_u(\lambda, \underline{v}, \widehat{\lambda})$ is

$$\Pi_u(\lambda, \underline{v}, \widehat{\lambda}) = R_u(\lambda, \underline{v}, \widehat{\lambda}) - \lambda s/(1 - G(\underline{v})). \tag{38}$$

Since $\beta(v; \underline{v}, \lambda)$ is strictly increasing in v, the marginal entrant $\underline{v}$ will win only if he is the sole entrant. Therefore, $\beta(\underline{v}; \underline{v}, \widehat{\lambda}) = 0$, and $\underline{v}$ satisfies the same entry condition (37).

An equilibrium with trade is characterized by $\lambda > 0$ and $\underline{v} < 1$ such that λ maximizes $\Pi_u(\lambda, \underline{v}, \widehat{\lambda})$ with $\widehat{\lambda} = \lambda$ and $\underline{v}$ satisfies (37).

The existence of an equilibrium here is somewhat more complicated than in Sections 2.2 and 3.2, since now $\underline{v}$ varies with λ. We do not analyze this case in full, but we conjecture that for sufficiently small s and c, equilibria with trade exist in both scenarios. Under this assumption, we compare the equilibrium outcomes in the limit as $s \to 0$.

Let $\lambda_i(s)$ and $\underline{v}_i(s)$ denote the equilibrium magnitudes in the equilibrium with maximal λ in the PO $(i = o)$ and PU $(i = u)$ scenarios, respectively.[33]

Claim 18 *(i) We have* $\lim_{s\to 0} \lambda_i(s) = -\ln c$ *for* $i = u$ *and* $i = o$.

(ii) In the limit, the total recruitment cost is higher in the PO scenario:

$$\lim_{s\to 0} \lambda_o(s) \frac{s}{1 - G(\underline{v}_o(s))} = (\ln c)^2 c > \lim_{s\to 0} \lambda_u(s) \frac{s}{1 - G(\underline{v}_u(s))}.$$

Thus, in the limit as $s \to 0$, both scenarios lead to the same level of effective participation, but the total recruitment cost is higher in the PO scenario. This cost ranking is the same as in the original setting, where bidders learn their values only after incurring c. We now proof the claim.

Part (i). In both scenarios, $\underline{v}_i(s) \to 1$ as $s \to 0$. Therefore, the entry condition $\underline{v}e^{-\lambda} = c$ for both scenarios implies $\lim_{s\to 0} \lambda_i(s) = -\ln c$.

[33] In the PO scenario, this is probably the unique equilibrium. However, we do not prove this, because a proof would essentially repeat the analysis in Section 2.2.

Part (ii). For a given s, the respective equilibria (with trade) of the two scenarios satisfy the first-order conditions $\partial\Pi_o(\lambda_o(s), \underline{v}_o(s))/\partial\lambda = 0$ and $\partial\Pi_u(\lambda_u(s), \underline{v}_u(s), \widehat{\lambda})/\partial\lambda|_{\widehat{\lambda}=\lambda_u(s)} = 0$, where

$$\partial R_o(\lambda_o(s), \underline{v}_o(s))/\partial\lambda = \frac{s}{1 - G(\underline{v}_o(s))} \tag{39}$$

and

$$\partial R_u(\lambda_o(s), \underline{v}_o(s), \widehat{\lambda})/\partial\lambda|_{\widehat{\lambda}=\lambda_o(s)} = \frac{s}{1 - G(\underline{v}_u(s))}. \tag{40}$$

Thus, in each of the scenarios, the total recruitment cost is

$$\lambda_i(s)\frac{s}{1 - G(\underline{v}_i(s))} = \lambda_i(s)\partial R_i/\partial\lambda. \tag{41}$$

By revenue equivalence, $R_o(\lambda, \underline{v})$ and hence $\partial R_o(\lambda, \underline{v}_o)/\partial\lambda$ are the same as they would be with the SPA for the same participation process. Let F^{SPA} denote the price distribution in the SPA; that is,

$$F^{SPA}(b|\lambda) = e^{-\frac{1-G(b)}{1-G(\underline{v})}\lambda} + e^{-\frac{(1-G(b))}{1-G(\underline{v})}\lambda}\frac{1 - G(b)}{1 - G(\underline{v})}\lambda \quad \text{for } b \geq \underline{v}$$

and $F^{SPA}(b|\lambda) = e^{-\lambda}(1 + \lambda)$ for $b \leq \underline{v}$. By revenue equivalence, $R_o(\lambda, \underline{v}) = \int_0^1 \left(1 - F^{SPA}(b|\lambda)\right) db$. Therefore,

$$\partial R_o(\lambda, \underline{v})/\partial\lambda = \lambda\underline{v}e^{-\lambda} + \int_{\underline{v}}^1 \left(\left(\frac{1 - G(b)}{1 - G(\underline{v})}\right)^2 \lambda e^{-\frac{1-G(b)}{1-G(\underline{v})}\lambda}\right) db.$$

Since $\underline{v}_o(s) \to 1$ as $s \to 0$, we have $\lim_{s\to 0} \partial R_o(\lambda_o(s), \underline{v}_o(s))/\partial\lambda = \lim_{s\to 0} \lambda_o(s)e^{-\lambda_o(s)} = -c\ln c$. Therefore, $\lim_{s\to 0} \lambda_o(s)\frac{s}{1-G(\underline{v}_o(s))} = (\ln c)^2 c$.

The inequality in Part (ii) of the claim will follow from $\lim_{s\to 0} \lambda_i(s) = -\ln c$ and (41) after we have established

$$\lim_{s\to 0} \partial R_u(\lambda, \underline{v}, \widehat{\lambda})/\partial\lambda|_{\widehat{\lambda}=\lambda} < \lim_{s\to 0} \partial R_o(\lambda, \underline{v})/\partial\lambda. \tag{42}$$

To prove (42), observe that by revenue equivalence, $R_o(\lambda, \underline{v}) = R_u(\lambda, \underline{v}, \lambda)$ and hence

$$\partial R_o(\lambda, \underline{v})/\partial\lambda = dR_u(\lambda, \underline{v}, \lambda)/d\lambda = \partial R_u(\lambda, \underline{v}, \widehat{\lambda})/\partial\lambda|_{\widehat{\lambda}=\lambda} + \partial R_u(\lambda, \underline{v}, \widehat{\lambda})/\partial\widehat{\lambda}|_{\widehat{\lambda}=\lambda}.$$

Then, by adapting the arguments used in Section 3.2, it can be shown that

$$\lim_{\underline{v}\to 1} \partial R_u(\lambda, \underline{v}, \widehat{\lambda})/\partial\widehat{\lambda}|_{\widehat{\lambda}=\lambda} =$$
$$\lim_{\underline{v}\to 1} \int_0^1 \left[e^{-\lambda[1-G(\beta^{-1}(b;\underline{v},\lambda))]/[1-G(\underline{v})]}\right] \frac{(G(\beta^{-1}(b;\underline{v},\lambda)) - G(\underline{v}))}{(1-G(\underline{v}))} db > 0,$$

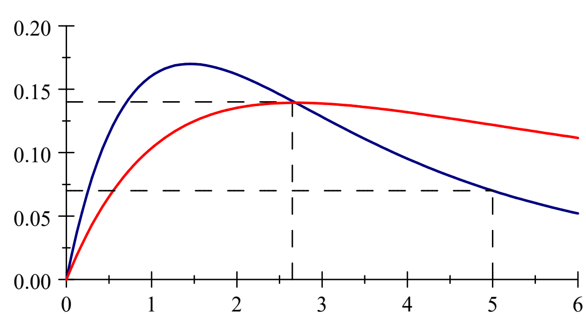
which implies (42) and hence Part (ii) of the claim.

A.4 Numerical Example: Uniform Distribution

We illustrate our finding with the uniform distribution, $G(v) = v$. The following formulas summarize the explicit expressions that we derived in the paper, after substituting for G:

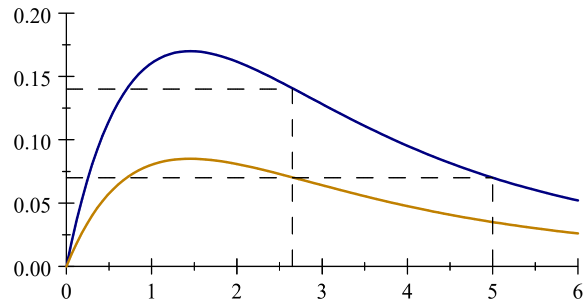
$$\begin{aligned}
U(\lambda) &= \int_0^1 e^{-(1-v)\lambda}[1-v]dv \\
W(\lambda) &= \int_0^1 \left[1 - e^{-(1-v)\lambda}\right] dv \\
R_o(\lambda) &= \int_0^1 \left[1 - e^{-(1-v)\lambda}\right] - \lambda e^{-(1-v)\lambda}[1-v]dv \\
R_o'(\lambda) &= \int_0^1 \lambda (1-v)^2 e^{-\lambda(1-v)} dv \\
\xi(\lambda) &= \int_0^1 (1-v) e^{-\lambda(1-v)} \left(1 - e^{-\lambda v}\right) dv.
\end{aligned}$$

For the following plots, we chose $c \approx 0.03$, implying $\bar{\lambda}^c \approx 5$. We first plot the analogue of Figure 4 for the PO scenario with a uniform value distribution.



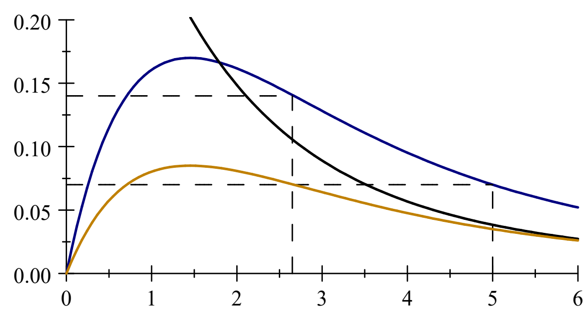
Marginal revenue and average revenue in the PO scenario with a uniform value distribution, with $\bar{s}_0 \approx 0.14$ and $\underline{\lambda}_o \approx 2.69$. For $c = 0.03$, we have $\bar{\lambda}^c \approx 5$.

To compare the PO and PU scenario, we plot the analogue of Figure 7. As can be seen, for s between 0.085 and 1.4, there is trade in the PO scenario but in the PU scenario, trade unravels.



Comparison of the PO and the PU scenario, showing R'_o (blue, top) and ξ (siena, bottom). The maximum of ξ is at about $\xi(1.45) \approx 0.085$.

Finally, we illustrate the welfare properties, plotting Figure 10 for this case. Recall that the welfare optimal λ^w satisfies $U(\lambda^w) = c + s$. As noted, in the PO scenario, $R'_o(\lambda) > U(\lambda)$ means that the seller recruits too many bidders for all $s \leq \bar{s}_o \approx 0.14$.



Illustrating the welfare properties, with $U(\lambda)$ in black, $R'_o(\lambda)$ in blue and $\xi(\lambda)$ in siena.

We include the ξ function to show the welfare properties of equilibrium in the PU case. In the range shown in the figure, for $\lambda \leq 6$, we have $\xi(\lambda) < U(\lambda)$.[34] Hence, when $c = 0$, then the seller will recruit too few bidders in the PU scenario.

In general, the numerical analysis suggests that $U - \xi$ is eventually monotone decreasing. Hence, for c small enough, this suggests that there is some cutoff $\hat{s}$ such that the seller recruits too few bidders in the PU scenario when s is above $\hat{s}$ and she recruits too many when s is smaller (in the equilibrium with larger participation).

[34] This remains the case for $\lambda \leq 50$, where U and ξ become numerically indistinguishable at 20 digits.