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# The Hold-up Problem with Flexible Unobservable Investments

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# The hold-up problem with flexible unobservable investments

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**Abstract**

The paper studies the canonical hold-up problem with one-sided investment by the buyer and full ex post bargaining power by the seller. The buyer can covertly choose any distribution of valuations at a cost and privately observes her valuation. The main result shows that in contrast to the well-understood case with linear costs, if investment costs are strictly convex in the buyer's valuation distribution, the buyer's equilibrium utility is strictly positive and total welfare is strictly higher than in the benchmark when valuations are public information, thus alleviating the hold-up problem. In fact, when costs are mean-based or display decreasing risk, the hold-up problem may disappear completely. Moreover, the buyer's equilibrium utility and total welfare might be non-monotone in costs. The paper utilizes an equilibrium characterization in terms of the Gateaux derivative of the cost function.

Keywords: Information Design, Hold-Up Problem, Unobservable Information
JEL: C61, D42, D82

## 1 Introduction

Consider the canonical hold-up problem where a buyer can make a costly, relation-specific ex ante investment that increases her valuation for the seller's good, and the seller has all bargaining power ex post. It is well-known that the hold-up problem is most severe if the buyer's valuation becomes public information ex post: in this case the seller will extract all gains from trade, and in anticipation of this, the buyer will not invest. In equilibrium, the buyer's utility is zero, and total

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welfare equals the gains from trade with no investment. In a seminal paper, Gul (2001) shows that when the buyer's investment decision induces a deterministic valuation, then the same welfare outcomes obtain even if the buyer's valuation remains her private information so that she can secure an information rent ex post: while some investment occurs in equilibrium, the buyer still gets zero utility overall because her ex post information rent is fully dissipated by her ex ante investment. In addition, total welfare remains the same as with public valuation because the privacy of the buyer's valuation implies inefficient bargaining ex post.

One feature of Gul's (2001) cost structure is that when the buyer chooses a probability distribution over valuations by adopting a mixed strategy over valuations, the cost of doing so is expected investment costs, and thus *linear* in the probability distribution. In this paper, I consider the hold-up problem when the buyer's cost of choosing a distribution over valuations is *convex* in the distribution. I show that if costs are strictly convex, then the buyer's equilibrium utility is strictly positive and total welfare exceeds the zero investment gains from trade, thus alleviating the hold-up problem. In fact, I identify environments where the first-best outcome obtains in equilibrium, and thus the hold-up problem disappears completely. Moreover, I show that both the buyer's utility and total welfare might increase with investment costs.

I adopt a framework where the buyer can covertly choose any distribution over an interval of valuations at a cost.[1] The cost function is convex in the distribution, nesting Gul's (2001) model with linear costs as a special case. The cost function is normalized in that there is a zero cost "default" distribution that corresponds to the buyer not investing. I further assume that the cost function is smooth in that it admits a functional derivative in the sense of a (linear) Gateaux differential. As is well known, the Gateaux differential is a directional derivative that captures the cost change when the buyer moves marginally from one distribution in the direction of another distribution.

As a methodological point, I provide an equilibrium characterization that states the conditions for a distribution by the buyer to be a best response to the seller's pricing strategy in terms of the Gateaux differential of the cost function.[2] This best response condition formally corresponds to the familiar condition of a distribution (or, a mixed strategy) to be a best response: Any valuation in the support of the distribution must yield the same payoff, and any valuation outside the support must not yield a higher payoff. The novelty is that with non-linear cost, the payoff corresponding to a valuation depends on the entire distribution itself (not just on the valuation). This implies that a best response is generally given only in implicit form.

Even though equilibria can, therefore, in general not be explicitly derived, it is remarkably

[1]"Covertly" means that the seller does not observe the distribution. This is the key difference between my paper and Condorelli and Szentes (2020) as explained in more detail below.

[2]Technically, I use a result in Georgiadis et al. (2023) which characterizes the optima of a concave functional that admits a linear Gateaux differential.

simple to derive equilibrium utilities and welfare through the best response conditions. First, I show that under the natural monotonicity assumption that the cost function respects first order stochastic dominance, the seller's equilibrium profit is always equal to the zero investment gains from trade and hence a constant.[3] It follows that total equilibrium welfare and the buyer's equilibrium utility differ by the size of the zero investment gains from trade.

Second, my result that the buyer's equilibrium utility is zero if her costs are linear, but positive if her costs are strictly convex, follows from a familiar marginal benefit versus marginal cost logic that, as I show, carries over to the current setting with flexible investments. Intuitively, the buyer's marginal benefit from increasing investment by "increasing" her valuation distribution is constant. The reason is that the benefit from choosing a distribution is simply the buyer's expected share of the trading surplus which is linear in the distribution. On the other hand, the marginal cost from increasing investment, which can be shown to be the Gateaux differential of the cost function, is increasing because costs are strictly convex.

Now, equilibrium implies that at the equilibrium distribution, marginal benefits are (weakly) larger than marginal costs for otherwise it would be profitable for the buyer to deviate in some direction. Intuitively, since marginal benefits are constant and marginal costs are increasing, marginal costs are strictly below marginal benefits when the buyer, starting from zero investment, increases investment up to the equilibrium distribution, resulting in strictly positive equilibrium utility. In contrast, when investment costs are linear, then both marginal benefits and marginal costs are constant, which implies that in equilibrium the buyer gets zero utility.

A direct corollary of these observations is that if the cost function is the sum of a linear and a scaled strictly convex part, then the buyer's utility (and thus total welfare) increases if the convex part is scaled up from zero to positive. In other words, the buyer's utility (and thus total welfare) locally increases with investment costs.

Explicit expressions for equilibrium strategies and utilities are hard to obtain in general. In a further part of the paper, I therefore impose more structure on the cost function. In particular, drawing on Cerreia-Vioglio et al. (2017), I impose assumptions on the Gateaux differential that amount to certain risk properties of the cost function. I distinguish three cases depending on whether costs only depend on the mean, or are decreasing or increasing with respect to the mean preserving spread order. In the terminology of Condorelli and Szentes (2020), costs are thus "mean-based", or decreasing, or increasing "in risk".[4] I also scale the cost function with a parameter that captures the magnitude of costs and marginal costs.

I show that in all cases, total welfare and the buyer's equilibrium utility increase as the cost

[3]To be precise, I impose an assumption on the cost function that implies strict monotonicity of the cost function and thus is slightly stronger than monotonicity.

[4]To be precise, Condorelli and Szentes (2020) impose these assumptions on the cost function directly, and my assumptions on the Gateaux differential are slightly stronger.

parameter increases from zero to a critical value. Moreover, unlike in the case with linear costs (and unless costs are prohibitively high), total welfare is strictly larger than in the benchmark case when the seller observes the buyer's valuation ex post. In this sense, the hold-up problem is alleviated when the buyer's valuation is private information and costs are strictly convex. In fact, if costs are mean-based or decreasing in risk, and the cost parameter is sufficiently large, the equilibrium outcome is efficient, that is, the hold-up problem disappears completely.[5]

The basic reason for these results lies in the nature of the buyer's commitment problem when choosing a distribution. As highlighted by Condorelli and Szentes (2020), the buyer would like to commit to a distribution so as to induce the seller to choose a low price. However, since the buyer's investment is covert in my setting, she cannot commit to a distribution. Thus, if the seller were to choose the low price from the commitment outcome, incentives arise for the buyer to secretely deviate to a different distribution. The strength of the deviation incentives depends on the comparison of marginal benefits and marginal costs. As mentioned earlier, the marginal benefits from a deviation are constant, because the benefit of choosing a distribution is simply the buyer's expected share of the ex post trading surplus which is linear in the distribution. If, in addition, marginal costs are constant (costs are linear), the buyer's overall deviation incentives are the same for any distribution. In fact, if the seller were to choose the low price from the commitment outcome, the buyer would want to deviate to the "highest possible" distribution, and in this sense, the buyer's commitment problem is most pronounced when costs are linear.[6]

If, on the other hand, costs are strictly convex, marginal costs are increasing, and thus larger deviations become more costly. In other words, convexity of costs attenuates the buyer's commitment problem, and this force moves the equilibrium outcome closer to the commitment outcome, thus increasing the buyer's utility (and total welfare). This effect is more pronounced the more increasing are marginal costs, or the more convex are costs. In my parameterization, an increase of the cost parameter makes costs more convex, and thus the buyer's utility (and total welfare) increases as the cost parameter is scaled up to a critical point.

When costs are mean-based or decreasing in risk, the equilibrium and the first-best outcome coincide once the parameter exceeds the critical point. This is easiest to illustrate when costs are decreasing in risk (the argument with mean-based costs is similar but slightly more involved). In this case, given a pricing strategy of the seller and holding the mean of the buyer's distribution fixed, the buyer prefers her distribution to be maximally risky, and consequently she puts all probability mass only on the smallest and largest possible valuations. In turn, the seller best responds by choosing either a high price equal to the largest possible valuation, or a low price

[5]When costs are mean-based, there might be multiple equilibria. I focus on Pareto-optimal equilibria, which are also buyer-optimal because in any equilibrium, the seller's profit is the equal to the zero investment gains from trade.

[6]This is the reason why, with linear costs, the commitment outcome of Condorelli and Szentes (2020) cannot be sustained when the seller does not observe the buyer's distribution, as in Gul (2001).

equal to the lowest possible valuation. But when the price is equal to the lowest possible valuation, trade is always efficient ex post, and the buyer is the residual claimant of the efficient surplus, leading her to choose the efficient investment distribution. Now, for the seller to actually best respond with the low price to the efficient investment distribution, this distribution must not place to much weight on large valuations, which is only the case if marginal investment costs are sufficiently high. In my parameterization, this occurs when the cost parameter is sufficiently large.

This reasoning does not apply when costs are increasing in risk. In this case, equilibrium involves a mixed pricing strategy by the seller, because given that the buyer prefers a minimally risky distribution, her best response to a deterministic price is to put all mass on a single valuation. But this is inconsistent with equilibrium because, anticipating that the buyer puts mass on a single valuation, the seller would then fully hold up the buyer, and the buyer would rather not invest. Because the seller mixes, equilibrium trade is inefficient ex post, and the equilibrium outcome then differs from the first-best.

My paper is most closely related to the abovementioned papers by Gul (2001) and Condorelli and Szentes (2020). Like Gul (2001), I consider the hold-up problem with unobservable investments, but allow for convex investment costs.[7,8] Non-linear costs are also considered in Condorelli and Szentes (2020), who, in contrast to my paper, consider the case when the buyer can commit to an investment distribution (or, equivalently, the seller can observe the distribution but not the realized valuation).[9]

From a technical point of view, I use a result in Georgiadis et al. (2023) which, in the context of a moral hazard problem with flexible effort choice by the agent, provides a first-order condition that characterizes the maximum of a concave, Gateaux-differentiable functional. In the second part of the paper, where I impose more structure on the cost function, I go beyond pure optimization and derive explicit equilibria of the investment game from this first-order condition.

My paper shares with Ravid et al. (2022) the feature that the seller does not observe the distribution of the buyer's valuation. The key difference is that in Ravid et al. (2022), the buyer's ex ante choice is to acquire information about, rather than invest in, her valuation. Ravid et al. (2022) show that in the limit when information acquisition costs are small, the buyer is worse

[7]Gul (2001) does not only consider the problem with one-shot ultimatum bargaining as I do here, but shows that efficiency can be restored by an appropriate dynamic bargaining protocol.

[8]Other papers that study versions of the hold-up problem with linear investment costs are Dilme (2019) and Lau (2008). Dilme (2019) considers a setting where the investment increases both parties' valuation, and the informed party makes a take-it or leave-it offer ex post, leading to endogenous adverse selection. He shows that the non-investing party gets less than when investment is observable. In Lau (2008), the seller obtains a truth-or-noise signal about the buyer's investment prior to making the take-it or leave-it offer. Lau (2008) shows that the information benefits the seller, while the buyer, as with no signal, still gets zero surplus.

[9]As a by-product, my analysis also shows that the commitment outcome might be first-best, because in those cases where my equilibrium outcome coincides with the first-best, it also coincides with the commitment outcome.

off than when information is for free. In contrast, I obtain welfare results away from the limit because the information acquisition constraint that the distribution be Bayesian consistent with a prior is missing from my framework and allows for more explicit equilibrium characterizations. In an extension, I show how my framework can be used to speak to information acquisition in the case that the buyer's true valuations can take on only two values.[10]

The paper is organized as follows. The next section presents an example with two possible buyer valuations. Section 3 presents the general model. Sections 4 and 5 contain the key equilibrium and welfare results. Section 6 derives explicit results when costs are mean-based, decreasing or increasing in risk, respectively. Section 7 discusses a connection to information acquisition. Section 8 concludes. All formal proofs are in the appendix.

## 2 Example

This section presents a simple example to illustrate the key intuitions of the paper. There is a buyer and a seller. The buyer can have two possible valuations $v \in \{\alpha, \omega\}$ for the seller's good where $0 < \alpha < \omega$. Ex ante, the buyer chooses a probability $f \in [0,1]$ with which the high valuation $\omega$ occurs. Ex post, after the valuation is realized, the seller makes a take-it or leave-it offer by choosing a price $p$. If the buyer rejects, both parties obtain zero payoff. If the buyer accepts, her payoff is valuation minus price, and the seller's payoff is the price.

Without loss, the seller chooses a price equal either to $\alpha$ or $\omega$. Allowing for mixed strategies, let $h \in [0,1]$ be the probability with which he chooses the high price. The buyer's cost of investing $f$ is $C(f) = \ell f + 1/2 \cdot \kappa f^2$, where $\kappa \geq 0$ and $\ell \in (0, \omega - \alpha)$.

If $\kappa = 0$, then costs are linear. This can be interpreted in the sense that the buyer has two pure strategies: "choose valuation $\alpha$" at cost 0, and "choose valuation $\omega$" at cost $\ell$. The linear costs $C(f) = \ell f$ therefore correspond to the costs of the mixed strategy where valuation $\alpha$ (resp. $\omega$) is chosen with probability $1 - f$ (resp. $f$).

The first-best investment level $f^{FB}$ maximizes the total surplus $(1-f)\alpha + f\omega - C(f)$ and is thus given by

$$f^{FB} = \min\left\{\frac{\omega - \alpha - \ell}{\kappa}, 1\right\}. \tag{1}$$

Next, I discuss equilibrium. I begin with the benchmark case that the buyer's valuation $v$ becomes publicly observable before the seller sets the price. In this case, the seller will always choose $p = v$. As a consequence, the buyer's ex post surplus is zero, and she will therefore choose zero investment $f = 0$. The seller's profit is $\alpha$, the buyer's utility is 0, and total welfare is $\alpha$.

[10]For the case with buyer commitment and information acquisition, see Roesler and Szentes (2017).

Next, I consider the case with unobservable investment where the seller observes neither the distribution $f$ nor the valuation $v$.[11] Consider first the seller's best response to $f$: if $f < \alpha/\omega$, the seller optimally chooses the low price ($h = 0$). If $f = \alpha/\omega$, the seller is indifferent between the high and low price ($h \in [0,1]$). And if $f > \alpha/\omega$, the seller optimally chooses the high price ($h = 1$).

Consider now the case with linear costs ($\kappa = 0$). In this case, the same welfare outcomes obtain as in the case with observable valuation. To see this, note that there is no equilibrium where the seller sets a deterministic price. The reason is that if the seller were to charge the low price ($h = 0$), the buyer's utility from investment $f$ is $f(\omega - \alpha) - \ell f$. Thus, her marginal investment benefit is $\omega - \alpha$, and her marginal investment cost is $\ell$. Since $\ell < \omega - \alpha$, the buyer's best response would be "full" investment $f = 1$. At that point, however, charging the low price would not be a best response for the seller.

Reversely, if the seller were to charge the high price ($h = 1$), the buyer's utility from investment $f$ is $0 - \ell f$ and her best response would be "zero" investment $f = 0$. At that point, however, charging the high price would not be a best response for the seller (recall that $\alpha > 0$).

Therefore, there is only a mixed strategy equilibrium where the buyer chooses $f = \alpha/\omega$ so as to keep the seller indifferent between both prices, and the seller chooses $h$ so as to keep the buyer indifferent between zero investment ($f = 0$) and full investment ($f = 1$).[12] Because the buyer is indifferent, her equilibrium utility is zero (her utility from zero investment). As the seller is indifferent, her equilibrium profit is $\alpha$ (her profit from charging the price $\alpha$), and total equilibrium welfare is $\alpha$. Therefore, payoffs are identical as in the case with observable investment: While there is now positive investment in equilibrium and the buyer obtains an ex post information rent, her rent is fully dissipated by her ex ante investment expenditures. Moreover, the positive investment does not improve welfare because trade is not efficient ex post.

Consider now what changes if $\kappa$ is larger than zero so that investment costs are convex. If the seller charges the low price ($h = 0$), the buyer's marginal investment benefit is still $\omega - \alpha$ but her marginal costs $\ell + \kappa f$ are now increasing in investment $f$. Therefore, if $\kappa$ is sufficiently large, the buyer's optimal choice of investment is below $\alpha/\omega$ so that charging the low price remains indeed optimal for the seller. In fact, if the seller charges the low price, trade is efficient ex post, and the buyer is the residual claimant of the efficient surplus, leading her to invest at the first-best level. Therefore, whenever $f^{FB}$ is below $\alpha/\omega$, the first-best outcome $f^{FB}$ obtains in equilibrium and there is no hold-up problem. Formally, it follows from (1) that $f^{FB} \leq \alpha/\omega$ if and only if

[11] The case with observable investment and unobservable valuation corresponds to the setting in Condorelli and Szentes (2020) where the buyer can commit to a distribution of valuations.

[12] More formally, given $h$, the buyer maximizes $f(1-h)(\omega-\alpha) - \ell f$. Thus, her best response is $f = 0$ if $h > 1 - \ell/(\omega - \alpha)$, and $f = 1$ if $h < 1 - \ell/(\omega - \alpha)$, and she is indifferent otherwise. Since $\ell < \omega - \alpha$, the only intersection of the buyer's and seller's best responses is at $f = \alpha/\omega$ and $h = 1 - \ell/(\omega - \alpha)$.

$\kappa \geq \hat{\kappa} = \omega/\alpha \cdot (\omega - \alpha - \ell)$. Thus, the first-best outcome prevails if $\kappa \geq \hat{\kappa}$.

What happens if $\kappa \in (0, \hat{\kappa})$? In this case, the first-best outcome does not obtain in equilibrium, because marginal costs are too low for the buyer not to deviate to a larger than the first-best investment $f^{FB}$ if the seller were to choose the low price $\alpha$. Similar to the case with linear cost, in equilibrium the buyer chooses $f = \alpha/\omega$ so as to keep the seller indifferent, and the seller chooses $h$ so as to render the buyer's choice of $\alpha/\omega$ optimal, that is, $(1-h)(\omega-\alpha) = C'(\alpha/\omega)$. In contrast to the case with linear cost, however, the buyer now obtains strictly positive utility. The reason is that up to the equilibrium investment level $f$, her marginal costs are strictly lower than her marginal benefits due to convexity of the costs and linearity of the benefits. Formally, the buyer's equilibrium utility in the range $\kappa \in (0, \hat{\kappa})$ calculates to

$$U_B = f(1-h)(\omega-\alpha) - C(f) = \frac{1}{2}\kappa\left(\frac{\alpha}{\omega}\right)^2. \tag{2}$$

Notice that this is increasing in $\kappa$. The reason is that in the range $\kappa \in (0, \hat{\kappa})$, the direct effect of facing higher investment costs is outweighed by the indirect strategic effect that the seller reduces the price (reduces $h$) as $\kappa$ increases. Moreover, as the seller is indifferent between the low and the high price, her profit is $\alpha$, irrespective of $\kappa$. Thus, total welfare is $U_B + \alpha$.

Overall, three insights emerge from these observations:

(i) The buyer's equilibrium utility and total welfare are non-monotone in costs. They increase up to $\hat{\kappa}$ and then decrease, as illustrated by Figure 1. The blue (solid) curve plots the buyer's utility as a function of $\kappa$.

(ii) Because the buyer's utility is strictly positive for $\kappa > 0$, total welfare is strictly larger than in the benchmark case when the buyer's valuation is public information.

(iii) For values $\kappa \geq \hat{\kappa}$, the equilibrium outcome is efficient, and the hold-up problem disappears, as illustrated by Figure 1. The red (dashed) curve plots first-best welfare minus the seller's equilibrium profits $\alpha$ and coincides with the buyer's equilibrium utility for $\kappa \geq \hat{\kappa}$. (Note that for $\kappa \leq \omega - \alpha - \ell$, we have that $f^{FB} = 1$. Thus, welfare is linear in $\kappa$ in this range.)

I now turn to the general model where the buyer's valuation can take any value in an interval. I will show that (i) and (ii) carry over to the general setting if investment costs are strictly convex. On the other hand, whether (iii) carries over depends on the structure of the cost function and obtains if the cost function is "mean-based" or displays "decreasing risk", but not if it displays "increasing risk".

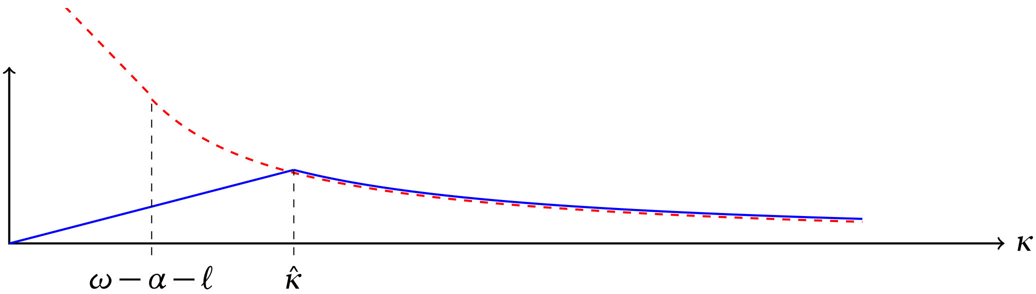


Figure 1: The figure shows $U_B$ as a function of $\kappa$ (blue) and total welfare minus $\alpha$ (red, dashed) for the values $\alpha = 1$, $\omega = 2$, $\ell = 1/2$, $\hat{\kappa} = 1$.

## 3 Model

There is a seller who has a good, and there is a buyer who can invest in her valuation for the good by choosing a cumulative distribution function (cdf) $F$ over the set of possible valuations $V = [\alpha, \omega]$, $0 \leq \alpha < \omega$, at cost $C(F)$. Let $\mathscr{F}$ denote the set of all cdf's over $V$. The timing is as follows: The seller chooses a price $p$, and the buyer simultaneously chooses a cdf $F \in \mathscr{F}$. Then the buyer privately observes her realized valuation $v$ and decides to accept or reject to trade at the price $p$. If she rejects, both parties get zero. If she accepts, the buyer's payoff is $v - p$, and the seller's payoff is $p$.

The buyer's strategy specifies a cdf $F$ and a decision to accept or reject, contingent on $p$. A (mixed) strategy for the seller is a cdf over prices. In a perfect Bayesian equilibrium (henceforth: equilibrium), the buyer accepts any price $p < v$ and rejects any price $p > v$, and her choice of cdf is optimal given the seller's pricing strategy, and the seller's strategy is optimal given the buyer's choice of cdf and acceptance/rejection decision.

It is a standard argument that in any equilibrium, the buyer accepts with probability 1 when indifferent ($p = v$) and that the seller never chooses a price strictly below the buyer's smallest possible valuation $\alpha$. Moreover, it is weakly dominated for the seller to choose a price strictly above the buyer's largest possible valuation $\omega$. To analyze the initial stage of the game, I therefore focus on seller strategies $H$ that are cdf's from the set $\mathscr{F}$.

The buyer's expected utility (net of investment costs) when valuation $v$ has realized and before having observed the price is

$$\overline{H}(v) = \int_\alpha^v (v-p)\, dH(p) = \int_\alpha^v H(p)\, dp, \tag{3}$$

where the second equality follows from integration by parts. The ex ante expected utilities for the buyer and seller are then given by[13]

$$U(F,H) = \int_V \overline{H}(v)\, dF - C(F), \qquad \Pi(H,F) = \int_V (1 - F(p^-))p\, dH(p). \tag{4}$$

With abuse of language, I refer to a combination $(F,H) \in \mathscr{F}^2$ as an equilibrium when $F$ and $H$ are mutual best responses given $U$ and $\Pi$.

I next state the assumptions on the cost function that I impose throughout the paper.

**A1** *$C : \mathscr{F} \to \mathbb{R}$ is convex and continuous*[14].

**A2** *$C$ is Gateaux differentiable, that is, for all $F, \tilde{F} \in \mathscr{F}$, the "Gateaux differential" of $F$ in the direction of $\tilde{F}$, given as the limit*

$$\delta C(F; \tilde{F} - F) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}[C(F + \epsilon(\tilde{F} - F)) - C(F)] \tag{5}$$

*exists. Moreover, there is a continuous "Gateaux derivative" $c_F : V \to \mathbb{R}$ so that*

$$\delta C(F; \tilde{F} - F) = \int_V c_F(v)\, d(\tilde{F} - F). \tag{6}$$

**A3** *$c_F(v)$ is strictly increasing in $v$ for all $F \in \mathscr{F}$.*

Convexity is an economically natural assumption. Note that a mixture of two distributions can always be generated by a two-stage process where in the first stage a random draw determines from which of the two distributions the valuation is drawn in the second stage. Convexity then simply captures the fact that the cost of generating the mixture distribution through this two-stage process is an upper bound on the least costly way to generate the simple counterpart of the mixture distribution.

A2 captures that costs are smooth. Intuitively, the Gateaux differential $\delta C(F; \tilde{F} - F)$ approximates the cost change $C(\tilde{F}) - C(F)$ if one moves from $F$ in the direction of $\tilde{F}$. Expression (6) means that the Gateaux differential is linear (in $\tilde{F} - F$) and thus amounts to a linear approximation of cost changes—analogous to a first-order Taylor approximation with $c_F$ corresponding to the gradient of $C$ at the point $F$.[15]

[13] $F(p^-)$ denotes the left limit $\lim_{q \uparrow p} F(q)$.

[14] More precisely, $C$ is assumed to be continuous in the topology of weak convergence, that is, if $F_n$ converges weakly to $F$, then $C(F_n)$ converges to $C(F)$. Because the set of cdfs $\mathscr{F}$ is compact, continuity of $C$ ensures existence of various optimizers below.

[15] To see the analogy, suppose $C : \mathbb{R} \to \mathbb{R}$ is uni-dimensional. Then by multiplying and dividing the Gateaux-

A3 implies that $C$ is monotone in the sense that it respects first order stochastic dominance, a natural property in the investment context considered here. In fact, monotonicity of $C$ is equivalent to $c_F$ being increasing for all $F \in \mathscr{F}$ (see Cerreia-Vioglio et al., 2017). While A3 is therefore slightly stronger than monotonicity of $C$, it simplifies some of my arguments, because, as I will show, it uniquely pins down the seller's equilibrium profits.[16]

A3 also implies that $C$ is uniquely minimized by the distribution which places full mass on the lowest valuation $\alpha$. I shall refer to this distribution as $F_{min}$, and normalize the cost of $F_{min}$ to zero to ensure that investment costs are non-negative:[17]

$$F_{min} = \mathbb{1}_{[\alpha,\omega]}, \quad C(F_{min}) = 0. \tag{7}$$

$F_{min}$ can thus be interpreted as the default distribution that arises if the buyer does "not invest".[18]

Finally, note that continuity of $C$ implies that there is a well-defined first-best distribution that maximizes the total surplus $\int_V v\, dF - C(F)$. If unique, I denote the first-best distribution as $F^{FB}$.

An important special case is the class of linear cost functions. $C$ is linear if and only if $C(F) = \int_V c(v)\, dF(v)$ for a continuous function $c : V \to \mathbb{R}$. In this case, the Gateaux derivative is $c_F = c$ at all points $F$. Linearity arises in a setting where the buyer can choose a valuation $v \in V$ at cost $c(v)$, and $V$ is the set of the buyer's pure strategies (see, e.g., Gul, 2001). $C(F)$ is then the cost of the mixed strategy that randomizes over $V$ according to $F$. For a linear cost function, we have

$$\min_{v \in V} c(v) = 0. \tag{8}$$

To see this, note that $C(F) = \int_V c(v)\, dF(v)$ is minimized by any cdf $F_{min}$ that places full mass on points $v$ were $c$ is minimal. Thus, $C(F_{min}) = \min_{v\in V} c(v)$ which is 0 by (7).

A useful benchmark for the analysis is the case in which the buyer's valuation becomes public information before the seller chooses the price. In equilibrium, the seller then chooses the price equal to the valuation and extracts the entire trading surplus. Anticipating this, the buyer chooses

differential with $\tilde{F} - F$, it can be written as

$$\delta C(F; \tilde{F} - F) = \lim_{\epsilon \to 0} \frac{1}{\epsilon(\tilde{F} - F)} [C(F + \epsilon(\tilde{F} - F)) - C(F)] \cdot (\tilde{F} - F) = C'(F) \cdot (\tilde{F} - F),$$

and hence $\delta C(F; \tilde{F} - F) \approx C(\tilde{F}) - C(F)$. It is well-known though that, in general, the Gateaux differential need not be linear, but only homogeneous.

[16] A3 implies that $C$ is strictly monotone in the sense that if $F$ first order stochastically dominates $G$ and does not coincide with $G$ almost everywhere, then $C(F) > C(G)$.

[17] $\mathbb{1}$ denotes the indicator function.

[18] That $F_{min}$ minimizes $C$ follows from monotonicity and since any distribution first order stochastically dominates $F_{min}$. Uniqueness follows from the fact that $c_F$ is strictly increasing. I omit the details.

the default distribution $F_{min}$. The resulting utilities and welfare are

$$U^{PUB} = 0, \quad \Pi^{PUB} = \alpha, \quad W^{PUB} = \int_V v \, dF_{min} = \alpha. \tag{9}$$

# 4 Equilibrium Analysis

My first proposition is the main equilibrium characterization of the paper.

**Proposition 1** *(i) There is an equilibrium.*

*(ii) $(F, H)$ is an equilibrium if and only if there are $\lambda$ and $\pi \geq \alpha$ such that*

$$\begin{aligned}
\overline{H}(v) - c_F(v) - \lambda &\leq 0 \quad \forall v \in V, & (10)\\
\overline{H}(v) - c_F(v) - \lambda &= 0 \quad \forall v \in supp(F), & (11)\\
(1 - F(p^-))p - \pi &\leq 0 \quad \forall p \in V, & (12)\\
(1 - F(p^-))p - \pi &= 0 \quad \forall p \in supp(H). & (13)
\end{aligned}$$

Part (i) follows from a standard fixed point argument along the same lines as in the existence proof in Ravid et al. (2022, footnote 22). Part (ii), more precisely, the conditions (10) and (11) which characterize the buyer's best response in terms of the Gateaux derivative, are somewhat non-standard. To shed light on part (ii), it is easiest to first consider the conditions (12) and (13). These conditions represent the familiar conditions for a (mixed) pricing strategy by the seller to be a best response to $F$: Any price in the support of the strategy must yield the same profit $\pi = (1 - F(p^-))p$, and any price outside the support must not yield a higher profit.

The conditions (10) and (11) are analogous conditions for the buyer. In fact, consider the special case of linear $C$, and recall the interpretation of $C$ as the cost of a mixed strategy when the buyer can choose a valuation $v$ at cost $c(v)$. In this case, the buyer's utility from the pure strategy $v$ is $\overline{H}(v) - c(v)$, and the conditions (10) and (11) therefore represent the conditions for a (mixed) strategy by the buyer to be a best response to $H$. The significance of part (ii) is that the same formal conditions characterize the buyer's best response even when the Gateaux derivative $c_F$ is not constant in $F$.

Notice, however, that when $c_F$ is not constant in $F$, (10) and (11) describe $F$ only implicitly, because $F$ appears on both sides. To see this more clearly, note that (10) and (11) imply that a point in $supp(F)$ is a maximizer of $\overline{H}(v) - c_F(v)$. Therefore, $F$ is a solution to (10) and (11) if and only if its support satisfies

$$supp(F) \subseteq \arg\max_{v \in V} \overline{H}(v) - c_F(v). \tag{14}$$

To establish (10) and (11), I can use Proposition 1 in Georgiadis et al. (2023) which shows that $F$ maximizes $U(G,H)=\int_V \overline{H}(v)dG - C(G)$ with respect to $G$, and is thus a best response to $H$, if and only if $F$ is the solution to the first-order condition[19]

$$\int_V \overline{H}(v) - c_F(v)\, dF \geq \int_V \overline{H}(v) - c_F(v)\, dG \quad \forall G \in \mathscr{F}. \tag{15}$$

With $\lambda = \int_V \overline{H}(v) - c_F(v)\, dF$, this writes

$$\int_V \overline{H}(v) - c_F(v) - \lambda\, dF = 0 \quad and \quad \int_V \overline{H}(v) - c_F(v) - \lambda\, dG \leq 0 \quad \forall G \in \mathscr{F}. \tag{16}$$

Because $F$ and $G$ are cdf's, this is equivalent to (10) and (11).

Finally, it is noteworthy that Proposition 1 does not use the monotonicity assumption A3.

# 5 Welfare Analysis

This section contains the key welfare results for general cost functions. The main result shows how the buyer's equilibrium utility and total welfare depend on the convexity of the cost function. To set the stage, I first show that in any equilibrium, the seller's profit is equal to $\alpha$ and thus coincides with his profit in the case when valuations are public.[20]

**Proposition 2** *The seller's equilibrium profit is $\alpha$.*

The argument is by contradiction. If the seller's profit was strictly larger than $\alpha$, then since the seller is indifferent between all prices in the support of the pricing distribution, the price $\alpha$ is not in the support. Thus, the smallest price, say $\underline{p}$, in the pricing distribution is strictly larger than $\alpha$. This implies that $\underline{p}$ cannot be in the support of the buyer's valuation distribution, because the buyer would benefit from redistributing probability mass from $v = \underline{p}$ to $v = \alpha$. This follows from the buyer's best response condition and the fact that $c_F$ is strictly increasing. But if $\underline{p}$ is not

[19]A similar characterization of the optimum of a concave functional in terms of its Gateaux derivative appears in Luenberger (1997, Lemma 1, p. 227).

[20]Proposition 2 does rely on A3. Without further assumptions on $c_F$, equilibrium profits are not uniquely pinned down because there can be multiple equilibria with different profits. This can be illustrated already in the linear case. Suppose that $\alpha = 0$, and that $c$ is strictly positive on the interval $(\alpha, \omega)$ and $c(\alpha) = c(\omega) = 0$. In this case, all distributions that place mass only on $\alpha$ or $\omega$ are costless for the buyer. There are therefore multiple equilibria: the buyer places probability $f$ on $\omega$ and $1-f$ on $\alpha$ for some $f \in [0,1]$, and the seller chooses $p = \omega$. The seller's profit is $f\omega$. This observation generalizes, and it can be shown that, in general, the seller's profit is bounded by

$$\min\left(\arg\min_{v \in V} c_F(v)\right) \leq \Pi \leq \max\left(\arg\min_{v \in V} c_F(v)\right). \tag{17}$$

in the support of the buyer's distribution, it cannot be an optimal price for the seller, because a slight price increase would make him better off.

Since the seller's profit is $\alpha$, total equilibrium welfare is $W = U_B + \alpha$ and thus pinned down by the buyer's equilibrium utility which is the object of the next proposition.

**Proposition 3** *Let $(F, H)$ be an equilibrium.*

*(i) The buyer's equilibrium utility is*[21]

$$U_B = \int_V c_F(v)\, dF(v) - C(F) - \min_{v \in V} c_F(v). \tag{18}$$

*(ii) If $C$ is linear, then $U_B = 0$.*

*(iii) If $C$ is strictly convex and $C(F) \neq 0$, then $U_B > 0$.*

The proof of part (i) shows that $\lambda$ in part (ii) of Proposition 1 is equal to $-\min_{v \in V} c_F(v)$. Once this is established, the expression for $U_B$ is immediate from plugging (11) into (4). Part (ii) is immediate from (18), the definition of linear costs and the fact that $\min_{v \in V} c(v) = 0$ by (8). Note that part (ii) replicates the result in Gul (2001) that when costs are linear, the welfare outcomes when the buyer's valuation is her private information coincide with those in the public information benchmark.

The argument behind part (iii), perhaps the key result of the paper, is economically intuitive, and it is instructive to elaborate on it. Imagine that, instead of choosing among all cdf's, the buyer chooses a uni-dimensional investment level $\tau \in [0, 1]$ by selecting a convex combination

$$F^\tau = \tau F + (1 - \tau) F_{min} \tag{19}$$

of the equilibrium distribution $F$ and the default distribution $F_{min}$. Given the seller's equilibrium distribution $H$, the buyer's benefit from investing $\tau$ is

$$\Psi(\tau) = \int_V \overline{H}(v)\, dF^\tau = \tau \int_V \overline{H}(v)\, d(F - F_{min}) + \int_V \overline{H}(v)\, dF_{min}, \tag{20}$$

and her cost is $\Phi(\tau) = C(F^\tau)$.

Because the buyer's equilibrium choice $F$ (in the equilibrium where she chooses among all cdf's) is $F^1$, her equilibrium utility is

$$U_B = \Psi(1) - \Phi(1). \tag{21}$$

[21] Note that $\min_{v \in V} c_F(v)$ is well-defined since $c_F$ is continuous by assumption.

I now use a marginal benefit and cost argument to show that this expression is strictly positive. Indeed, because the investment benefit $\Psi(\tau)$ is linear in $\tau$, marginal benefits $\Psi'(\tau) = \int_V \overline{H}(v)\, d(F - F_{min})$ are constant. On the other hand, note that

$$F^{\tau+\epsilon} = F^{\tau} + \epsilon(F - F_{\min}). \tag{22}$$

Therefore, marginal costs at $\tau$ correspond exactly to the Gateaux differential at $F^{\tau}$ in the direction $F - F_{min}$. Formally:[22]

$$\Phi'(\tau) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}[C(F^{\tau} + \epsilon(F - F_{\min})) - C(F^{\tau})] = \delta C(F^{\tau}, F - F_{min}) = \int_V c_{F^{\tau}}(v)\, d(F - F_{\min}). \tag{23}$$

Now, the fact that $F$ is a best response implies that at $\tau = 1$, marginal benefits (weakly) exceed marginal costs, because otherwise, the buyer could profitably deviate by marginally lowering $\tau$. Formally, the equilibrium conditions (10) and (11) imply that at $\tau = 1$:

$$\begin{aligned}
\Psi'(1) - \Phi'(1) &= \int_V \overline{H}(v) - c_F(v)\, d(F - F_{min}) && (24)\\
&= \int_V \lambda\, dF - \int_V \overline{H}(v) - c_F(v)\, dF_{min} && (25)\\
&= -\int_V \overline{H}(v) - c_F(v) - \lambda\, dF_{min} && (26)\\
&\geq 0. && (27)
\end{aligned}$$

Crucially, because $C$ is strictly convex, so is $\Phi$,[23] and thus $\Phi'$ is strictly increasing. Therefore, because marginal benefits are constant, marginal benefits strictly exceed marginal costs for all $\tau < 1$: $\psi'(\tau) > \phi'(\tau)$.

Finally, note that the buyer can guarantee herself a weakly positive utility by investing $\tau = 0$ which corresponds to choosing the default distribution $F_{min} = F^0$.[24]

Therefore, because the buyer's marginal utility is strictly positive up to $\tau = 1$, her overall

[22] This observation is also used in Chew and Nishimura, 1992, page 300.

[23] To see this, let $\zeta \in (0,1)$, then

$$\begin{aligned}
\Phi(\zeta\tau + (1-\zeta)\tau') &= C(F_{min} + (\zeta\tau + (1-\zeta)\tau')(F - F_{min})) && (28)\\
&= C(\zeta(F_{min} + \tau(F - F_{min})) + (1-\zeta)(F_{min} + \tau'(F - F_{min}))) && (29)\\
&< \zeta C(F_{min} + \tau(F - F_{min})) + (1-\zeta)C(F_{min} + \tau'(F - F_{min})) && (30)\\
&= \zeta\Phi(\tau) + (1-\zeta)\Phi(\tau'). && (31)
\end{aligned}$$

[24] This would result in utility $\Psi(0) - \Phi(0) = \int_V \overline{H}(v)\, dF_{min} - C(F_{min})$ which is weakly positive because $\overline{H}$ is positive and $C(F_{min}) = 0$ by (7).

utility is strictly positive. Formally:[25]

$$U_B = \Psi(1) - \Phi(1) = \int_0^1 \Psi'(\tau) - \Phi'(\tau)\, d\tau + \Psi(0) - \Phi(0) > 0. \tag{32}$$

Before I turn to economic implications of Proposition 3, it is noteworthy that the proposition does not rely on the monotonicity assumption A3. Moreover, the expression for the buyer's utility is quite general and does not depend on the details of the hold-up problem considered here, but carries over to games where players are engaged in some strategic interaction ex post and (some) players can choose an investment ex ante. The equilibrium of the ex post game pins down the best response conditions analogous to (10) and (11) from which players' utilities can be deduced as in (18).

The next corollary uses the results obtained so far to derive welfare changes when a strictly convex part is added to a linear cost function. Part (i) says that the buyer's equilibrium utility and total welfare may increase as costs increase. Part (ii) says that with strictly convex costs, the welfare when valuations are private is strictly larger than in the benchmark case with public valuations. In this sense, the hold-up problem is alleviated with unobservable investments.

**Corollary 1** *Consider a cost function that is a combination of a linear and a strictly convex cost function, that is,* $C(F) = \int_V \ell(v)\, dF + \kappa\Gamma(F)$ *where* $\kappa \geq 0$ *and* $\Gamma$ *is strictly convex and satisfies A2-A3. Suppose there is* $\tilde{\kappa} > 0$ *and an equilibrium* $(F_{\tilde{\kappa}}, H_{\tilde{\kappa}})$ *with* $C(F_{\tilde{\kappa}}) \neq 0$*. Then we have:*

*(i) The buyer's equilibrium utility and total welfare is strictly larger at* $\tilde{\kappa}$ *than at* $\kappa = 0$*.*

*(ii) Total welfare when valuations are private information is strictly larger than when they are public information at* $\tilde{\kappa}$ *but the same at* $\kappa = 0$*:* $W \mid_{\kappa=\tilde{\kappa}} > W^{PUB} \mid_{\kappa=\tilde{\kappa}}$ *and* $W \mid_{\kappa=0} = W^{PUB} \mid_{\kappa=0}$*.*

Part (i) is a direct implication of Propositions 2 and 3. Part (ii) follows from (i) and (9). In the next section, I impose more structure on the cost function under which there will be an interval of $\tilde{\kappa}$'s that satisfy the assumption in Corollary 1 that $C(F_{\tilde{\kappa}}) \neq 0$ in equilibrium.

# 6 Cost specifications

From now on, I assume that $C$ can be written as

$$C(F) = \kappa\Gamma(F), \tag{33}$$

[25]For the special case that the cost function is of the form $C(F) = \tilde{C}(\int \mu(v)\, dF)$, $\tilde{C}$, $\mu$ both convex, Georgiadis et al. (2023) analogously show that in a flexible moral hazard problem, the agent obtains a strictly positive rent if and only if $\tilde{C}$ is not linear.

where $\kappa > 0$ and $\Gamma$ is strictly convex and satisfies A2-A3 and has Gateaux derivative $\gamma_F$.[26]

Moreover, I shall distinguish cost functions according to their risk properties. By Cerreia-Vioglio et al. (2017), the risk properties of a cost function are closely connected to the shape of the Gateaux derivative (just like the risk attitudes of an expected utility maximizer are connected to the shape of her Bernoulli utility function).

In particular, I distinguish between the following cases.[27]

**A4** *$\Gamma(F) = \Gamma_0(M_F)$ depends only on the mean $M_F$ of $F$ where $\Gamma_0 : [\alpha, \omega] \to \mathbb{R}$ is strictly convex and differentiable.*

**A5** *$\gamma_F$ is strictly concave for all $F \in \mathscr{F}$.*

**A6** *$\gamma_F$ is strictly convex and differentiable for all $F \in \mathscr{F}$.*

Under A4, all that matters is the mean of a distribution but not its risk. Notice that the Gateaux derivative $\gamma_F(v) = \Gamma_0'(M_F)v$ is linear, thus excluding the other two cases. Moreover, strict convexity of $\Gamma_0$ ensures that $\Gamma$ is strictly convex while differentiability is only imposed to simplify the exposition. By Cerreia-Vioglio et al. (2017), A5 implies that $\Gamma$ (and thus $C$) increases if $F$ becomes less risky in the sense of the mean preserving spread order, and A6 implies that $\Gamma$ increases if $F$ becomes more risky in the sense of the mean preserving spread order. Differentiability of $\gamma_F$ in A6 is only imposed to simplify the exposition. I follow Condorelli and Szentes (2020) and refer to the three cases respectively also as mean-based, decreasing in risk, and increasing in risk (even though this is somewhat imprecise, as they impose these assumptions on $C$, not on $c_F$).

Finally, throughout this section, I assume that $\alpha > 0$,[28] and I denote by $T_f$ the "two-point distribution" that places mass $(1-f)$ on $\alpha$ and mass $f$ on $\omega$.

The main results of this section characterize welfare outcomes as a function of the cost parameter $\kappa$. I show that in all three cases, there is a critical $\hat{\kappa}$ so that the buyer's equilibrium utility and total welfare increase in $\kappa$ for all $\kappa \in (0, \hat{\kappa})$, and welfare under private information is strictly larger than when the buyer's valuation becomes public information. Moreover, under A4 and A5, equilibrium welfare coincides with first-best welfare for $\kappa$ larger than $\hat{\kappa}$. Importantly, this result is not driven simply by costs being prohibitive and holds for a range of $\kappa$'s where the default distribution $F_{min}$ is not first-best. In other words, the hold-up problem disappears in this case. I

[26] The analysis carries over without much substantial change, but requires more case distinctions, if $C(F)$ is of the form $\int_V \ell \, dF + \kappa\Gamma(F)$ as in Corollary 1 as long as $c_F = \ell + \kappa\gamma_F$ satisfies A3 and, respectively, A4, A5, or A6.

[27] Most of the analysis does not rely on the "strictness" properties A5 and A6. Strictness simplifies the analysis because it rules out multiplicity of equilibria at various places.

[28] The case $\alpha = 0$ is special in that the seller cannot guarantee himself a positive profit by charging a price equal to the lowest possible valuation. In this case, there are "mis-coordination" equilibria where the buyer does not invest and the seller charges a high price. The assumption $\alpha > 0$ rules out these uninteresting equilibria. Gul (2001) imposes a similar assumption.

also show that, in contrast, under A6 the equilibrium outcome differs from the first-best (unless costs are prohibitive so that the default distribution is first-best.)

I begin the analysis with noting a general property of the buyer's equilibrium distribution that follows directly from the fact that the seller's equilibrium profit is $\alpha$. Define for $\pi, \beta \in [\alpha, \omega]$, the distribution[29]

$$K_{\pi}^{\beta}(v) = \begin{cases} 0 & if \quad v < \pi \\ 1 - \pi/v & if \quad v \in [\pi, \beta) \\ 1 & if \quad v \geq \beta \end{cases} . \tag{34}$$

$K_{\pi}^{\beta}$ is known as an "equal revenue distribution" because if the seller faces "demand" $K_{\pi}^{\beta}$, then any price $p \in [\pi, \beta]$ gives him the same revenue $\pi$. Expressed differently, a distribution $F$ allows the seller to get profit larger than $\pi$ if it is located below $K_{\pi}^{\omega}$ at some point $v$, because the price $p = v$ yields the seller profit $v(1 - F(v^{-})) > v(1 - K_{\pi}^{\omega}(v)) = \pi$. This observation readily implies that because the seller's equilibrium profit is $\alpha$ by Proposition 2, the buyer's equilibrium distribution must be located (weakly) above $K_{\alpha}^{\omega}$:

**Lemma 1** *Let $(F, H)$ be an equilibrium. Then $F$ is first order stochastically dominated by $K_{\alpha}^{\omega}$, that is, $F(v) \geq K_{\alpha}^{\omega}(v)$ for all $v \in V$.*

## 6.1 Mean-based costs

In this section, I assume that A4 holds. Since $C(F) = \kappa \Gamma_0(M_F)$, the Gateaux derivative is

$$\gamma_F(v) = \Gamma_0'(M_F)v, \tag{35}$$

where $\Gamma_0' > 0$ due to A3. I begin by characterizing the first-best distribution.

**Lemma 2** *Let A4 hold, and define*

$$\kappa_0 = \frac{1}{\Gamma_0'(\omega)}, \quad \kappa_1 = \frac{1}{\Gamma_0'(\alpha)}. \tag{36}$$

*Then any distribution with mean $M^{FB} = M^{FB}(\kappa)$ is first-best, where*

$$M^{FB} = \begin{cases} \omega & if \quad \kappa \leq \kappa_0 \\ \Gamma_0'^{-1}(1/\kappa) & if \quad \kappa \in (\kappa_0, \kappa_1) \\ \alpha & if \quad \kappa_1 \leq \kappa \end{cases} . \tag{37}$$

[29] This distribution plays a key role also in Gul (2001), Roesler and Szentes (2017), Condorelli and Szentes (2020), Ravid et al. (2022).

To understand the lemma, notice that a first-best distribution maximizes the total surplus

$$\int_V v\, dF - C(F) = M_F - \kappa\Gamma_0(M_F), \tag{38}$$

which depends only on the mean, because costs are mean-based. Because $M_F \in [\alpha, \omega]$, the claim follows from the first-order condition for the maximizer of (38). Note also that because $\Gamma_0$ is strictly convex, the maximizer is unique whereas the first-distribution is unique only for $\kappa$ outside the interval $(\kappa_0, \kappa_1)$. For $\kappa \in (\kappa_0, \kappa_1)$, any distribution $F$ with $M_F = M^{FB}$ is a first-best distribution. Finally, observe that strict convexity of $\Gamma_0$ also implies that $M^{FB}$ is strictly decreasing within $(\kappa_0, \kappa_1)$.

Next, I characterize equilibrium.

**Proposition 4** *Under A4, $(F, H)$ is an equilibrium only if the seller's pricing distribution is a two-point distribution $H = T_h$ with $h < 1$, and $F(v) \geq K_\omega^\alpha(v)$ for all $v \in V$. Moreover:*

*(i) $(F, T_h)$ is an equilibrium with $h > 0$ if and only if $F(\omega^-) = 1 - \alpha/\omega$, $\kappa\Gamma_0'(M_F) < 1$, and $h = 1 - \kappa\Gamma_0'(M_F)$.*

*(ii) $(F, T_h)$ is an equilibrium with $h = 0$ if and only if $F$ is a first-best distribution.*

That $F$ is first order stochastically dominated by $K_\alpha^\omega$ is Lemma 1. To see that the seller's pricing distribution is a two-point distribution, recall from (14) that in equilibrium every valuation in the buyer's support maximizes the function $\overline{H}(v) - c_F(v)$. Because $\overline{H}$ is convex by definition, and $c_F$ is linear for mean-based costs, either (1) $\overline{H}(v) - c_F(v)$ is maximized at a corner point $\alpha$ or $\omega$, or (2) any $v \in [\alpha, \omega]$ is a maximizer, and $\overline{H}$ is, in fact, linear itself. In case (1), because the buyer has no valuation in $(\alpha, \omega)$, setting a price in $(\alpha, \omega)$ is strictly suboptimal. Case (2) implies, by definition, that $\overline{H}' = H$ is constant on $(\alpha, \omega)$, and thus $H$ has no support point in $(\alpha, \omega)$.

Moreover, there cannot be an equilibrium where the seller charges $p = \omega$ with probability 1 ($h = 1$). The reason is that if $p = \omega$ the buyer's utility is $0 - C(F)$ and her best response would be the default distribution. But then $p = \omega$ would yield zero profit and is not a best response for the seller. Hence, $h < 1$ in equilibrium.

Part (i) describes a (candidate) equilibrium where the seller randomizes between the prices $\alpha$ and $\omega$. For the seller to be indifferent between $\alpha$ and $\beta$, the buyer needs to put mass $\alpha/\omega$ on $\omega$, hence $1 - F(\omega^-) = \alpha/\omega$. Moreover, if the seller randomizes between $\alpha$ and $\omega$, the buyer obtains utility $(1-h)(M_F - \alpha) - \kappa\Gamma_0(M_F)$, because costs are mean-based. The first-order condition for the (mean of the) buyer's best response thus implies $1 - h = \kappa\Gamma_0'(M_F)$.

Part (ii) of the proposition describes a (candidate) equilibrium where the seller charges the low price $\alpha$ with probability 1. Ex post trade is then efficient, and the buyer is the residual claimant of the efficient surplus. Thus, choosing a first-best distribution is a best response.

When costs are mean-based, there can be multiple equilibria where the buyer's distribution has different means. To see this, consider condition (i) of Proposition 4. For the seller to be indifferent between the prices $\alpha$ and $\omega$, all that is needed is that the buyer's distribution puts mass $\alpha/\omega$ on $\omega$. On the other hand, given the seller chooses price $\omega$ with probability $h$, it follows from the buyer's best response condition that because $\gamma_F$ is linear, any distribution with mean such that $1-h=\kappa\Gamma_0'(M_F)$ is a best response. In general, there are multiple combinations $(h, M_F)$ that satisfy these requirements.[30]

Even though there might be multiple equilibria, in any equilibrium, the seller's profit is $\alpha$ by Proposition 2. Therefore, an equilibrium which maximizes the buyer's utility also maximizes total welfare, and is also Pareto-optimal. It is therefore natural to focus on Pareto-optimal equilibria which I characterize next.

**Proposition 5** *Under A4, we have:*

*(i) If $M_{K_\alpha^\omega} < M^{FB}$, then:*

*(a) There is a unique Pareto-optimal equilibrium $(F, H)$ with $F = K_\alpha^\omega$ and $H = T_h$ with $h = 1-\kappa\Gamma_0'(M_{K_\alpha^\omega})$.*

*(b) In this equilibrium, the buyer's utility is $U_B = \kappa[\Gamma_0'(M_{K_\alpha^\omega})(M_{K_\alpha^\omega}-\alpha)-(\Gamma_0(M_{K_\alpha^\omega})-\Gamma_0(\alpha))]$.*

*(ii) If $M_{K_\alpha^\omega} \geq M^{FB}$, then:*

*(a) There is a first-best distribution $F$ so that $F$ and $H = T_h$ with $h = 0$ is a Pareto-optimal equilibrium.*

*(b) In any Pareto-optimal equilibrium, the buyer's utility is $U_B = W^{FB} - \alpha$, where $W^{FB}$ is first-best welfare.*

Intuitively, if $M_{K_\alpha^\omega} < M^{FB}$, then no first-best distribution can be first order stochastically dominated by $K_\alpha^\omega$, and hence there is no equilibrium as in part (ii) of Proposition 4. Moreover, the characterization of the first-best in Lemma 2 together with the fact that $\Gamma_0'$ is strictly increasing implies that $\kappa\Gamma_0'(M_{K_\alpha^\omega}) < 1$ if $M_{K_\alpha^\omega} < M^{FB}$. Therefore, by part (i) of Proposition 4, $F = K_\alpha^\omega$ with the respective $T_h$ is an equilibrium. Finally, it is Pareto-optimal because among all buyer distributions that are equilibrium candidates, $K_\alpha^\omega$ has the maximal mean.

If, on the other hand, $M_{K_\alpha^\omega} \geq M^{FB}$, then there is a first-best distribution with $F^{FB} \geq K_\alpha^\omega$. For example, by an intermediate value argument, there is an equal revenue distribution $K_\alpha^\beta$ with

[30] Among all distributions $F$ that put mass $\alpha/\omega$ on $\omega$ and satisfy $F \geq K_\alpha^\omega$, the distribution $T_{\alpha/\omega}$ has the smallest mean, and the distribution $K_\alpha^\omega$ has the largest mean. Therefore, any distribution $F$ with $K_\alpha^\omega \leq F \leq T_{\alpha/\omega}$ and $\kappa\Gamma_0'(M_F) < 1$ is consistent with equilibrium.

largest support point $\beta < \omega$ such that $M_{K_\alpha^\beta} = M^{FB}$. Since costs are mean-based, $K_\alpha^\beta$ is a first-best distribution. Thus there is an equilibrium as in part (ii) of Proposition 4. Because the buyer chooses a first-best distribution in such an equilibrium, it is clearly Pareto-optimal.

The next corollary restates the previous proposition as a comparative statics result in terms of $\kappa$.

**Corollary 2** *Let A4 hold and let*

$$\hat{\kappa} = \frac{1}{\Gamma_0'(M_{K_\alpha^\omega})}. \tag{39}$$

*Then $\hat{\kappa} \in (\kappa_0, \kappa_1)$, and along any selection of Pareto-optimal equilibria $(F_\kappa, H_\kappa)$ we have:*

*(i) The buyer's utility $U_B$ and total welfare is strictly increasing in $\kappa$ for $\kappa \in (0, \hat{\kappa})$.*

*(ii) Total welfare is equal to first-best welfare for $\kappa \geq \hat{\kappa}$.*

*(iii) Total welfare when valuations are private information is strictly larger than when they are public for all $\kappa < \kappa_1$.*

To understand the result, recall from Lemma 2 that when $\kappa$ is small ($\kappa < \kappa_0$), the first-best distribution puts all mass on $\omega$, and $M^{FB} = \omega$. Thus, for small values of $\kappa$, we are in the parameter region of part (i) of Proposition 4. As $\kappa$ increases within the range $(0, \hat{\kappa})$, there are two effects on the buyer's utility. On the one hand, there is a price effect, as the seller increases the probability of charging the low price $p = \alpha$. On the other hand, the buyer faces higher investment costs.

In the range $(0, \hat{\kappa})$, the price effect outweighs the cost effect, and $U_B$ increases with $\kappa$. To see this, note that since costs are mean-based, we can think of the buyer as simply choosing a (uni-dimensional) mean $M \in [\alpha, \omega]$. Since the buyer's utility is $(1-h)(M-\alpha) - \kappa\Gamma_0(M)$, the marginal benefit of doing so is the probability of the low price $1-h$. Thus, given $1-h$, the buyer increases the mean until marginal costs are equal to marginal benefits: $1-h = \kappa\Gamma_0'(M_{K_\alpha^\omega})$. On the other hand, for "inframarginal" units $M < M_{K_\alpha^\omega}$, marginal benefits are strictly larger than marginal costs, since costs are strictly convex, thus generating strictly positive utility for the buyer.

Therefore, as $\kappa$ increases, the price effect exactly compensates the buyer for the cost effect at the margin $M_{K_\alpha^\omega}$. At the same time, the difference between marginal benefits and marginal costs for the inframarginal units becomes more pronounced. Thus, the price effect outweighs the cost effect overall. As a result, the buyer's utility as well as total welfare increase in $\kappa$. Formally, strict convexity of $\Gamma_0$ implies that the term in square brackets in the expression for the buyer's utility in part (i),(b) of Proposition 4 is strictly positive.

To see part (ii) of the lemma, observe that as $\kappa$ increases, $M^{FB}$ decreases until it reaches the level $M_{K_\alpha^\omega}$ at $\kappa = \hat{\kappa}$ at which point we move into the parameter region of part (ii) of Proposition

4. The equilibrium outcome then coincides with the first-best, and this explains part (ii) of the lemma. Notice that $\hat{\kappa} < \kappa_1$. Hence, for $\kappa \in (\hat{\kappa}, \kappa_1)$, the hold-up problem disappears even though the first-best distribution in this region differs from the default distribution.

Finally, part (iii) is a direct consequence of the fact that the buyer's utility is strictly positive for all $\kappa < \kappa_1$.

## 6.2 Strictly concave Gateaux derivative

In this section, I assume A5. I begin with the characterization of the first-best distribution.

**Lemma 3** *Under A5, there is a unique first-best distribution given by the two-point distribution* $F^{FB} = T_{f^{FB}}$ *where* $f^{FB}$ *minimizes* $f(\omega - \alpha) - C(T_f)$ *over* $f \in [0,1]$.

To understand the intuition, consider the problem of maximizing total surplus for a given mean:

$$\max_F \int_V v\, dF - C(F) = M_F - C(F) \quad s.t. \quad M_F = M. \tag{40}$$

Because $C$ is strictly decreasing in risk, $F$ minimizes costs by maximizing risk, that is, by putting mass only on the support bounds $\alpha$ and $\omega$. Thus, the solution to (40) is a two-point distribution $T_f$ and induces total surplus $f(\omega - \alpha) - C(T_f)$.

More formally, the result follows from a general characterization of the first-best distribution that I give in the appendix. This characterization is formally analogous to the buyer's best response conditions (10) and (11) with the difference that the buyer's gross benefit $\overline{H}(v)$ is replaced by the first-best gross benefit $v$. In particular, any point in the support of the first-best distribution maximizes $v - \kappa\gamma_F(v)$. Because of strict concavity of $\gamma_F$, the only possible maximizers are $\alpha$ or $\omega$, and thus, the only points in the support of a first-best distribution are $\alpha$ or $\omega$.

I next derive necessary conditions for equilibrium. First, the same argument as in the previous paragraph implies that the buyer's equilibrium distribution is a two-point distribution $T_f$ when the Gateaux derivative is strictly concave. More specifically, by (14), any point in the support of the buyer's equilibrium distribution maximizes $\overline{H}(v) - \kappa\gamma_F(v)$. Because of convexity of $\overline{H}$ and strict concavity of $\gamma_F$, the only possible maximizers are $\alpha$ or $\omega$. Moreover, by Lemma 1, $F$ is first order stochastically dominated by $K_\alpha^\omega$. For a two-point distribution $F = T_f$, this is the case if and only if $f \leq \alpha/\omega$ or, equivalently, $T_f \geq T_{\alpha/\omega}$. Finally, because the buyer's distribution has only $\alpha$ or $\omega$ in its support, it is strictly suboptimal for the seller to charge a price strictly in between $\alpha$ and $\omega$. Thus, $H$ is a two-point distribution $T_h$. As with Proposition 4, we have $h < 1$. The following proposition summarizes.

**Proposition 6** *Under A5, $(F,H)$ is an equilibrium only if $F$ and $H$ are two-point distributions $H = T_h$ with $h < 1$, and $F = T_f$ with $F \geq T_{\alpha/\omega}$.*

By Lemma 3 and Proposition 6, the search for first-best and the buyer's equilibrium distributions can be restricted to two-point distributions $T_f$. Moreover, the mean $M = (1-f)\alpha + f\omega$ of a two-point distribution uniquely pins down $f = \frac{M-\alpha}{\omega-\alpha}$. Therefore, instead of searching for first-best or equilibrium values of $f$, one can as well search for first-best or equilibrium values of $M \in [\alpha, \omega]$. With this change of variables, the setting becomes effectively mean-based, and the results from the previous section essentially carry over. More specifically, define the cost of choosing the two-point distribution $T_f$ with mean $M$ as

$$\Gamma_0(M) = \Gamma(T_{\frac{M-\alpha}{\omega-\alpha}}). \tag{41}$$

$\Gamma_0$ is then increasing and strictly convex with derivative[31]

$$\Gamma_0'(M) = \frac{\gamma_{T_{\frac{M-\alpha}{\omega-\alpha}}}(\omega) - \gamma_{T_{\frac{M-\alpha}{\omega-\alpha}}}(\alpha)}{\omega - \alpha}. \tag{43}$$

The characterization of the mean of the first-best distribution is then identical as in Lemma 2. Likewise, given that the seller chooses a two-point distribution $T_h$ in equilibrium, the buyer's utility from $T_f$ in terms of its mean is $(1-h)(M-\alpha) - \kappa\Gamma_0(M)$ which is formally identical as in the case with mean-based costs in the previous section. Therefore, Proposition 4 as well as Proposition 5 and Corollary 2 carry over verbatim with the only difference that the distribution $K_\alpha^\omega$ is replaced by the two-point distribution $T_{\alpha/\omega}$. The reason is that now the buyer's equilibrium distribution $F$ is in the class of two-point distributions with $F = T_f \geq T_{\alpha/\omega}$.

**Remark** Because under A4 and A5, the equilibrium outcome is first-best for $\kappa \geq \hat{\kappa}$, the equilibrium outcome also coincides with the buyer-optimal commitment outcome as analyzed in Condorelli and Szentes (2020). The reason is that seller's profit when the buyer has commitment is never smaller than $\alpha$ because the seller can guarantee himself $\alpha$ by charging the price $\alpha$ with probability 1. Therefore, if $\kappa \geq \hat{\kappa}$, then even with commitment, the buyer cannot attain higher utility than his equilibrium utility $W^{FB} - \alpha$.

[31]To see this, note that $T_{f+\epsilon} = T_f + \epsilon(T_1 - T_0)$, and thus

$$\frac{d}{df}\Gamma(T_f) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\Gamma(T_f + \epsilon(T_1 - T_0)) - \Gamma(T_f)\right] = \delta\Gamma(T_f; T_1 - T_0) = \int_V \gamma_{T_f}\, d(T_1 - T_0) = \gamma_{T_f}(\omega) - \gamma_{T_f}(\alpha). \tag{42}$$

The expression for $\Gamma_0'(M)$ now follows with the chain rule.

## 6.3 Strictly convex Gateaux derivative

In this section, I assume A6. I begin by characterizing the first-best.

**Lemma 4** *Under A6, a first-best distribution is a deterministic distribution $F^{FB} = \mathbb{1}_{[v^{FB},\omega]}$ that puts all mass on $v^{FB}$ where $v^{FB}$ maximizes $v - \kappa\Gamma(\mathbb{1}_{[v,\omega]})$ over $v \in [\alpha, \omega]$. In particular, the default distribution is first-best if and only if*

$$\kappa \geq \kappa_1 = \frac{1}{\gamma'_{\mathbb{1}_{[\alpha,\omega]}}(\alpha)}. \tag{44}$$

Intuitively, consider again problem (40). Because now $C$ is strictly increasing in risk, $F$ minimizes costs by minimizing risk, that is, by concentrating all mass on a single point. Thus, the solution to (40) is deterministic. Moreover, the total surplus induced by a distribution that puts all mass on $v$ is $v - \kappa\Gamma(\mathbb{1}_{[v,\omega]})$.

The characterization for when the default distribution is first-best will be used below when I compare equilibrium and first-best. The characterization follows from the fact that any point in the support of the first-best distribution maximizes $v - \kappa\gamma_{F^{FB}}(v)$, as explained after Lemma 3. Therefore, the default distribution $\mathbb{1}_{[\alpha,\omega]}$ is first-best if and only if the only point in its support, $\alpha$, maximizes $v - \kappa\gamma_{\mathbb{1}_{[\alpha,\omega]}}(v)$. This the case if and only if $1 - \kappa\gamma'_{\mathbb{1}_{[\alpha,\omega]}}(\alpha) \leq 0$, that is, (44).

I next characterize equilibrium. Strict convexity of $\gamma_F$ implies that the function $v - \kappa\gamma_F(v)$ has a unique maximizer $v^*(F)$ on $[\alpha, \omega]$. Recall also the definition of $K_\pi^\beta$ from (34).

**Proposition 7** *Under A6, $(F, H)$ is an equilibrium if and only if*

$$F = K_\alpha^{v^*(F)}, \quad H(p) = \kappa\gamma'_F(p)\mathbb{1}_{[\alpha,v^*(F))}(p) + \mathbb{1}_{[v^*(F),\omega]}(p). \tag{45}$$

Proposition 7 says that the buyer's distribution is an equal revenue distribution and that the seller's pricing distribution is essentially equal to the derivative of the Gateaux derivative. In particular, the supports $[\alpha, v^*(F)]$ of both distributions are convex and identical. The proof of Proposition 7 follows from the same arguments as in the proof of Proposition 1 in Gul (2001). The only difference is that Gul (2001) considers the case with linear cost $C$ so that $c_F$ does not depend on $F$. This does not, however, matter for the argument.[32]

Proposition 7 characterizes the equilibrium only implicitly because $F$ depends on $v^*(F)$. A more explicit characterization can be obtained by noting that $v^*(F) = \beta$ is the upper support bound of an equal revenue distributions $F = K_\alpha^\beta$. Thus, equilibrium is characterized by the solutions to the equation $v^*(K_\alpha^\beta) = \beta$. Since $v^*(F)$ maximizes the function $v - \kappa\gamma_F(v)$, $v^*(F)$ is

[32] As in Gul (2001), differentiability of $\gamma_F$ is not needed for the result. In this case, the derivative of $\gamma_F$ in (46) has to be replaced by the right derivative.

the solution to the respective first order condition. This pins down the equilibrium value of $\beta$ as stated in the next proposition.

**Proposition 8** *Under A6, $(F, H)$ is an equilibrium if and only if*

$$F = K_\alpha^\beta, \quad H(p) = \kappa\gamma'_{K_\alpha^\beta}(p)\mathbb{1}_{[\alpha,\beta)}(p) + \mathbb{1}_{[\beta,\omega]}(p), \tag{46}$$

*and $\beta$ is any value that satisfies:*

$$\kappa\gamma'_{K_\alpha^\beta}(\beta) = 1, \quad \textit{or} \quad \beta = \alpha \textit{ and } \kappa\gamma'_{K_\alpha^\alpha}(\alpha) \geq 1, \quad \textit{or} \quad \beta = \omega \textit{ and } \kappa\gamma'_{K_\alpha^\omega}(\omega) \leq 1. \tag{47}$$

In general, there might be multiple equilibria, because there might be multiple solutions $\beta$ to (47). A sufficient condition for there to be a unique solution $\beta$ is that $\gamma'_{K_\alpha^\beta}(v)$ is strictly increasing in $\beta$ for all $v \in V$. To see this, note that the convexity of $\gamma$ then implies that the function $\gamma'_{K_\alpha^x}(x)$ is strictly increasing in $x$, and an intermediate value argument implies that (47) has a unique solution.[33] One class of cost functions which satisfies this property is the class of "moment-based" cost functions $\Gamma(F) = \tilde{\Gamma}(\int_V \mu(v)\, dF)$ where $\tilde{\Gamma}$ is differentiable, increasing and convex, and $\mu$ is differentiable, increasing and strictly convex.[34]

For the purpose of comparative statics with respect to $\kappa$, I shall now assume that equilibrium is unique. Recall that $\kappa_1$ is the critical value from which on the default distribution becomes first-best.

**Corollary 3** *Suppose that A6 holds, and let*

$$\hat{\kappa} = \frac{1}{\gamma'_{K_\alpha^\omega}(\omega)}. \tag{48}$$

*Moreover, suppose that there is a unique equilibrium. Then, we have:*

*(i) The buyer's utility $U_B$ and total welfare is strictly increasing in $\kappa$ for $\kappa \in (0, \hat{\kappa})$.*

*(ii) Total welfare when valuations are private information is strictly larger than when they are public for all $\kappa < \kappa_1$.*

*(iii) First-best welfare is strictly larger than equilibrium welfare for all $\kappa < \kappa_1$, that is, unless the default distribution is first-best.*

[33] Recall that a differentiable convex function is also continuously differentiable.

[34] In this case, $\gamma_F(v) = \tilde{\Gamma}'(\int_V \mu(v) dF)\mu(v)$, and $\gamma'_F(v) = \tilde{\Gamma}'(\int_V \mu(v) dF)\mu'(v)$. As $F$ increases in the first order stochastic dominance sense, so does $\tilde{\Gamma}'(\int_V \mu(v)\, dF)$ as well as $\gamma'_F(v)$. Because $K_\alpha^\beta$ increases in the first order stochastic dominance sense as $\beta$ increases, the claim follows.

The intuition behind part (i) of the corollary is similar as in Corollary 2. As $\kappa$ increases in the range $(0, \hat{\kappa})$, the direct effect of facing higher investment costs is outweighed by the indirect strategic effect that the seller reduces the price (in the first order sense). More precisely, observe that for values $\kappa \leq \hat{\kappa}$, the buyer chooses $F = K_\alpha^\omega$ in equilibrium. Therefore, the Gateaux derivative $\kappa \gamma_{K_\alpha^\omega}$ becomes steeper as $\kappa$ increases. By Proposition 7, this means that the seller's pricing distribution decreases in the first order sense. This price effect outweighs the direct cost effect because costs are strictly convex.

Part (ii) follows from the fact that for $\kappa < \kappa_1$ the equilibrium distribution is different from the default distribution. Therefore, by Proposition 3, the buyer's utility is strictly positive, and thus total welfare is strictly larger than when valuations are private information.

While parts (i) and (ii) are analogous to the case with linear or strictly concave Gateaux derivative, part (iii) is different. The reason is that unless the buyer's equilibrium distribution is the default distribution, the buyer's and the seller's distribution have the same (non-degenerate) interval support $[\alpha, \beta]$, leading to trade inefficiencies ex post. The equilibrium outcome is therefore not efficient.

I conclude this section with a parameterized example that sheds light on the difference between equilibrium and first-best welfare as a function of $\kappa$.

### 6.3.1 Example

Consider the second moment $Q_F = \int_V v^2 \, dF$, and define

$$\Gamma(F) = \frac{1}{4} Q_F^2 - \frac{1}{4} \alpha^4 \tag{49}$$

with Gateaux derivative

$$\gamma_F(v) = \frac{1}{2} Q_F v^2. \tag{50}$$

$\Gamma$ is evidently convex. Since $F_{min}$ places all mass on $\alpha$, the second term in (49) ensures that $\Gamma(F_{min}) = 0$, in line with the normalization in (7).

With an eye on applying Proposition 8, note that since $K_\alpha^\beta$ has density $\alpha/v$ on $[\alpha, \beta)$ and a mass point of size $\alpha/\beta$ at $\beta$, we have

$$Q_{K_\alpha^\beta} = \int_\alpha^\beta v^2 \frac{\alpha}{v^2} \, dv + \beta^2 \cdot \frac{\alpha}{\beta} = 2\alpha\beta - \alpha^2 \tag{51}$$

so that $\gamma'_{K^{\beta}_{\alpha}}(v) = [2\alpha\beta - \alpha^2]v$. Condition (47) for the equilibrium value $\beta$ therefore writes

$$\kappa[2\alpha\beta^2 - \alpha^2\beta] = 1, \quad \text{or} \quad \beta = \alpha \text{ and } \kappa\alpha^3 \geq 1, \quad \text{or} \quad \beta = \omega \text{ and } \kappa[2\alpha\omega^2 - \alpha^2\omega] \leq 1. \tag{52}$$

Thus, I obtain the following equilibrium characterization where I calculate the buyer's utility using (18).

**Lemma 5** *Let $\Gamma$ be given by (49). Let $\hat{\beta}$ be the positive solution to the quadratic equation*

$$2\alpha\kappa\hat{\beta}^2 - \alpha^2\kappa\hat{\beta} - 1 \;=\; 0. \tag{53}$$

*Define $\hat{\kappa} = \frac{1}{\alpha\omega(2\omega-\alpha)}$ and $\kappa_1 = \frac{1}{\alpha^3}$. Then the equilibrium value $\beta$ in Proposition 8 is*

$$\beta = \begin{cases} \omega & if \quad \kappa \leq \hat{\kappa} \\ \hat{\beta} & if \quad \kappa \in (\hat{\kappa}, \kappa_1) \\ \alpha & if \quad \kappa \geq \kappa_1 \end{cases}. \tag{54}$$

*Moreover, the buyer's equilibrium utility is $U_B = \kappa\alpha^2(\beta - \alpha)^2$.*

The blue solid line in Figure 2 illustrates the typical shape of equilibrium welfare $U_B + \alpha$ as a function of $\kappa$. It increases linearly in the range $\kappa \in (0, \hat{\kappa})$ and then decreases. At $\kappa_1$, investment costs become prohibitive, and $F = F_{min}$ in equilibrium.

Next, I characterize the first-best which follows straightforwardly from Lemma 4.

**Lemma 6** *Let $\Gamma$ be given by (49). Let $\kappa_0 = 1/\omega^3$. Then the first-best distribution is $F^{FB} = \mathbb{1}_{[v^{FB},\omega]}$ where*

$$v^{FB} = \begin{cases} \omega & if \quad \kappa \leq \kappa_0 \\ \kappa^{-\frac{1}{3}} & if \quad \kappa \in (\kappa_0, \kappa_1) \\ \alpha & if \quad \kappa \geq \kappa_1 \end{cases}. \tag{55}$$

Figure 2 plots the first-best welfare (red, dashed line). The difference between first-best and equilibrium welfare is largest at $\kappa = 0$ (corresponding to linear costs) and then decreases with $\kappa$ until it becomes zero at the level $\kappa_1$ where costs become prohibitive so that zero investment is efficient.

# 7 Investment and information acquisition

My model can be extended to allow an interpretation where it is costly for the buyer not only to invest in, but also to learn about, her valuation. Ravid et al. (2022) consider a framework where

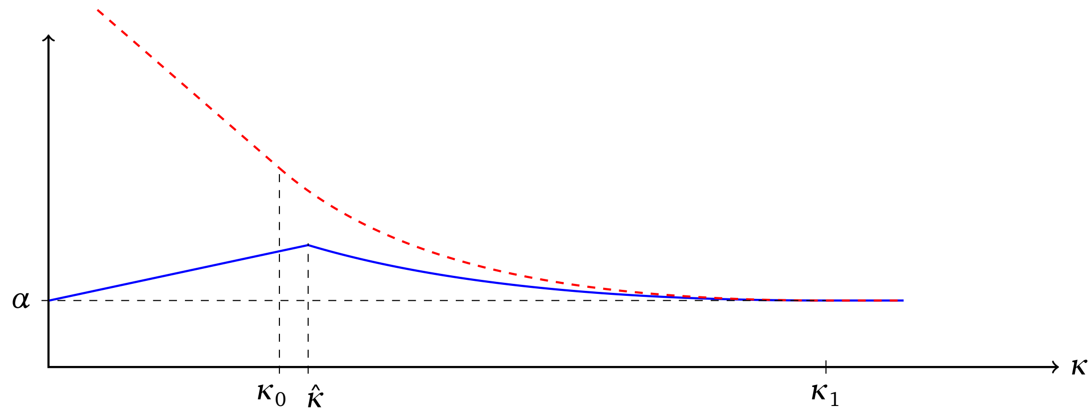


Figure 2: Total equilibrium welfare (blue) and first-best welfare (red, dashed) for the cost specification (49) as a function of $\kappa$ for the values $\alpha = 1$, $\omega = 3/2$, $\kappa_0 = 1/12$, $\kappa_1 = 1/2$.

the buyer's true valuation is a value $\theta$ in a compact interval $\Theta$. Initially, the buyer only has a prior belief $F_0 \in \mathscr{F}$ over $\Theta$, but she can acquire a signal about her valuation at a cost before trading. Since the buyer's preferences for the good are linear in the valuation, a signal corresponds to a distribution $F \in \mathscr{F}$ of posterior means that is a mean preserving contraction (MPC) of the prior $F_0$. Optimizing over a functional subject to the MPC constraint is, by now, a well-studied problem when the functional is linear (Dworczak and Martini, 2019, Kleiner et al., 2021), but is difficult when the functional is non-linear such as when information acquisition costs are non-linear.

The approach presented in this paper can however be applied to include information acquisition when one restricts the space $\Theta$ of the buyer's true valuations to consist of only two possible values, as I now illustrate. Suppose that $\Theta = \{\alpha, \omega\}$. A prior then corresponds simply to a mean $M_0 \in V = [\alpha, \omega]$, and any signal corresponds to a cdf $F \in \mathscr{F}$ over posterior means $v \in V$ with the simplified MPC constraint that $F$ has mean $M_0$, that is, $\int_V v dF = M_0$.

Specifically, suppose that without investing, the buyer's valuation is equal to the lowest possible valuation $\alpha$ with probability 1. The buyer can invest to increase the mean $M_F$ of the valuation distribution $F$ at a cost $\rho\varphi(M_F)$, $\rho \geq 0$, where $\varphi$ is strictly increasing and convex. The mean $M_F$ then corresponds to the prior, and the buyer can learn about the true valuation given the prior at a cost. Specifically, consider a strictly convex function $r : V \to \mathbb{R}$ and let $\sigma \int_V r(v) - r(M_F)\, dF$, $\sigma \geq 0$, be the ("posterior-separable") cost of information acquisition. Since $r$ is convex, information acquisition costs increase in the mean preserving spread order, or equivalently, in Blackwell informativeness. Moreover, acquiring no information, which corresponds to choosing the degenerate distribution $\mathbb{1}_{[M,\omega]}$ that places probability 1 on $M$, is costless.

The cost function

$$C(F) = \rho\varphi(M_F) + \sigma \int_V r(v) - r(M_F)\, dF \tag{56}$$

therefore combines the cost of investing and the cost of information acquisition. Suppose that $\rho\varphi - \sigma r$ is strictly convex. Then $C$ is convex, and if $r$ is strictly increasing, then the Gateaux derivative

$$c_F(v) = [\rho\varphi'(M_F) - \sigma r'(M_F)]v + \sigma r(v) \tag{57}$$

is strictly increasing and strictly convex. Therefore, Proposition 8 applies unchanged with $\kappa\gamma_F$ replaced by $c_F$. Moreover, as explained after the statement of Proposition 8, equilibrium is unique if $c'_{K_\alpha^\beta}(v)$ is strictly increasing in $\beta$ for all $v \in V$. This is satisfied here because $\rho\varphi - \sigma r$ is strictly convex.

I now use these observations to obtain the following comparative statics result.

**Lemma 7** *Let $C$ be given by (56). Let $\varphi$, $r$ be strictly increasing, differentiable, and strictly convex, and let $\rho\varphi - \sigma r$ be strictly convex. Then there are $\hat{\rho}, \hat{\sigma}(\rho)$ so that for all $(\rho, \sigma)$ with $\rho < \hat{\rho}$ and $\sigma < \hat{\sigma}(\rho)$, the buyer's equilibrium utility and total welfare is strictly increasing in $\rho$ and strictly decreasing in $\sigma$.*

The formal reason behind this result is similar to the reason behind Corollaries 2 and 3. As $\rho$ increases or $\sigma$ decreases, the convex part of the cost function $C$ goes up. Thus, an increase in $\rho$ or a decrease in $\sigma$ corresponds to an increase in $\kappa$ in Corollaries 2 and 3.

Economically, Lemma 7 suggests that the comparative statics with respect to investment and information acquisition costs are opposed to one another. While higher investment costs (higher $\rho$) attenuate the hold-up problem, higher information acquisition costs (higher $\sigma$) aggravate the hold-up problem.

# 8 Conclusion

In this paper, I reconsider the hold-up problem with unobservable investments when the buyer's investment costs are convex in the investment distribution. The main results are that in contrast to the case with linear costs, the buyer's utility is positive in equilibrium, and welfare with privately observable valuation is strictly larger than when valuations are public. Moreover, buyer utility and total welfare might increase with costs. Finally, when costs are mean-based or decreasing in risk, the hold-up problem may disappear, since the equilibrium outcome is efficient in some cost range. The equilibrium characterization I derive is portable to other applications with flexible pre-investments.

# A Appendix

**Proof of Proposition 1** I only show part (ii). (Part (i) follows with the same arguments as the proof in footnote 22 in Ravid et al., 2022.) Note first that it is a standard argument that the (mixed) strategy $H$ is a best response to $F$ for the seller if and only (12) and (13) hold, where $\pi$ is the seller's best response profit. Because the seller can guarantee himself the profit $\alpha$ by choosing the price $\alpha$ with probability 1, we have $\pi \geq \alpha$.

That the buyer's best response to $H$ is characterized by (10) and (11) is shown in the main text. QED

**Proof of Proposition 2** Note first that by setting the price $\alpha$ with probability 1, the seller can ensure profit $\alpha$. Thus, $\alpha \leq \Pi$. To see that $\alpha \geq \Pi$, assume to the contrary that $\alpha < \Pi$. Let $\underline{p} = \min supp(H)$ be the lower support bound of the seller's pricing distribution.

I first show that $\underline{p} \notin supp(F)$. Indeed, since $\alpha < \Pi = (1 - F(\underline{p}^-))\underline{p}$, we have $\alpha < \underline{p}$. Because $\overline{H}(\alpha) = \overline{H}(\underline{p}) = 0$ by definition of $\overline{H}$, and since $c_F(\alpha) < c_F(\underline{p})$ by A3, it follows

$$\overline{H}(\alpha) - c_F(\alpha) - \lambda = 0 - c_F(\alpha) - \lambda > 0 - c_F(\underline{p}) - \lambda = \overline{H}(\underline{p}) - c_F(\underline{p}) - \lambda. \tag{58}$$

Therefore, (10) and (11) imply that $\underline{p} \notin supp(F)$.

Now distinguish two cases:

(a) $F(\underline{p}^-) < 1$. Since $F(\omega) = 1$ and $\underline{p} \notin supp(F)$, this implies that $\underline{p} < \omega$. Therefore, since $\underline{p} \notin supp(F)$, there is $q > \underline{p}$ with $F(\underline{p}^-) = F(q)$, and hence the seller could increase profits by increasing the price from $\underline{p}$ to $q$, contradicting that $\underline{p} \in supp(H)$.

(b) $F(\underline{p}^-) = 1$. Then the seller's profit from price $\underline{p}$ is zero, and hence, since $\underline{p} \in supp(H)$, we have that $\Pi = 0$. This contradicts that $\alpha < \Pi$. QED

**Proof of Proposition 3** To show the generality of the proposition, I prove it without invoking A3.

As to (i). I first show that $\lambda = -\min_{v \in V} c_F(v)$. Indeed, let $\underline{v} = \min supp(F)$, $\underline{p} = \min supp(H)$ be the lower support bounds.

Observe first that $\underline{v} \leq \underline{p}$. Otherwise, if $\underline{p} < \underline{v}$, then $F(\underline{p}^-) = 0$ so that the seller's profit at $\underline{p}$ is $(1 - F(\underline{p}^-))\underline{p} = \underline{p}$. But since $F(\underline{v}^-) = 0$ by definition of $\underline{v}$, the seller could strictly increase his profit by deviating to the price $p = \underline{v}$. By (12) and (13), this contradicts that $\underline{p}$ is in $supp(H)$.

Next, I show that $\underline{v} \in \arg\min_{v \in V} c_F(v)$. To the contrary, suppose $c_F(\underline{v}) > c_F(\hat{v})$ where $\hat{v} \in \arg\min_{v \in V} c_F(v)$. Because $\underline{v} \leq \underline{p}$, we have that $\overline{H}(\underline{v}) = 0$. Therefore, because (trivially) $\overline{H}(\hat{v}) \geq 0$, we have

$$\overline{H}(\underline{v}) - c_F(\underline{v}) - \lambda < \overline{H}(\hat{v}) - c_F(\hat{v}) - \lambda. \tag{59}$$

By (10) and (11), this contradicts that $\underline{v}$ is in $supp(F)$.

Therefore, because $\underline{v} \leq \underline{p}$ implies $\overline{H}(\underline{v}) = 0$, and because $c_F(\underline{v}) = \min_{v \in V} c_F(v)$, we infer from (11) that

$$0 = \overline{H}(\underline{v}) - c_F(\underline{v}) - \lambda = -\min_{v \in V} c_F(v) - \lambda, \tag{60}$$

as desired.

To see the expression for $U_B$, recall from (4) that $U_B(H, F) = \int_V \overline{H}(v)\, dF(v) - C(F)$. Thus, plugging in $\overline{H}$ from (11) yields (18).

As to (ii). Note that when $C$ is linear, we have $\int_V c_F(v)\, dF = \int_V c(v)\, dF = C(F)$. Thus, $U_B = -\min_{v \in V} c(v) = 0$ by (8).

As to (iii). The proof is given in the main text. QED

**Proof of Corollary 1** The proof is given in the main text. QED

**Proof of Lemma 1** The proof is given in the main text. QED

**Proof of Lemma 2** The proof is given in the main text. QED

**Proof of Proposition 4** Let $(F, H)$ be an equilibrium. The proof for the first part of the statement is given in the main text. It remains to show the equilibrium conditions (i) and (ii).

As to (i). We have that $h \in (0, 1)$ in equilibrium if and only if the seller is indifferent between the prices $\alpha$ and $\omega$ which is the case if and only if $F(\omega^-) = 1 - \alpha/\omega$. Further, $F$ is a best response by the buyer if and only if (10) and (11) hold. Note that $\overline{H}(v) = (1-h)v$ since $H = T_h$, and $c_F(v) = \kappa \Gamma_0'(M_F) v$. Hence, (10) and (11) write

$$\begin{aligned} (1-h)v - \kappa \Gamma_0'(M_F) v - \lambda &= 0 \quad \forall v \in supp(F), & (61)\\ (1-h)v - \kappa \Gamma_0'(M_F) v - \lambda &\leq 0 \quad \forall v \in V. & (62) \end{aligned}$$

Since $F(\omega^-) = 1 - \alpha/\omega$, $F$ has a mass point of mass $\alpha/\omega$ at $\omega$. Because $\alpha/\omega < 1$, $F$ has at least one other point in its support, and thus (61) is true for at least two points in $V$. Since the function on the left hand side is linear, it follows that (61) is actually true for all points in $V$. Thus, (61) and (62) are equivalent to

$$(1-h)v - \kappa \Gamma_0'(M_F) v - \lambda = 0 \quad \forall v \in V. \tag{63}$$

This is equivalent to $h = 1 - \kappa \Gamma_0'(M_F)$ and $\lambda = 0$. Thus, $h \in (0, 1)$ is equivalent to $\kappa \Gamma_0'(M_F) < 1$. This completes the proof of (i).

As to (ii). To see the "only if"-part, let $(F, T_h)$ be an equilibrium with $h = 0$, that is, $p = \alpha$ with probability 1. Then the buyer's utility is $M_F - \alpha - \kappa\Gamma_0(M_F)$ which is equal to the full surplus minus the constant $\alpha$. Therefore, the buyer's best response $F$ is a first-best distribution.

To see the "if"-part, let $F$ be a first-best distribution with $F(v) \geq K_\alpha^\omega(v)$ for all $v \in V$. Then, in particular $F(\omega^-) \geq K_\alpha^\omega(\omega^-) = K_\alpha^\omega(\omega) = 1 - \alpha/\omega$. In other words, the mass on $\omega$ is less than $\alpha/\omega$. Thus, the seller weakly prefers $p = \alpha$ over $p = \omega$. Hence, $h = 0$ is a best response by the seller. Moreover, given $h = 0$, $F$ is a best response by the buyer as argued in the previous paragraph, and this completes the proof. QED

**Proof of Proposition 5** As to (i),(a). Let $M_{K_\alpha^\omega} < M^{FB}$. Then there is no first-best distribution $F^{FB}$ that is first order stochastically dominated by $K_\alpha^\omega$. Hence, $F^{FB} \not\geq K_\alpha^\omega$. Hence, by part (ii) of Proposition 4, there is no equilibrium with $h = 0$.

Next, I show that $(K_\alpha^\omega, T_h)$ with $h = 1 - \Gamma_0'(M_{K_\alpha^\omega})$ is an equilibrium. By part (i) of Proposition 4, it is sufficient to show that $\Gamma_0'(M_{K_\alpha^\omega}) < 1$. Indeed, because $\alpha \leq M_{K_\alpha^\omega} < M^{FB}$, Lemma 2 implies that either $M^{FB} = \omega$ and thus $\kappa\Gamma_0'(\omega) = \kappa\Gamma_0'(M^{FB}) < 1$, or $M^{FB} = \Gamma_0'^{-1}(1/\kappa)$ and thus $\kappa\Gamma_0'(M^{FB}) = 1$. Since $\Gamma_0$ is strictly convex, $\Gamma_0'$ is strictly increasing, and so the fact that $M_{K_\alpha^\omega} < M^{FB}$ implies that $\Gamma_0'(M_{K_\alpha^\omega}) < 1$, as desired.

To complete the proof of part (a), I now argue that $(K_\alpha^\omega, T_h)$ with $h = 1 - \Gamma_0'(M_{K_\alpha^\omega})$ is uniquely Pareto-optimal. Indeed, let $(\tilde{F}, T_{\tilde{h}})$ be another equilibrium. As remarked above, there is no equilibrium with $\tilde{h} = 0$. Hence, $\tilde{h} = 1 - \Gamma_0'(M_{\tilde{F}})$ by part (i) of Proposition 4. The buyer's utility in $(\tilde{F}, T_{\tilde{h}})$ is

$$\tilde{U}_B = (1 - h)(M_{\tilde{F}} - \alpha) - \Gamma_0(M_{\tilde{F}}) = \Gamma_0'(M_{\tilde{F}})(M_{\tilde{F}} - \alpha) - \Gamma_0(M_{\tilde{F}}). \tag{64}$$

The derivative with respect to $M_{\tilde{F}}$ is $\Gamma_0''(M_{\tilde{F}})(M_{\tilde{F}} - \alpha)$ which is positive since $\Gamma_0$ is strictly convex. Therefore, the Pareto-optimal equilibrium maximizes $M_{\tilde{F}}$. Recall from Proposition 4 that $\tilde{F} \geq K_\alpha^\omega$ in any equilibrium. Therefore, $M_{\tilde{F}} < M_{K_\alpha^\omega}$ for any $\tilde{F} \neq K_\alpha^\omega$, and hence $(K_\alpha^\omega, T_h)$ is uniquely Pareto-optimal.

To complete the proof of part (i), it remains to show (b). Note that since the default distribution puts all mass on $\alpha$, we have $\Gamma_0(\alpha) = 0$ by (7). Therefore part (b) follows from (64).

As to (ii),(a). I first show that there is an equilibrium $(F, T_h)$ with $h = 0$ and $F$ a first-best distribution. Indeed, because $M_{K_\alpha^\omega} \geq M^{FB} \geq \alpha = M_{K_\alpha^\alpha}$, an intermediate value argument delivers that there is $\hat{v} \leq \omega$ so that $M_{K_\alpha^{\hat{v}}} = M^{FB}$, and hence $K_\alpha^{\hat{v}}$ is a first-best distribution by Lemma 2. Moreover, $K_\alpha^{\hat{v}} \geq K_\alpha^\omega$ by definition, and thus $(K_\alpha^{\hat{v}}, T_h)$ with $h = 0$ is an equilibrium by part (ii) of Proposition 4.

Since the seller chooses $p = \alpha$ with probability 1 in this equilibrium, and $K_\alpha^{\hat{v}}$ is a first-best distribution, the buyer extracts the residual first-best surplus $W^{FB} - \alpha$. Since the seller gets $\alpha$ in any equilibrium by Proposition 2, there is no equilibrium in which the buyer gets a higher utility. It follows that $(K_\alpha^{\hat{v}}, T_0)$ is a Pareto-optimal equilibrium.

Part (b) is obvious, and this completes the proof. QED

**Proof of Corollary 2** That $\hat{\kappa} \in (\kappa_0, \kappa_1)$ follows from (36) and the fact that $M_{K_\alpha^\omega} \in (\alpha, \omega)$ and that $\frac{1}{\Gamma_0'(v)}$ is strictly decreasing in $v$ due to strict convexity of $\Gamma_0$.

Moreover, by (37), we have that $M_{K_\alpha^\omega} = M^{FB}$ if and only if $\kappa = \hat{\kappa}$, and since $M^{FB}$ is decreasing in $\kappa$ (and strictly so in $(\kappa_0, \kappa_1)$), it follows that $M_{K_\alpha^\omega} < M^{FB}$ for $\kappa < \hat{\kappa}$ and $M_{K_\alpha^\omega} > M^{FB}$ for $\kappa > \hat{\kappa}$.

I now show part (i) of the corollary. The previous paragraph implies that for $\kappa < \hat{\kappa}$, part (i) of Proposition 5 applies, and the buyer's utility is $U_B = \kappa[\Gamma_0'(M_{K_\alpha^\omega})(M_{K_\alpha^\omega} - \alpha) - (\Gamma_0(M_{K_\alpha^\omega}) - \Gamma_0(\alpha))]$. Note that the term in the square brackets is strictly positive because strict convexity of $\Gamma_0$ implies $\Gamma_0'(x)(x-\alpha) > \Gamma_0(x) - \Gamma_0(x)$ for all $x > \alpha$. Therefore, $U_B$ is strictly increasing in $\kappa$.

As for part (ii) of the corollary, for $\kappa \geq \hat{\kappa}$, part (ii) of Proposition 5 applies, and therefore welfare in a Pareto-optimal equilibrium is first best.

Finally, part (iii) follows from part (ii) and (9). QED

**Proof of Lemma 3** To prove the lemma, I first characterize the first-best distribution for general $c_F$.

**Lemma A.1** *$F^{FB}$ maximizes $\int_V v\, dG - C(G)$ if and only if there is $\lambda^{FB}$ such that*

$$v - c_{F^{FB}}(v) - \lambda^{FB} \leq 0 \quad \forall v \in V, \tag{65}$$

$$v - c_{F^{FB}}(v) - \lambda^{FB} = 0 \quad \forall v \in supp(F^{FB}). \tag{66}$$

The proof of Lemma A.1 is identical to the proof that establishes the best response conditions (10) and (11) for the buyer. The only difference is that in the objective function, the buyer's expected gross benefit $\int_V \overline{H}(v)\, dF$ is replaced by $\int_V v\, dF$. QED

I can now prove Lemma 3. Because $c_F$ is strictly concave, $v - c_F(v)$ is strictly convex and thus maximized at the points $\alpha$ or $\omega$. Thus, a distribution that satisfies (65) and (66) must be a two-point distribution $T_f$ for some $f$, and the optimal $f^{FB}$ maximizes $\int_V v\, dT_f - C(T_f) = f(\omega - \alpha) + \alpha - C(T_f)$. Since $C$ is strictly convex by assumption, $C(T_f)$ is strictly convex in $f$, and thus $f^{FB}$ is unique. QED

**Proof of Proposition 6** The proof is given in the main text. QED

**Proof of Lemma 4** By Lemma A.1, a first-best distribution is characterized by the conditions (65) and (66). Because $c_F$ is strictly convex for all $F$, the function $v - c_F(v)$ has a unique maximizer for all $F$. Therefore, the conditions (65) and (66) can be satisfied only for a distribution $F^{FB}$ which is a degenerate distribution $\mathbb{1}_{[v^{FB},\omega]}$ for some point $v^{FB}$. Since a degenerate distribution that puts all mass on $v$ generates total welfare $v - C(\mathbb{1}_{[v,\omega]})$, $v^{FB}$ maximizes this expression.

The argument behind expression (44) is in the main text. QED

**Proof of Proposition 7** The proof is the same as the proof of Proposition 1 in Gul (2001). QED

**Proof of Proposition 8** Because $v^*(K_\alpha^\beta)$ maximizes $v - \kappa\gamma_{K_\alpha^\beta}(v)$, we have that $v^*(K_\alpha^\beta)$ is given as the solution $v^*$ to the first order condition

$$\kappa\gamma'_{K_\alpha^\beta}(v^*) = 1, \quad \text{or} \quad v^* = \alpha \text{ and } \kappa\gamma'_{K_\alpha^\beta}(\alpha) \geq 1, \quad \text{or} \quad v^* = \omega \text{ and } \kappa\gamma'_{K_\alpha^\beta}(\omega) \leq 1. \tag{67}$$

By Proposition 7, the equilibrium value of $\beta$ is given by $v^*(K_\alpha^\beta) = \beta$. Inserting this in (67) yields the claim. QED

**Proof of Corollary 3** As to (i). Since $\kappa < \hat{\kappa}$, Proposition 8 implies that $F = K_\alpha^\omega$ in equilibrium. By part (i) of Proposition 3, the buyer's equilibrium utility is thus

$$U_B \;=\; \kappa\left(\int_V \gamma_{K_\alpha^\omega}(v)\, dK_\alpha^\omega(v) - \Gamma(K_\alpha^\omega) - \min_{v\in V}\gamma_{K_\alpha^\omega}(v)\right). \tag{68}$$

It follows with the same arguments as in part (iii) of Proposition 3 that the term in brackets is strictly positive. Thus, since $K_\alpha^\omega$ is independent of $\kappa$, $U_B$ is strictly increasing in $\kappa$ for all $\kappa < \hat{\kappa}$.

As to (ii). Recall that $F_{min}$ is uniquely given by the distribution $\mathbb{1}_{[\alpha,\omega]}$ that places mass 1 on $\alpha$. For $\kappa < \kappa_1$, Proposition 8 implies that $F = K_\alpha^\beta$ with $\beta > \alpha$ in equilibrium. By part (iii) of Proposition 3, it follows that $U_B > 0$. Hence, since seller profit is $\alpha$ by Proposition 2, total welfare $U_B + \alpha$ is strictly larger than $\alpha = W^{PUB}$.

As to (iii). By Lemma 4 and Proposition 8, the equilibrium distribution $F$ coincides with the first-best if and only if $F = K_\alpha^\alpha = \mathbb{1}_{[\alpha,\omega]}$ and $F^{FB} = \mathbb{1}_{[\alpha,\omega]}$. The former is the case if $\kappa \geq \kappa_1$, and the latter is the case if $\alpha$ maximizes $v - \kappa\gamma_{\mathbb{1}_{[\alpha,\omega]}}(v)$ which is also equivalent to $\kappa \geq \kappa_1$. Therefore, first-best welfare is strictly larger than equilibrium welfare if and only if $\kappa < \kappa_1$. QED

**Proof of Lemma 5** The characterization of the equilibrium value $\beta$ follows by straightforward algebra from Proposition 8 and (52).

It remains to calculate $U_B$. By (18) and (50):

$$\int_V c_F(v)\, dF(v) \;=\; \frac{1}{2}\kappa Q_F \int_V v^2\, dF(v) = \frac{1}{2}\kappa Q_F^2, \tag{69}$$

and $\min_{v\in V} c_F(v) = c_F(\alpha) = \frac{1}{2}\kappa Q_F \alpha^2$. Hence, by (18),

$$
\begin{aligned}
U_B &= \int_V c_F(v)\, dF(v) - C(F) - \min_{v\in V} c_F(v) && (70)\\
&= \frac{1}{2}\kappa Q_F^2 - \frac{1}{4}\kappa Q_F^2 + \frac{1}{4}\kappa\alpha^4 - \frac{1}{2}\kappa Q_F \alpha^2 && (71)\\
&= \frac{1}{4}\kappa\left(Q_F^2 + \alpha^4 - 2Q_F\alpha^2\right) && (72)\\
&= \frac{1}{4}\kappa\left(Q_F - \alpha^2\right)^2. && (73)
\end{aligned}
$$

Plugging in $Q_F = 2\alpha\beta - \alpha^2$ for $F = K_\alpha^\beta$ from (51) yields the claim. QED

**Proof of Lemma 6** By Lemma 4, $v^{FB}$ maximizes

$$
v - \kappa\Gamma(\mathbb{1}_{[v,\omega]}) = v - \frac{1}{4}\kappa v^4 + \frac{1}{4}\kappa\alpha^4. \tag{74}
$$

Expression (55) now follows from the first order condition for this maximization problem. QED

**Proof of Lemma 7** Notice first that since $c_F$ is strictly increasing so that A3 holds, the default distribution puts all mass on $\alpha$. Therefore, by (7),

$$
C(F_{min}) = \varphi(\alpha) = 0. \tag{75}
$$

Moreover, by Proposition 2, A3 also implies that the seller's profit is $\alpha$ so that total welfare is $U_B + \alpha$. Hence, it is enough to show the claim for $U_B$ only. To do so, define

$$
\hat{\rho} = \frac{1}{\varphi'(M_{K_\alpha^\omega})}, \qquad \hat{\sigma}(\rho) = \frac{1 - \rho\varphi'(M_{K_\alpha^\omega})}{r'(\omega) - r'(M_{K_\alpha^\omega})}. \tag{76}
$$

By definition, we then have for all $(\rho, \sigma)$ with $\rho < \hat{\rho}$ and $\sigma < \hat{\sigma}(\rho)$ that

$$
c'_{K_\alpha^\omega}(\omega) \le 1, \tag{77}
$$

and hence, Proposition 8 implies that $K_\alpha^\omega$ is the buyer's equilibrium distribution. By part (i) of

Proposition 3, because $c_{K_\alpha^\omega}(v)$ is minimized at $\alpha$, the buyer's equilibrium utility is thus

$$
\begin{aligned}
U_B &= \int_V c_{K_\alpha^\omega}(v)\, dK_\alpha^\omega(v) - C(K_\alpha^\omega) - \min_{v\in V} c_{K_\alpha^\omega}(v) && (78)\\
&= \rho\left[\int_V \varphi'(M_{K_\alpha^\omega}) v\, dK_\alpha^\omega(v) - \varphi(M_{K_\alpha^\omega}) - \varphi(\alpha)\right] && (79)\\
&\quad -\sigma\left[\int_V r'(M_{K_\alpha^\omega}) v\, dK_\alpha^\omega(v) - r(M_{K_\alpha^\omega}) - r(\alpha)\right] && (80)\\
&\quad -\sigma r(\alpha) && (81)\\
&= \rho\left[\varphi'(M_{K_\alpha^\omega})M_{K_\alpha^\omega} - \varphi(M_{K_\alpha^\omega}) - \varphi(\alpha)\right] - \sigma\left[r'(M_{K_\alpha^\omega})M_{K_\alpha^\omega} - r(M_{K_\alpha^\omega})\right]. && (82)
\end{aligned}
$$

Because $\varphi(\alpha) = 0$ by (75), strict convexity of $\varphi$ and $r$ implies that the square brackets are strictly positive. Thus, the buyer's utility is strictly increasing in $\rho$ and strictly decreasing in $\sigma$. QED

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