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and the Balance Sheet of the Private Sector

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Fiscal Policy and the Balance Sheet of the Private Sector*

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Abstract

This paper characterizes optimal fiscal policy in a growth model with incomplete markets, heterogeneous agents (households and entrepreneurs), and idiosyncratic productivity risk. Entrepreneurs run risky production, which they cannot finance optimally because of an agency problem. In the second-best optimum, they issue continuously traded bonds. Households invest in private and public debt. The government has to finance an exogenous expenditure flow and maximizes a weighted sum of the welfare of entrepreneurs and households. We show that any constrained Pareto optimal allocation can be decentralized as a competitive equilibrium by issuing an appropriate amount of public debt, combined with suitable wealth taxation. Positive public expenditure shocks leave the optimal debt-to-GDP ratio unaffected and increase tax rates. Such shocks also determine whether $r < g$ or $r > g$ in equilibrium, with different dynamics and fiscal sustainability in the two regimes.

Keywords: Incomplete Financial Markets, Heterogeneous Agents, Debt, Taxes, Interest, Growth, Ponzi Games

JEL Classification: D31, D43, D52, D63, E21, E44, E62.

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1 Introduction

This paper studies fiscal policy in the presence of incomplete financial markets, heterogeneous agents with conflicting interests, and non-insurable idiosyncratic production risk. Our approach is different from previous work because we ask why markets are incomplete in the first place and what this implies for firms' capital structure and investment. In such a framework, we ask which fiscal instruments the government should use to finance public spending and, in particular, how much of it should be financed through taxes compared to debt. What are the distributional consequences of issuing public debt, and how can one engineer Pareto improvements? How do these choices depend on the weight of different interests in the economy, and how do they affect equilibrium interest rates, savings, growth, and fiscal sustainability in the long run? These are particularly relevant questions today, as new large public spending programs are on the political agenda in many countries.

To answer these questions, we develop an analytically tractable dynamic macroeconomic model along classical lines of Merton (1971), Dumas (1989), and more recently Angeletos (2007), He and Krishnamurty (2012) and Brunnermeier and Sannikov (2014). It features incomplete financial markets and two types of risk-averse agents: households and entrepreneurs subject to idiosyncratic productivity shocks. Entrepreneurs want to finance their investments by raising outside funds, in particular from households, but face the problem that their revenues are private information and standard outside equity finance is therefore impossible. Putting this corporate finance perspective at center stage allows us to address the classic macroeconomic problems outlined above from a new angle.

A benevolent social planner in our economy faces the task of redistributing unobservable output among entrepreneurs and households such as to share idiosyncratic production risk and optimize intertemporal production and consumption. Intuitively, she must design a dynamic multi-agent mechanism that rewards entrepreneurs with high output for sharing some of this output with low performing entrepreneurs and households, without leaving them so much surplus as to jeopardize the risk-sharing objective. Building on the mechanism-design approach in our companion paper Biais et al. (2023), the present paper shows that constrained optimal allocations can be decentralized if entrepreneurs can issue short-term debt, by giving the government three fiscal instruments: redistribution of initial endowments across households and entrepreneurs, issuance of public debt, and linear taxation of wealth of entrepreneurs and households at different rates.

In fact, if their debt is very short-term, entrepreneurs can react flexibly to output shocks by issuing or retiring debt, even when the basic information asymmetry between entrepreneurs and outsiders prevents them from issuing equity. We assume that such debt transactions are frictionless. Furthermore, entrepreneurs can use inside equity as a further margin of adjustment, and we show that they optimally use both margins simultaneously in order to downsize the balance sheet after negative production shocks and size up after positive shocks. Short-term debt therefore is a "*flexible hard claim*" that provides second-best managerial incentives in the spirit of Grossman and Hart (1983)'s theory of debt as an incentive mechanism. It is flexible because it can be traded continuously and without frictions, and it is hard because, knowing that a firm can adjust its leverage flexibly, outsiders can enforce the remaining claims perfectly. In fact, since idiosyncratic production shocks follow a Brownian motion, debt is safe, as in much of the classic asset pricing literature following Merton (1971). However, as discussed below, this is a departure from the contingent-claims finance literature building on Leland (1994), which assumes costless outside equity issuance and long-term debt that requires costly adjustments.

The reason why the three fiscal instruments achieve optimal decentralization is that the entrepreneurs' scaling decisions in response to their production shocks follow the same logic as the social planner's redistribution policy: entrepreneurs with higher instantaneous output can scale up (by buying capital from distressed entrepreneurs), while entrepreneurs with unfavorable production shocks must scale down to avoid bankruptcy (which they will, because bankruptcy is less attractive than continuation even at very small size). Since the entrepreneurs' production decisions depend on their net worth, there is a role for public debt in affecting firms' balance sheets. To this end, the government optimally mimics private debt and issues safe short-term debt, too, which is a perfect substitute to private debt. Hence, the only tradeable security in this economy is safe short-term debt.

As we discuss in Section 5, the decentralization result is akin to the Second Welfare Theorem. Just as in classic complete-market settings à la Arrow-Debreu, the public planner can redistribute resources through appropriate taxes or transfers and affect production decisions. In the equilibrium of the decentralized economy, fiscal policy affects the aggregate balance sheet of the private sector, and in particular private leverage, through public bonds and taxes. Issuing public debt and distributing the proceeds to entrepreneurs and households has three effects: a balance sheet effect, an interest rate effect, and a growth effect. The balance sheet effect reduces the net leverage of entrepreneurs and increases their incentives to undertake risky investments at any given risk-free interest rate. This directly benefits entrepreneurs. But to clear the market for capital, the increased supply of bonds increases the risk-free interest rate. This partially counteracts the balance sheet effect for entrepreneurs, as they have to pay higher interest on their lower net debt, and it directly benefits households. Finally, issuing public debt increases the aggregate wealth of the private sector, which stimulates aggregate consumption, which in turn has a negative impact on output growth. The optimal level of government debt is always positive and balances these different effects, depending on the weight of entrepreneurs and households in the social welfare function. In particular, Ricardian equivalence does not hold, because changing the firms' budget constraints has real effects.[1]

The Pareto optima in our model involve redistribution and therefore are not always Pareto improvements over the Laisser-Faire allocation of no government intervention. However, once we control for redistribution and start out with an allocation in which taxes and transfers are used optimally, but the government budget is balanced, we obtain the remarkable result that optimal public debt issuance increases the expected utility of households and entrepreneurs at every point in time and for all realizations of individual productivity shocks. Public debt issuance thus constitutes what Aguiar et al. (2023) call a "Robust Pareto Improvement" over a situation in which only taxes are used to finance government expenditures. In fact, government debt acts like a rising tide lifting all boats, by reducing the entrepreneurs' exposure to household lending and thus their leverage, which increases investment incentives in an endogenously risky business. This translates into universal Pareto improvements because the government can use taxes and transfers to re-distribute the realized gains from improved investment incentives.

Equipped with this version of the Second Welfare Theorem, we can proceed and address the central questions asked in the first paragraph above. In particular, questions about the optimal public debt to GDP ratio or the role of optimal taxes versus public debt can be answered simply and explicitly. Interestingly, changing the weight of the productive sector (i.e., the welfare weight of entrepreneurs) in the social welfare function fundamentally changes the optimal structure of public finances and the answers to the above questions. When this weight is small and only household interests matter, the growth rate approaches the Modified Golden Rule rate, in line with the benchmark result by Aiyagari (1994). The reason is that buffering the uninsurable productivity shocks of entrepreneurs is of little direct importance for welfare, and thus public debt is low, entrepreneurs' equity is relatively small, and leverage is large. This implies low investment demand by entrepreneurs and thus low interest rates. The two effects combined yield a regime in which the growth rate exceeds the interest rate, $g > r$.

When the welfare weight of entrepreneurs is greater, buffering their shocks becomes more important. Thus, more public debt is optimally issued, equity increases, and leverage declines. Higher equity causes interest rates to rise. Greater wealth in the economy triggers more consumption and thus growth declines. Thus, r increases

[1]We note that many of the usual assumptions for Ricardian equivalence hold: agents are fully rational and forward looking, everybody can borrow and lend at the same safe interest rate, and the path of government expenditures is fixed. But as Barro (1974) himself points out in his classical paper, a further assumption needed is that "the marginal net-wealth effect of government bonds is close to zero." The whole point of the optimal policy considered here is to violate this assumption.

and g decreases, and the regime can switch from $r < g$ to $r > g$. However, at some level of the welfare weight of entrepreneurs, the economy reaches a level of growth that is so small and a level of debt to GDP that is so large that the wealth effect from issuing public debt is dominated by the ensuing reduction of growth. If the welfare weight of entrepreneurs increases beyond that level, their losses in case of negative productivity shocks are less and less severe for them, since their leverage is low and their equity is high. Hence, it is optimal to operate with lower public debt, as this reduces current consumption by all agents and stimulates growth. In this case, the wealth effect from issuing public debt is less important than the reduction of growth, and the government runs an eternal primary surplus.

In terms of fiscal policy, we use this model to revisit the classic question of how to fund public expenditure between debt and taxes. In the model, public expenditure is exogenous, constant, and needed to provide public infrastructure. Hence, spending needs only change over time if this change is unanticipated (we do not consider aggregate risk). Recently, many countries have arguably experienced large positive public spending shocks in the sense of having to invest heavily in disaster prevention against the consequences of climate change or in military expenditure in a world of increased political instability and international aggression. To the extent that they were difficult to anticipate, our model yields a strong neutrality result on the impact of such shocks on fiscal policy, which we term the "Fiscal Separation Principle" (see Proposition 8 below). In any welfare-optimal equilibrium of the model, fiscal policy is conducted such that a positive public spending shock increases taxes and reduces growth, but does not affect the debt-to-GDP ratio. Public debt is a matter of long-term policy towards incomplete markets, while current public goods are financed by taxes. Again, such a shock may cause a regime change from $g > r$ to $g < r$. While this result certainly depends on the structure of our simple model, in particular on the use of logarithmic utility, we believe that its basic message is more general.

Our work builds on and contributes to different strands of the macroeconomic literature on fiscal policy with agent heterogeneity, which we review in the next section. However, it is worth emphasizing one issue of more than recent interest. As our model yields explicit analytic expressions for the equilibrium growth and interest rates, the analysis can directly address the question of the structural determinants of the difference $r - g$.[2] As noted above, we find that the interest rate may exceed the growth rate of GDP in same parameter constellations, in particular when entrepreneurs' weight in social welfare is large, and the opposite occurs when it is low.[3] In the first case ($g < r$), the intertemporal budget constraint of the

[2]Some of the important recent contributions to the $r - g$ debate are Barro (2023), Blanchard (2019), Brunnermeier et al. (2021), Cochrane (2019), Cochrane (2022), Dumas et al. (2022), Reis (2021).

[3]The full comparative statics of the determinants of this tradeoff is given in Proposition 10.

government binds, and the value of outstanding debt at each date is equal to the net present value of all future primary surpluses. In the second case, $r < g$ and there is a permanent and growing primary deficit. In this case, the government's intertemporal budget constraint does not bind and, perhaps surprisingly, the public debt-to-GDP ratio is small. The government optimally runs a Ponzi scheme: it eternally repays old debt by issuing new one.

The remainder of this paper is organized as follows. In the next section, we provide a more detailed discussion of the literature. The basic model in terms of preferences, technology, and endowments is set out in Section 3, where we also present a welfare analysis based on dynamic mechanism design and characterize (constrained) optimal allocations. In Section 4, we introduce a market environment, characterize individually optimal decisions and show how to decentralize the decisions of the optimal mechanism. The general equilibrium analysis is presented in Section 5, which also presents the full implementation of the optimal mechanism, derives the Pareto frontier of the distributional problem between households and entrepreneurs, and discusses Robust Pareto Improvements in the sense of Aguiar et al. (2023). In Section 6, we discuss the roles of debt and taxes under optimal fiscal policy. The implications of fiscal policy for optimal growth, interest rates, and the government budget are explored in Section 7. The conclusion in Section 8 presents a brief outlook on how to build on this paper in further research. A few formal proofs are relegated to the appendix, and a detailed proof of individual optimization outcomes is added as an online appendix.

2 Relation to the literature

Our paper is related to several strands of the literature.

First, since the early overlapping generations models dating back to Diamond (1965), a sizable literature has examined how fiscal policy influences the relations between three crucial macroeconomic variables: the real interest rate (r), the growth rate of output (g) and the marginal product of capital (μ).[4] A recent strand of this literature re-examines this question in settings with infinitely-lived agents, using continuous-time methods from asset pricing. It also provides ways of endogenizing r, g and μ. Building on the seminal contributions by He and Kr-

[4] In the context of the overlapping generations framework, a recent debate about the sustainability of fiscal policy has focused on when and why governments can run prolonged deficits without being forced to rely on taxation when $r < g$ – so-called "Ponzi schemes" – in the presence of uncertain production returns (Blanchard and Weil (2001), Blanchard (2019), Jiang et al. (2019) Dumas et al. (2022)). Abel et al. (1989) and Hellwig (2021) examine whether the conditions for dynamic inefficiency have to be based on the returns of all assets or only on the return of the safe asset. Dumas et al. (2022) endogenize the structural deficit in the form of an underfunded social security scheme and characterizes debt capacity limits, and Brumm and Hussman (2023) provide a quantitative theory of optimal and maximal debt to GDP.

ishnamurty (2012), Brunnermeier and Sannikov (2014), and Di Tella (2017), this literature considers economies with aggregate risk and studies the emergence and amplification of financial crises, as well as the role of intermediaries in this dynamic. Our work is complementary, as we focus on the long-rung behavior of the economy and optimal fiscal policy when fiscal policy is limited by informational frictions between firms and outside investors and there are conflicting interests between households and firms. Unlike Basak and Cuoco (1998) and Brunnermeier and Sannikov (2014, 2016), we follow the more traditional approach in macroeconomics and assume that production, rather than capital accumulation, is risky (both approaches are in general largely equivalent). This is because this approach lends itself better to the information-based view of corporate finance, which is at the heart of our model.

Regarding the welfare-enhancing role of public debt, our work builds on Brunnermeier et al. (2021), who focus on how to integrate a bubble term representing government expenditures – without ever raising taxes for them – into the fiscal theory of the price level in the presence of idiosyncratic risks and incomplete markets. They determine what they call the optimal "bubble mining rate", which is the optimal rate of issuing government debt. Brunnermeier et al. (2022) extend this approach and resolve the "public debt valuation puzzle", by showing that the price of debt is procyclical, since the bubble term rises in bad times. Reis (2021) considers a model in which households are hit by idiosyncratic depreciation shocks to their capital and face borrowing constraints. This creates a misallocation of resources and, together with non-insurable idiosyncratic risks, a demand for public debt as a safe asset and as an alternative form of savings. Reis (2021) identifies the determinants of the upper limit of spending that can be financed by debt. We also use a model with uninsurable idiosyncratic productivity risk, but there are no borrowing constraints and debt markets are frictionless. Different from Reis (2021), public debt in our model is not a new asset, it is a perfect substitute for safe private debt. Yet, it boosts the wealth of entrepreneurs and raises r.

This body of work builds on the broad strand of macroeconomic theory analyzing uninsurable idiosyncratic income shocks and incomplete financial markets. Prominent theoretical references are Bewley (1983), Imrohoroğlu (1989), Huggett (1993) and in particular Aiyagari (1994). On the empirical side, Dyrda and Pedroni (2022) provide a calibration of a corresponding Ramsey problem to US data and evaluate redistribution and welfare through public debt, capital, and labour taxes numerically. The seminal paper of Aiyagari (1994), in which households self-insure against idiosyncratic income fluctuations by buying shares of aggregated capital, is widely used to examine the impact of household heterogeneity when markets are incomplete. This literature is large and was surveyed by Heathcote et al. (2009) and Krueger et al. (2016). We follow Angeletos (2007), who has enriched the neoclassical growth model by uninsured idiosyncratic investment risk and characterized the macroeconomic effects of this feature. In our model, only firms are subject to uninsurable idiosyncratic productivity risks and we study optimal fiscal policy. If there is no public debt, the leverage of firms and the equilibrium risk-free interest rate are entirely determined by the wealth distribution within the private sector. By issuing public debt, the government can modify the aggregate balance sheet of the private sector and change the portfolio problem of firms, such that their owners face less risk.

The new heterogeneous-agents macro models with incomplete markets are a natural framework to analyze fiscal policy. An emerging literature has indeed made important progress with such models, which often have a rich institutional structure (see, in particular, Le Grand and Ragot (2023), Dyrda and Pedroni (2022)), and Greulich et al. (2023)). We are complementary to this work by developing a tractable growth model with heterogeneous agents, in which we can fully address the optimality of fiscal policy based on a version of the Second Welfare Theorem. The model features a finance-growth channel that is different from, in particular, Aguiar et al. (2023) who develop the concept of Robust Pareto Optimality in the framework of a generalized Aiyagari-Huggett model, as discussed above. Their generality comes at the expense of some restrictions that we can avoid in our theory, such as the assumptions that the form of tax schedules (linear) and transfers (lump-sum) is exogenously given, that equilibria are stationary, that there are exogenous borrowing constraints, or that $r < g$.

A classic part of the literature relevant for our work examines the role of government debt as an asset that can help overcome financial frictions. Woodford (1990) shows how issuing highly liquid public debt can increase the flexibility of liquidity-constrained agents to respond to variations in income and spending opportunities, thereby increasing economic efficiency. Aiyagari and McGrattan (1998) develop a model in which households face a borrowing constraint, which generates a precautionary savings motive. Government debt loosens borrowing constraints and enhances the liquidity of households, which improves consumption smoothing. These papers also stress the cost of higher government debt via adverse wealth distribution, incentive effects, and crowding-out effects on investment. In a similar vein, Angeletos et al. (2023) explore how public debt can be used as collateral or a liquidity buffer in order to ease financial frictions. Since public debt lowers the liquidity premium but increases the cost of borrowing for the government, there exists a long-run optimal level of public debt. In our paper, there are neither borrowing nor liquidity constraints, and public debt gives rise to a tradeoff between risk reduction of firms and higher interest rates, that can be interpreted and solved as a multi-agent mechanism design problem. As in the present paper, Holmström and Tirole (1998) take a corporate finance perspective, which they use to study optimal contracts between firms and outside investors in the presence of managerial moral hazard. They show that the public supply of liquidity is not necessary in an economy with no aggregate uncertainty and in which intermediaries coordinate the use of scarce private liquidity. Our work challenges this conclusion and argues that the underlying distributional problems necessitate a government with the power to tax and re-distribute.

Last but not least, our work is naturally linked to the literature on continuous-time corporate finance. Building on Leland (1994), the contingent claims literature evaluating the tradeoff between bankruptcy costs and tax benefits of debt typically assumes costless equity issuance and long-term debt with positive adjustment costs, exactly opposite to our assumptions in the present paper.[5] As pointed out, e.g., by Abel (2018) and Bolton et al. (2021), this structure is difficult to reconcile with some of the empirical evidence on leverage dynamics, which is one reason why these authors model capital structure with instantaneous, frictionless debt, as in the Merton-based approach of our paper. Different from all these papers we take a strict agency view, which rules out outside equity as a source of funding. Although this makes our model unsuitable for standard tax-bankruptcy cost analyses, in terms of capital structure theory it makes the case for having riskless instantaneous debt in a model of risky corporate earnings. As discussed above, this leads to the concept of short-term debt as a flexible hard claim and links our result on the implementation of the optimal funding mechanism of entrepreneurs to the theory of debt as an optimal incentive device in the spirit of Grossman and Hart (1983). On a more general note, when replacing "outside equity" with "inside equity", models of costless equity funding such as the one developed in the paper by Bolton et al. (2021) become similar to our model in several respects. The main difference in terms of corporate finance is that our work explores the macroeconomic consequences of such financial choices under informational frictions. This is a useful new perspective, as corporate finance models typically are partial equilibrium models and assume $r > g$, an assumption that is not always warranted, as our analysis shows.

3 The Model

3.1 The Macroeconomic Set-Up

The economy features two classes of agents, a mass 1 continuum of entrepreneurs and a mass 1 continuum of households, plus a government. Time is continuous: $t \in [0, \infty)$. There is only one physical good that can be consumed or invested.

Entrepreneurs run their own firms that are risky in the sense that they produce random output at each point in time. These random outputs are not publicly ob-

[5] Two prominent papers among many others in this tradition are Goldstein et al. (2001) and Strebulaev (2007).

servable, which implies that entrepreneurs cannot share their risks and that their equity cannot be traded.[6] The physical good is initially held by households and entrepreneurs, but only entrepreneurs can invest in productive technologies. Households cannot. For simplicity we also assume that households do not work and that they live off their savings. They are identical, and are not subject to individual shocks. Without loss of generality, we can therefore aggregate them into a single representative household (the "household sector"). We cannot do this for the productive sector. Even if starting out identically, entrepreneurs become heterogeneous after $t = 0$ because of their idiosyncratic productivity shocks: lucky entrepreneurs grow bigger while unlucky ones operate less capital.

The government must finance an exogenous stream of public expenditures and can redistribute wealth between households and entrepreneurs, as long as it respects the informational constraints described above.

The model presented here is an extension of Biais et al. (2023), where we characterize constrained efficient allocations in a dynamic contracting model between one principal and many agents. The two main changes with respect to that model are, first, that we here have a government with a fixed level of public expenditures instead of a self-interested principal, and second, that we add a household sector that consumes but does not produce.

3.2 Technology, Preferences, and Endowments

Entrepreneurs are indexed by $i \in [0, 1]$, with initial endowments (equity) $\tilde{e}^i$. Their aggregate initial endowment is

$$\tilde{E} = \int_0^1 \tilde{e}^i di.$$

The representative household has initial endowment $\tilde{H}$.

Entrepreneurs have the same technology and produce random output at each point in time: if k_t^i is the volume of capital managed by entrepreneur i at date t, her instantaneous output net of depreciation is

$$dy_t^i = k_t^i[\mu dt + \sigma dz_t^i]$$

where $\mu > 0$ is the average instantaneous return net of depreciation, $\sigma \geq 0$ is the volatility of the instantaneous return, and the (z_t^i) are firm-specific i.i.d. Brownian motions, whose increments can be interpreted as idiosyncratic productivity shocks. Let $(\mathcal{F}_t^i)_{t \geq 0}$ be the filtration generated by (z_t^i). All processes associated

[6] This is as in Basak and Cuoco (1998) and much of the subsequent literature, the difference being that in our work the trading restriction is explicitly modelled and derived from an underlying informational friction, whereas it is exogenous in the cited literature. Brunnermeier and Sannikov (2016) generalize the existing literature by assuming that firms can sell equity, but must hold an exogenous minimum fraction of it.

with entrepreneur i are assumed to be square-integrable and adapted to $(\mathcal{F}_t^i)$. The instantaneous output dy_t^i can only be observed by the entrepreneur.

To simplify the presentation, we assume that capital does not depreciate. Since production shocks are independent, they wash out in the aggregate due to the law of large numbers,[7] and aggregate production (gross domestic product) at time t is

$$dY_t = \mu K_t dt \tag{1}$$

where

$$K_t = \int_0^1 k_t^i di$$

denotes aggregate capital at time t. Since aggregate output is deterministic, also K_t is non-random. At each time t, the government must finance public goods that cost γK_t, where γ is an exogenous fraction of the aggregate capital stock. Government expenditures as a share of GDP at date t are thus an exogenous constant γ/μ.[8]

Entrepreneurs (E) and households (H) have logarithmic preferences over consumption streams $(c_t^i)_{t=0}^{\infty}$ and $(c_t^H)_{t=0}^{\infty}$. At each time t, they maximize

$$\mathbb{E}_t \int_t^{\infty} e^{-\rho s} \log c_s^k ds, \quad k = i, H,$$

where the expectations operator for firm i refers to the probability measure and the associated filtration of (z_t^i). $\rho > 0$ is the common discount rate.

3.3 Constrained Optimal Allocations

By definition, an allocation consists of a set of consumption paths (c_t^i) for entrepreneurs $i \in [0, 1]$ and (c_t^H) for the representative household, together with capital allocations (k_t^i) for entrepreneurs. Note that all these paths are a priori random. Since their output is only privately observable, entrepreneurs can divert part of this output and consume it secretly, which is inefficient for aggregate risk-sharing. To avoid this diversion, a social planner must impose incentive-compatibility conditions for firms.

Although our set-up is slightly different, we can apply the results in Biais et al. (2023) that imply the following characterization of constrained optimal allocations.[9] There is a number $x > 0$ such that

[7] We show this explicitly in Biais et al. (2023) by considering the limit of the corresponding economy with a finite number of firms.

[8] We do not model the social utility generated by these expenditures explicitly and, therefore, say nothing about their optimal level.

[9] The analysis in Biais et al. (2023) is technically involved and develops a guess-and-verify method using mean-field game theory. See Achdou et al. (2014) for a pioneering contribution and Cardaliaguet (2012) for a mathematical introduction.

- capital is initially allocated equally between entrepreneurs: $k_0^i = K_0$ for all i, and is continually reallocated as a function of performance as follows:

$$\frac{dk_t^i}{k_t^i} = g_t dt + \sigma x dz_t^i, \tag{2}$$

- entrepreneurs consume an amount of output equal to the fraction ρ/x of the capital they manage:

$$c_t^i = \frac{\rho}{x} k_t^i. \tag{3}$$

In contrast, it is straightforward to see that household consumption must be certain, because there is no aggregate risk in the economy. Hence, we can express it as a proportion of aggregate capital:

$$c_t^H = \chi_t^H K_t, \tag{4}$$

where $\chi_t^H > 0$.

Optimality requires that aggregate capital K_t satisfies

$$\dot{K}_t = (\mu - \gamma) K_t - c_t^H - \int_0^1 c_t^i di, \tag{5}$$

where the dot represents the time derivative. Condition (5) simply states that at each time, total instantaneous output dY_t is either used by the government, consumed by households or entrepreneurs, or invested by entrepreneurs. Nothing is thrown away. For the same reason, all initial endowments must be used in production at time $t = 0$:

$$K_0 = \widetilde{E} + \widetilde{H}. \tag{6}$$

Together with the above properties, equation (5) immediately implies that the aggregate growth rate of capital in (2) satisfies

$$g_t = \mu - \gamma - \chi_t^H - \frac{\rho}{x}. \tag{7}$$

3.4 Welfare Function and Constrained Welfare Optima

Using the above characterization, we can compute the intertemporal utility of the representative household as

$$\begin{aligned} \rho V^H &= \rho \int_0^\infty e^{-\rho t} \log\left(\chi_t^H K_t\right) dt \\ &= \rho \int_0^\infty e^{-\rho t} \log \chi_t^H dt + \log K_0 + \frac{\mu - \gamma}{\rho} - \frac{1}{x} - \int_0^\infty e^{-\rho t} \chi_t^H dt. \end{aligned}$$

The expected utility of each entrepreneur at time 0 has a similar expression, except that it incorporates a risk premium.[10] Of course, as noted above, after time

[10] For completeness we provide this proof in Appendix A.1.

0, entrepreneurs will be heterogeneous. But from an ex ante perspective, since technologies are i.i.d., they all have the same expected utility, given by

$$\begin{aligned}\rho V^E &= \rho \mathbb{E} \int_0^\infty e^{-\rho t} \log\left(\frac{\rho}{x} k_t^i\right) dt \\ &= \log\frac{\rho}{x} + \log K_0 + \frac{\mu-\gamma}{\rho} - \frac{1}{x} - \int_0^\infty e^{-\rho t} \chi_t^H dt - \frac{\sigma^2 x^2}{2\rho}.\end{aligned}$$

The social planner chooses x and (χ_t^H) so as to maximize the welfare function

$$\begin{aligned}W &= \alpha V^E + (1-\alpha) V^H \\ &= \log K_0 + \frac{\mu-\gamma}{\rho} - \frac{1}{x} + \alpha \log\frac{\rho}{x} \\ &\quad -\alpha \frac{\sigma^2 x^2}{2\rho} + \int_0^\infty e^{-\rho t} \left[(1-\alpha)\rho \log \chi_t^H - \chi_t^H\right] dt,\end{aligned} \tag{8}$$

where $\alpha \in (0,1)$ is the welfare weight of entrepreneurs. The expression under the integral in (8) is bounded and can be maximized pointwise. Hence, W is maximized for a constant value of χ_t^H, say χ^*, and an x^* that are uniquely determined by the first-order conditions[11]

$$\chi^* = \rho(1-\alpha) \tag{9}$$

$$\frac{\sigma^2}{\rho} x^{*\,3} + x^* - \frac{1}{\alpha} = 0. \tag{10}$$

This leads to the following characterization of the welfare optimum.

Proposition 1. *The unique constrained optimal allocation that maximizes welfare W satisfies conditions (2), (3), and (4), with the parameter values x^* and χ^* given by (9)-(10). The optimal growth rate of aggregate capital is constant and equal to*

$$g^* = \mu - \gamma - \rho - \frac{\rho \sigma^2 x^*}{\rho + \sigma^2 x^{*\,2}} \tag{11}$$

The optimal mechanism is surprisingly simple. It is stationary, and all consumption and investment allocations are controlled by two numbers, χ^H and x, that are given by two explicit simple formulas.

4 Decentralized Decisions

4.1 The Market Environment

We now specify a market environment in which the welfare maximizing allocation of Proposition 1 can be decentralized as the unique rational expectations equilibrium of a private property market economy. As noted above, there can be no market for

[11] It is straightforward to verify that the first-order conditions determine the unique global maximum.

claims based on the entrepreneurs' output, as this is private information. Hence, entrepreneurs can only issue risk-free debt. In order to be able to react to their private output shocks, they must be able to adjust their leverage flexibly. We assume that such instantaneous debt markets exist and are frictionless. Let r_t be the (instantaneous) interest on this debt. Hence, at each moment t, one unit of the good (or capital) can be exchanged costlessly against one unit of safe debt, which instantaneously (i.e., at time $t + dt$) pays r_t.

The government impacts the economy by using three fiscal instruments, which are consistent with its informational constraints. First, it can at all times issue or retire safe short-term public debt B_t, which is a perfect substitute for private debt. Second, it can make initial transfers L^E and L^H to entrepreneurs and households, respectively. And third, it can continuously tax the wealth of entrepreneurs and households by means of linear taxes, at rates τ_t^E and τ_t^H, respectively. Initial transfers and instantaneous taxes can all be negative, thus allowing for taxes and transfers at all stages. Note that these instruments are by no means general. As we shall see, however, they suffice to implement the second-best.

At time $t = 0$, after the government's initial intervention, net wealth is H_0 for the representative household, e_0^i for entrepreneur i, and aggregate equity in the productive sector is

$$E_0 = \int_0^1 e_0^i di.$$

The aggregate wealth of the private sector is

$$E_0 + H_0 = K_0 + B_0$$

where K_0 is the initial stock of physical capital in (6). Thus the government can modify the balance sheet of the private sector by shifting goods from households to entrepreneurs and by issuing and distributing public debt, which is a fiat claim. However, the government cannot produce any output, so that the aggregate capital stock of the economy is still K_0.[12]

Government debt evolves according to

$$\dot{B}_t = \gamma K_t + r_t B_t - T_t, \tag{12}$$

where T_t is net aggregate tax revenue (tax revenue minus subsidies) at time $t > 0$.

We now describe the individual decision problems of households and entrepreneurs. These are standard and yield well-known solutions going back to Merton (1971).

4.2 Individual Decisions

After the initial government intervention, the representative household has initial net worth $H_0 > 0$ at time $t = 0$, no further income later, and saves via private

[12]In our model, there is no need for one-time lump-sum spending at date 0, which is implicit in the literature on maximum public deficit capacity (see, e.g., Reis (2021) or Brumm and Hussman (2023)).

and public bonds, which are perfect substitutes. There is no other form of savings, since the good cannot be stored. Hence, the household chooses a consumption path $c^H = \left(c_t^H\right)_{t\geq 0}$ that solves

$$\max_{c^H} \int_0^\infty e^{-\rho t} \log c_t^H dt,$$

subject to the equation of motion of wealth

$$\dot{H}_t = (r_t - \tau_t^H)H_t - c_t^H. \tag{13}$$

This is a standard dynamic programming problem. The solution is

$$c_t^H = \rho H_t \tag{14}$$

for all $t \in [0, \infty)$.

The entrepreneurs' problem under decentralization is a bit more complex. At time t, entrepreneur i has wealth e_t^i, chooses productive capital k_t^i, and adjusts her debt level d_t^i such as to satisfy her balance sheet constraint

$$k_t^i = d_t^i + e_t^i. \tag{15}$$

We allow debt d_t^i to be negative, in which case the entrepreneur has no debt but invests in bonds issued by other entrepreneurs or the government. The entrepreneur's flow of funds is given by

$$k_t^i[\mu dt + \sigma dz_t^i] = [r_t d_t^i + \tau_t^E e_t^i + c_t^i]dt + de_t^i, \tag{16}$$

where the left-hand side represents earnings before interest and taxes and the right-hand side is the sum of interest payments, taxes, private consumption (dividends), and the change in equity as a residual. (16) thus reflects the simple accounting identity:

$$\text{EBIT} = \text{interest} + \text{taxes} + \text{dividends} + \text{retained earnings}.$$

The entrepreneur chooses a path of k_t^i, d_t^i, c_t^i, $t \geq 0$ that solves

$$\max_{k^i, d^i, c^i} \mathbb{E} \int_0^\infty e^{-\rho t} \log c_t^i dt$$

subject to the balance sheet constraint (15) and the law of motion (16) for each $t \geq 0$. The Bellman Equation yields the standard solution

$$c_t^i = \rho e_t^i \tag{17}$$

$$k_t^i = \frac{\mu - r_t}{\sigma^2} e_t^i \tag{18}$$

and the stochastic law of motion for entrepreneurial equity

$$de_t^i = \left[\left(\frac{\mu - r_t}{\sigma} \right)^2 + r_t - \tau_t^E - \rho \right] e_t^i dt + \frac{\mu - r_t}{\sigma} e_t^i dz_t^i. \tag{19}$$

Note that the flow of funds equation (16) assumes that the entrepreneur is always able and willing to pay the interest on her debt. If she decided to default, her assets could be seized, entailing zero consumption and negatively infinite utility ever after. Hence, if this reaction happens with some probability, strategic default is not an issue.[13] Moreover, the stochastic differential equation (19) describes a Geometric Brownian Motion:[14]

$$e_t^i = e_0^i exp \left(\int_0^t \left[r_s - \tau_s^E - \rho + \frac{1}{2} \left(\frac{\mu - r_s}{\sigma} \right)^2 \right] ds + \int_0^t \frac{\mu - r_s}{\sigma} dz_s^i \right) > 0$$

Therefore, equity is always positive for all entrepreneurs: involuntary default thus is no issue either. The capital k_t^i managed by entrepreneur i evolves as a function of her performance, but she never defaults in equilibrium. In fact, condition (18) implies that the capital-to-equity ratio

$$x_t = \frac{k_t^i}{e_t^i} \tag{20}$$

is identical across entrepreneurs. Using this same leverage target, all entrepreneurs adjust their debt and capital continuously in response to their earnings shocks. After a positive shock, they invest more and issue more debt; after a negative shock, they do the opposite. Hence, Brownian productivity shocks are not enough to drive entrepreneurs into bankruptcy.[15]

4.3 Optimal Policy Decisions

Suppose a social planner wants to decentralize the optimal allocation derived in Section 3, and consider the pair (χ^H, x) given by (9) and (10).[16]

By (4) and (14), decentralizing household consumption requires

$$\chi^H K_t = \rho H_t \tag{21}$$

for all t. In particular, initial household wealth must be

$$H_0 = \frac{\chi^H}{\rho} K_0 \tag{22}$$

[13]There is a large literature on strategic default, which we do not need to discuss here. See Hart and Moore (1998) or Bolton and Scharfstein (1996) for foundational work and Fan and Sundaresan (2000) for an early classic in continuous time.

[14]See, e.g., Shreve (2004), p. 147-8.

[15]This is why Abel (2018) assumes discrete earnings shocks and Bolton et al. (2021) a jump-diffusion process.

[16]The decentralization argument in this subsection holds for any such pair, not just the optimal one given in (9) and (10).

and τ_t^H must be chosen such that H_t grows at the same rate as K_t, namely g.[17] Thus (7), (13), and (14) imply that

$$\tau_t^H = r_t - g - \rho. \tag{23}$$

Decentralizing the constrained optimal decisions for entrepreneurs and implementing (2) and (3) by means of (17) and (18) requires setting

$$\frac{k_t^i}{e_t^i} = \frac{\mu - r_t}{\sigma^2} = x, \tag{24}$$

which implies that the interest rate must be constant:

$$r_t = r \equiv \mu - \sigma^2 x \tag{25}$$

Moreover, aggregate entrepreneurial equity must grow at the same rate as aggregate capital K_t. Hence, by (19) the tax rate on entrepreneurs' equity must satisfy

$$\begin{aligned} \tau_t^E &= r_t - g - \rho + \left(\frac{\mu - r_t}{\sigma}\right)^2 \\ &= \tau_t^H + \sigma^2 x^2. \end{aligned} \tag{26}$$

Finally, initial equity must be equalized across entrepreneurs and make (3) and (17) consistent, which implies

$$e_0^i = E_0 = \frac{K_0}{x}. \tag{27}$$

Using (10) to eliminate α, (9), (11), (25), and rearranging, we can summarise the implementation results (22), (23), (26), and (27) in the following proposition.

Proposition 2. *For the constrained optimal allocation characterized in subsection 3.4 to be the outcome of decentralized individual choices, it is necessary to set*

$$\tau_t^H = \tau^H = \gamma - \frac{\sigma^4 x^3}{\rho + \sigma^2 x^2} \tag{28}$$

$$\tau_t^E = \tau^E = \gamma - \frac{\sigma^4 x^3}{\rho + \sigma^2 x^2} + \sigma^2 x^2 \tag{29}$$

$$L^H = H_0 - \widetilde{H} = (1-\alpha)\widetilde{E} - \alpha \widetilde{H} \tag{30}$$

$$L^E = E_0 - \widetilde{E} = \frac{1}{x}\left(\widetilde{H} - (x-1)\widetilde{E}\right) \tag{31}$$

where $x = x^*$ as given by (10). If $r = r^* \equiv \mu - \sigma^2 x^*$, (28)-(31) are also sufficient.

[17]Note that the planner wants households to hold the fraction of capital that corresponds to their weight $(1-\alpha)$ in the welfare function. The rest of the capital plus all the government bonds should be held by the entrepreneurs. In fact, private and public bonds are perfect substitutes, so the composition of households' portfolio is irrelevant. But the total volume of private bonds must equal H_t

The implementation of the optimal incentive mechanism of Section 3.3 through short-term debt in (24) and Proposition 2 is remarkable from a corporate finance perspective. Short-term debt has two different roles in this optimality result. First, the variable x controls the entrepreneurs' incentives for investment and payout in (2) and (3). And second, the leverage ratio x provides a hard, but flexible individual claim that implements these incentives through the entrepreneurs' balance sheet. Importantly, short-term debt is a hard claim in the sense of, e.g., Hart and Moore (1995), as each entrepreneur i must and will pay out $r_t d_t^i$ to outsiders at all times. On the other hand, the claim is highly flexible as it reacts instantly to the entrepreneur's liquidity dy_t^i. Hence, the variable x controls incentives as well as payout obligations, in the spirit of Grossman and Hart (1983)'s theory of debt as an incentive mechanism.

5 Macroeconomic Equilibrium

This section determines the equilibria of the market economy defined in Section 4 when government taxes and transfers are given by (28)-(31). Let (r_t) be a trajectory of interest rates.

5.1 The Aggregate Balance Sheet

As noted in Section 4.2, households' aggregate wealth H_t is deterministic and optimally follows the law of motion

$$\dot{H}_t = \left(r_t - \tau^H - \rho\right) H_t. \tag{32}$$

This wealth is entirely invested in risk-free debt, and the household is indifferent between public debt and corporate debt. Let D_t^H and B_t^H denote the households' holdings of private and public debt, respectively. The households' balance sheet then is

$$H_t = D_t^H + B_t^H. \tag{33}$$

Optimal individual balance sheets of entrepreneurs follow random trajectories but thanks to the Law of Large Numbers, the aggregate balance sheet of the productive sector is deterministic. Denoting by B_t^E the entrepreneurs' aggregate holdings of public debt (which may be negative), it is given by:

$$\begin{array}{c|c} \text{Assets} & \text{Liabilities} \\ \hline K_t & D_t^H \\ B_t^E & E_t \end{array} \tag{34}$$

where E_t is total wealth of entrepreneurs, with dynamics implied by (19):

$$\dot{E}_t = \left[\left(\frac{\mu - r_t}{\sigma}\right)^2 + r_t - \tau^E - \rho\right] E_t \tag{35}$$

Government debt is $B_t = B_t^H + B_t^E$ and must evolve according to (12). Making taxes explicit, this gives

$$\dot{B}_t = \gamma K_t + r_t B_t - \tau^H H_t - \tau^E E_t. \tag{36}$$

Note that we allow B_t to be negative. Consolidating the aggregate firm balance sheet (34) with the households' balance sheet equation (33) yields the private sector's aggregate balance sheet,

Assets	Liabilities
K_t	H_t
B_t	E_t

(37)

Equilibrium requires markets to clear at all times, given the fiscal policy in place. The following definition makes this precise.

Definition. *Given tax rates (τ^H, τ^E), lump-sum transfers (L^H, L^E), and a public debt trajectory $(B_t)_{t\geq 0}$, a* ***General Equilibrium with Fiscal Policy*** *(GEFP) is an interest rate trajectory $(r_t)_{t\geq 0}$ and a dynamic consumption-investment allocation such that*

1. *firms and households behave optimally given $(r_t)_{t\geq 0}$ and the policy,*
2. *the government's budget evolves according to (36),*
3. *the debt market clears at each $t \geq 0$.*

Note that market clearing for private and public debt implies that the aggregate balance sheet constraint (37) holds at each $t \geq 0$. Note also that (37) pins down the initial amount of public debt B_0 and thus imposes consistency of fiscal policy. Indeed, by using (6),

$$\begin{aligned} B_0 &= \widetilde{H} + L^H + \widetilde{E} + L^E - K_0 \\ &= L^H + L^E \end{aligned} \tag{38}$$

It is possible to characterize GEFPs fairly generally, including existence and asymptotic behavior. However, this is of limited interest in our context, as the stationary welfare optimal equilibria implied by Proposition 2 can be characterized explicitly. Hence, existence follows directly from the implementation result in the next section.

5.2 Implementation

Proposition 2 has shown that the welfare optimal allocation can be individually optimal in a market environment with a particular given interest rate, and has identified the unique policy necessary to achieve this. The following proposition generalizes the implementation result in Biais et al. (2023) and shows that this allocation is the unique equilibrium outcome in the GEFP with this policy.

Proposition 3. *Suppose fiscal policy follows the taxation rules (28)-(31) and the debt policy* $B_t = (L^H + L^E)e^{g^* t}$ *for all* $t \geq 0$*, where* g^* *is given by (11). Then,* $(r_t)_{t\geq 0}$ *is an equilibrium interest rate trajectory if and only if* $r_t = r^* = \mu - \sigma^2 x^*$ *for all* $t \geq 0$*.*

Proposition 2 and the "if"-part of Proposition 3 imply that the fiscal policy of Proposition 3 implements the constrained welfare optimum of Proposition 1 as a general equilibrium outcome. The "only-if" part of Proposition 3 states that this implementation is unique. The proof, which is given in Appendix A2, relies on the dynamic structure of the problem and uses the Picard-Lindelöf uniqueness theorem from the theory of ordinary differential equations.

It is worth emphasizing the strong double uniqueness result here. By Proposition 2 there is exactly one fiscal policy to implement the optimum. By Proposition 3 the general equilibrium of the resulting market economy is unique. Proposition 3 is the counterpart of the Second Welfare Theorem in Arrow-Debreu economies for our incomplete markets environment.[18] Note that lump-sum transfers at $t = 0$ alone are insufficient: optimal implementation also needs flat taxes for households and entrepreneurs, together with an optimal public debt trajectory.

We can now formulate our main result on the relationship between optimal risk-sharing, incentive provision, and public debt.

Proposition 4. *When* $\sigma > 0$*, the implementation of the welfare optimum requires public debt to be strictly positive:*

$$B_0 = \frac{1 - \alpha x^*}{x^*} K_0 > 0. \tag{39}$$

Proof. Evaluating (38) for (30) and (31) yields the identity in (39). The polynomial determining x^* in (10) is increasing, and strictly positive for all $x \geq 1/\alpha$. Hence, $x^* < 1/\alpha$, implying a strictly positive B_0. □

Hence, the welfare optimum is not compatible with balanced budgets, and a government wishing to implement this optimum through fiscal policy must issue a positive amount of public debt. As we discuss in the next section, the reason is that the private sector does not issue sufficient debt due to missing risk-sharing opportunities. Note that the welfare optimum (x^*, χ^*) of Proposition 1 is independent of the government expenditure coefficient γ, while the taxes needed to implement it are not.

5.3 Pareto Optimality

The optimal allocation (9)-(10) is parameterized by the welfare weight α. By eliminating α between the two equations, we can obtain the (constrained) Pareto frontier

[18] It is stronger because of the uniqueness result.

between households and entrepreneurs in (χ^H, x)-space. But since we are interested in the economy's balance sheet and the use of public debt to influence it, we rather propose a different characterization of this welfare tradeoff. Remember that the common incentive-risk control variable x is implemented by a variable that resembles the entrepreneurs' common debt-equity ratio in the decentralized economy. In fact, from (20), their optimal debt-equity ratio is

$$\frac{d_t^i}{e_t^i} = \frac{k_t^i - e_t^i}{e_t^i} = x_t - 1$$

iff $k_t^i \geq e_t^i$.[19] Otherwise, it is 0 (and the entrepreneur holds debt on the asset side of its balance sheet). Note that $x_t > 1$ if and only if $D_t^H - B_t^E > 0$, i.e. if entrepreneurs are net borrowers.[20] For simplicity of exposition, we often refer to x_t as "firm leverage".

To mirror $x_t = K_t/E_t$ we therefore introduce the variable $h_t = H_t/E_t$ to describe the Pareto conflict between entrepreneurs and households. While x_t is informative about risk and return of the productive sector, h_t is informative about the relative wealth of the household sector. By direct substitution of (4) into (14) we have, even out of equilibrium,

$$h_t = \chi_t^H x_t/\rho \tag{40}$$

An advantage of this parametrization is that it yields a simple normalization of the economy's aggregate balance sheet (37):

Assets	Liabilities
x_t	h_t
$1 + h_t - x_t$	1

Hence, public debt is positive iff $1 + h_t - x_t > 0$. The Pareto frontier describing the welfare conflict between households and entrepreneurs can now be graphically represented in (x, h)-space by eliminating α between (9), (10), and (40).[21] This yields

$$h(x) = x - \frac{\rho}{\rho + \sigma^2 x^2} \tag{41}$$

for $x \geq x_{min}$, where x_{min} is the lower bound below which h would be negative and is given by the unique root of

$$\sigma^2 x^3 + \rho x - \rho = 0. \tag{42}$$

Figure 1 shows the Pareto Frontier in the (x, h) plane, for some specific values of ρ, σ, and γ. The figure also shows the "Zero-Debt-Line" $h + 1 - x = 0$, below

[19] In welfare optimal equilibria, entrepreneurs' optimal debt-to-equity ratios are not only identical, but even time-independent, which is the second equality in (24).

[20] This can only happen if $B_t > H_t$, which means that public debt exceeds the total wealth of households.

[21] Remember that welfare optimal equilibria are stationary.

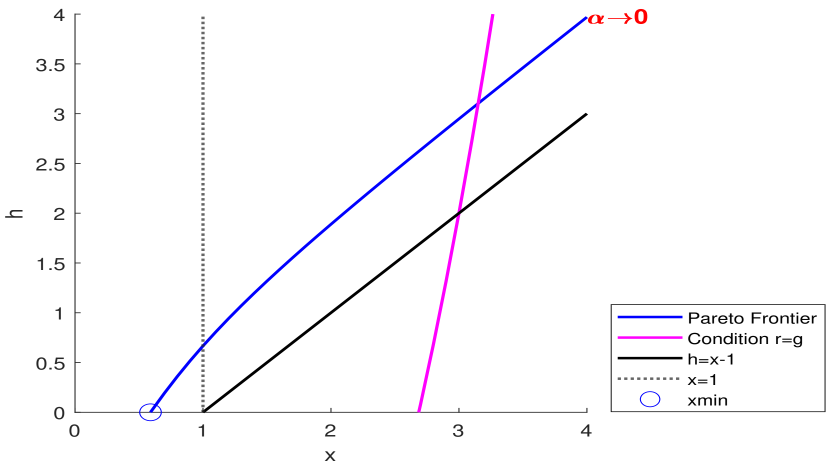

Figure 1: The Pareto Frontier in (x, h) space for $\rho = 0.01, \sigma = 0.2, \gamma = 0.1$.

which the allocation involves negative public debt, and the locus of points for which $r = g$, which we will discuss in detail in Section 7. The Zero-Debt-Line corresponds to the unconstrained Pareto Frontier: when there are no frictions, idiosyncratic risks can be completely eliminated, which is equivalent to taking $\sigma = 0$. In this case, optimal public debt is zero.

When $\sigma > 0$, the Pareto Frontier lies entirely above the Zero-Debt-Line, and it converges to the diagonal $h = x$ for $x \to \infty$. A simple inspection shows that the lower bound for the Pareto Frontier satisfies $x_{min} < 1$. Hence, there are Pareto Optima with $K^* < E^*$, i.e. in which entrepreneurs are net lenders on average. This means that the situation mentioned in footnote 20 not only can occur, but can even be optimal. This is the case if α is large, i.e. if fiscal policy caters strongly to the interests of entrepreneurs.

Pareto optimal equilibria are stationary, and taxation and redistribution ensure that the wealth of entrepreneurs and of households increases at the same rate on average. However, there is a second source of heterogeneity in this economy, which leads to a potential second Pareto conflict. This heterogeneity arises from the growing inequality among entrepreneurs. Indeed, suppose that all entrepreneurs start out with equity $e_0^i = e_0$ at time 0, and that, in an equilibrium given by x, their aggregate equity grows at the economy's optimal growth rate g. Then, by the standard theory of Brownian motion, individual equity e_t^i at time t as given by (19) is log-normally distributed with mean and variance

$$\begin{aligned} E[e_t] &= e_0 \exp gt \\ var(e_t) &= e_0^2 \left[\exp 2gt\right] \left[\exp\left(\sigma^2 x^2 t\right) - 1\right]. \end{aligned}$$

Thus the coefficient of dispersion of entrepreneurial wealth grows over time:

$$\frac{\sqrt{var(e_t)}}{E[e_t]} = \sqrt{\exp(\sigma^2 x^2 t) - 1}.$$

The heterogeneity of entrepreneurs is endogenous in our economy: even if the initial redistribution of capital equalizes initial wealth among them, the impossibility to tax individual profits implies that the coefficient of dispersion of the distribution of their equity wealth necessarily grows over time. In equilibrium, in the longer run, there must be some very rich entrepreneurs with large firms, and some who are doing much worse than the representative household, with small firms and little income.

But while taxes and lump-sum redistribution are ineffective in addressing this distributional problem, the issuance of public debt helps. Interestingly, it helps all entrepreneurs alike, regardless of their fortunes along their stochastic individual growth paths. The following thought experiment illustrates this basic result, by explicitly constructing the Pareto improvement that is possible in a situation where the government uses taxes optimally but issues no debt. Without loss of generality, we restrict attention to steady states.

As discussed above, under balanced budgets (BB), $h = x - 1$, which by (40) implies

$$\chi^H = \rho \frac{x-1}{x}.$$

Hence, the welfare function (8) becomes

$$\begin{aligned} \rho W_{BB} &= \log K_0 + \frac{\mu - \gamma - \chi^H}{\rho} - \frac{1}{x} + \alpha(\log \frac{\rho}{x} - \frac{\sigma^2 x^2}{2\rho}) + (1-\alpha)\log \chi^H \\ &= \log K_0 + \frac{\mu - \gamma}{\rho} - 1 + \log \rho - \log x + (1-\alpha)\log(x-1) - \alpha \frac{\sigma^2 x^2}{2\rho} \end{aligned}$$

Maximizing W_{BB} yields the maximum welfare that can be achieved without issuing public debt. The unique maximizer $x_{BB} > 1$ is given by the first-order condition

$$\frac{\sigma^2}{\rho} x^3 - \frac{\sigma^2}{\rho} x^2 + x - \frac{1}{\alpha} = 0.$$

Note the similarity with the equation defining x^*, (10), and that private leverage is higher here: $x_{BB} > x^*$. It has to partially compensate the missing public debt. x_{BB} corresponds to the following allocation of initial wealth: $E_0 = \frac{K_0}{x_{BB}}$ and $H_0 = K_0 - E_0$. The growth rate of output in this restricted optimum is $g_{BB} = \mu - \gamma - \rho$, and up to additive constants, the expected continuation utilities can be written as follows:

$$\begin{aligned} \rho V_{BB}^H &= \log H_0 - H_0 - E_0 \\ \rho V_{BB}^E &= \log E_0 - H_0 - E_0 - \frac{\sigma^2}{2\rho E_0^2}. \end{aligned}$$

To understand how this allocation can be Pareto improved, suppose that the government issues a small amount of debt and distributes it to the two categories of agents, so that both ΔE_0 and ΔH_0 are positive. The government also adjusts the tax rates, so that the economy remains in the new steady state. The first order change in households' utility is such that

$$\rho \Delta V_{BB}^H = \frac{\Delta H_0}{H_0} - \Delta(H_0 + E_0),$$

corresponding to the difference between the relative wealth increase and the total wealth increase (equal to new government debt), which reduces growth. The equivalent term for entrepreneurial equity is

$$\rho \Delta V_{BB}^E = \frac{\Delta E_0}{E_0} - \Delta(H_0 + E_0) + \frac{\sigma^2}{\rho E_0^3} \Delta E_0,$$

where a new term appears, corresponding to the reduction in the risk premium that follows the decrease in private leverage. Hence, it is possible to distribute the additional wealth created by the government in such a way that both types of agents benefit, as long as the following two conditions hold:

$$h_0 < \frac{\Delta H_0}{\Delta E_0} < h_0 + \frac{\sigma^2 x_0^3}{\rho}.$$

This is possible in an economy with frictions, where $\sigma > 0$. As discussed above, in a frictionless economy, the Pareto-improving role of government debt disappears. Importantly, the welfare gain from increasing public debt above 0 benefits households and *all* entrepreneurs in *all* instances and at *all* times. This "balance sheet effect" of public debt, as we will call it in the next section, thus operates like a rising tide that lifts all the boats. Such a universal Pareto improvement has been termed "Robust Pareto Improvement" by Aguiar et al. (2023).

Proposition 5. *Suppose that the government balances its budget and uses initial transfers and linear wealth taxes in an optimal way. In this situation, issuing public debt constitutes a Robust Pareto Improvement in the sense of Aguiar et al. (2023).*

5.4 Laisser-Faire

As an extreme benchmark, this subsection briefly characterizes a fully passive government, which does not engage in fiscal policy or redistribution. Such a Laisser-Faire (LF) policy has three features: (i) $B_t = 0$ for all t (balanced budget), (ii) $L^H = L^E = 0$ (no lump-sum redistribution), and (iii) $\tau_t^H = \tau_t^E = \tau_t$ (equal taxation).

Laisser-Faire therefore implies $T_t = \tau_t(H_t + E_t) = \tau_t K_t$. Together with the balanced-budget constraint $T_t = \gamma K_t$, this implies $\tau_t = \gamma$.

Furthermore, without continuing corrective taxation, the economy is not kept in steady state. The individual optimization results (17) and (18) imply that the entrepreneurs' individual capital-to-equity ratios $\frac{k_t^i}{e_t^i} = \frac{\mu - r_t}{\sigma^2}$ are still independent of i, but they now depend on t. Hence, under LF, the economy evolves on a trajectory (x_t, h_t) that is entirely contained in the Zero-Debt-Line $x = h + 1$ and starts at $x_0 > 1$. One can show that in Laisser-Faire equilibrium (x_t, h_t) converges monotonically to $(1, 0)$.

Note that Pareto Optima are not necessarily Pareto improvements over the Laisser-Faire. If we denote the expected utility of households and entrepreneurs, respectively, at the optimal allocation $x^*(\alpha)$ by $V^k(\alpha)$, $k = H, E$, this follows from the following two properties, which are a consequence of (9) and (10):

$$\lim_{\alpha \to 0} V^E(\alpha) = \lim_{\alpha \to 1} V^H(\alpha) = -\infty.$$

Hence, the allocation that maximizes total welfare W is a Pareto improvement over Laisser-Faire when α is intermediary. When α is large, households strictly prefer Laisser-Faire to the welfare optimum, while firms strictly prefer Laisser-Faire when α is small.

6 Taxes and Debt

As noted in the introduction, issuing public debt and distributing it to the private sector has three effects that jointly affect the consumption and investment decisions of the private sector and feed back into each other: a balance sheet effect, an interest rate effect, and a growth effect. The balance sheet effect reduces the net leverage of entrepreneurs and increases their incentives to make risky investments. The interest rate is affected because the increased supply of bonds increases the risk-free interest rate. This partially counteracts the balance sheet effect, as the entrepreneurs have to pay higher interest on their lower net debt, and it benefits households. Finally, growth is affected because issuing public debt increases the aggregate wealth of the private sector, which stimulates aggregate consumption, which in turn reduces output growth by the classic Ramsey-Solow logic.

It is worth emphasizing that public debt does not instantaneously "crowd out" private investment in the traditional sense (see Blanchard (2008)). "Crowding out" usually refers to the substitution of private investment by public spending, which by assumption is impossible in our model, where government expenditure is exogenous. Yet, there is "dynamic crowding out" if higher public debt reduces the growth rate of the economy and capital accumulation, which lowers investment in the long-term.

Since the wealth increases generated by public debt must directly accrue to entrepreneurs in order to trigger the balance sheet effect, it is necessary to balance them by continuously redistributing wealth from entrepreneurs to households to maintain optimal growth. Hence, there is a further consequence of public debt.

Since its issuance directly benefits entrepreneurs, it must be complemented by redistribution through ongoing taxation.

6.1 Taxes

To clarify the role of taxes in our economy, consider the following thought experiment. Suppose there are no financial frictions such that all idiosyncratic risks can be diversified away and we can effectively take $\sigma = 0$. The optimal allocation then is simply implemented by redistributing initial wealth in proportions α and $1-\alpha$, having no government debt, taxing entrepreneurs and households equally at $\tau_t^E = \tau_t^H = \gamma$, and thus keeping the economy at $E_t = \alpha K_t$ and $H_t = (1-\alpha)K_t$ at all times, as required by (10).

Suppose now that at date 0, when aggregate capital is at the level K_0, frictions appear, such that it is not possible to eliminate idiosyncratic risks anymore and $\sigma > 0$.[22] By (39), the optimal response of the government to this shock is to issue an amount $B_0 = (\frac{1}{x^*} - \alpha)K_0$ of debt and to distribute it exclusively to the entrepreneurs. Indeed, by (22), $H_0 = (1-\alpha)K_0$. Together with the aggregate balance sheet identity (37), this implies

$$E_0 = \alpha K_0 + B_0.$$

Thus the entrepreneurs are initially the only direct beneficiaries of government intervention. The following result shows that in any optimal allocation, households are subsidized afterwards through ongoing taxation.

Proposition 6. *To implement the welfare optimal allocation, households must be subsidized, in the sense that they contribute less in taxes than their share of public expenditures in the social welfare function:* $\tau^H H_t < (1-\alpha)\gamma K_t$ *for all* t*.*

Proof. Evaluating the claimed inequality at the welfare optimal values shows that it is equivalent to $\tau^H < \gamma$. This follows directly from (28). □

Proposition 6 states that households contribute less than their "fair share" of ongoing public expenditures.[23]

[22] This thought experiment corresponds to the traditional experiments in macroeconomic classics, such as Kiyotaki and Moore (1997), where a stationary equilibrium is shocked unexpectedly. The specific shock analyzed here is the same as in Di Tella (2017). In fact, citing from his paper, introducing "an aggregate uncertainty shock that increases idiosyncratic risk in the economy ... can create balance sheet recessions." Different from Di Tella (2017), we are interested in the long-run consequences of market imperfections rather than in booms and recessions.

[23] The proposition does not say that $\tau^{H*} < 0$. However, the proposition implies that this is the case if γ is sufficiently small. In this case, households receive continuous subsidies.

6.2 Debt

Since the optimal issue of public debt is continuously supported by redistributive taxation, it must depend on the welfare weights in the population. We now ask how. A standard measure of government indebtedness is the debt-to-GDP ratio, which in our model is given by

$$\delta_t \equiv \frac{B_t}{dY_t} dt = \frac{1 + h_t - x_t}{\mu x_t}.$$

Evaluated at the welfare optimum and noting that the relationship between B_t and Y_t is constant, the optimal debt-to-GDP ratio is

$$\delta^* = \frac{\sigma^2 x^*}{\mu(\rho + \sigma^2 x^{*2})} \tag{43}$$

which is strictly positive by Proposition 4. Two simple observations now show immediately how δ^* depends on the welfare weighting. First, by differentiating (10), x^* is strictly decreasing in α. Second, an inspection of (43) shows that δ^* is a strictly quasiconcave function of x^*. Hence, the optimal debt-to-GDP ratio is also single-peaked in α, which is summarized in the following proposition.

Proposition 7. *The optimal debt-to-GDP ratio is a strictly quasiconcave function of the political weight of entrepreneurs, with maximum at $\widehat{\alpha} = \min(1, \frac{\sigma}{2\sqrt{\rho}})$. It converges to 0 for $\alpha \to 0$.*

Proof. Differentiating (43) shows that δ^*, as a function of x, is strictly quasiconcave, with maximum at $x = \sqrt{\rho}/\sigma$. An inspection of (42) shows that $x_{\min} \geq \sqrt{\rho}/\sigma$ if and only if $\sqrt{\rho}/\sigma \leq \frac{1}{2}$. Since $x^* \in [x_{\min}, \infty)$, this shows that δ^*, as given by (43), is strictly decreasing in x^* if $\sqrt{\rho} \leq \frac{\sigma}{2}$ and strictly quasiconcave with maximum at $\sqrt{\rho}/\sigma$ otherwise. The rest of the proposition follows because $\frac{dx^*}{d\alpha} < 0$ and by inserting $x^* = \sqrt{\rho}/\sigma$ into equation (10). □

Hence, as long as the weight of entrepreneurs in the welfare function is not too large, an increase of this weight increases the debt-to-GDP ratio. This is mainly driven by the balance sheet effect discussed in the previous section: public debt, whether held directly by entrepreneurs or indirectly, when held by the household sector, makes entrepreneurs less risky and thus increases investment, increasing wealth in the whole economy. The more entrepreneurs matter, the more useful is debt. This effect is counteracted by the growth effect that takes over when entrepreneurs' interests are so dominant that further increases of debt (relative to GDP) would decrease growth too much, compared to the current creation of wealth.

Figure 2 illustrates Proposition 7 by plotting the optimal debt-to-GDP ratio as a function of the welfare weight of entrepreneurs. The figure uses values for ρ and μ that are in the standard range of the literature, and shows how sensitive the optimal debt-to-GDP ratio is to different values of the volatility of idiosyncratic productivity risk. Of course, it is not obvious how to calibrate σ in the present model. Nevertheless, calibrations for idiosyncratic productivity shocks have been the subject of various studies, and recent work, for instance, by Bloom et al. (2018) or Arellano et al. (2019), has provided estimates for such shocks. Bloom et al. (2018) report that the yearly variance of plant-establishment-level TFP shocks in the US in a non-recession time was 0.198. In order to use these estimates for a numerical illustration, one needs additional information about how much of the volatility is not insurable, which is hard to assess. But the value can serve as an upper bound.

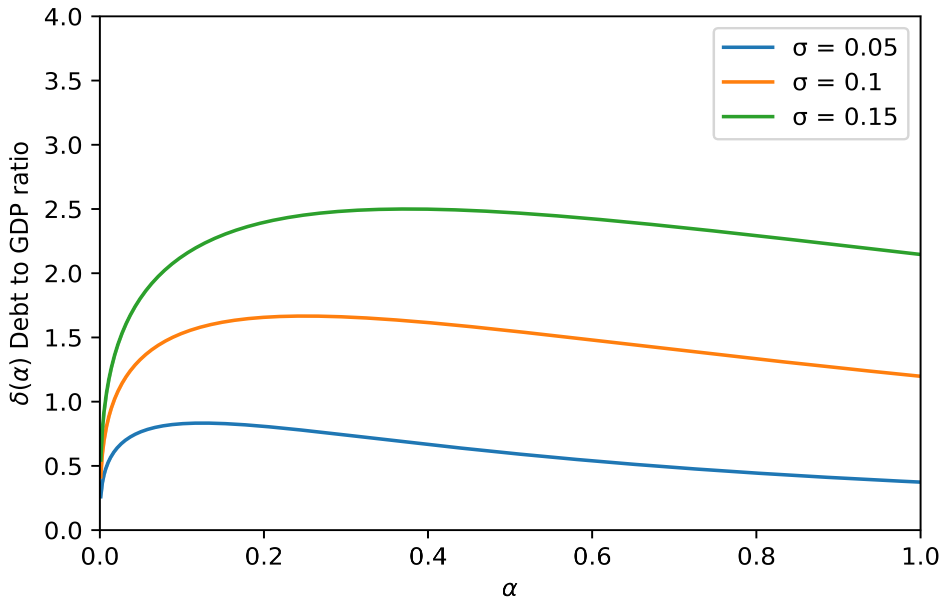

Figure 2: Debt-to-GDP ratio for $\rho = 0.04, \mu = 0.15$ and different values for σ.

When $\sigma < 2\sqrt{\rho}$, we have $\widehat{\alpha} < 1$ in Proposition 7. Given the preceding discussion, this seems to be the empirically relevant range in our framework.[24] Figure 2 therefore displays the inverse U-shape to be expected according to Proposition 7. The debt-to-GDP ratio is largest if the interests in the economy are relatively balanced, and decreases if one group becomes more and more dominant.

In our model, firm debt is safe because steady state equity follows a geometric Brownian motion and therefore never reaches zero: entrepreneurs do not default. Hence, when the government issues public debt, it does *not* create a new type of (safe) asset: government debt is exactly as good as existing private debt. However, public debt is valuable because there is not enough private debt due to the agency

[24]For example, it comprises all combinations $\rho \geq 0.02$ and $\sigma \leq 0.28$.

problem in corporate finance. Additional public debt therefore allows entrepreneurs to reduce their risk exposure. Of course, a necessary requirement for our analysis is the credibility of the government's promise to never default. But since the government is assumed to maximize social welfare, which is achieved in the steady state with sustainable debt issuance, there is neither a reason for the government to default nor for the private sector to believe that the government will default. Not defaulting is time-consistent for our benevolent government.[25]

6.3 Public Expenditure Changes

This section addresses a core question of this paper: how should the government finance public spending shocks? Our model provides a surprisingly clear and simple answer.

The model describes fiscal policy with exogenous public spending needs that are considered stable over the foreseeable future. In this section we vary these spending needs. This corresponds to a thought experiment akin to the one of Section 6.1, with an unexpected shock at some date that changes long-run public spending needs and thus moves the economy from one steady state to another.[26] This is consistent with recent experience, as governments all over the globe have decided or been forced to spend large sums on security threats posed by new wars and international conflicts, or on the environmental disasters resulting from climate change and the degradation of the natural environment. Both these kinds of shocks reflect long-term challenges and seem to be more permanent.[27] Such public expenditure shocks have a straightforward interpretation in our model as an increase of γ (there could well be negative spending shocks, too, of course). For this interpretation, it is immaterial whether a higher γ provides a higher quality of public services that improves the population's well-being, or whether simply more resources are needed to maintain the public infrastructure that supports the current state of activities. In this latter perspective, a higher γ is required to maintain the productivity in the private sector, captured in our model by the parameter μ. Alternatively, one can use our model to directly study the link between γ and μ, if the above crisis scenarios generate such a tradeoff.

[25]Extending our model, though, in the spirit of the seminal papers of Calvo (1988) and Cole and Kehoe (2000), one can ask nevertheless whether default can be a problem. Suppose for example that for whatever reason – for instance, coordination failures in debt issuance auctions –, there is a chance at some point in time t that the private sector will refuse to roll over public debt. But since the government relies on taxation of wealth, even this would not cause default. By the basic balance sheet identity, $B_t = H_t + E_t - K_t$, which is strictly smaller than $H_t + E_t$. Hence, off the equilibrium, the government can confiscate sufficient private wealth via emergency taxation to stop such a debt run in the first place.

[26]But remember that these spending needs are exogenous in our model. We do not consider discretionary government spending.

[27]The same is true for the large-scale funding needs necessary to decarbonize the global economy. However, the need for drastic action has been well-known for more than 20 years, so it is more difficult to consider this as an example of an unexpected large spending shock.

Equations (9), (10), (40), (28), (29), and (43) provide a clear answer to these two questions. If γ changes, the private sector optimum x^* remains constant, while the tax rates τ^H and τ^E needed to implement it increase one to one. This implies that debt and taxes have two very different roles in fiscal policy in our economy, which we highlight in the following proposition.

Proposition 8 (Fiscal Separation Principle)**.** *The optimal debt-to-GDP ratio is independent of the government's permanent spending needs γ. Any change in government spending needs must be financed one to one by a change in tax rates.*

Hence, higher public spending needs are financed by increasing taxes, and the structural variables x and h, as well as the debt-to-GDP ratio, remain unaffected. Furthermore, there is no instantaneous re-distribution after such a shock, as the aggregate private balance sheet does not need to change. This does *not* mean that a higher γ has no welfare costs. In fact, higher (exogenous) government spending crowds out private investments, which makes the economy poorer and the growth rate decline. But Proposition 8 states that public borrowing provides no structural remedy against this. The composition of the economy's aggregate balance sheet is determined by the financial friction σ, the economic discount rate ρ, and the distributional preferences as given by α. They alone determine how the intertemporal tradeoff between wealth today and growth tomorrow stemming from market incompleteness is resolved through public borrowing. All remaining current expenses are covered by current taxes.[28]

7 Interest and Growth

Formulas (11) and (25) show how growth and interest depend on optimal fiscal policy in our model. In this section, we investigate this link and how it depends on the underlying parameters of the economy.

7.1 Interest

The following proposition follows directly from differentiating (25).

Proposition 9. *The optimal interest rate $r^* = \mu - \sigma^2 x^*$ is an increasing function of μ and α and a decreasing function of ρ. It is negative if μ or α are sufficiently small.*

[28] Clearly, the simplicity of our prescription is due to the simplicity of our model. In particular, the fact that private productivity and public expenditure enter the social welfare function only in terms of their difference $\mu - \gamma$ is due to the assumption of constant returns to scale and the lack of any direct impact of γ on μ. Similarly, the assumption of log preferences excludes non-trivial intertemporal substitution patterns coming from the demand side. But the separation of intertemporal market completion through public debt and the funding of current public expenditures is a more general insight.

Proposition 9 sheds some interesting light on the recent debate about the observation that real interest rates have fallen over the last decades and have reached negative territory in a variety of industrialized countries, already before the recent inflationary hump. At the center of most explanations for this phenomenon is the observation that the amount of savings, relative to investment demand, has changed. While some explanations put emphasis on the origin of changes in savings, others put more emphasis on changes in productivity or put emphasis on both. One prominent voice is Rachel and Summers (2019), who stress that these secular movements are for a larger part a reflection of changes in saving and investment propensities. They argue that the industrialized world will probably face a longer period of secular stagnation, with sluggish growth and low real interest rates.[29]

Our results point to structural factors that might contribute to low real interest rates. For instance, and consistent with Proposition 9, permanent shifts in the objectives of policy-making with respect to risk-bearing versus non-risk-bearing agents can induce a secular decline and even negative values of real interest rates. Proposition 9 is also consistent with the suggested link between aggregate productivity and interest rates.

7.2 The Sustainability of Fiscal Policy

The preceding results make it possible to characterize the relation between growth and interest rate in equilibrium with optimal fiscal policy. Surprisingly, our simple model makes an explicit but non-trivial prediction about this widely debated relation.

Proposition 10. *In welfare optimal equilibrium, $g^* > r^*$ if and only if*

$$2\alpha\,(\rho+\gamma) + (\rho+\gamma+\alpha)\sqrt{\alpha\left(1+\frac{\gamma}{\rho}\right)} < \sigma^2. \tag{44}$$

The proof of Proposition 10 is in the appendix. The proposition makes precise predictions about the determinants of the difference between r and g at the welfare optimum, involving four of the five exogenous variables of the model. As discussed in the introduction, at least in recent history, the case $g > r$ seems to have been more relevant than the opposite case. In this respect, the prediction of Proposition 10 is that the growth rate will optimally exceed the interest rate when the private propensity to consume ρ, public expenditures γ, and the political weight of entrepreneur interests α are small, and when idiosyncratic production risk σ is large. These predictions are independent of the productivity of capital, μ.

[29] For discussions (and evidence) on how to differentiate whether rising income inequality or an aging of the population can have contributed to an increase in savings see e.g. Mian et al. (2021); von Weizsäcker and Krämer (2019) discuss how technological progress and demography may have jointly contributed to a secular decline in real interest rates.

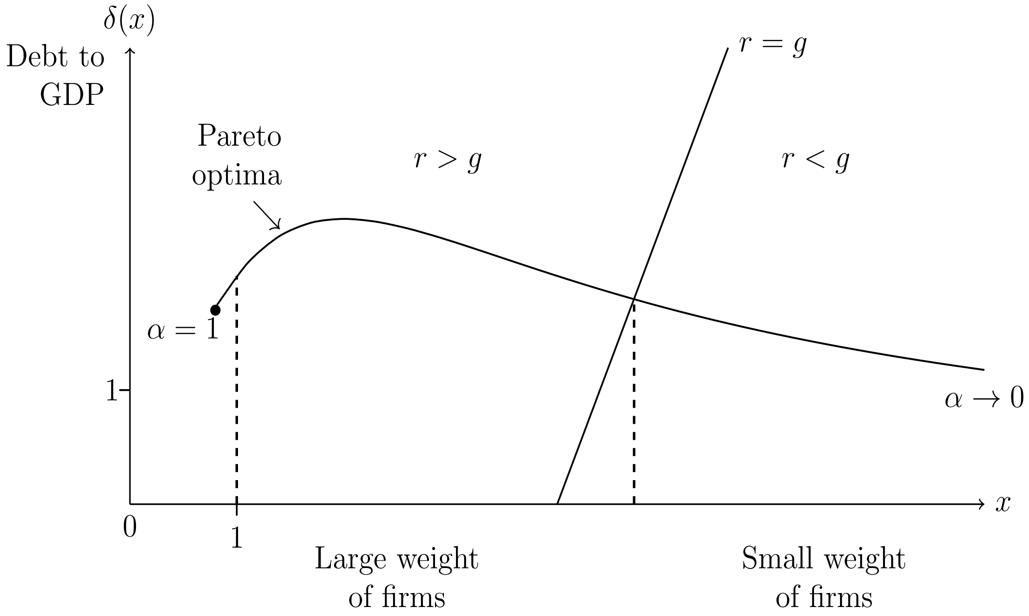

Figure 3: Regimes for parameter values $\rho = 0.04, \sigma = 0.1, \mu = 0.1$.

Figure 3, which mirrors Figure 2, illustrates the insight of Proposition 10 that depending on the welfare weight of entrepreneurs, the economy can be in different regimes $r - g > 0$ or $r - g < 0$. Whether $r < g$ or not has been a central question in recent debates about the sustainability of the US' and other countries' fiscal policy. From an asset pricing perspective, Cochrane (2019) describes the limits of public deficits by noting that in models with infinitely-lived agents, "[t]he market value of government debt equals the present discounted value of primary surpluses." Consistent with the prediction of Proposition 10, Cochrane (2022) argues that under complete financial markets ($\sigma = 0$ in our model), a permanent relationship $r < g$ is theoretically implausible, and empirically unlikely when r and g are measured correctly.[30] On the other hand, Blanchard (2019) adopts a more positive view on the theoretical possibility of $r < g$ and investigates the potential and limitations of a fiscal expansion at little or no fiscal cost.

In our model of an economy with endogenous growth, idiosyncratic production risk, and imperfect macroeconomic risk-sharing, the return on safe debt r can fall below g. If buffering the losses of entrepreneur owners has less weight in the welfare function, public debt issuance and the reduction of corporate leverage are less important. As a consequence, entrepreneurs are only willing to invest in risky production if the real interest rate is sufficiently low. Hence, as summarized in Figure 3 above, there is a role for government policy to actively reduce r in such cases.

Our analysis can reconcile both views about the dynamics of the government

[30]Cochrane (2022) provides a comprehensive account how the $r < g$ – debate is connected to the fiscal theory of the price level.

budget in a single model. The government's flow budget constraint at date t, (12), can be written as

$$\dot{B}_t = \gamma K_t + r_t B_t - T_t = r B_t - S_t, \tag{45}$$

where S_t is the primary surplus. Consider an arbitrary steady state (not necessarily optimal) and let r and g be the associated interest and growth rates, respectively. Discounting and integrating (45) between dates 0 and some later date T yields:[31]

$$B_0 = \int_0^T S_t e^{-rt} dt + B_T e^{-rT}.$$

This relation can be viewed as the balance sheet identity for the public sector, with liabilities B_0 and two types of assets as follows:

Assets	Liabilities
$X_0 := \int_0^T S_t e^{-rt} dt$	B_0
$W_0 := B_T e^{-rT}$	

where we let $T \to \infty$.

As in our previous discussion, we can distinguish two cases. The first case is $r > g$. Then W_0 tends to zero when T tends to ∞, and we obtain the standard relationship that the value of debt equals the net present value of future primary surpluses, as argued, e.g., by Cochrane (2019). The second case is $r < g$. Then W_0 tends to $+\infty$ and X_0 tends to $-\infty$. Hence, in the limit, the balance sheet identity $X_0 + W_0 = B_0$ is not well defined. However, we can interpret $B_T e^{-rT}$ as a form of intangible asset for the government, which can be attributed to its capacity to borrow again in the future and may be called government "goodwill". In fact, our analysis shows that it is rather the government's "eternal power to issue safe debt" that creates this intangible asset. As long as the government can convince investors of its capacity to sustain a high enough level of growth, this intangible asset has a positive value.

In light of the results of the last two sections and under the assumption that σ is not too large,[32] we can therefore distinguish two polar cases for the importance of entrepreneurial interests on the sustainability of government deficits. First, if α is small, $g > r$ in equilibrium, and the government runs increasing budget deficits that it covers by taxes and by rolling over ever-increasing public debt. Nevertheless, by Proposition 7 the public debt-to-GDP ratio is small. Second, if α is large, we have $g < r$ in equilibrium, "[t]he market value of government debt equals the present

[31] Which discount rate should be used for the government budget constraint has been the subject of recent work. Brunnermeier et al. (2021), and Reis (2021) offer particular rationales for using discount rates different from r.

[32] We need the inequality in (44) to be reversed for $\alpha = 1$, which implies an upper bound on σ. This is consistent with our discussion of plausible parameter ranges in Section 6.2, and in particular with the assumption $\sigma < 2\sqrt{\rho}$ that ensures $\widehat{\alpha} < 1$ in Proposition 7. If σ is large (which seems implausible empirically), we have $r < g$ for all α.

discounted value of primary surpluses" (Cochrane (2019)), and the public debt-to-GDP ratio is intermediary. For medium values of α, the public debt-to-GDP ratio is large, the growth rate is low, and the sign of $r - g$ depends on γ, σ, and γ as given by (44). Hence, government deficits have a "Cochranian" interpretation or a "Blanchardian" one, depending on α.

8 Conclusion and Applications

We have presented a simple model in which government debt issuance affects corporate leverage, and thus the investment and growth dynamics of the economy, through changes in the mix of private and public debt. It highlights how the weights of different private agents in the government welfare function impact the relationship between r and g. In this sense, interest, growth, and public debt are a matter of redistributionary political trade-offs.

Our model allows many extensions and a variety of applications. First, we have used it as a building block of a theory of money and banking in Gersbach et al. (2023), in which central bank reserves play a safety role for commercial banks, while controlling monetary policy, similar to the role of government debt in the current model. Since the Great Financial Crisis of 2007-2009, the reserves of commercial banks in the US, the UK, Japan, and in the Euro Area have strongly increased, albeit to different degrees. Our preliminary results support the argument that banks' holding large amounts of central bank reserves can be desirable from a welfare perspective when banks face significant uninsurable idiosyncratic risks.

Second, looking forward, it is promising to endogenize public expenditures in our model and to address the public goods problem of government expenditure explicitly. This would allow to study the impact of preference shocks for public good provision on the mix of taxes and debt. In particular, it is likely that such a theory will provide more nuanced results on the desirability of public debt.

Third, the present paper has implications for normative macroeconomic theories in which government debt serves a socially desirable purpose. To what extent is the rise of government debt over the past decades an optimal response to changing fundamentals? For instance Yared (2019) provides a comprehensive account of political economy theories on government debt and discusses how these theories may explain a substantial part of the long-term trend in government debt accumulation. Our model could be a natural starting point to introduce political constraints such as participation constraints of firms, upper limits on taxation of firms and lower bounds of consumption utility of households. When such constraints become binding, the balance between taxes and debt issuance may shift. Moreover, adding political competition and political turnover directly may be fruitfully studied in our framework when augmenting it by an explicit election framework.

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A Appendix

A.1 Derivation of entrepreneurs' expected utility

Under the incentive-compatible mechanism described in Section 3.3, entrepreneurs' expected utility is

$$\begin{aligned}
\rho V^E &= \rho \mathbb{E} \int_0^\infty e^{-\rho t} \log \left(\frac{\rho}{x} k_t^i \right) dt \\
&= \log \frac{\rho}{x} + \rho \mathbb{E} \int_0^\infty e^{-\rho t} \log k_t^i dt
\end{aligned}$$

By the Itô-Doeblin Lemma and using (2),[33]

$$\begin{aligned}
\log k_t^i &= \log k_0^i + \int_0^t \frac{1}{k_s^i} dk_s^i - \frac{1}{2} \int_0^t \frac{1}{(k_s^i)^2} \sigma^2 x^2 (k_s^i)^2 ds \\
&= \log k_0^i + \int_0^t \left(g_s - \frac{1}{2} \sigma^2 x^2 \right) ds + \int_0^t \sigma x dz_s^i
\end{aligned}$$

Integrating partially and inserting (7) yields

$$\rho \mathbb{E} \int_0^\infty e^{-\rho t} \log k_t^i dt = \log k_0 + \frac{1}{\rho} \left(\mu - \gamma - \frac{\rho}{x} - \frac{1}{2} \sigma^2 x^2 \right) - \int_0^\infty e^{-\rho t} \chi_t^H dt$$

A.2 Proof of Proposition 3

A.2.1 "If"

We must verify the three properties of a GEFP.

The first property follows directly from Proposition 2. The third property follows by the construction of (B_t): by (38) and the definition of (B_t), the aggregate balance sheet holds at time $t = 0$, and it holds for all $t > 0$ because B_t grows at the rate g^*, just as H_t, E_t, and K_t. To verify the second property, we must show that $B_t = H_t + E_t - K_t$ satisfies (36), i.e. that

$$\dot{H}_t + \dot{E}_t - \dot{K}_t = \gamma K_t + r(H_t + E_t - K_t) - \tau^H H_t - \tau^E E_t. \tag{A1}$$

Using (23) and (26) and rearranging, (A1) is equivalent to

$$(r - \gamma - g^*) K_t = \rho H_t + \left(\rho - \left(\frac{\mu - r}{\sigma} \right)^2 \right) E_t \tag{A2}$$

We know from (20) that $K_t = x E_t$. Using this together with $H_t = (1 - \alpha) K_t$, we can substitute out for H_t and E_t by K_t. Substituting r from (25) then shows

[33] See, e.g., Shreve (2004), p. 187.

that (A2) is equivalent to

$$\rho - \sigma^2 x^2 - \rho\alpha x + \frac{\sigma^4 x^4}{\rho + \sigma^2 x^2} = 0,$$

which holds for $x = x^*$ by (10).

A.2.2 "Only If"

Suppose fiscal policy follows the taxation and transfer rules (28)-(31) and the debt policy $B_t = (L^H + L^E)e^{g^* t}$ for all $t \geq 0$. Suppose also that $(r_t)_{t\geq 0}$ is an equilibrium interest rate trajectory. We must show that $r_t = r^* = \mu - \sigma^2 x^*$ for all $t \geq 0$.

The aggregate balance sheet constraint (37) together with (32), (35), (18), and (36) implies

$$\begin{aligned} \dot{K}_t &= \dot{H}_t + \dot{E}_t - \dot{B}_t \\ &= \left(r_t - \tau^H - \rho\right) H_t + \left(r_t - \tau^E - \rho\right) E_t + (\mu - r_t)K_t - \gamma K_t - r_t B_t + T_t \\ &= (\mu - \gamma)K_t - \rho(H_t + E_t), \end{aligned} \tag{A3}$$

which is the economy's IS equation (equality of investment and net savings).[34]

At each date t, the four aggregate variables K_t, B_t, E_t, H_t are linked by the balance sheet identity (37). In fact, by the homogeneity of the entrepreneurs' investment problem, ratios of the state variables are sufficient to characterize equilibrium. We pick here the capital-equity ratio x_t as defined in (20), and $h_t \equiv \frac{H_t}{E_t}$, the ratio of household wealth over entrepreneurial equity.[35] The trajectories of the two state variables (x_t, h_t) completely determine all aggregate variables (output, consumption, and investment) in equilibrium. In fact, by (A3), the equilibrium growth rate g_t of capital is

$$g_t = \frac{\dot{K}_t}{K_t} = \mu - \gamma - \rho\frac{h_t + 1}{x_t}. \tag{A4}$$

By (32), aggregate household wealth grows according to

$$\frac{\dot{H}_t}{H_t} = \mu - \rho - \tau^H - \sigma^2 x_t \tag{A5}$$

and aggregate equity of entrepreneurs, similarly, according to

$$\frac{\dot{E}_t}{E_t} = \mu - \rho - \tau^E - \sigma^2 x_t(1 - x_t) \tag{A6}$$

Given this direct relation between equilibria and the $x_t - h_t$ trajectories, we now characterize these trajectories.

[34](A3) is the counterpart of the optimality condition (5) in the mechanism design problem.
[35]See the motivation and discussion in Section 5.3.

The initial values of the system are given by the lump sum transfers at date 0:

$$h_0 = \frac{H_0}{E_0} = \frac{\tilde{H} + L^H}{\tilde{E} + L^E}, \tag{A7}$$

$$x_0 = \frac{K_0}{E_0} = \frac{\tilde{H} + \tilde{E}}{\tilde{E} + L^E}. \tag{A8}$$

The dynamics of the state variables for $t > 0$ are then determined by the instantaneous tax rates. Using the definition of h_t, (A5) and (A6) imply

$$\dot{h}_t = (\tau^E - \tau^H - \sigma^2 x_t^2) h_t. \tag{A9}$$

Similarly, using (A4) and (A6),

$$\dot{x}_t = (\sigma^2 x_t^2 - \rho)(1 - x_t) + (\tau^E - \gamma) x_t - \rho h_t. \tag{A10}$$

If the system (A7)-(A10) has a solution that stays in the interior of the positive (x, h) quadrant, then this solution yields a unique general equilibrium for the given fiscal policy, as shown above. Using (28)-(31) and the definition of x^* in (10), one easily verifies that the constant trajectory $(x_t, h_t) = (x^*, (1-\alpha)x^*)$ solves (A7)-(A10). By the Picard-Lindelöf Theorem from the theory of ordinary differential equations (see, e.g., Hirsch and Smale (1974)), the system only has one solution, which is maximal. Hence, the interest rate trajectory $(r_t)_{t \geq 0}$ we started out with must be the constant trajectory $r_t = r^*$.

A.3 Proof of Proposition 10

From (11) and (25) we have

$$r^* - g^* = \frac{\rho}{x^*} - \sigma^2 x^* + \rho(1-\alpha) + \gamma.$$

Using (10), this implies

$$\frac{x^{*2}}{\rho}(r^* - g^*) = \left(1 - \alpha + \frac{\gamma}{\rho}\right) x^{*2} + 2x^* - \frac{1}{\alpha}.$$

Hence, we have $r^* < g^*$ iff $x^* < \widetilde{x}$, where $\widetilde{x}$ is the unique positive solution to

$$x^2 + \frac{2}{y} x - \frac{1}{\alpha y} = 0, \tag{A11}$$

i.e.

$$\widetilde{x} = \frac{1}{y}\left[\sqrt{1 + \frac{y}{\alpha}} - 1\right],$$

where $y \equiv 1 - \alpha + \frac{\gamma}{\rho}$. Using the definition of x^* and (A11), the condition $x^* < \widetilde{x}$ is equivalent to

$$\left(4\alpha + y + \frac{\rho\alpha}{\sigma^2} y^2\right)\left[\sqrt{1 + \frac{y}{\alpha}} - 1\right] > 2y + \frac{\rho}{\sigma^2} y^3. \tag{A12}$$

In a number of straightforward steps, (A12) can be re-written as (44) in the proposition.

B Online Appendix: The Individual Decision Problems

For Online Publication

For completeness, this appendix provides a detailed solution to the individual optimization problems of Section 4.2 that were only sketched in the main text.

B.1 Households

Suppose that the representative household has initial net worth n_0^H at time $t = 0$, no further income later, and can only save via safe debt. Consider the variation of the household's decision problem in which the household starts out at time $t \geq 0$ with net worth $n > 0$. It chooses a consumption path c_s^H, $s \geq t$, to solve the standard consumption problem

$$\begin{aligned} &\max_{c^H} \int_t^\infty e^{-\rho s} \log c_s^H ds \\ &dn_s^H = \left((r_s - \tau_s^H) n_s^H - c_s^H\right) ds \qquad \text{(B1)} \\ &n_t^H = n \\ &n_s^H \geq 0. \end{aligned}$$

Denote the optimal consumption path for this problem by $c_s^H(t, n)$.

Remark 1. *The problem is homogeneous and invariant to scaling. Hence, if $c_s^H = c_s^H(t, n)$, $s \geq t$, is an optimal path for the problem with initial condition $n_t^H = n$, then αc_s^H, $s \geq t$, is an optimal path for the problem with initial condition $n_t^H = \alpha n$, for $\alpha > 0$.*

Hence, any optimal path satisfies

$$c_s^H(t, n) = c_s^H(t, 1) n.$$

Let $V^H(t, n)$ be the value function of the problem. Homogeneity implies

$$\begin{aligned} V^H(t, n) &= \int_t^\infty e^{-\rho s} \log c_s^H(t, n) ds \\ &= \frac{e^{-\rho t}}{\rho} \log n + v^H(t), \qquad \text{(B2)} \end{aligned}$$

where

$$v^H(t) = \int_t^\infty e^{-\rho s} \log c_s^H(t, 1) ds \qquad \text{(B3)}$$

is independent of n.

Ignoring the non-negativity conditions (which will be satisfied at the optimum), the Bellman Equation of the household's problem is

$$\frac{\partial V^H}{\partial t} + \max_c \left[e^{-\rho t} \log c + \frac{\partial V^H}{\partial n} \left((r_t - \tau_t^H) n - c \right) \right] = 0.$$

From (B2), we have

$$\frac{\partial V^H}{\partial n} = \frac{e^{-\rho t}}{\rho n},$$

such that the Bellman Equation becomes

$$\frac{\partial V^H}{\partial t} + \max_c \left[e^{-\rho t} \log c + \frac{e^{-\rho t}}{\rho n} \left((r_t - \tau_t^H) n - c \right) \right] = 0. \tag{B4}$$

It is easy to see that the first-order condition

$$c = \rho n \tag{B5}$$

is necessary and sufficient for the maximization problem in (B4). In particular, (B5) implies that $c > 0$. The Bellman Equation thus is equivalent to

$$-e^{-\rho t} \log n + \dot{v}^H(t) + e^{-\rho t} \left[\log \rho n - 1 + \frac{r_t - \tau_t^H}{\rho} \right] = 0,$$

which is equivalent to

$$\dot{v}^H(t) = \frac{e^{-\rho t}}{\rho} \left[\rho - \rho \log \rho - r_t + \tau_t^H \right].$$

This can be integrated explicitly to yield

$$\rho v^H(t) = (1 - \log \rho) \left(1 - e^{-\rho t} \right) - \int_0^t e^{-\rho s} (r_s - \tau_s^H) ds + \rho v^H(0). \tag{B6}$$

By (B5), if $n_s^H(t, n)$ is on the trajectory generated by $c_s^H(t, n)$, $s \geq t$, the optimal policy is

$$c_s^H(t, n) = \rho n_s^H(t, n). \tag{B7}$$

Hence, inserting (B7) into (B1) yields the law of motion for household savings with initial value 1 at time $t = 0$, $n_s^H(0, 1)$, as

$$\frac{dn_s^H(0,1)}{ds} = (r_s - \tau_s^H - \rho) n_s^H(0, 1).$$

Integrating yields

$$\log n_s^H(0, 1) = \int_0^s (r_\tau - \tau_\tau^H - \rho) d\tau, \tag{B8}$$

where the constant of integration in (B8) is $\log n_0^H(0, 1) = \log 1 = 0$, by the construction of v.

Inserting (B7) and (B8) into (B3) yields, for $t = 0$,

$$\begin{aligned} v^H(0) &= \int_0^\infty e^{-\rho s}(\log\rho + \log n_s^H(0,1))ds \\ &= \frac{\log\rho}{\rho} + \int_0^\infty e^{-\rho s}\int_0^s (r_\tau - \tau_\tau^H - \rho)d\tau ds \\ &= \frac{\log\rho}{\rho} - \frac{1}{\rho} + \frac{1}{\rho}\int_0^\infty e^{-\rho\tau}(r_\tau - \tau_\tau^H)d\tau. \end{aligned}$$

Combining this with (B6) yields

$$\rho v^H(t) = -(1-\log\rho)e^{-\rho t} + \int_t^\infty e^{-\rho s}(r_s - \tau_s^H)ds,$$

which together with (B2) yields the households' value function as

$$\rho V^H(t,n) = e^{-\rho t}\left(\log(\rho n) - 1\right) + \int_t^\infty e^{-\rho s}\left(r_s - \tau_s^H\right)ds.$$

B.2 Entrepreneurs

Net of initial lump sum taxes, at time $t = 0$ entrepreneur i has an initial equity position $e_0^i > 0$. Consider the variation where she starts at time t with equity $e^i > 0$. She chooses a path k_s^i, e_s^i, c_s^i, $s \geq t$ such as to

$$\begin{aligned} &\max_{k^i,e^i,c^i} \mathbb{E}\int_t^\infty e^{-\rho s}\log c_s^i ds \\ &de_s^i = \left[(\mu - r_s)k_s^i + (r_s - \tau_s^E)e_s^i - c_s^i\right]ds + \sigma k_s^i dz_s^i \qquad \text{(B9)} \\ &e_t^i = e^i \\ &e_s^i \geq 0, \end{aligned}$$

where equation (B9) is the flow of funds equation (16) in the main text. Denote the value function of the problem by $V^E(t,e^i)$.

Since, as in the household problem, the feasible set is homogeneous, any solution is invariant to scaling, and we must have, at the optimum,

$$(k_s^i(t,e^i), c_s^i(t,e^i)) = (k_s^i(t,1)e^i, c_s^i(t,1)e^i).$$

Therefore,

$$V^E(t,e^i) = \frac{e^{-\rho t}}{\rho}\log e^i + v^E(t), \qquad \text{(B10)}$$

where

$$v^E(t) = \mathbb{E}\int_t^\infty e^{-\rho s}\log c_s^i(t,1)ds \qquad \text{(B11)}$$

is independent of e^i.

We first solve the unconstrained problem, in which we ignore the non-negativity constraint on e_s^i. In this case, the Bellman Equation is

$$\frac{\partial V^E}{\partial t} + \max_{k,c}\left[e^{-\rho t}\log c + \frac{\partial V^E}{\partial e}\left((\mu - r_t)k + (r_t - \tau_t^E)e^i - c\right) + \frac{\partial^2 V^E}{\partial e^2}\frac{\sigma^2}{2}k^2\right] = 0.$$

From (B10), we have

$$\begin{aligned}\frac{\partial V^E}{\partial e} &= \frac{e^{-\rho t}}{\rho e^i}\\ \frac{\partial^2 V^E}{\partial e^2} &= -\frac{e^{-\rho t}}{\rho (e^i)^2}.\end{aligned}$$

The Bellman Equation therefore becomes

$$\frac{\partial V^E}{\partial t} + max_{k,c}e^{-\rho t}\left[\log c + \frac{1}{\rho e^i}\left((\mu - r_t)k + (r_t - \tau_t^E)e^i - c\right) - \frac{1}{2\rho(e^i)^2}\sigma^2 k^2\right] = 0 \tag{B12}$$

and the first-order conditions

$$c = \rho e^i \tag{B13}$$

$$k = \frac{\mu - r_t}{\sigma^2}e^i \tag{B14}$$

are necessary and sufficient for the maximum in (B12). In particular, (B13) implies that $c > 0$. The Bellman Equation therefore is equivalent to

$$\begin{aligned}-e^{-\rho t}\log e^i + \dot{v}^E(t) + e^{-\rho t}\left[\log \rho e^i - 1 + \frac{r_t - \tau_t^E}{\rho} + \frac{(\mu - r_t)^2}{2\rho\sigma^2}\right] &= 0\\ \Leftrightarrow \dot{v}^E(t) = e^{-\rho t}\left[1 - \log\rho - \frac{r_t - \tau_t^E}{\rho} - \frac{(\mu - r_t)^2}{2\rho\sigma^2}\right].\end{aligned}$$

This is a deterministic ODE that can be integrated explicitly to yield

$$\rho v^E(t) = (1 - \log\rho)\left(1 - e^{-\rho t}\right) - \int_0^t e^{-\rho s}\left(r_s - \tau_s^E + \frac{(\mu - r_s)^2}{2\sigma^2}\right)ds + \rho v^E(0) \tag{B15}$$

From (B13)–(B14), if $e_s^i = e_s^i(t, e^i)$ is on a trajectory generated by $c_s^i(t, e^i)$ and $k_s^i(t, e^i)$, $s \geq t$, the optimal policy is

$$c_s^i(t, e^i) = \rho e_s^i \tag{B16}$$

$$k_s^i(t, e^i) = \frac{\mu - r_s}{\sigma^2}e_s^i. \tag{B17}$$

Hence, inserting (B16) and (B17) into the equation of motion (B9) yields the (random) law of motion for entrepreneur equity, with $s \geq t$ and $e_t^i = e_t^i(t, e^i) = e^i$, as

$$de_s^i = \left[\left(\frac{\mu - r_s}{\sigma}\right)^2 + r_s - \tau_s^E - \rho\right]e_s^i ds + \frac{\mu - r_s}{\sigma}e_s^i dz_s^i \tag{B18}$$

$$\equiv (\beta_s - \rho)e_s^i ds + \gamma_s e_s^i dz_s^i, \tag{B19}$$

where we have set, for simplicity,

$$\beta_s = \left(\frac{\mu - r_s}{\sigma}\right)^2 + r_s - \tau_s^E \tag{B20}$$

$$\gamma_s = \frac{\mu - r_s}{\sigma}. \tag{B21}$$

We must determine $v^E(0)$. From (B11), using (B17), we have

$$\begin{aligned} v^E(0) &= \mathbb{E}\int_0^\infty e^{-\rho s}\log \rho e_s^i(0,1)ds \\ &= \frac{\log\rho}{\rho} + \mathbb{E}\int_0^\infty e^{-\rho s}\log e_s^i(0,1)ds. \end{aligned} \tag{B22}$$

Applying the Itô-Doeblin formula (Shreve (2004), p. 187) to (B19) yields

$$\begin{aligned} d\log e_s^i &= \frac{1}{e_s^i}de_s^i - \frac{1}{2\left(e_s^i\right)^2}\gamma_s^2\left(e_s^i\right)^2 ds \\ &= \left(\beta_s - \rho - \frac{1}{2}\gamma_s^2\right)ds + \gamma_s dz_s^i \end{aligned}$$

For $e_s^i = e_s^i(0,1)$, where by definition $e_0^i = 1$, this means with probability 1,

$$\log e_s^i(0,1) = \int_0^s \left(\beta_\tau - \rho - \frac{1}{2}\gamma_\tau^2\right)d\tau + \int_0^s \gamma_\tau dz_\tau^i.$$

By the definition of the stochastic integral, under standard integrability assumptions for r_s, the expectation in (B22) then is

$$\begin{aligned} \mathbb{E}\int_0^\infty e^{-\rho s}\log e_s^i(0,1)ds &= \int_0^\infty e^{-\rho s}\int_0^s\left(\beta_\tau - \rho - \frac{1}{2}\gamma_\tau^2\right)d\tau ds \\ &= -\frac{1}{\rho} + \int_0^\infty e^{-\rho s}\int_0^s\left(\beta_\tau - \frac{1}{2}\gamma_\tau^2\right)d\tau ds \\ &= -\frac{1}{\rho} + \frac{1}{\rho}\int_0^\infty e^{-\rho\tau}\left(\beta_\tau - \frac{1}{2}\gamma_\tau^2\right)d\tau. \end{aligned}$$

Inserting this into (B22) and using (B20)–(B21),

$$\rho v^E(0) = \log\rho - 1 + \int_0^\infty e^{-\rho s}\left(r_s - \tau_s^E + \frac{(\mu - r_s)^2}{2\sigma^2}\right)ds.$$

Combining this with (B15) yields

$$\rho v^E(t) = -e^{-\rho t}(1 - \log\rho) + \int_t^\infty e^{-\rho s}\left(r_s - \tau_s^E + \frac{(\mu - r_s)^2}{2\sigma^2}\right)ds. \tag{B23}$$

Finally, inserting (B23) into the value function (B10), yields

$$\rho V^E(t, e^i) = e^{-\rho t}(\log\rho e^i - 1) + \int_t^\infty e^{-\rho s}\left(r_s - \tau_s^E + \frac{(\mu - r_s)^2}{2\sigma^2}\right)ds.$$