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The Preference Lattice

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THE PREFERENCE LATTICE*

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Abstract

Most comparisons of preferences are instances of *single-crossing dominance*. We examine the lattice structure of single-crossing dominance, proving characterisation, existence and uniqueness results for minimum upper bounds of arbitrary sets of preferences. We apply these theorems to derive new comparative statics theorems for collective choice and under analyst uncertainty, to characterise a general ‘maxmin’ class of uncertainty-averse preferences over Savage acts, and to revisit the tension between liberalism and Pareto efficiency in social choice.

1 Introduction

Comparisons of preferences are ubiquitous in economics: examples include ‘more risk-averse/uncertainty-averse than’ (in decision theory),¹ ‘takes larger actions than’ (in monotone comparative statics),² ‘more impatient than’ (in dynamic problems),³ and ‘more self-controlled than’ (in models of temptation).⁴ All of these preference comparisons, and many others besides, are special cases of *single-crossing dominance*, a general unified way of comparing preferences.

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¹Yaari (1969), Epstein (1999) and Ghirardato and Marinacci (2002).

²Topkis (1978), Milgrom and Shannon (1994) and Quah and Strulovici (2009).

³Horowitz (1992) and Benoît and Ok (2007).

⁴Gül and Pesendorfer (2001) and Dekel and Lipman (2008).

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In this paper, we investigate the lattice structure of single-crossing dominance. Our results characterise the minimum upper bounds of arbitrary sets of preferences, and furnish necessary and sufficient conditions for their existence and uniqueness.

We use these theorems to derive new insights in a variety of economic settings that feature some of the aforementioned preference comparisons. First, we derive comparative-statics theorems for collective choice and for problems in which the analyst has only partial knowledge of a decision-maker's preference. Secondly, we characterise a general class of maxmin preferences over Savage acts as minimum upper bounds with respect to 'more uncertainty-averse than'. Finally, we delineate when Pareto efficient aggregation of individual preferences is compatible with liberalism.

1.1 Overview of the theory

The abstract environment consists of a non-empty set \mathcal{X} of alternatives equipped with a partial order \succeq .⁵ A *preference* is a complete and transitive binary relation on \mathcal{X} . We write \mathcal{P} for the set of all preferences. *Single-crossing dominance* captures a greater appetite for \succeq -larger alternatives:

Definition 1. For two preferences $\succeq, \succeq' \in \mathcal{P}$, we write $\succeq' S \succeq$ iff for any pair $x \succeq y$ in \mathcal{X} , $x \succeq(\succ) y$ implies $x \succeq'(\succ') y$. We say that \succeq' *single-crossing dominates* \succeq .⁶

Given a set $P \subseteq \mathcal{P}$ of preferences, a preference $\succeq' \in \mathcal{P}$ is an *upper bound* of P iff $\succeq' S \succeq$ for every $\succeq \in P$, and a *minimum* upper bound iff in addition $\succeq'' S \succeq'$ for every (other) upper bound \succeq'' of P . Intuitively, a minimum upper bound of P is a preference that likes large alternatives more than does any preference in P , but only just. Maximum lower bounds are defined analogously.

In §2, we study the *lattice structure* of (\mathcal{P}, S) by developing characterisation, existence and uniqueness results for minimum upper bounds. Our characterisation theorem (§2.1) describes the minimum upper bounds of arbitrary sets $P \subseteq \mathcal{P}$ of preferences. Our existence theorem (§2.2) identifies the condition on \succeq , called *crown- and diamond-freeness*, that is necessary and sufficient for every set $P \subseteq \mathcal{P}$ to possess a minimum upper bound. Finally,

⁵Definitions of standard terms like 'partial order' are collected in appendix A.

⁶This definition is from Milgrom and Shannon (1994), in the context of monotone comparative statics. This type of preference comparison has been used at least since Yaari's (1969) definition of 'more risk-averse than' (see §4.4). It also plays a role in the social choice literature following Grandmont (1978).

our uniqueness proposition (§2.3) asserts that every set $P \subseteq \mathcal{P}$ has a *unique* minimum upper bound precisely if \triangleright is *complete*. We extend our results to maximum lower bounds in appendix F.

1.2 Overview of the applications

We employ our theorems to answer a broad range of economic questions.

Application to monotone comparative statics (§3). In monotone comparative statics, an agent chooses an alternative from a set $\mathcal{X} \subseteq \mathbf{R}$. The canonical result states that if the agent’s preference $\succeq \in \mathcal{P}$ increases in the sense of single-crossing dominance S , then her optimal choices $X(\succeq) = \{x \in \mathcal{X} : x \succeq y \text{ for every } y \in \mathcal{X}\}$ increase in the the strong set order.

We first study collective choice: there is a group of agents, with preferences $P \subseteq \mathcal{P}$. We prove that when the set P increases in the strong set order, so does the *consensus*: the set $C(P) = \bigcap_{\succeq \in P} X(\succeq)$ of alternatives that every agent considers optimal. Generalising, we characterise comparative statics for the set $C_k(P)$ of alternatives that every individual considers at least k^{th} -best: it increases in the strong set order whenever P does also if $k = 2$ or (trivially) if $k = |\mathcal{X}|$, but can strictly *decrease* if $2 < k < |\mathcal{X}|$. The proofs of these results make extensive use of the existence and characterisation theorems.⁷

Secondly, we consider comparative-statics predictions by an analyst who knows only that the agent’s preference belongs to a set $P \subseteq \mathcal{P}$. The possibly-optimal choices from a menu $M \subseteq \mathcal{X}$ are $X_M(P) := \bigcup_{\succeq \in P} X_M(\succeq)$, where $X_M(\succeq)$ are the optimal choices for $\succeq \in \mathcal{P}$. Under a richness assumption on P , possible choices are sharply bounded by the choices of the minimum upper bound \succeq^* of P : $\max X_M(P) = \max X_M(\succeq^*)$ for any menu $M \subseteq \mathcal{X}$. The proof turns on the characterisation theorem, and the existence of \succeq^* is ensured by the existence theorem. Comparative statics follow: for a shift of the uncertainty P to increase $\min X_M(P)$ and $\max X_M(P)$ whatever the menu $M \subseteq \mathcal{X}$, it is necessary and sufficient that P ’s minimum upper bound and maximum lower bound both increase.

Application to uncertainty- and risk-aversion (§4). In the Savage framework, there are payoff-relevant *consequences* and possible *states of the world*, and a decision-maker has preferences over *acts*, meaning maps from states to consequences. Let \mathcal{X} be all acts, and \mathcal{P} all preferences over acts.

⁷Furthermore, the very possibility of comparing sets $P, P' \subseteq \mathcal{P}$ by the strong set order relies on the uniqueness proposition, since the strong set order is defined in terms of minimum upper bounds and maximum lower bounds.

One preference is called *more uncertainty-averse than* another iff whenever the latter (strictly) prefers an unambiguous act to some other act, so does the former. ‘Unambiguous act’ can mean a constant act, or more generally one that is measurable with respect to an (exogenously-given) collection of events deemed unambiguous. For this summary, assume that only constant acts are unambiguous, and that consequences are monetary prizes.

We consider preferences that are *monotone* (more money for sure is better than less money for sure) and *solvable* (every act has a certainty equivalent). For a monotone and solvable preference $\succeq \in \mathcal{P}$, write $e(\succeq, x)$ for the (by monotonicity, unique) certainty equivalent of an act $x \in \mathcal{X}$.

A *maxmin preference* is one that is represented by $x \mapsto \min_{\succeq \in P} e(\succeq, x)$ for some set $P \subseteq \mathcal{P}$ of monotone and solvable preferences. Intuitively, such a preference cautiously values each act $x \in \mathcal{X}$ according to its worst certainty equivalent among the preferences in P . Maxmin expected utility (Gilboa & Schmeidler, 1989) is the special case in which P comprises only expected-utility preferences with common risk attitude (but different beliefs).

We characterise maxmin preferences by proving that P is a maxmin representation of \succeq^* iff \succeq^* is a minimum upper bound of P with respect to ‘more uncertainty-averse than’. A comparative-statics result follows: when P increases in an appropriate sense, \succeq^* becomes more uncertainty-averse.

This result carries over to risk-aversion: *cautious* preferences over lotteries are precisely minimum upper bounds with respect to ‘more risk-averse than’.

Application to social choice (§5). A group of individuals, each with preferences over a set \mathcal{X} of social alternatives, must aggregate their preferences into a social preference. Two attractive normative principles are (*Pareto efficiency* and *liberalism*). Efficiency requires that whenever every individual (strictly) prefers one alternative to another, so does the social preference.

Liberalism means that individuals have rights: certain social decisions can be made only with the assent of every individual. (For example, increasing state surveillance may be permissible only with unanimous consent.) We formalise liberalism by supposing that for certain pairs $x, y \in \mathcal{X}$ of alternatives, a liberal society may rank y above x only with the consent of every individual. Write $x \triangleright y$ when this is the case, and call \triangleright a *liberal constraint*.

We study the compatibility of efficiency and liberalism. Our basic insight, established using the characterisation theorem, is that efficiency and liberalism together demand that the social preference over alternatives be a minimum upper bound of the set of individual preferences. It follows that efficiency and liberalism conflict unless the liberal constraint \triangleright is crown- and diamond-free, since otherwise, by the existence theorem, minimum upper

bounds fail to exist for some constellations of individual preferences. This is an ‘impossibility of a Paretian liberal’ result in the vein of Sen (1970).

But we also find good news. We prove an (im)possibility theorem that characterises all liberal constraints \succeq that admit an efficient and liberal social welfare function. The theorem’s necessary and sufficient condition is stringent (implying crown- and diamond-freeness), but not hopelessly so.

1.3 Related literature

Our work relates to the combinatorics literature on permutation lattices (e.g. Bennett & Birkhoff, 1994; Markowsky, 1994; Duquenne & Cherfouh, 1994). Here the alternatives are $\mathcal{X} = \{1, \dots, n\}$ for some $n \in \mathbf{N}$, and the partial order \succeq is the ordinary inequality. Thus \succeq is complete and \mathcal{X} is finite.

In this context, anti-symmetric (i.e. never-indifferent) preferences may be thought of as *permutations*, and single-crossing dominance is known as the *weak order* (or *permutohedron order*). It has been known since Guilbaud and Rosenstiehl (1963) and Yanagimoto and Okamoto (1969) that the set of all permutations equipped with the weak order is a (complete) lattice. Our uniqueness proposition is a result along these lines.

Since this literature assumes that \succeq is complete, it certainly contains no analogue of our existence theorem. We are not aware of any analogue of our characterisation theorem, either. Besides avoiding the restrictive assumption that \succeq is complete, we differ from this literature by allowing for preferences with indifferences and by permitting \mathcal{X} to be of unrestricted cardinality.

Crown- and diamond-freeness are standard concepts in combinatorics (e.g. Baker, Fishburn & Roberts, 1972; Griggs, Li & Lu, 2012; Lu, 2014). Ball, Pultr and Sichler (2006) show that crown- and diamond-freeness together are *nearly* equivalent to the absence of weak cycles from the Hasse diagram. This latter property appears in the probability literature, where it characterises those posets (\mathcal{X}, \succeq) on which every first-order stochastically increasing family of probability measures may be realised by an a.s. increasing process on \mathcal{X} (Fill & Machida, 2001). Brooks, Frankel and Kamenica (2021) prove the analogous result for second-order stochastic dominance, and apply this to beliefs and information structures.

2 Theory

In this section, we develop our general results about the lattice structure of single-crossing: our characterisation theorem (§2.1), our existence theorem

(§2.2) and our uniqueness proposition (§2.3). Recall from §1.1 the abstract environment and basic definitions.

2.1 Characterisation of minimum upper bounds

Our characterisation will be in terms of *P-chains*, defined as follows.

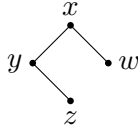
Definition 2. For a set $P \subseteq \mathcal{P}$ of preferences and two alternatives $x \succeq y$ in \mathcal{X} , a *P-chain* from x to y is a finite sequence $(w_k)_{k=1}^K$ in \mathcal{X} such that

- (i) $w_1 = x$ and $w_K = y$,
- (ii) for every $k < K$, $w_k \succeq w_{k+1}$, and
- (iii) for every $k < K$, $w_k \succ w_{k+1}$ for some $\succ \in P$.

A *strict P-chain* is a *P-chain* with $w_k \succ w_{k+1}$ for some $k < K$ and $\succ \in P$.

In a word, *P-chain* is a \succeq -decreasing sequence of alternatives along which, at each juncture (each $k < K$), some preference in P prefers the previous (\succeq -larger) alternative to the subsequent (\succeq -smaller) one. Clearly a *P-chain* of length $K \geq 3$ is simply the concatenation of $K - 1$ *P-chains* of length 2.

Example 1. Consider $\mathcal{X} = \{x, y, z, w\}$, with \succeq such that $x \triangleright w$ and $x \triangleright y \triangleright z$ (so $x \triangleright z$), and w, y and w, z are incomparable. The partial order \succeq may be depicted graphically as



In this (‘Hasse’) diagram, there is path from a down to b iff $a \succeq b$. We will use diagrams of this sort throughout.

Consider $P = \{\succeq_1, \succeq_2\}$, $z \succ_1 w \succ_1 x \succ_1 y$ and $y \succ_2 z \succ_2 w \succ_2 x$. By inspection, (x, y) and (y, z) are strict *P-chains*. Thus there is a strict *P-chain* from x to z , namely (x, y, z) . Note, however, that (x, z) is *not* a *P-chain*.

Although $x \succeq w$, there is no *P-chain* from x to w : the only candidate is (x, w) , and it fails to be a *P-chain* since neither preference favours x over w .

The following asserts that an upper bound of $P \subseteq \mathcal{P}$ is precisely a preference that (strictly) prefers a larger alternative to a smaller one whenever there is a (strict) *P-chain* between them:

UB characterisation lemma. For a preference $\succeq' \in \mathcal{P}$ and a set $P \subseteq \mathcal{P}$ of preferences, the following are equivalent:

- (1) \succeq' is an upper bound of P .
- (2) \succeq' satisfies: for any \succeq -comparable $x, y \in \mathcal{X}$, wlog $x \succeq y$,
 - (i) $x \succeq' y$ if there is a P -chain from x to y , and
 - (ii) $y \succeq' x$ only if there is no strict P -chain from x to y .

Example 1 (continued). The P -chains, all of them strict, are (x, y) , (y, z) and (x, y, z) . Thus by the UB characterisation lemma, a preference $\succeq' \in \mathcal{P}$ is an upper bound of P iff $x \succ' y \succ' z$ (and $x \succ' z$). Thus $x \succ'_a y \succ'_a z \succ'_a w$ and $w \succ'_b x \succ'_b y \succ'_b z$ are both upper bounds.

Proof. (2) implies (1): Let $\succeq' \in \mathcal{P}$ satisfy condition (2); we wish to show that $\succeq' \succeq \succeq$ for any $\succeq \in P$. To that end, fix a \succeq -comparable pair $x, y \in \mathcal{X}$, wlog $x \succeq y$, and suppose that $x \succeq(\succ) y$ for some $\succeq \in P$; we must show that $x \succeq'(\succ') y$. This is immediate since (x, y) is a (strict) P -chain.

(1) implies (2): Let \succeq' be an upper bound of P . Fix a \succeq -comparable pair $x, y \in \mathcal{X}$, wlog $x \succeq y$; we must show that if there is a (strict) P -chain from x to y , then $x \succeq'(\succ') y$.

Suppose that there exists a P -chain $(w_k)_{k=1}^K$ from x to y . For each $k < K$, we have $w_k \succeq w_{k+1}$ as well as $w_k \succ_k w_{k+1}$ for some $\succ_k \in P$. Because \succeq' is an upper bound of P , it must be that $w_k \succeq' w_{k+1}$ for each $k < K$. Since \succeq' is transitive (because it lives in \mathcal{P}), it follows that $x \succeq' y$.

Suppose there is a strict P -chain $(w_k)_{k=1}^K$ from x to y . As in the weak case, we must have $w_k \succeq' w_{k+1}$ for each $k < K$. Moreover, since the P -chain is strict, we have $w_k \succ w_{k+1}$ for some $k < K$ and $\succ \in P$; hence $w_k \succ' w_{k+1}$ since \succeq' is an upper bound of P . Thus $x \succ' y$ by the transitivity of \succeq' . ■

The UB characterisation lemma says that an upper bound must have a (strict) ‘upward’ preference whenever there is a (strict) P -chain. Our characterisation theorem says that the *minimum* upper bounds are those which have a (strict) ‘upward’ preference *only* when there is a (strict) P -chain:

Characterisation theorem. For a preference $\succeq^* \in \mathcal{P}$ and a set $P \subseteq \mathcal{P}$ of preferences, the following are equivalent:

- (1) \succeq^* is a minimum upper bound of P .
- (2) \succeq^* satisfies: for any \succeq -comparable $x, y \in \mathcal{X}$, wlog $x \succeq y$,

- (\star) $x \succ^* y$ iff there is a P -chain from x to y , and
- ($\star\star$) $y \succ^* x$ iff there is no strict P -chain from x to y .

The analogous result for maximum lower bounds is given in appendix F.

Example 1 (continued). A minimum upper bound $\succeq^* \in \mathcal{P}$ must satisfy $x \succ^* y \succ^* z$ (and $x \succ^* z$) since it is an upper bound. Since $x \succeq w$ but there is no P -chain from x to w , minimumhood requires that $w \succ^* x$ by the characterisation theorem. In sum, $\succeq^* \in \mathcal{P}$ is a minimum upper bound iff $w \succ^* x \succ^* y \succ^* z$.

One direction of the proof is straightforward:

Proof that (2) implies (1). Fix a subset P of \mathcal{P} and a $\succeq^* \in \mathcal{P}$ that satisfies (\star)–($\star\star$). It is immediate from the UB characterisation lemma that \succeq^* is an upper bound of P . To see that \succeq^* is a minimum of the upper bounds of P , let \succeq' be any upper bound of P . Fix a \succeq -comparable pair $x, y \in \mathcal{X}$, wlog $x \succeq y$, and suppose that $x \succeq^*(\succ^*) y$. By property (\star) (property ($\star\star$)), there must be a (strict) P -chain from x to y . Since \succeq' is an upper bound of P , it follows by the UB characterisation lemma that $x \succeq'(\succ') y$. Since $x, y \in \mathcal{X}$ were arbitrary, this establishes that $\succeq' S \succeq^*$. ■

The other direction relies the following lemma, whose proof is given in appendix C.

Lemma 1. Let P be a set of preferences, and let $x, y \in \mathcal{X}$ be \succeq -comparable, wlog $x \succeq y$. If there is no (strict) P -chain from x to y , then there exists an upper bound \succeq'' of P with $x \not\succeq''(\not\succeq'') y$.

Proof that (1) implies (2). Fix a set $P \subseteq \mathcal{P}$ of preferences and a preference $\succeq' \in \mathcal{P}$. We will prove the contra-positive: if \succeq' violates (\star)–($\star\star$), then it cannot be a minimum upper bound of P . If a preference \succeq' violates the ‘if’ part of (\star) or the ‘only if’ part of ($\star\star$), then it fails to be an upper bound of P by the UB characterisation lemma.

Suppose that a preference \succeq' violates the ‘only if’ part of (\star) (the ‘if’ part of ($\star\star$)): there are \succeq -comparable $x, y \in \mathcal{X}$, wlog $x \succeq y$, such that there is no (strict) P -chain from x to y , and yet $x \succeq'(\succ') y$. By Lemma 1, there is an upper bound \succeq'' of P such that $x \not\succeq''(\not\succeq'') y$. Then $\succeq'' \not\leq \succeq'$, so \succeq' fails to be a minimum of the upper bounds of P . ■

2.2 Existence of minimum upper bounds

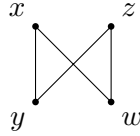
In this section, we provide a necessary and sufficient condition on \succeq for minimum upper bounds to exist for every set of preferences. This condition rules out two special subsets: *crowns* and *diamonds*.

In §2.2.1, we define crowns and show that \succeq must be free of them if every set of preferences is to possess a minimum upper bound. In §2.2.2, we do the same for diamonds. In §2.2.3, we give the existence theorem, which asserts in addition that crown- and diamond-freeness of \succeq is *sufficient* for existence.

2.2.1 Crowns

The following example shows how existence can fail.

Crown example. Consider $\mathcal{X} = \{x, y, z, w\}$ with the following order \succeq :



(That is, each of x and z \succeq -dominates each of y and w , but x, z and are \succeq -incomparable, as are y, w .) Let $P = \{\succeq_1, \succeq_2\} \subseteq \mathcal{P}$, where $w \succ_1 x \succ_1 y \succ_1 z$ and $y \succ_2 z \succ_2 w \succ_2 x$. We have $x \succeq y$ and $z \succeq w$, and there is a strict P -chain from x to y and from z to w . On the other hand, $x \succeq w$ and $z \succeq y$, but there is no P -chain from x to w or from z to y . So by the characterisation theorem, a minimum upper bound \succeq^* of P must have $x \succ^* y \succ^* z \succ^* w \succ^* x$. Such a \succeq^* cannot be transitive, so cannot live in \mathcal{P} . It follows that no minimum upper bound exists. (To illustrate, consider the preferences $\succeq', \succeq'' \in \mathcal{P}$ given by $x \succ' y \succ' z \succ' w$ and $z \succ'' w \succ'' x \succ'' y$. Both are upper bounds, but neither is minimum since $\succeq' \not\leq \succeq'' \not\leq \succeq'$.)

The problem is that \succeq features a *crown*, defined as follows.

Definition 3. Let \sqsubseteq be a binary relation on a set \mathcal{A} . For $K \geq 4$ even, a K -*crown* is a sequence $(a_k)_{k=1}^K$ in \mathcal{A} such that non-adjacent $a_k, a_{k'}$ are \sqsubseteq -incomparable, and $a_{k-1} \sqsubseteq a_k \sqsubseteq a_{k+1}$ for each $1 < k \leq K$ even (where $a_{K+1} := a_1$ by convention). A *crown* is a K -crown for some $K \geq 4$ even. The relation \sqsubseteq is *crown-free* iff it features no crowns.

Some crowns are drawn in Figure 1.

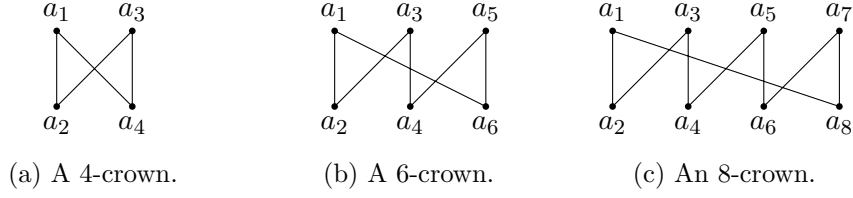
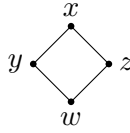


Figure 1: Crowns.

Crown-freeness rules out a specific form of incompleteness. A strong sufficient condition is completeness. A weaker sufficient condition is that the comparability relation \sqsubseteq be transitive.⁸ Neither is necessary:

Diamond example. Consider \mathcal{X} with the partial order \supseteq given by



(I.e. $x \supseteq y \supseteq w$ and $x \supseteq z \supseteq w$, but y, z are \supseteq -incomparable.) \supseteq is not complete since y, z are \supseteq -incomparable. Nor is comparability \boxtimes transitive, since $y \boxtimes x \boxtimes z$ but y, z are \supseteq -incomparable. But \supseteq is manifestly crown-free.

Lemma 2 (necessity of crown-freeness). If every pair of preferences possesses a minimum upper bound, then \supseteq is crown-free.

Proof. We prove the contra-positive. Suppose that \supseteq features a crown (x_1, \dots, x_K) . Consider $P = \{\succeq_a, \succeq_b\} \subseteq \mathcal{P}$, where

$$x_K \succ_a x_1 \succ_a x_2 \succ_a \cdots \succ_a x_{K-2} \succ_a x_{K-1}$$

$$x_2 \succ_b x_3 \succ_b \cdots \succ_b x_{K-1} \succ_b x_K \succ_b x_1.$$

For $1 < k \leq K$ even, $x_{k-1} \triangleright x_k$ and $x_{k-1} \succ x_k$ for some $\succeq \in P$ (in particular, \succeq_a for $k < K$ even, \succeq_b for $k > 1$ even). Hence (x_{k-1}, x_k) is a strict P -chain, so by the characterisation theorem, $x_{k-1} \succ^* x_k$ for any minimum upper bound \succeq^* of P .

Moreover, for $1 < k \leq K$ even, $x_{k+1} \triangleright x_k$, and $x_{k+1} \not\succeq x_k$ for all $\succeq \in P$. (This is apparent, separately, for $k < K$ even and for $k = K$.) Hence there is no P -chain from x_{k+1} to x_k , so by the characterisation theorem, $x_k \succ^* x_{k+1}$ for any minimum upper bound of \succeq^* of P .

⁸For a binary relation \sqsubseteq , comparability \sqsubseteq is defined by $a \sqsubseteq b$ iff either $a \sqsubseteq b$ or $b \sqsubseteq a$.

It follows that any minimum upper bound \succeq^* of P must satisfy $x_1 \succ^* x_2 \succ^* \dots \succ^* x_{K-1} \succ^* x_K \succ^* x_1$. Such a \succeq^* cannot be transitive, so cannot live in \mathcal{P} ; hence P admits no minimum upper bound. ■

2.2.2 Diamonds

Existence can fail even in the absence of crowns:

Diamond example (continued). Let $P = \{\succeq_1, \succeq_2\} \subseteq \mathcal{P}$, where $y \succ_1 w \succ_1 z \succ_1 x$ and $w \succ_2 z \succ_2 x \succ_2 y$. Evidently (x, y) and (y, w) are strict P -chains. We have $z \succeq w$ but no P -chain from z to w , and $x \succeq z$ but no P -chain from x to z . So by the characterisation theorem, a minimum upper bound \succeq^* of P must satisfy $x \succ^* y \succ^* w \succ^* z \succ^* x$. Since such a relation \succeq^* cannot be transitive, it follows that P admits no minimum upper bound.

The trouble is that \succeq features a *diamond*:

Definition 4. Let \sqsubseteq be a partial order on a set \mathcal{A} . A *diamond* is four elements (a, b, c, d) of \mathcal{A} such that $a \sqsubseteq b \sqsubseteq d$ and $a \sqsubseteq c \sqsubseteq d$, but b, c are \sqsubseteq -incomparable. The order \sqsubseteq is *diamond-free* iff it features no diamonds.

Diamond-freeness is strong, e.g. the only diamond-free lattices are chains.⁹ Like crown-freeness, diamond-freeness rules out a specific form of incompleteness, and is implied by strong forms of ‘limited incompleteness’ such as completeness of \sqsubseteq or transitivity of the comparability relation \sqsubseteq . Neither of these conditions is necessary for diamond-freeness, nor is crown-freeness:

Crown example (continued). By inspection, there are no diamonds. But there is a crown, and \succeq is not complete, nor is \boxtimes is transitive.

Lemma 3 (necessity of diamond-freeness). If every pair of preferences possesses a minimum upper bound, then \succeq is diamond-free.

The proof is almost exactly the diamond example, so we omit it.

2.2.3 The existence theorem

Our existence theorem asserts that crown- and diamond-freeness are not only necessary, but also sufficient, for the existence of minimum upper bounds.

Existence theorem. The following are equivalent:

⁹If (\mathcal{X}, \succeq) is a lattice but not a chain, then there are \succeq -incomparable $x, y \in \mathcal{X}$, in which case $(x \wedge y, x, y, x \vee y)$ is a diamond.

- (1) Every *set* of preferences has a minimum upper bound.
- (2) Every *pair* of preferences has a minimum upper bound.
- (3) \succeq is crown- and diamond-free.

By way of illustration, crown- and diamond-freeness fails in the crown and diamond examples, but is satisfied in Example 1. More generally, it holds whenever there are three or fewer alternatives, and fails for any lattice (\mathcal{X}, \succeq) that is not a chain (see footnote 9). Crown- and diamond-freeness is arguably a strong property, but it does hold in several applications (see §3–5).

In appendix F, we show further that these properties are equivalent to every set (or two-element set) of preferences possessing a maximum lower bound. By analogy with complete lattices, call (\mathcal{P}, S) a *pre-lattice* iff every two-element set $P \subseteq \mathcal{P}$ possesses a minimum upper bound and maximum lower bound, and a *complete pre-lattice* if this holds for every set $P \subseteq \mathcal{P}$.¹⁰ The elaborated existence theorem in appendix F implies that (\mathcal{P}, S) is a complete pre-lattice iff it is a pre-lattice iff \succeq features no crowns or diamonds.

As for the proof, it is immediate that (1) implies (2), and we have already established in Lemmata 2 and 3 that (2) implies (3). Proving that (3) implies (1) is more difficult; we do this in appendix D. The idea is as follows. Let \succeq° be the minimal binary relation (in general incomplete) that satisfies properties (\star) – $(\star\star)$ in the characterisation theorem. We show first that absent diamonds in \succeq , \succeq° must be ‘weakly transitive’. We then show that when there are no crowns in \succeq , weak transitivity of \succeq° implies that it satisfies a stronger transitivity-type property called *Suzumura consistency*. This permits us to invoke an extension theorem due to Richter (1966) and Suzumura (1976) to conclude that \succeq° may be extended to a complete and transitive relation (i.e. a preference). This preference is a minimum upper bound by the characterisation theorem.

2.3 Uniqueness of minimum upper bounds

When minimum upper bounds exist, they need not be unique:

Example 2. Consider $\mathcal{X} = \{x, y\}$ with the empty partial order \succeq , so that x, y are \succeq -incomparable. Let $P = \{\succeq_1, \succeq_2\} \subseteq \mathcal{P}$, where $x \succ_1 y$ and $y \succ_2 x$.

Since all alternatives are \succeq -incomparable, there are no P -chains. Conditions (\star) – $(\star\star)$ in the characterisation theorem are therefore (vacuously)

¹⁰ (\mathcal{P}, S) is not a complete lattice because S is a mere pre-order, i.e. it need not be anti-symmetric. The failure of anti-symmetry means that minimum upper bounds and maximum lower bounds need not be unique; we study this issue in §2.3.

satisfied by any preference. So by the characterisation theorem, every preference is a minimum upper bound of P .

The message of Example 2 is that preferences that disagree only on \succeq -incomparable pairs of alternatives S -dominate each other, leading to a multiplicity of minimum upper bounds. This is a general lesson:

Uniqueness proposition. The following are equivalent:

- (1) Every set of preferences has *at most* one minimum upper bound.
- (2) Every set of preferences has *exactly* one minimum upper bound.
- (3) \succeq is complete.

The analogue for maximum lower bounds is given in appendix F. Together, these two results imply that (\mathcal{P}, S) is a complete lattice iff \succeq is complete.

Proof that (3) implies (2). Suppose that \succeq is complete, and fix a set $P \subseteq \mathcal{P}$. Since completeness implies crown- and diamond-freeness, P has at least one minimum upper bound by the existence theorem.

To show uniqueness, let $\succeq', \succeq'' \in \mathcal{P}$ be minimum upper bounds of $P \subseteq \mathcal{P}$. Then by the characterisation theorem, \succeq' and \succeq'' must agree on all \succeq -comparable pairs of alternatives. Since \succeq is complete, it follows that \succeq' and \succeq'' agree on all pairs of alternatives, i.e. that they are identical. ■

It is immediate that (2) implies (1). Our proof that (1) implies (3) relies on the following lemma, proved in appendix E.

Lemma 4. Let $x, y \in \mathcal{X}$ be \succeq -incomparable. Then any set $P \subseteq \mathcal{P}$ has upper bounds $\succeq', \succeq'' \in \mathcal{P}$ such that $x \succ' y$ and $y \succ'' x$.

Proof that (1) implies (3). We prove the contra-positive. Suppose that \succeq is incomplete. Then the grand set \mathcal{P} has multiple upper bounds by Lemma 4, and clearly each of these is a minimum upper bound. ■

3 Application to monotone comparative statics

In this section, we use our theorems to extend the theory of monotone comparative statics¹¹ in two directions.

¹¹See e.g. Topkis (1978), Milgrom and Shannon (1994) and Quah and Strulovici (2009), and the textbook treatment by Topkis (1998).

In §3.2, we study the comparative statics of collective choice. We focus on the *consensus*: the set of alternatives that every individual in a group $P \subseteq \mathcal{P}$ considers optimal. We show that when P increases in the strong set order, so does the consensus. We go on fully to characterise the comparative-statics properties of the set of alternatives that each individual considers at least k^{th} -best: this set increases in the strong set order as P does also if $k = 2$ or (trivially) if $k = |\mathcal{X}|$, but can strictly *decrease* if $2 < k < |\mathcal{X}|$.

In §3.3, we consider the problem of an analyst who wishes to predict how an agent will choose from any given menu $M \subseteq \mathcal{X}$ of alternatives, but is uncertain about the agent's preference: all she knows is that it belongs to a set $P \subseteq \mathcal{P}$. We show that the minimum upper bound and maximum lower bound of P provide tight bounds on possible choice, and sharply characterise how possible choices vary with the analyst's uncertainty P .

3.1 The canonical theory

We begin with a brief recap of the canonical theory of monotone comparative statics. An agent chooses an alternative x from a set $\mathcal{X} \subseteq \mathbf{R}$ ordered by the usual inequality \geq . The agent chooses optimally with respect to her preference $\succeq \in \mathcal{P}$. Denote by $X(\succeq)$ the (possibly empty) set of optimal alternatives for preference $\succeq \in \mathcal{P}$:

$$X(\succeq) := \{x \in \mathcal{X} : x \succeq y \text{ for every } y \in \mathcal{X}\}.$$

Definition 5. Consider a lattice $(\mathcal{A}, \sqsupseteq)$ and two subsets $A, B \subseteq \mathcal{A}$. A dominates B in the $(\sqsupseteq\text{-induced})$ strong set order iff for any $a \in A$ and $b \in B$, we have $a \vee b \in A$ and $a \wedge b \in B$.

Note that A dominates B in the strong set order if either set is empty.

MCS theorem.¹² Let $\succeq, \succeq' \in \mathcal{P}$ be preferences. If $\succeq' S \succeq$, then $X(\succeq')$ dominates $X(\succeq)$ in the $(\geq\text{-induced})$ strong set order.

That is, when the agent's preference increases in the sense of S , the set of optimal alternatives increases in the sense of the strong set order.

Remark 1. The most general MCS theorem allows the set of actions to be any lattice (\mathcal{X}, \supseteq) . This added generality is not useful for our purposes because in order to apply the existence theorem, we shall require that \supseteq be crown- and diamond-free, and chains are the only diamond-free lattices.

¹²Due to Milgrom and Shannon (1994) and LiCalzi and Veinott (1992).

3.2 Comparative statics for collective choice

There is a group of agents, each with a preference $\succeq \in \mathcal{P}$. Write $P \subseteq \mathcal{P}$ for the set of preferences in the group. The *consensus* $C(P)$ is the set of alternatives that every individual in the group finds optimal: $C(P) := \bigcap_{\succeq \in P} X(\succeq)$.

Since \geq is complete, (\mathcal{P}, S) is a lattice by the uniqueness proposition. We may therefore use the (S -induced) strong set order to compare sets of preferences.

Proposition 1 (consensus comparative statics). Let \mathcal{X} be a subset of \mathbf{R} ordered by inequality \geq , and let $P, P' \subseteq \mathcal{P}$ be sets of preferences. If P' dominates P in the (S -induced) strong set order, then $C(P')$ dominates $C(P)$ in the (\geq -induced) strong set order.

In other words, when agents' preferences shift up in the sense of the strong set order, so does the consensus. Note that it may be that either $C(P)$ or $C(P')$ is empty, in which case the conclusion holds automatically.

Proof. Fix $P, P' \subseteq \mathcal{P}$ such that P' dominates P in the S -induced strong set order. The conclusion is immediate if either $C(P)$ or $C(P')$ is empty, so suppose not. Take $x \in C(P)$ and $x' \in C(P')$; we must show that $x \wedge x'$ lies in $C(P)$ and that $x \vee x'$ lies in $C(P')$. We will prove the former; the proof of the latter is similar.

Take any $\succeq \in P$ and $\succeq' \in P'$. Since the order \geq on \mathcal{X} is complete, the set $\{\succeq, \succeq'\}$ possesses a minimum upper bound \succeq^* by the existence theorem. Since P' dominates P in the S -induced strong set order, the minimum upper bound \succeq^* lies in P' . Because $\succeq^* S \succeq$, the MCS theorem implies that $X(\succeq^*)$ dominates $X(\succeq)$ in the \geq -induced strong set order. Since $x \in C(P) \subseteq X(\succeq)$ and $x' \in C(P') \subseteq X(\succeq^*)$, it follows that $x \wedge x' \in X(\succeq)$. Since $\succeq \in P$ was arbitrary, this shows that $x \wedge x' \in C(P)$. ■

Proposition 1 can be used to study comparative statics for social choice. A *preference profile* is an element $\pi = (\succeq_1, \dots, \succeq_k)$ of $\bigcup_{n \in \mathbf{N}} \mathcal{P}^n$, and its *support* is the set $\text{supp } \pi := \{\succeq_1, \dots, \succeq_k\}$ of all preferences represented in it. A *social choice function (SCF)* is a map $\phi : \bigcup_{n \in \mathbf{N}} \mathcal{P}^n \rightarrow \mathcal{X}$ that picks an alternative for each preference profile. It is *monotone* iff $\phi(\pi') \geq \phi(\pi)$ whenever $\text{supp } \pi'$ dominates $\text{supp } \pi$ in the (S -induced) strong set order, and *respects unanimity* iff $\phi(\pi) \in C(\text{supp } \pi)$ whenever the latter is non-empty. The following corollary of Proposition 1 is proved in appendix G:

Corollary 1. Let $\mathcal{X} \subseteq \mathbf{R}$ be compact. Then there exists a monotone SCF that respects unanimity.

When the consensus is empty, it is natural to consider individuals' *second-favourite* alternatives. Let the *second consensus* be $C_2(P) := \bigcap_{\succeq \in P} X_2(\succeq)$, where $X_2(\succeq)$ are the alternatives that $\succeq \in \mathcal{P}$ considers at least second-best:

$$X_2(\succeq) = \{x \in \mathcal{X} : y \succ x \text{ for at most one } y \in \mathcal{X}\}.$$

Proposition 2 (second consensus comparative statics). Let \mathcal{X} be a subset of \mathbf{R} ordered by inequality \geq , and let $P, P' \subseteq \mathcal{P}$ be sets of preferences. If P' dominates P in the (S -induced) strong set order and $C(P) = C(P')$, then $C_2(P') \setminus C(P')$ dominates $C_2(P) \setminus C(P)$ in the (\geq -induced) strong set order.

In particular, the second consensus increases ($C_2(P')$ dominates $C_2(P)$ in the strong set order) whenever the consensus is empty ($C(P) = \emptyset = C(P')$). Outside of that case, however, the second consensus need not increase.¹³

The proof is in appendix H. The argument is rather intricate, and makes extensive use of minimum upper bounds and maximum lower bounds.

Surprisingly, for $2 < k < |\mathcal{X}|$, there is no analogous comparative-statics result for the set $C_k(P)$ of alternatives that are at least k^{th} -best according to every $\succeq \in P$. The following counter-example is for $k = 3$ and $|\mathcal{X}| = 4$, but extends easily to any k and $|\mathcal{X}|$ such that $2 < k < |\mathcal{X}|$.

Example 3. Let $\mathcal{X} = \{1, 2, 3, 4\}$. For any labelling $\{x, y, z, w\} = \mathcal{X}$, write ' $xyzw$ ' for the preference $\succeq \in \mathcal{P}$ satisfying $x \succ y \succ z \succ w$. Let

$$P := \{1234, 1324, 1342, 2134, 2314, 2341, 3214, 3241, 3421\} \quad \text{and} \\ P' := \{4321, 4231, 4213, 3421, 3241, 2341, 3214, 2314, 2134\}.$$

Then P' dominates P in the (S -induced) strong set order,¹⁴ and $C(P) = C(P') = C_2(P) = C_2(P') = \emptyset$, but $C_3(P) = \{3\}$ and $C_3(P') = \{2\}$.

Remark 2. The results of this section remain true if agents are constrained to choose from a (possibly small) menu $M \subseteq \mathcal{X}$, provided the menu is *order-convex* in the sense that $x, z \in M$, $y \in \mathcal{X}$ and $x \leq y \leq z$ imply $y \in M$. For example, Proposition 1 generalises as follows: under the same hypothesis, for any non-empty and order-convex $M \subseteq \mathcal{X}$, $C_M(P')$ dominates $C_M(P)$ in the (\geq -induced) strong set order, where $C_M(P) := \bigcap_{\succeq \in P} X_M(\succeq)$ and $X_M(\succeq) := \{x \in M : x \succeq y \text{ for every } y \in M\}$. To see why, write \succeq_M for the restriction of a preference $\succeq \in \mathcal{P}$ to M ,¹⁵ and for any $P \subseteq \mathcal{P}$

¹³If $\mathcal{X} = \{1, 2, 3\}$, $P = \{\succeq\}$ where $1 \succ 2 \succ 3$ and $P' = \{\succeq, \succeq'\}$ where $1 \succ' 3 \succ' 2$, then $C(P) = \{1\} = C(P')$, $C_2(P) = \{1, 2\}$ and $C_2(P') = \{1\}$.

¹⁴Moreover, P is a *sublattice* (it dominates itself in the strong set order), and so is P' .

¹⁵That is, \succeq_M is the binary relation on M such that for any $x, y \in M$, $x \succeq_M y$ iff $x \succeq y$.

let $P_M := \{\sqsupseteq : \sqsupseteq = \succeq_M \text{ for some } \succeq \in P\}$. Since M is order-convex, the hypothesis of Proposition 1 implies that P'_M dominates P_M in the strong set order. Thus applying Proposition 1 (with M in place of \mathcal{X}) yields that

$$\begin{aligned} \{x \in M : x \sqsupseteq y \text{ for all } y \in M \text{ and } \sqsupseteq \in P'_M\} & \text{ dominates} \\ \{x \in M : x \sqsupseteq y \text{ for all } y \in M \text{ and } \sqsupseteq \in P_M\} & \text{ in the strong set order.} \end{aligned}$$

By inspection, the former set equals $C_M(P')$, and the latter is $C_M(P)$.

3.3 Robust comparative statics

Consider an analyst who knows only that the agent's preference belongs to a set $P \subseteq \mathcal{P}$, and wishes to predict choice across menus $M \subseteq \mathcal{X}$. For a non-empty menu $M \subseteq \mathcal{X}$ of alternatives, the agent's possible choices are $X_M(P) := \bigcup_{\succeq \in P} X_M(\succeq)$, where

$$X_M(\succeq) := \{x \in M : x \succeq y \text{ for every } y \in M\}$$

are the \succeq -best alternatives in the menu M . The analyst seeks to bound the agent's choices across menus M , and to predict how these choices vary with the uncertainty P . We focus on *rich* uncertainty, formalised as follows.

Definition 6. A set $P \subseteq \mathcal{P}$ is *rich* iff for any $x_0, x_1, \dots, x_K \in \mathcal{X}$, if there are preferences $\succeq_1, \dots, \succeq_K \in P$ such that $x_0 \succeq_1 x_1 \succeq_2 x_2 \succeq_3 \dots \succeq_K x_K$, then there is a preference $\succeq \in P$ such that $x_0 \succeq x_k$ for every $k \in \{1, \dots, K\}$.

Since \succeq is complete, the existence theorem guarantees that the (arbitrary) set $P \subseteq \mathcal{P}$ of preferences has a minimum upper bound and a maximum lower bound. By the uniqueness proposition, they are unique.

Proposition 3. Let \mathcal{X} be a finite subset of \mathbf{R} ordered by inequality \succeq , and let $P \subseteq \mathcal{P}$ be non-empty and rich. Then $\max X_M(P) = \max X_M(\succeq^*)$ and $\min X_M(P) = \min X_M(\succeq_*)$ for every non-empty menu $M \subseteq \mathcal{X}$, where \succeq^* is the minimum upper bound of P and \succeq_* is the maximum lower bound.

In other words, the choices made by \succeq^* and \succeq_* are tight bounds on the set of possible choices $X_M(P)$, across all non-empty menus $M \subseteq \mathcal{X}$.

Proof. We prove the claim about maxima, omitting the symmetric argument for minima. Fix a non-empty menu $M \subseteq \mathcal{X}$. We first show that $\max X_M(P) \leq \max X_M(\succeq^*)$. For every $\succeq \in P$, we have $\succeq^* \succeq \succeq$ since \succeq^* is an upper bound of P , so that $X_M(\succeq^*)$ dominates $X_M(\succeq)$ in the \succeq -induced strong set order

by the MCS theorem, implying that $\max X_M(\succeq^*) \geq \max X_M(\succeq)$. Hence $\max X_M(\succeq^*) \geq \max \bigcup_{\succeq \in P} X_M(\succeq) = \max X_M(P)$.

For the reverse inequality, it suffices to exhibit a preference $\succeq \in P$ such that $\max X_M(\succeq^*) \in X_M(\succeq)$. To that end, enumerate the elements of the menu as $M = \{x_0, \dots, x_N\}$ where for each $n < N$, we have either $x_n \succ^* x_{n+1}$ or $x_n \succeq^* x_{n+1} \succeq^* x_n$ and $x_n > x_{n+1}$. Evidently $x_0 = \max X_M(\succeq^*)$.

Claim. For each $n < N$, P contains a preference \succeq_n such that $x_n \succeq_n x_{n+1}$.

Proof of the claim. Fix an $n < N$. Suppose first that $x_{n+1} > x_n$. Then by the characterisation theorem, there is no strict P -chain from x_{n+1} to x_n ; in particular, (x_{n+1}, x_n) is not a strict P -chain. Thus we may choose \succeq_n to be any preference in P , since all of them prefer x_{n+1} to x_n .

Suppose instead that $x_n > x_{n+1}$. Then by the characterisation theorem, there must be a P -chain $(y^k)_{k=1}^K$ from x_n to x_{n+1} , so $x_n = y^0 \succeq^1 y^1 \succeq^2 \dots \succeq^K y^K = x_{n+1}$ for some preferences $\succeq^1, \dots, \succeq^K \in P$. Since P is rich, it follows that there is a $\succeq_n \in P$ such that $x_n \succeq_n x_{n+1}$. \square

Since $n < N$ was arbitrary, we have shown that there are preferences $\succeq_1, \dots, \succeq_N \in P$ such that $x_0 \succeq_1 x_1 \succeq_2 x_2 \succeq_3 \dots \succeq_N x_N$. Thus since P is rich, it contains a preference \succeq such that $x_0 \succeq x_n$ for every $n \in \{1, \dots, N\}$, which is to say that $x_0 \in X_M(\succeq)$. \blacksquare

Proposition 3 delivers a characterisation of comparative statics:

Corollary 2 (robust comparative statics). Let \mathcal{X} be a finite subset of \mathbf{R} ordered by inequality \geq , and let $P, P' \subseteq \mathcal{P}$ be non-empty and rich. Write \succeq^* (\succeq'^*) for the minimum upper bound of P (of P'), and \succeq_* (\succeq'_*) for the maximum lower bound. The following are equivalent:

- (1) $\succeq'^* S \succeq^*$ and $\succeq'_* S \succeq_*$.
- (2) $\max X_M(P') \geq \max X_M(P)$ and $\min X_M(P') \geq \min X_M(P)$ for every non-empty menu $M \subseteq \mathcal{X}$.

In other words, the shifts of the analyst's uncertainty P which lead her to predict higher choices from every menu are exactly those that increase the minimum upper bound and maximum lower bound.

Proof. If $\succeq'^* S \succeq^*$ and $\succeq'_* S \succeq_*$, then for any non-empty $M \subseteq \mathcal{X}$, $X_M(\succeq'^*)$ dominates $X_M(\succeq^*)$ and $X_M(\succeq'_*)$ dominates $X_M(\succeq_*)$ in the \geq -induced strong set order by the MCS theorem, whence

$$\max X_M(P') = \max X_M(\succeq'^*) \geq \max X_M(\succeq^*) = \max X_M(P)$$

by Proposition 3, and similarly $\min X_M(P') \geq \min X_M(P)$.

For the converse, suppose that $\succeq^{*'} \not\mathcal{S} \succeq^*$ or $\succeq'_* \not\mathcal{S} \succeq_*$; without loss of generality, the former. Then by definition of S , there is a binary menu $M = \{x, y\} \subseteq \mathcal{X}$ with $x > y$ such that either $X_M(\succeq^{*'}) \not\ni x \in X_M(\succeq^*)$ or $X_M(\succeq^{*'}) \ni y \notin X_M(\succeq^*)$. So by Proposition 3, we have either

$$\max X_M(P') = \max X_M(\succeq^{*'}) = y < x = \max X_M(\succeq^*) = \max X_M(P)$$

or (similarly) $\min X_M(P') < \min X_M(P)$. ■

Our analysis has focussed on rich preference uncertainty $P \subseteq \mathcal{P}$. The following refinement of Proposition 3 shows that richness is indispensable: without it, choice cannot be bounded across menus, whether by \succeq^* and \succeq_* or by any other pair of preferences.

Proposition 3*. For a finite set \mathcal{X} of alternatives and a non-empty set $P \subseteq \mathcal{P}$ of preferences over \mathcal{X} , the following are equivalent:

- (1) P is rich.
- (2) For any total order \succeq on \mathcal{X} , there is a preference $\succeq \in \mathcal{P}$ such that $\max X_M(P) = \max X_M(\succeq)$ for every non-empty $M \subseteq \mathcal{X}$.
- (3) For any total order \succeq on \mathcal{X} , $\max X_M(P) = \max X_M(\succeq^*)$ for every non-empty $M \subseteq \mathcal{X}$, where \succeq^* is the minimum upper bound of P .

The proof is in appendix I. (1)–(3) are also equivalent to the analogues of (2) and (3) for $\min X_M(\cdot)$ and the maximum lower bound \succeq_* of P .

4 Application to uncertainty- and risk-aversion

In this section, we apply our results to uncertainty-aversion.¹⁶ We identify a natural compactness condition under which ‘maxmin’ preferences are precisely minimum upper bounds with respect to ‘more uncertainty-averse than’, and without which the equivalence generally fails.

In §4.2, we introduce general *maxmin preferences*, which nest maxmin expected utility (Gilboa & Schmeidler, 1989). We then (§4.3) use our theorems to characterise maxmin preferences as minimum upper bounds, and deduce uncertainty-aversion comparative statics for maxmin preferences. Finally, in §4.4, we translate these insights to risk-aversion in choice among lotteries.

¹⁶See e.g. Ellsberg (1961), Schmeidler (1989), Gilboa and Schmeidler (1989), Klibanoff, Marinacci and Mukerji (2005) and Maccheroni, Marinacci and Rustichini (2006).

4.1 Environment

There is a state space $(\mathcal{S}, \mathcal{E})$ comprising a non-empty set \mathcal{S} of *states of the world* and a σ -algebra \mathcal{E} of subsets of \mathcal{S} , whose members are called *events*. There is also a non-empty set \mathcal{C} of payoff-relevant *consequences*. A (*Savage*) *act* is a finite-ranged \mathcal{E} -measurable map $\mathcal{S} \rightarrow \mathcal{C}$.

Following Epstein (1999), we suppose that there is an exogenously-given collection $\mathcal{E}^\circ \subseteq \mathcal{E}$ of *unambiguous* events. The unambiguous events are those to which a decision-maker is able to assign probabilities; they are ‘understood’. The collection \mathcal{E}° is assumed to contain the universal event \mathcal{S} and to be closed under complementation and countable disjoint union.¹⁷ An *unambiguous act* is one that is \mathcal{E}° -measurable; all other acts are called *ambiguous*.

Let \mathcal{X} be the set of all acts, with typical elements $x, y \in \mathcal{X}$. Write $\mathcal{X}^\circ \subseteq \mathcal{X}$ for those that are unambiguous. Let \mathcal{P} be all preferences over \mathcal{X} .

Definition 7. For two preferences $\succeq, \succeq' \in \mathcal{P}$, we say that \succeq' is *more uncertainty-averse than* \succeq iff for any unambiguous act $x^\circ \in \mathcal{X}^\circ$ and any act $x \in \mathcal{X}$, $x^\circ \succeq(\succ) x$ implies $x^\circ \succeq'(\succ') x$.

This definition is from Epstein (1999). Ghirardato and Marinacci (2002) studied the case in which only trivial events are unambiguous ($\mathcal{E}^\circ = \{\mathcal{S}, \emptyset\}$), so that the unambiguous acts are precisely the constant ones.

Given the interpretation of the ‘unambiguous’ events \mathcal{E}° as those to which a decision-maker is able to assign probabilities, it only makes sense to consider preferences that are consistent with a probabilistic belief about \mathcal{E}° . That is, we must restrict attention to the set \mathcal{P}° of preferences $\succeq \in \mathcal{P}$ that are *probabilistically sophisticated* on \mathcal{X}° , meaning that their restriction $\succeq|_{\mathcal{X}^\circ}$ to the unambiguous acts \mathcal{X}° is ordinally represented by $x \mapsto U(\mu \circ x^{-1})$ for some map $U : \Delta(\mathcal{C}) \rightarrow \mathbf{R}$ and some probability measure $\mu : \mathcal{E}^\circ \rightarrow [0, 1]$,¹⁸ where $\Delta(\mathcal{C})$ denotes the set of finite-support lotteries over \mathcal{C} and $\mu \circ x^{-1} \in \Delta(\mathcal{C})$ is the pushforward lottery: $(\mu \circ x^{-1})(C) := \mu(\{s \in \mathcal{S} : x(s) \in C\})$ for each finite $C \subseteq \mathcal{C}$. The interpretation is that the decision-maker has a probabilistic *belief* μ about the state and cares only about the distribution of consequences, so evaluates each act x purely on the basis of its induced (subjective) lottery $\mu \circ x^{-1}$ over consequences. The shape of U captures *risk attitude*; subjective expected utility is the special case in which U is linear.

An *unambiguous equivalent* for $\succeq \in \mathcal{P}^\circ$ of an act $x \in \mathcal{X}$ is an unambiguous act $x^\circ \in \mathcal{X}^\circ$ such that $x \succeq x^\circ \succeq x$. A preference is called *solvable* iff it has an

¹⁷ $\mathcal{S} \setminus E \in \mathcal{E}^\circ$ for any $E \in \mathcal{E}^\circ$, and $\bigcup_{n=1}^\infty E_n \in \mathcal{E}^\circ$ for any pairwise disjoint $(E_n)_{n=1}^\infty \subseteq \mathcal{E}^\circ$.

¹⁸A relation \sqsupseteq on a set \mathcal{A} is *ordinally represented* by $f : \mathcal{A} \rightarrow \mathbf{R}$ iff for any $a, b \in \mathcal{A}$, $a \sqsupseteq b$ iff $f(a) \geq f(b)$.

unambiguous equivalent for every act. When considering solvable preferences, it is natural (though not necessary) to assume that either the space \mathcal{C} of consequences or the collection \mathcal{E}° of unambiguous events is ‘rich’, since otherwise solvability is stringent. Arbitrarily fix a map $e : \mathcal{P}^\circ \times \mathcal{X} \rightarrow \mathcal{X}^\circ$ such that $e(\succeq, x)$ is an unambiguous equivalent for \succeq of x for each solvable preference $\succeq \in \mathcal{P}^\circ$ and each act $x \in \mathcal{X}$, with $e(\succeq, x) = x$ in case $x \in \mathcal{X}^\circ$.

Let $\mathcal{P}^{\mu, U}$ be the set of all solvable preferences $\succeq \in \mathcal{P}^\circ$ such that $\succeq|_{\mathcal{X}^\circ}$ has belief μ and risk attitude U . Each preference $\succeq \in \mathcal{P}^{\mu, U}$ may be viewed as a map $\mathcal{X} \rightarrow \mathbf{R}$, viz. the canonical utility representation $x \mapsto U(\mu \circ e(\succeq, x)^{-1})$, and so we may speak of *pointwise compact* sets $P \subseteq \mathcal{P}^{\mu, U}$ of preferences.¹⁹

4.2 Maxmin preferences

Definition 8. Given a belief $\mu : \mathcal{E}^\circ \rightarrow [0, 1]$ and a risk attitude $U : \Delta(\mathcal{C}) \rightarrow \mathbf{R}$, a pointwise compact set $P \subseteq \mathcal{P}^{\mu, U}$ is a *maxmin representation* of a preference $\succeq^* \in \mathcal{P}$ iff $x \mapsto \min_{\succeq \in P} U(\mu \circ e(\succeq, x)^{-1})$ ordinally represents \succeq^* .

Intuitively, such a decision-maker is unsure which preference $\succeq \in P$ to evaluate acts according to, so cautiously values acts at their worst unambiguous equivalent among $\succeq \in P$. An alternative interpretation is that there is a group of agents with preferences P , and that collective decisions are made according to the ‘Rawlsian’ maxmin criterion.

In case no non-trivial event is unambiguous ($\mathcal{E}^\circ = \{\mathcal{S}, \emptyset\}$), unambiguous acts are precisely constants acts, and thus unambiguous equivalents are *certainty equivalents*; then after we identify each consequence $c \in \mathcal{C}$ with the act $x_c \in \mathcal{X}^\circ$ constant at c and with the lottery $\delta_c \in \Delta(\mathcal{C})$ degenerate at c , a maxmin representation becomes $x \mapsto \min_{\succeq \in P} U(e(\succeq, x))$. If in addition consequences are monetary prizes ($\mathcal{C} \subseteq \mathbf{R}$), then provided U is strictly increasing, we may assume without loss that it is the identity $c \mapsto c$, so that a maxmin representation is simply $x \mapsto \min_{\succeq \in P} e(\succeq, x)$.

A special case of a maxmin representation is *maxmin expected utility* (Gilboa & Schmeidler, 1989), where P consists entirely of preferences that are probabilistically sophisticated (across all acts \mathcal{X} , not just the unambiguous ones) with a common, linear risk attitude. Precisely: each $\succeq \in P$ is ordinally represented by $x \mapsto \int_{\mathcal{C}} u d(\mu_{\succeq} \circ x^{-1}) = \int_{\mathcal{S}} (u \circ x) d\mu_{\succeq}$ for some belief $\mu_{\succeq} : \mathcal{E} \rightarrow [0, 1]$, where $u : \mathcal{C} \rightarrow \mathbf{R}$ is bounded and $\{\mu_{\succeq} : \succeq \in P\}$ is closed in the product topology.²⁰

¹⁹Recall that a set \mathcal{F} of maps $\mathcal{A} \rightarrow \mathbf{R}$ is called *pointwise compact* exactly if $\{f(a) : f \in \mathcal{F}\}$ is a compact subset of \mathbf{R} for every $a \in \mathcal{A}$.

²⁰Then P is pointwise compact: for any act $x \in \mathcal{X}$, the set $\{\int_{\mathcal{S}} (u \circ x) d\mu_{\succeq} : \succeq \in P\}$ is

4.3 Characterisation of maxmin preferences

Proposition 4 (maxmin characterisation). Fix a belief $\mu : \mathcal{E}^\circ \rightarrow [0, 1]$ and a risk attitude $U : \Delta(\mathcal{C}) \rightarrow \mathbf{R}$. For a pointwise compact set $P \subseteq \mathcal{P}^{\mu, U}$ and a preference $\succeq^* \in \mathcal{P}$, the following are equivalent:

- (1) P is a maxmin representation of \succeq^* .
- (2) \succeq^* is a minimum upper bound of P with respect to ‘more uncertainty-averse than’.

Proposition 4 furnishes an intuitive interpretation of maxmin preferences: a preference with maxmin representation P is precisely one that is more uncertainty-averse than any preference in P , but no more uncertainty-averse than that. It also implies comparative statics for uncertainty-aversion:

Corollary 3 (maxmin comparative statics). Fix a belief $\mu : \mathcal{E}^\circ \rightarrow [0, 1]$ and a risk attitude $U : \Delta(\mathcal{C}) \rightarrow \mathbf{R}$, and let $\succeq, \succeq' \in \mathcal{P}$ admit maxmin representations $P, P' \subseteq \mathcal{P}^{\mu, U}$. If P' contains P , or if P' dominates P in the strong set order,²¹ then \succeq' is more uncertainty-averse than \succeq .

Another consequence of Proposition 4 is that every pointwise compact set $P \subseteq \mathcal{P}^{\mu, U}$ possesses exactly one minimum upper bound with respect to ‘more uncertainty-averse than’. Existence can actually be proved directly using the existence theorem, without assuming solvability or pointwise compactness. Let \succeq° be the preference on \mathcal{X}° represented by $x \mapsto U(\mu \circ x^{-1})$, and define a binary relation \supseteq on \mathcal{X} by $x \supseteq y$ iff either (i) $x = y$, (ii) $x \in \mathcal{X}^\circ \not\supseteq y$ or (iii) $x, y \in \mathcal{X}^\circ$ and $x \succeq^\circ y$. Minimum upper bounds with respect to ‘more uncertainty-averse than’ are precisely minimum upper bounds with respect to the single-crossing dominance relation S induced by \supseteq .²² And clearly \supseteq is crown- and diamond-free; thus we may invoke the existence theorem.²³

Proof of Proposition 4. Let \succeq° and \supseteq be as the previous paragraph. Fix a pointwise compact set $P \subseteq \mathcal{P}^{\mu, U}$ and a preference $\succeq^* \in \mathcal{P}$.

Suppose that P is a maxmin representation of \succeq^* . By the characterisation theorem, it suffices to show that for all $x \triangleright y$ in \mathcal{X} , $x \succeq^*(\succ^*) y$ iff there is

bounded since u is, and is closed because $\int_{\mathcal{S}}(u \circ x)d\mu_n \rightarrow \int_{\mathcal{S}}(u \circ x)d\mu$ whenever $\mu_n \rightarrow \mu$ in the product topology, since $u \circ x$ is bounded.

²¹The strong set order was defined in §3.1.

²²If $\succeq'|_{\mathcal{X}^\circ} = \succeq^\circ$, then \succeq' is more uncertainty-averse than \succeq iff $\succeq' S \succeq$. And clearly $\succeq'|_{\mathcal{X}^\circ} = \succeq^\circ$ for any upper bound \succeq' of P with respect to ‘more uncertainty-averse than’.

²³A detail: \supseteq need not be a partial order, as it may fail to be anti-symmetric on \mathcal{X}° . But our theorems apply to the modified set of alternatives in which each \succeq° -equivalence class of unambiguous acts is treated as a single alternative.

a (strict) P -chain from x to y . This holds when $x \notin \mathcal{X}^\circ$ since then x, y are \triangleright -incomparable, and holds for $x, y \in \mathcal{X}^\circ$ since $\triangleright|_{\mathcal{X}^\circ} = \succeq^*|_{\mathcal{X}^\circ} = \succeq|_{\mathcal{X}^\circ} = \succeq^\circ$ for every $\succeq \in P$. For $x \in \mathcal{X}^\circ \not\preceq y$, there is a (strict) P -chain from x to y iff $x \succeq(\succeq) y$ for some $\succeq \in P$ iff $x \succeq^\circ(\succeq^\circ) e(\succeq, y)$ for some $\succeq \in P$ iff $U(\mu \circ x^{-1}) \geq (>) \min_{\succeq \in P} U(\mu \circ e(\succeq, y)^{-1})$ iff $x \succeq^*(\succeq^*) y$.

For the converse, suppose that \succeq^* is a minimum upper bound of P .

Claim. \succeq^* is solvable. (And hence belongs to $\mathcal{P}^{\mu, U}$.)

Proof of the claim. Fix an ambiguous act $x \in \mathcal{X} \setminus \mathcal{X}^\circ$; we seek an unambiguous equivalent $y \in \mathcal{X}^\circ$. Since P is pointwise compact, there is a $\succeq' \in P$ such that $y := e(\succeq', x)$ satisfies $U(\mu \circ y^{-1}) = \min_{\succeq \in P} U(\mu \circ e(\succeq, x)^{-1})$. Clearly $y \triangleright x$. We have $y \succeq^* x$ since $y \succeq' x$ and \succeq^* is an upper bound of $P \ni \succeq'$.

Choose $\succeq'' \in \mathcal{P}$ such that P is a maxmin representation of \succeq'' . Then $x \succeq'' y$ (and $y \succeq'' x$) by definition of \succeq' . As shown above, \succeq'' must be an upper bound of P ; thus $\succeq'' S \succeq^*$. It follows that $x \succeq^* y$. \square

Now, since \succeq^* is solvable and a minimum upper bound, $e(\succeq, \cdot) \succeq^\circ e(\succeq^*, \cdot)$ for every $\succeq \in \mathcal{P}$, and $e(\succeq^*, \cdot) \succeq^\circ e(\succeq', \cdot)$ for every $\succeq' \in \mathcal{P}$ such that $e(\succeq, \cdot) \succeq^\circ e(\succeq', \cdot)$ for every $\succeq \in \mathcal{P}$. Equivalently, $x \mapsto U(\mu \circ e(\succeq^*, x)^{-1})$ is the pointwise greatest map $\mathcal{X} \rightarrow \mathbf{R}$ that is pointwise smaller than $x \mapsto U(\mu \circ e(\succeq, x)^{-1})$ for every $\succeq \in P$; in other words, it is $x \mapsto \min_{\succeq \in P} U(\mu \circ e(\succeq, x)^{-1})$. \blacksquare

There is one subtlety in the (perhaps) intuitive equivalence asserted by Proposition 4: the role of pointwise compactness. The equivalence generally fails when P is not pointwise compact. Firstly, P can be a maxmin representation of a solvable preference, and yet admit no solvable minimum upper bound; then (1) \implies (2) fails, and (2) \implies (1) fails since P has a minimum upper bound by the existence theorem (as argued above).²⁴ Secondly, P can admit several minimum upper bounds; then (2) \implies (1) fails.²⁵

4.4 Risk-aversion and caution

In this section, we translate to lotteries, characterising *cautious* preferences as minimum upper bounds with respect to ‘more risk-averse than’.

There is a set \mathcal{C} of consequences. $\mathcal{X} := \Delta(\mathcal{C})$ denotes all lotteries over consequences, and \mathcal{P} is the set of all preferences over \mathcal{X} . We identify each degenerate lottery with the consequence that it delivers.

²⁴Example: $\mathcal{E}^\circ = \{\mathcal{S}, \emptyset\}$, $\mathcal{C} = [0, 1]$, $U(x) = x$ for each $x \in \mathcal{C}$, and for some fixed $y \in \mathcal{X} \setminus \mathcal{C}$, $P := \{\succeq_\varepsilon : \varepsilon \in (0, 1]\}$, where $\varepsilon \succeq_\varepsilon y \succ_\varepsilon 0$ for each $\varepsilon \in (0, 1]$. Here P is not pointwise closed.

²⁵Example: $\mathcal{E}^\circ = \{\mathcal{S}, \emptyset\}$, $\mathcal{C} = \mathbf{N}$, $U(x) = -x$ for each $x \in \mathcal{C}$, and $P := \{\succeq_n : n \in \mathbf{N}\}$, where each \succeq_n satisfies $e(\succeq_n, x) = n$ for all $x \in \mathcal{X} \setminus \mathcal{C}$. Here P is not pointwise bounded.

Definition 9 (Yaari, 1969). For two preferences $\succeq, \succeq' \in \mathcal{P}$, we say that \succeq' is *more risk-averse than* \succeq iff for any consequence $c \in \mathcal{C}$ and any lottery $x \in \mathcal{X}$, $c \succeq(\succ) x$ implies $c \succeq'(\succ') x$.

Fix a function $e : \mathcal{P} \times \mathcal{X} \rightarrow \mathcal{C}$ such that $e(\succeq, x)$ is a certainty equivalent for \succeq of x for each solvable $\succeq \in \mathcal{P}$ and each $x \in \mathcal{X}$, with $e(\succeq, x) = x$ if $x \in \mathcal{X}^\circ$. Given any $u : \mathcal{C} \rightarrow \mathbf{R}$, let \mathcal{P}^u be all solvable preferences $\succeq \in \mathcal{P}$ whose restriction $\succeq|_{\mathcal{C}}$ to consequences is ordinally represented by u . Since each $\succeq \in \mathcal{P}^u$ may be viewed as a map $\mathcal{X} \rightarrow \mathbf{R}$, namely its canonical representation $x \mapsto u(e(\succeq, x))$, we may speak of pointwise compact sets $P \subseteq \mathcal{P}^u$.

Definition 10. Given a utility function $u : \mathcal{C} \rightarrow \mathbf{R}$ over consequences, a pointwise compact set $P \subseteq \mathcal{P}^u$ is a *cautious representation* of a preference $\succeq^* \in \mathcal{P}$ iff $x \mapsto \min_{\succeq \in P} u(e(\succeq, x))$ ordinally represents \succeq^* .

If consequences are monetary prizes ($\mathcal{C} \subseteq \mathbf{R}$), then provided u is strictly increasing, we may assume without loss that it is the identity $c \mapsto c$, in which case a cautious representation is simply $x \mapsto \min_{\succeq \in P} e(\succeq, x)$.

Cautious expected utility (Cerreia-Vioglio, Dillenberger & Ortoleva, 2015) is the special case in which $P \subseteq \mathcal{P}^u$ contains only expected-utility preferences.

Corollary 4. Fix a utility function $u : \mathcal{C} \rightarrow \mathbf{R}$. For a pointwise compact set $P \subseteq \mathcal{P}^u$ and a preference $\succeq^* \in \mathcal{P}$, P is a cautious representation of \succeq^* iff \succeq^* is a minimum upper bound of P with respect to ‘more risk-averse than’.

Corollary 4 implies comparative statics along the lines of Corollary 3, as well as the existence and uniqueness of minimum upper bounds of arbitrary subsets of \mathcal{P}^u . Existence does not require solvability or pointwise compactness.

5 Application to social choice

In this section, we apply our theorems to social choice. We consider a liberal principle which requires unanimous consent in order for certain alternatives to be ranked above others. Following Sen (1970), we ask whether such a liberal constraint can be reconciled with a Pareto efficiency criterion.

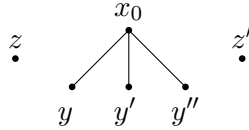
We show (§5.2) that liberalism and efficiency jointly require that the social preference be a minimum upper bound of the set of individual preferences. We build on this result in §5.3 to prove an (im)possibility theorem that characterises when efficiency and liberalism can be reconciled. Finally, we discuss the relationship with Sen’s theorem in §5.4.

5.1 Environment

There is a society composed of n individuals, each with a preference over a set \mathcal{X} of social alternatives. For certain pairs $x, y \in \mathcal{X}$, the liberal principle precludes ranking y (strictly) above x unless all members of society consent; we write $x \succeq y$ when this is the case. We assume that the liberal constraint \succeq is a partial order (in particular, transitive).

For example, it may be that $x \succeq y$ if alternative y involves more surveillance or social control than does x . Alternatively, consider a community deciding what amenities to provide at public expense. If alternative y is much like x except for the addition of a luxury such as a swimming pool or gym, then liberalism may allow the community to favour y over x only if every individual consents, i.e. $x \succeq y$. Or it could be that \succeq reflects the status quo:

Example 4. There are old alternatives \mathcal{Y} and new alternatives \mathcal{Z} , so that $\mathcal{X} = \mathcal{Y} \cup \mathcal{Z}$. When only alternatives \mathcal{Y} were available, $x_0 \in \mathcal{Y}$ was chosen. Departures from the status quo, in the form of some $y \in \mathcal{Y}$ being ranked above x_0 , require unanimous consent: $x_0 \succeq y$ for every other $y \in \mathcal{Y}$. There are no other constraints. The partial order \succeq thus looks as follows:



A *social welfare function* (SWF) is a map $f : \mathcal{P}^n \rightarrow \mathcal{P}$ that carries each preference profile $\pi \in \mathcal{P}^n$ into a (social) preference $f(\pi)$. This specification encodes the assumptions of *transitivity* (social preference must be transitive) and *universal domain* (all profiles of individual preferences are considered).

Definition 11. A SWF f is *liberal* iff for any profile $\pi = (\succeq_1, \dots, \succeq_n)$ and any alternatives $x \succeq y$, $x f(\pi) y$ (and not $y f(\pi) x$) whenever $x \succeq_i (\succ_i) y$ for some individual i .

Observe that a SWF f is liberal precisely if $f(\pi)$ is an upper bound of π for every profile π of preferences.²⁶ By the UB characterisation lemma (p. 7), this means precisely that for $x \succeq y$, $x f(\pi) y$ (and not $y f(\pi) x$) whenever there is a (strict) π -chain from x to y .

A SWF is f (*Pareto*) *efficient* iff for any profile $\pi = (\succeq_1, \dots, \succeq_n)$, whenever $x \succeq_i (\succ_i) y$ for every individual i , we have $x f(\pi) y$ (and not $y f(\pi) x$). Unsurprisingly, efficiency typically clashes with liberalism:

²⁶We abuse notation throughout by identifying each profile $\pi = (\succeq_1, \dots, \succeq_n)$ with the set $P := \{\succeq_1, \dots, \succeq_n\}$ of preferences in it.

Example 5. Let $\mathcal{X} = \{x, y, z\}$ with $x \triangleright y \triangleright z$, and let $\pi = (\succeq_1, \succeq_2)$, where $z \succ_1 x \succ_1 y$ and $y \succ_2 z \succ_2 x$. If f is a liberal SWF, then $f(\pi)$ must rank x above y since $x \triangleright y$ and $x \succ_1 y$. $f(\pi)$ must similarly rank y above z , and thus rank x above z by transitivity. But then f is not efficient, since z is unanimously strictly preferred to x .

We therefore consider a weaker efficiency property that requires unanimity to be respected only when this does not obviously conflict with liberalism:

Definition 12. A SWF is f *weakly efficient* (*w-efficient*) iff for any profile $\pi = (\succeq_1, \dots, \succeq_n)$, whenever $x \succeq_i (\succ_i) y$ for every individual i and there is no strict π -chain (no π -chain) from y to x , we have $x f(\pi) y$ (and not $y f(\pi) x$).

5.2 w-efficiency and liberalism may conflict

Lemma 5 (only MUBs are w-efficient and liberal). If f is w-efficient and liberal, then $f(\pi)$ is a minimum upper bound of π for every profile $\pi \in \mathcal{P}^n$.

It follows by the existence theorem that a necessary condition for the existence of a w-efficient and liberal SWF is that \triangleright be crown- and diamond-free. For instance, the preferences in the crown and diamond examples (pp. 9 and 11) cannot be aggregated in a w-efficient and liberal way.

Proof. Let f be w-efficient and liberal, fix a profile $\pi = (\succeq_1, \dots, \succeq_n) \in \mathcal{P}^n$, and write $\succeq := f(\pi)$. Since f is liberal, \succeq is an upper bound of π .

For minimumhood, let \succeq' be a (nother) upper bound of π , and fix alternatives $x, y \in \mathcal{X}$ such that $x \triangleright y$; we must show that $x \succeq (\succ) y$ implies $x \succeq' (\succ') y$. We prove the contra-positive: we assume that $y \succ' (\succeq') x$, and will show that $y \succ (\succeq) x$. Since \succeq' is an upper bound of π , and $x \triangleright y$ and $y \succ' (\succeq') x$, the UB characterisation lemma (p. 7) implies that there is no (strict) π -chain from x to y . Thus in particular, (x, y) is not a (strict) π -chain, which is to say that $y \succ_i (\succeq_i) x$ for every individual i . By the w-efficiency of f , it follows that $y \succ (\succeq) x$. ■

Crown- and diamond-freeness of \triangleright is not *sufficient* for w-efficient and liberal preference aggregation, however:

Example 1 (continued from p. 6). Clearly \triangleright is crown- and diamond-free. Let there be $n = 2$ individuals, and let f be a SWF; we will show that f cannot be w-efficient and liberal. Consider $\pi = (\succeq_1, \succeq_2)$, where \succeq_1 and \succeq_2 are as given previously (p. 6). Write $\succeq^* := f(\pi)$.

By Lemma 5, a necessary condition for f to be w-efficient and liberal is that \succeq^* be a minimum upper bound of π . We saw (p. 8) that this requires

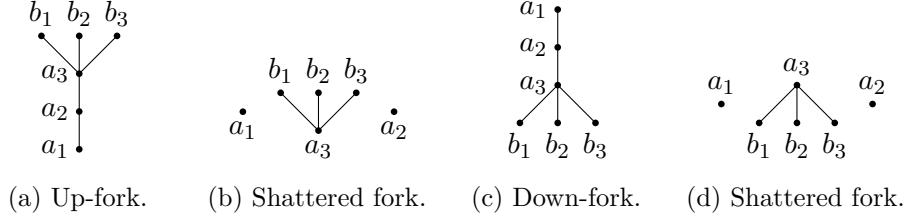


Figure 2: Forks and shattered forks.

precisely that $w \succ^* x \succ^* y \succ^* z$. But then f fails to be w -efficient: z is unanimously strictly preferred to w , and there is no P -chain from w to z , and yet $f(\pi) = \succeq^*$ ranks w above z .

5.3 (Im)possibility theorem

Crown- and diamond-freeness is necessary but not sufficient for the existence of a w -efficient and liberal SWF, and it can be shown that completeness is sufficient but not necessary. Our (im)possibility theorem furnishes the necessary and sufficient condition: (\mathcal{X}, \succeq) must be either a *fork* or a *shattered fork*. These are drawn in Figure 2, and formally defined as follows.

Definition 13. Let \sqsubseteq be a binary relation on a set \mathcal{A} . Call an element $a \in \mathcal{A}$ *isolated* iff it is \sqsubseteq -incomparable to every other element of \mathcal{A} . A set $A \subseteq \mathcal{A}$ is an *upward (downward) fork head* iff it has a minimum (maximum) element $a \in A$ and $A \setminus \{a\}$ is an anti-chain, and simply a *fork head* iff it is either an upward or a downward fork head.

The poset $(\mathcal{A}, \sqsubseteq)$ is an *up-fork (down-fork)* iff \mathcal{A} can be partitioned as $\mathcal{A} = A \cup \{a\} \cup B$ where $A \cup \{a\}$ is an upward (downward) fork head with minimum (maximum) element a and $\{a\} \cup B$ is a chain with maximum (minimum) element a , and simply a *fork* iff it is an up-fork or a down-fork.

The poset $(\mathcal{A}, \sqsubseteq)$ is a *shattered fork* iff \mathcal{A} can be partitioned as $\mathcal{A} = A \cup B$, where A is a fork head and each $b \in B$ is isolated.

(Im)possibility theorem. The following are equivalent:

- (1) There exists a w -efficient and liberal SWF.
- (2) (\mathcal{X}, \succeq) is either a fork or a shattered fork.

The proof is in appendix J. There, we also provide a ‘forbidden subposet’ variant of the (im)possibility theorem according to which the existence of a w -efficient and liberal SWF is equivalent to \succeq being free of certain subposets.

The (im)possibility theorem is in part a negative result: the necessary and sufficient condition for w-efficient and liberal preference aggregation is stringent. In the crown and diamond examples, \succeq fails to be crown- and diamond-free, so (\mathcal{X}, \succeq) is certainly not a fork or shattered fork. Further:

Example 1 (continued). \succeq is crown- and diamond-free, but (\mathcal{X}, \succeq) is neither a fork nor a shattered fork. We saw that no w-efficient and liberal SWF exists.

But the (im)possibility theorem also has a positive side, because some natural liberal constraints are either forks or shattered forks:

Example 4 (continued). By inspection, the liberal constraint \succeq is a shattered (down-)fork. Thus there is a w-efficient and liberal SWF.

5.4 Relation to Sen’s ‘impossibility of a Paretian liberal’

Our (im)possibility result is complementary to Sen’s (1970) ‘impossibility of a Paretian liberal’, which asserts that when there are $n \geq 2$ individuals and $|\mathcal{X}| \geq 4$ alternatives, there exists no efficient and minimally liberal SWF. ‘Minimally liberal’ here means that there exist two individuals $i \neq j$ and four distinct alternatives x_i, y_i, x_j, y_j such that individual i (j) dictates the social preference on x_i, y_i (x_j, y_j). The idea is that x_i and y_i differ solely in respects relevant only to individual i (such as the colour of i ’s wallpaper), and that liberalism demands that such decisions be left to individuals.

Our notion of liberalism is better-suited to settings such as surveillance, where social alternatives do not specify individual matters such as wallpaper colour. Our (im)possibility theorem provides a somewhat more encouraging conclusion about the compatibility of efficiency with liberalism.

Appendices

A Standard definitions

This appendix collects definitions of standard order-theoretic concepts used in this paper. Let \mathcal{A} be a non-empty set, and \sqsubseteq a binary relation on it.

For $a, b \in \mathcal{A}$, we write $a \not\sqsubseteq b$ iff it is not the case that $b \sqsubseteq a$, and $a \sqsupset b$ iff $a \sqsubseteq b$ and $a \not\sqsubseteq b$, and we say that a, b are \sqsubseteq -(in)comparable iff (n)either $a \sqsubseteq b$ (n)or $b \sqsubseteq a$.

\sqsubseteq is *complete* iff every pair $a, b \in \mathcal{A}$ is \sqsubseteq -comparable, *transitive* iff $a \sqsubseteq b \sqsubseteq c$ implies $a \sqsubseteq c$ for $a, b, c \in \mathcal{A}$, *reflexive* iff $a \sqsubseteq a$ for any $a \in \mathcal{A}$, and

anti-symmetric iff $a \sqsupseteq b \sqsupseteq a$ implies $a = b$ for $a, b \in \mathcal{A}$. \sqsupseteq is a *partial order* iff it is transitive, reflexive and anti-symmetric; $(\mathcal{A}, \sqsupseteq)$ is then called a *poset* (partially ordered set). If \sqsupseteq is both complete and a partial order, then it is called a *total order*.

Fix a subset $A \subseteq \mathcal{A}$. A is an (*anti*-)*chain* iff all pairs $a, b \in \mathcal{A}$ of distinct elements are \sqsupseteq -(in)comparable. An element $a \in A$ is a *minimum* (*maximum*) of A iff $b \sqsupseteq a$ ($a \sqsupseteq b$) for every $b \in A$.

An element $b \in \mathcal{A}$ is an *upper bound* of a set $A \subseteq \mathcal{A}$ iff $b \sqsupseteq a$ for each $a \in A$, and a *minimum upper bound* iff in addition it is a minimum of the set of upper bounds of A . Note that if \sqsupseteq is anti-symmetric, then the minimum upper bound of a set is unique if it exists. (Maximum) lower bounds are defined analogously. Minimum upper bounds and maximum lower bounds are also known as *joins* and *meets*, or *suprema* and *infima*.

A partially ordered set $(\mathcal{A}, \sqsupseteq)$ is a *complete lattice* iff every subset of \mathcal{A} has a minimum upper bound and a maximum lower bound, and simply a *lattice* iff this is true for every two-element subset.

B Extension theorems

Extension theorems will be key to proving Lemma 1 (appendix C), the existence theorem (appendix D), and Lemma 4 (appendix E).

Definition 14. Let \sqsupseteq and \sqsupseteq' be binary relations on a set \mathcal{A} . \sqsupseteq' is an *extension* of \sqsupseteq iff for $a, b \in \mathcal{A}$, $b \sqsupseteq(\sqsupseteq) a$ implies $b \sqsupseteq'(\sqsupseteq') a$.

Definition 15. A binary relation \sqsupseteq on a set \mathcal{A} is *Suzumura consistent* iff for $a_1, \dots, a_K \in \mathcal{A}$, $a_1 \sqsupseteq a_2 \sqsupseteq \dots \sqsupseteq a_{K-1} \sqsupseteq a_K$ implies that either $a_1 \sqsupseteq a_K$ or a_1, a_K are \sqsupseteq -incomparable.

Richter–Suzumura extension theorem. A binary relation admits a complete and transitive extension iff it is Suzumura consistent.

This result is due to Suzumura (1976), and is a consequence of Richter’s theorem (below). We use the Richter–Suzumura extension theorem directly in the proof of the existence theorem (appendix D). In proving Lemma 4 (appendix E), we rely on the following corollary.

Suzumura corollary. Let \sqsupseteq be a transitive binary relation on a set \mathcal{A} , and let $a, b \in \mathcal{A}$ be such that $b \not\sqsupseteq a$. Then \sqsupseteq admits a complete and transitive extension \sqsupseteq' such that $a \sqsupseteq' b$.

Proof. Let \sqsupseteq^Δ be the binary relation on \mathcal{A} such that, for any $c, d \in \mathcal{A}$, $c \sqsupseteq^\Delta d$ iff either (i) $c \sqsupseteq d$ or (ii) $c = a$ and $d = b$. It suffices to show that \sqsupseteq^Δ

admits a complete and transitive extension to \mathcal{A} . So by the Richter–Suzumura extension theorem, what we must show is that \sqsupseteq^Δ is Suzumura consistent.

To this end, let $a_1 \sqsupseteq^\Delta \dots \sqsupseteq^\Delta a_K$ in \mathcal{A} ; we must establish that $a_K \not\succ^\Delta a_1$. Let $\mathcal{I} = \{k \leq K : a_k = a \text{ and } a_{k+1} = b\}$, where $K + 1$ is treated as 1 by convention. If \mathcal{I} is empty, then $a_1 \sqsupseteq^\Delta a_K$ by transitivity of \sqsupseteq . If \mathcal{I} is a singleton $\mathcal{I} = \{k\}$, suppose toward a contradiction that $a_K \sqsupseteq a_1$; then $b = a_{k+1} \sqsupseteq a_k = a$ by transitivity of \sqsupseteq , contradicting the hypothesis that $b \not\sqsupseteq a$. Finally, suppose that $|\mathcal{I}| > 1$. Then there exist $k_1 < k_2$ such that $a_{k_1} = b$, $a_{k_2} = a$, and $a_k \sqsupseteq a_{k+1}$ for all $k_1 \leq k < k_2$. It follows by transitivity of \sqsupseteq that $b \sqsupseteq a$, a contradiction with $b \not\sqsupseteq a$ —thus $|\mathcal{I}| \leq 1$. ■

The proof of Lemma 1 (appendix C) uses a more general theorem.

Definition 16. An *order pair* on a set \mathcal{A} is a pair $(\sqsupseteq, \sqsupseteq')$ where \sqsupseteq and \sqsupseteq' are binary relations on \mathcal{A} and for any $a, b \in \mathcal{A}$, $a \sqsupseteq' b$ implies $a \sqsupseteq b$.

Definition 17. Let $(\sqsupseteq_1, \sqsupseteq'_1)$ and $(\sqsupseteq_2, \sqsupseteq'_2)$ be order pairs on a set \mathcal{A} . $(\sqsupseteq_2, \sqsupseteq'_2)$ is an *order-pair extension* of $(\sqsupseteq_1, \sqsupseteq'_1)$ iff for any $a, b \in \mathcal{A}$, $a \sqsupseteq_1 b$ implies $a \sqsupseteq_2 b$ and $a \sqsupseteq'_1 b$ implies $a \sqsupseteq'_2 b$.

Definition 18. Let $(\sqsupseteq, \sqsupseteq')$ be an order pair on a set \mathcal{A} . A *cycle* is a sequence $(a_k)_{k=1}^K$ in \mathcal{A} such that $a_1 \sqsupseteq a_2 \sqsupseteq \dots \sqsupseteq a_{K-1} \sqsupseteq a_K \sqsupseteq' a_1$.

Richter’s theorem. An order pair $(\sqsupseteq, \sqsupseteq')$ on a set \mathcal{A} features no cycles iff it admits an order-pair extension (\succeq, \succ) where \succeq is a preference on \mathcal{A} .

This result is due to Richter (1966). See Chambers and Echenique (2016, pp. 7–8) for a proof of the ‘only if’ direction (the ‘if’ direction is trivial).

C Proof of Lemma 1 (§2.1, p. 8)

We prove the contra-positive, using Richter’s theorem (appendix B).²⁷ Suppose that all upper bounds \succeq'' of P satisfy $x \succeq'' (\succ'') y$; we shall exhibit a (strict) P -chain from x to y . To that end, let \sqsupseteq be the binary relation on \mathcal{X} such that for all $z, w \in \mathcal{X}$, $z \sqsupseteq w$ iff either (i) $z \triangleright w$ and there is a P -chain from z to w or (ii) $z = y$ and $w = x$. Let \sqsupseteq' be the binary relation on \mathcal{X} such that for all $z, w \in \mathcal{X}$, $z \sqsupseteq' w$ iff either (i) $z \triangleright w$ and there is a strict P -chain from z to w or (ii) $z = y$ and $w = x$ ($z \sqsupseteq' w$ iff $z \triangleright w$ and there is a strict P -chain from z to w). Clearly $(\sqsupseteq, \sqsupseteq')$ is an order pair.

Any $\succeq'' \in \mathcal{P}$ such that (\succeq'', \succ'') is an order-pair extension of $(\sqsupseteq, \sqsupseteq')$ must be an upper bound of P by the UB characterisation lemma (p. 7),

²⁷We thank an anonymous referee for suggesting this short argument.

and must satisfy $y \succ''(\succeq'') x$ by construction of \sqsupseteq' (of \sqsupseteq). By hypothesis, no \succeq'' with these properties exists. Hence by Richter's theorem, the order pair $(\sqsupseteq, \sqsupseteq')$ features cycles. Let $(z_k)_{k=1}^K$ be a cycle of minimal length. Since $(z_k)_{k=1}^K$ is a cycle, there must be a $k' \leq (<) K$ at which $z_{k'} = y$ and $z_{k'+1} = x$ (where $z_{K+1} := z_1$ by convention), since otherwise $z_1 \triangleright \cdots \triangleright z_K \triangleright z_1$. Since $(z_k)_{k=1}^K$ is of minimal length, the index k' is unique, and thus by definition of \sqsupseteq there is a P -chain from z_k to z_{k+1} for every $k \neq k'$ in $\{1, \dots, K-1\}$ (and by definition of \sqsupseteq' there is a strict P -chain from z_K to z_1). Hence $(w_k)_{k=1}^K := (z_{k+k'})_{k=1}^K$, where the indices are interpreted modulo K , is a (strict) P -chain from x to y . ■

D Proof that (3) implies (1) in the existence theorem (§2.2.3, p. 11)

For a given $P \subseteq \mathcal{P}$, let \succeq° be the (in general, incomplete) binary relation that satisfies conditions (\star) – $(\star\star)$ in the characterisation theorem for \triangleright -comparable pairs of alternatives, and that does not rank \triangleright -incomparable pairs of alternatives.²⁸ For each P , \succeq° clearly exists and is unique.

In light of the characterisation theorem, property (1) in the existence theorem requires precisely that \succeq° admit a complete and transitive extension (i.e. an extension that lives in \mathcal{P}) for any $P \subseteq \mathcal{P}$.²⁹ Our task is therefore to show that whenever \triangleright is crown- and diamond-free, \succeq° admits a complete and transitive extension for every $P \subseteq \mathcal{P}$. We will use the Richter–Suzumura extension theorem (appendix B).

We first state two lemmata, then use them to show that (3) implies (1). The remainder of this appendix is devoted to proving the lemmata.

Definition 19. A binary relation \sqsupseteq on a set \mathcal{A} is *weakly transitive* iff for $a, b, c \in \mathcal{A}$, if $a \sqsupseteq b \sqsupseteq c$ and a, c are \sqsupseteq -comparable, then $a \sqsupseteq c$.

Lemma 6 (weak transitivity of \succeq°). Suppose that \triangleright is diamond-free. Then \succeq° is weakly transitive for any $P \subseteq \mathcal{P}$.

Weak transitivity is implied by Suzumura consistency (take $K = 3$). The converse is false in general,³⁰ but true for \succeq° when \triangleright has no crowns:

²⁸Explicitly: for any $x, y \in \mathcal{X}$, if x, y are \triangleright -incomparable then $x \not\succeq^\circ y \not\succeq^\circ x$, and if x, y are \triangleright -comparable, wlog $x \triangleright y$, then $x \succeq^\circ y$ iff there is a P -chain from x to y , and $y \succeq^\circ x$ iff there is no strict P -chain from x to y .

²⁹The term ‘extension’ was defined in appendix B.

³⁰Consider $\mathcal{A} = \{a, b, c, d\}$ and the binary relation \sqsupseteq such that $a \sqsupseteq b \sqsupseteq c \sqsupseteq d \sqsupseteq a$ and no other pairs are \sqsupseteq -comparable. This relation satisfies weak transitivity, but clearly violates Suzumura consistency.

Lemma 7 (Suzumura consistency of \succeq°). Suppose that \succeq is crown-free. Then if \succeq° is weakly transitive, it is Suzumura consistent.

Proof that (3) implies (1). Suppose that \succeq is crown- and diamond-free, and fix any $P \subseteq \mathcal{P}$. Since \succeq is diamond-free, \succeq° is weakly transitive by Lemma 6. Since \succeq is crown-free, it follows by Lemma 7 that \succeq° is Suzumura consistent. Invoking the Richter–Suzumura extension theorem (appendix B), we conclude that \succeq° admits a complete and transitive extension. ■

It remains to prove Lemmata 6 and 7. Begin with the former. The role of diamond-freeness is to ensure a ‘crossing’ property of decreasing sequences:

Observation 1. Suppose that \succeq is diamond-free, and consider $x, y, z \in \mathcal{X}$ with $x \triangleright z \triangleright y$. Let $(w_k)_{k=1}^K$ be a finite decreasing sequence with $w_1 = x$ and $w_K = y$, and let k' be the last $k < K$ such that $w_k \triangleright z$. Then $z \triangleright w_{k'+1}$.

Proof. Since $w_1 = x \triangleright z$, there exist $k < K$ such that $w_k \triangleright z$, so k' is well-defined. It cannot be that $w_{k'+1} \triangleright z$, by definition of k' . Nor can it be that $z, w_{k'+1}$ are \succeq -incomparable, for then $(x, z, w_{k'+1}, y)$ is a diamond: $x \triangleright z \triangleright y$ (by hypothesis), $x \triangleright w_{k'+1} \triangleright y$ (since $(w_k)_{k=1}^K$ is decreasing from x to y), and $z, w_{k'+1}$ are \succeq -incomparable. Hence $z \triangleright w_{k'+1}$ by anti-symmetry of \succeq . ■

We will use the following piece of notation: given a set \mathcal{A} , a sequence $(a_n)_{n=1}^N$ in \mathcal{A} and an element $b \in \mathcal{A}$, we write $(b) \cup (a_n)_{n=1}^N$ for the sequence (b, a_1, \dots, a_N) , and similarly $(a_n)_{n=1}^N \cup (b)$ for the sequence (a_1, \dots, a_N, b) .

Proof of Lemma 6. Suppose that \succeq features no diamonds, and fix a $P \subseteq \mathcal{P}$ and $x, y, z \in \mathcal{X}$ such that $x \succeq^\circ y \succeq^\circ z$ and x, z are \succeq° -comparable. We must show that $x \succeq^\circ z$. This is immediate if x, y, z are not distinct, so suppose that they are. Then by anti-symmetry of \succeq , there are six cases to check: one for each strict ordering by \succeq of x, y, z .

Case 1: $x \triangleright y \triangleright z$. Since $x \succeq^\circ y$ and $x \triangleright y$, by (\star) there is a P -chain from x to y . Similarly there is a P -chain from y to z . The concatenation of these two P -chains is a P -chain from x to z ; hence $x \succeq^\circ z$ by (\star) .

Case 2: $x \triangleright z \triangleright y$. Since $x \succeq y$ and $x \succeq^\circ y$, there is a P -chain $(w_k)_{k=1}^K$ from x to y by (\star) . Let k' be the last $k < K$ for which $w_k \triangleright z$, so that $w_{k'} \triangleright z \triangleright w_{k'+1}$ by Observation 1. By definition of a P -chain, there is a preference \succeq in P such that $w_{k'} \succeq w_{k'+1}$. It must be that $w_{k'+1} \succeq z$, for otherwise $(z) \cup (w_k)_{k=k'+1}^K$ would be a strict P -chain from z to y , in which case $y \not\succeq^\circ z$ by $(\star\star)$, a contradiction. So we have $w_{k'} \succeq w_{k'+1} \succeq z$, which by

transitivity of \succeq yields $w_{k'} \succeq z$. It follows that $(w_k)_{k=1}^{k'} \cup (z)$ is a P -chain from x to z , so that $x \succeq^\circ z$ by (\star) .

Case 3: $y \triangleright x \triangleright z$. This case is similar to the second. Since $y \triangleright z$ and $y \succeq^\circ z$, there is a P -chain $(w_k)_{k=1}^K$ from y to z by (\star) . Let k' be the last $k < K$ for which $w_k \triangleright x$, so that $w_{k'} \triangleright x \triangleright w_{k'+1}$ by Observation 1. By definition of a P -chain, there is a preference \succeq in P such that $w_{k'} \succeq w_{k'+1}$. It must be that $x \succeq w_{k'}$, for otherwise $(w_k)_{k=1}^{k'} \cup (x)$ would be a strict P -chain from y to x , in which case $x \not\succeq^\circ y$ by $(\star\star)$, a contradiction. So we have $x \succeq w_{k'} \succeq w_{k'+1}$, which by transitivity of \succeq yields $x \succeq w_{k'+1}$. It follows that $(x) \cup (w_k)_{k=k'+1}^K$ is a P -chain from x to z , so that $x \succeq^\circ z$ by (\star) .

Case 4: $y \triangleright z \triangleright x$. Suppose toward a contradiction that $x \not\succeq^\circ z$. Then since $z \triangleright x$, by $(\star\star)$ there exists a strict P -chain from z to x . Since $y \triangleright z$ and $y \succeq^\circ z$, there is a P -chain from y to z by (\star) . Concatenating these two P -chains yields a strict P -chain from y to x , so that $x \not\succeq^\circ y$ by $(\star\star)$, a contradiction.

Case 5: $z \triangleright x \triangleright y$. This case is similar to the fourth. Suppose toward a contradiction that $x \not\succeq^\circ z$. Then since $z \triangleright x$, by $(\star\star)$ there exists a strict P -chain from z to x . Since $x \triangleright y$ and $x \succeq^\circ y$, there is a P -chain from x to y by (\star) . Concatenating these two P -chains yields a strict P -chain from z to y , so that $y \not\succeq^\circ z$ by $(\star\star)$, a contradiction.

Case 6: $z \triangleright y \triangleright x$. Suppose toward a contradiction that $x \not\succeq^\circ z$. Then by $(\star\star)$, there is a strict P -chain $(w_k)_{k=1}^K$ from z to x . Let k' be the last $k < K$ for which $w_k \triangleright y$, so that $w_{k'} \triangleright y \triangleright w_{k'+1}$ by Observation 1. By definition of a P -chain, there is a preference \succeq in P such that $w_{k'} \succeq w_{k'+1}$. Since $(w_k)_{k=1}^K$ is a strict P -chain, there are k'' and $\succeq'' \in P$ be such that $w_{k''} \succ'' w_{k''+1}$, with $\succ'' = \succ$ if $k'' = k'$.

Case 6, sub-case A: $k'' < k'$. It must be that $y \succ w_{k'}$, for otherwise $(w_k)_{k=1}^{k'} \cup (y)$ would be a strict P -chain from z to y , in which case $y \not\succeq^\circ z$ by $(\star\star)$, a contradiction. So we have $y \succ w_{k'} \succeq w_{k'+1}$, which by transitivity of \succeq yields $y \succ w_{k'+1}$. It follows that $(y) \cup (w_k)_{k=k'+1}^K$ is a strict P -chain from y to x , so that $x \not\succeq^\circ y$ by $(\star\star)$, a contradiction.

Case 6, sub-case B: $k'' \geq k'$. It must be that $y \succeq w_{k'}$, for otherwise $(w_k)_{k=1}^{k'} \cup (y)$ would be a strict P -chain from z to y , in which case $y \not\succeq^\circ z$ by $(\star\star)$, a contradiction. So we have $y \succeq w_{k'} \succeq w_{k'+1}$, which by transitivity of \succeq yields $y \succeq w_{k'+1}$, and $y \succ w_{k'+1}$ if $k'' = k'$. It follows that $(y) \cup (w_k)_{k=k'+1}^K$ is a strict P -chain from y to x , so that $x \not\succeq^\circ y$ by $(\star\star)$, a contradiction. ■

It remains to prove Lemma 7.

Definition 20. For a binary relation \sqsubseteq on a set \mathcal{A} , a *weak cycle* is a

finite sequence $(a_k)_{k=1}^K$ of distinct elements of \mathcal{A} such that a_k, a_{k+1} are \sqsubseteq -comparable for each $1 \leq k \leq K$, where a_{K+1} is understood as a_1 .

Note that crowns and diamonds are both examples of weak cycles. The role of crown-freeness is to deliver a ‘shortcut’ property of weak cycles:

Observation 2. Let \sqsubseteq be a transitive and crown-free binary relation on a set \mathcal{A} . Then for any weak cycle (a_1, \dots, a_K) in \sqsubseteq , (a_1, a_k, a_K) is a weak cycle in \sqsubseteq for some $1 < k < K$.

Proof. We prove the contra-positive. Suppose that \sqsubseteq is transitive and that there is a weak cycle (a_1, \dots, a_K) such that (a_1, a_k, a_K) is not a weak cycle for any $1 < k < K$. In particular, choose (a_1, \dots, a_K) to be a weak cycle of *minimal length* with this property. Clearly its length K is ≥ 4 . We will show that (a_1, \dots, a_K) is a crown.

There are two cases, $a_1 \sqsupset a_2$ and $a_1 \sqsubset a_2$. We consider the former case only; the latter is analogous. It must be that $a_2 \sqsubset a_3$, for if $a_2 \sqsupset a_3$ then a_1, a_3 are \sqsubseteq -comparable by transitivity of \sqsubseteq , so (a_1, a_3, \dots, a_K) is a weak cycle for which (a_1, a_k, a_K) is not a weak cycle for any $3 \leq k < K$, contradicting the minimality of (a_1, \dots, a_K) . Proceeding using the same argument, we conclude that $a_{k-1} \sqsupset a_k \sqsubset a_{k+1}$ for every $1 < k < K$ even, where by convention $a_{K+1} := a_1$. (This is apparent, separately, for $k < K$ even and for $k = K$.)

Furthermore, it must be that for non-adjacent $k' < k''$ (i.e. those with $2 \leq k'' - k' \leq K - 2$), $a_{k'}, a_{k''}$ are \sqsubseteq -incomparable, for if they were comparable then $(a_1, \dots, a_{k'}, a_{k''}, \dots, a_K)$ would be a weak cycle for which (a_1, a_k, a_K) is not a weak cycle for any $k \notin \{1, K\}$, contradicting the minimality of (a_1, \dots, a_K) . It follows that (a_1, \dots, a_K) is a crown. ■

Proof of Lemma 7. Suppose that \supseteq is crown-free and that \succeq° is weakly transitive. We must show that for every $K \geq 3$, the following claim holds:

for any $x_1, \dots, x_K \in \mathcal{X}$, $x_1 \succeq^\circ \dots \succeq^\circ x_K \succeq^\circ x_1$ implies $x_1 \succeq^\circ x_K$. $C(K)$

We proceed by strong induction on K . In the base case $K = 3$, $C(3)$ is immediate from weak transitivity of \succeq° .

Now fix any $K \geq 4$, and suppose that $C(K')$ holds for all $K' \leq K - 1$; we will establish $C(K)$. Take any $x_1, \dots, x_K \in \mathcal{X}$ with $x_1 \succeq^\circ \dots \succeq^\circ x_K \succeq^\circ x_1$, wlog distinct. Since \succeq° compares all and only \supseteq -comparable pairs of alternatives, (x_1, \dots, x_K) is a weak cycle in \supseteq . Since \supseteq (transitive and crown-free, Observation 2 implies the existence of a $1 < k < K$ such that (x_1, x_k, x_K) is a weak cycle in \supseteq . We consider three cases.

Case 1: $k = 2$. Since $x_2 \succeq^\circ \dots \succeq^\circ x_K \succeq^\circ x_2$, the inductive hypothesis (in particular, $C(K - 1)$) implies that $x_2 \succeq^\circ x_K$. Since $x_1 \succeq^\circ x_2$, and x_1, x_K are \succeq° -comparable because they are \succeq -comparable, it follows by weak transitivity that $x_1 \succeq^\circ x_K$.

Case 2: $k = K - 1$. This case is analogous to the first.

Case 3: $2 < k < K - 1$. Since $x_1 \succeq^\circ \dots \succeq^\circ x_k$ and x_1, x_k are \succeq° -comparable (because they are \succeq -comparable), the inductive hypothesis (in particular, $C(k)$) implies that $x_1 \succeq^\circ x_k$. Similarly, $x_k \succeq^\circ \dots \succeq^\circ x_K$, the \succeq -comparability of x_k, x_K and the inductive hypothesis yield $x_k \succeq^\circ x_K$. Since x_1, x_K are \succeq° -comparable (because they are \succeq -comparable), it follows by weak transitivity that $x_1 \succeq^\circ x_K$. ■

E Proof of Lemma 4 (§2.3, p. 13)

Since \succeq is transitive, and x, y are \succeq -incomparable, the Suzumura corollary (appendix B) implies that \succeq admits complete and transitive extensions $\succeq', \succeq'' \in \mathcal{P}$ such that $x \succ' y$ and $y \succ'' x$. \succeq' is an upper bound of \mathcal{P} (and hence of any $P \subseteq \mathcal{P}$) because whenever $z \succeq (\succ) w$, we have $z \succeq' (\succ') w$ since \succeq' extends \succeq ; thus the consequent in the definition of ‘ $\succeq' S \succeq'$ ’ (p. 2) is satisfied for any $z, w \in \mathcal{X}$ and $\succeq \in \mathcal{P}$. Similarly for \succeq'' .

F Results for maximum lower bounds

The *inverse* of a binary relation \sqsupseteq on a set \mathcal{A} is the binary relation \sqsubseteq such that for $a, b \in \mathcal{A}$, $a \sqsubseteq b$ iff $b \sqsupseteq a$.

Observation 3. If $S^\sqsupseteq (S^\sqsubseteq)$ is the single-crossing-dominance relation induced by (the inverse of) the primitive order \succeq on \mathcal{X} , then S^\sqsubseteq is the inverse of S^\sqsupseteq .

Hence a maximum lower bound with respect to S^\sqsupseteq is precisely a minimum upper bound with respect to S^\sqsubseteq . Since \sqsubseteq is crown- and diamond-free (complete) iff \succeq is, the existence theorem (uniqueness proposition) delivers:

Corollary 5 (existence). The following are equivalent:

- (1) Every set of preferences has a minimum upper bound.
- (2) Every set of preferences has a maximum lower bound.
- (3) \succeq is crown- and diamond-free.

Corollary 6 (uniqueness). The following are equivalent:

- (1) Every set of preferences has *at most* one minimum upper bound.
- (2) Every set of preferences has *exactly* one minimum upper bound.

- (3) Every set of preferences has *at most* one maximum lower bound.
- (4) Every set of preferences has *exactly* one maximum lower bound.
- (5) \succeq is complete.

Finally, the analogue of the characterisation theorem is as follows.

Definition 21. For a set $P \subseteq \mathcal{P}$ of preferences and two alternatives $y \succeq x$ in \mathcal{X} , a *reverse P -chain* from y to x is a finite sequence $(w_k)_{k=1}^K$ in \mathcal{X} such that (i) $w_1 = y$ and $w_K = x$, (ii) for every $k < K$, $w_k \preceq w_{k+1}$, and (iii) for every $k < K$, $w_k \succeq w_{k+1}$ for some $\succeq \in P$. A reverse P -chain is *strict* iff $w_k \succ w_{k+1}$ for some $k < K$ and $\succeq \in P$.

Corollary 7 (characterisation). For a preference $\succeq_\star \in \mathcal{P}$ and a set $P \subseteq \mathcal{P}$ of preferences, the following are equivalent:

- (1) \succeq_\star is a maximum lower bound of P .
- (2) \succeq_\star satisfies: for any \succeq -comparable $x, y \in \mathcal{X}$, wlog $y \succeq x$,
 - $x \succeq_\star y$ iff there is a reverse P -chain from x to y , and
 - $y \succeq_\star x$ iff there is no strict reverse P -chain from x to y .

G Proof of Corollary 1 (§3.2, p. 15)

Equip the space $2^{\mathcal{P}} \setminus \{\emptyset\}$ of non-empty sets of preferences with the S -induced strong set order \succsim . It is easily verified that \succsim is a partial order. The consensus C is a correspondence $2^{\mathcal{P}} \setminus \{\emptyset\} \rightrightarrows \mathcal{X}$, and Proposition 1 says precisely that it is increasing.

Let \mathcal{Q} be the set of non-empty $P \subseteq \mathcal{P}$ at which $C(P)$ is non-empty. The restriction of C to \mathcal{Q} is a non-empty-valued increasing correspondence into $\mathcal{X} \subseteq \mathbf{R}$. It follows by Theorem 2.7 in Kukushkin (2013) that it admits an increasing selection $c : \mathcal{Q} \rightarrow \mathcal{X}$.³¹

Let $\mathcal{U}(P) := \{P'' \in \mathcal{Q} : P'' \succsim P\}$ for each non-empty $P \subseteq \mathcal{P}$, and define $d : 2^{\mathcal{P}} \setminus \{\emptyset\} \rightarrow \mathcal{X}$ by $d(P) := \inf\{c(P'') : P'' \in \mathcal{U}(P)\}$ if $\mathcal{U}(P) \neq \emptyset$ and $d(P) := \sup \mathcal{X}$ otherwise. The map d is well-defined, and really does map into \mathcal{X} , because (\mathcal{X}, \succeq) is a complete lattice by the compactness of \mathcal{X} and the Frink–Birkhoff theorem.³² Define an SCF $\phi : \bigcup_{n \in \mathbf{N}} \mathcal{P}^n \rightarrow \mathcal{X}$ by $\phi(\pi) := d(\text{supp } \pi)$ for every preference profile $\pi \in \bigcup_{n \in \mathbf{N}} \mathcal{P}^n$.

Since $d = c$ on \mathcal{Q} (because c is increasing) and c is a selection from C , we have that $d(P) \in C(P)$ whenever the latter is non-empty, which is to say that the SCF ϕ respects unanimity.

³¹This step is non-trivial because \mathcal{X} and \mathcal{P} need not be finite, nor even countable.

³²See e.g. Topkis (1998, Theorem 2.3.1).

To see that ϕ is monotone, consider $P, P' \in \mathcal{P}$ with $P' \succsim P$; we must show that $d(P') \geq d(P)$. Observe that $\mathcal{U}(P') \subseteq \mathcal{U}(P)$. If $\mathcal{U}(P')$ is empty then $d(P') = \sup \mathcal{X} \geq d(P)$ since $d(P) \in \mathcal{X}$. If $\mathcal{U}(P)$ is empty then so is $\mathcal{U}(P')$, putting us in the previous case. If neither is empty then

$$d(P') = \inf\{c(P'') : P'' \in \mathcal{U}(P')\} \geq \inf\{c(P'') : P'' \in \mathcal{U}(P)\} = d(P). \quad \blacksquare$$

H Proof of Proposition 2 (§3.2, p. 16)

Fix $P, P' \subseteq \mathcal{P}$ such that P' dominates P in the S -induced strong set order and $C(P) = C(P')$. Suppose toward a contradiction that there are alternatives $x > x'$ such that x belongs to $C_2(P) \setminus C(P)$ and x' belongs to $C_2(P') \setminus C(P')$, but $x' \notin C_2(P) \setminus C(P)$. (The other case, in which $x \notin C_2(P') \setminus C(P')$ rather than $x' \notin C_2(P) \setminus C(P)$, is symmetric.)

We have $x \notin C_2(P')$ because $x \notin C(P) = C(P')$ and $x \notin C_2(P') \setminus C(P')$. Since $x' \in C_2(P')$, it follows that there is an alternative $y \neq x'$ and a preference $\succeq' \in P'$ such that $x' \succ' x$ and $y \succ' x$.

First claim. $x \succ z$ for every $\succ \in P$ and each $z \in \mathcal{X}$ such that $x' \neq z < x$.

Proof of the first claim. Suppose toward a contradiction that $z \succ x$ and $x' \neq z < x$ for some $\succ \in P$. Since also $x' \succ' x$ and $x' < x$, the maximum lower bound \succeq_* of $\{\succ, \succ'\}$ must satisfy both $z \succ_* x$ and $x' \succ_* x$, so that $x \notin X_2(\succeq_*)$. The preference \succeq_* belongs to P since P' dominates P in the S -induced strong set order, so $C_2(P) \subseteq X_2(\succeq_*)$. But then $x \notin C_2(P)$, a contradiction. \square

Second claim. $x' \succeq x$ for every $\succeq \in P$.

Proof of the second claim. We first show that $y > x$. Suppose toward a contradiction that $y < x$. On the one hand, the facts that $y \succ' x$, that \succ' belongs to P' , and that P' dominates P in the S -induced strong set order together imply that there must be a $\succeq \in P$ such that $y \succ x$. On the other hand, since $x' \neq y < x$, the first claim requires that $x \succeq y$. Contradiction!

Now, to prove the claim, suppose toward a contradiction that $x \succ x'$ for some $\succeq \in P$. Since $y \succ' x$ and $y > x > x'$, it follows that the minimum upper bound \succeq^* of $\{\succeq, \succ'\}$ satisfies $y \succ^* x \succ^* x'$, so that $x' \notin X_2(\succeq^*)$. The preference \succeq^* belongs to P' since P' dominates P in the S -induced strong set order, so $C_2(P') \subseteq X_2(\succeq^*)$. But then $x' \notin C_2(P')$, a contradiction. \square

Since $x' \notin C(P') = C(P)$ by hypothesis, there is an alternative $z \in \mathcal{X}$ and a preference $\succeq \in P$ such that $z \succ x'$, whence $z \succ x$ by the second claim. Then $z > x$ by the first claim, and $X(\succeq) = \{z\}$ since $x \in C_2(P) \subseteq X_2(\succeq)$.

Third claim. There is a preference $\succeq'' \in P'$ such that $x' \succ'' x$ and $z \succ'' w$ for every alternative $w \in \mathcal{X}$ satisfying $x' \neq w < z$.

Proof of the third claim. Let \mathcal{Z} denote the \succeq' -best alternatives in $\mathcal{X} \setminus \{x'\}$, and let z' be the \geq -smallest element of \mathcal{Z} . If $z' = z$, then since $x' \succ' x$, we may take $\succeq'' := \succeq'$. Assume for the remainder that $z' \neq z$.

Note that $z' \succ' x$ since $x' \neq y \succ' x$. This implies that $z' > x$, since $z' < x$ together with the fact that P' dominates P in the S -induced strong set order would imply the existence of a preference $\succeq^\dagger \in P$ such that $z' \succ^\dagger x$, which is impossible by the first claim.

Write \succeq^* for the minimum upper bound of $\{\succeq, \succeq'\}$; we will show that we may take $\succeq'' := \succeq^*$. Since P' dominates P in the S -induced strong set order, \succeq^* lies in P' . It cannot be that $z' > z$, because this together with $z' \succeq' z$ (by $z \neq x'$), $z > x$ and $z \succ x$ would imply $z' \succeq^* z \succ^* x'$, in which case $x' \notin X_2(\succeq^*) \supseteq C_2(P')$, a contradiction. Hence $z > z' > x$, which since $z \succ z' \succ' x$ implies that $z \succ^* z' \succ^* x$. Then $x' \succ^* x$ since $x' \in C_2(P') \subseteq X_2(\succeq^*)$. Furthermore, $z \succ^* w$ for any $w < z$ since $z \succ w$. \square

Choose $\succeq'' \in P'$ as per the third claim. Let \succeq_* be the maximum lower bound of $\{\succeq, \succeq''\}$, and note that it belongs to P since P' dominates P in the S -induced strong set order. We have $x' \succ_* x$ since $x' < x$ and $x' \succ'' x$. We furthermore have $z \succ_* x$ by the characterisation theorem (the version for maximum lower bounds, given in appendix F), since $X(\succeq) = \{z\}$ and $z \succ'' w$ for every alternative $w \in \mathcal{X}$ satisfying $z > w \geq x$, so that there is no reverse $\{\succeq, \succeq''\}$ -chain in from x to z . Thus $x \notin X_2(\succeq_*) \supseteq C_2(P)$, a contradiction. \blacksquare

I Proof of Proposition 3* (§3.3, p. 19)

Let \succeq^* denote the minimum upper bound of P . We shall show that (1) is equivalent to (3) and (separately) that (3) is equivalent to (2).

(1) implies (3) by Proposition 3. For the converse, suppose that P is not rich; we will show that (3) fails. By hypothesis, there exists a non-empty menu $M = \{x_0, x_1, \dots, x_K\} \subseteq \mathcal{X}$ and preferences $\succeq_1, \dots, \succeq_K \in P$ such that $x_0 \succeq_1 x_1 \succeq_2 x_2 \succeq_3 \dots \succeq_K x_K$, and yet $x_0 \notin X_M(\succeq)$ for every $\succeq \in P$. Let \triangleright be a total order on \mathcal{X} such that $x_0 \triangleright \dots \triangleright x_K$. Then (x_0, x_1, \dots, x_K) is a P -chain, so $x_0 \in X_M(\succeq^*)$ by the characterisation theorem. It follows that $\max X_M(\succeq^*) = x_0 > \max X_M(P)$, so (3) fails.

(3) immediately implies (2). To prove the converse, we shall demonstrate that for any given total order \triangleright on \mathcal{X} , if there is a preference $\succeq \in \mathcal{P}$ such that $\max X_M(P) = \max X_M(\succeq)$ for every non-empty menu $M \subseteq \mathcal{X}$,

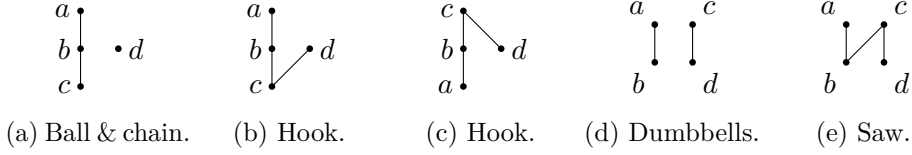


Figure 3: The posets in Definition 22.

then \succeq^* is such a preference. So fix a total order \supseteq on \mathcal{X} , and suppose toward a contradiction that $\succeq \in \mathcal{P}$ has the requisite property whereas \succeq^* does not. Then there is a non-empty menu $M \subseteq \mathcal{X}$ such that $x := \max X_M(\succeq) = \max X_M(P) \neq \max X_M(\succeq^*) =: y$. It must be that $x < y$, since (as argued in the proof of Proposition 3) we have $x \leq y$ by the MCS theorem and the fact that \succeq^* is an upper bound of P . It follows by the characterisation theorem that there is a P -chain $(w_k)_{k=1}^K$ from y to x , so that $\max X_{\{w_k, w_{k+1}\}}(P) = \{w_k\}$ for every $k \in \{1, \dots, K-1\}$. Then because (by hypothesis) $\max X_{M'}(\succeq) = \max X_{M'}(P)$ for every non-empty menu $M' \subseteq \mathcal{X}$, it must be that $y = w_1 \succeq w_2 \succeq \dots \succeq w_K = x$. Since $x \in X_M(\succeq)$ by hypothesis and $y \in M$, it follows that $y \in X_M(\succeq)$. But then $\max X_M(\succeq) \geq y > x = \max X_M(\succeq)$, which is absurd. ■

J Proof of the (im)possibility theorem (§5.3, p. 27)

In this appendix, we add a third ‘forbidden subposet’ equivalence to the (im)possibility theorem (§J.1), then give the proof (§J.2).

J.1 Elaboration of the (im)possibility theorem

The ‘forbidden subposet’ version of the (im)possibility theorem features the four-element posets drawn in Figure 3, which are explicitly defined as follows.

Definition 22. Let \sqsubseteq be a binary relation on a set \mathcal{A} , and let $a, b, c, d \in \mathcal{A}$ be distinct.

- (1) (a, b, c, d) is a *ball and chain* iff $a \sqsupset b \sqsupset c$ and $a \sqsupset c$, and no other pair of elements is \sqsubseteq -comparable.
- (2) (a, b, c, d) is a *hook* iff either $a \sqsupset b \sqsupset c \sqsubset d$ and $a \sqsupset c$ or (dually) $a \sqsubset b \sqsubset c \sqsupset d$ and $a \sqsupset c$, and no other pair of elements is \sqsubseteq -comparable.
- (3) (a, b, c, d) is a *pair of dumbbells* iff $a \sqsupset b$ and $c \sqsupset d$, and no other pair of elements is \sqsubseteq -comparable.

- (4) (a, b, c, d) is a saw iff $a \sqsubset b \sqsubset c \sqsubset d$, and no other pair of elements is \sqsubseteq -comparable.

Elaborated (im)possibility theorem. The following are equivalent:

- (1) There exists a w-efficient and liberal SWF.
- (2) (\mathcal{X}, \succeq) is either a fork or a shattered fork.
- (3) \succeq is free from 4-crowns, diamonds, balls-and-chains, hooks, dumbbells and saws.

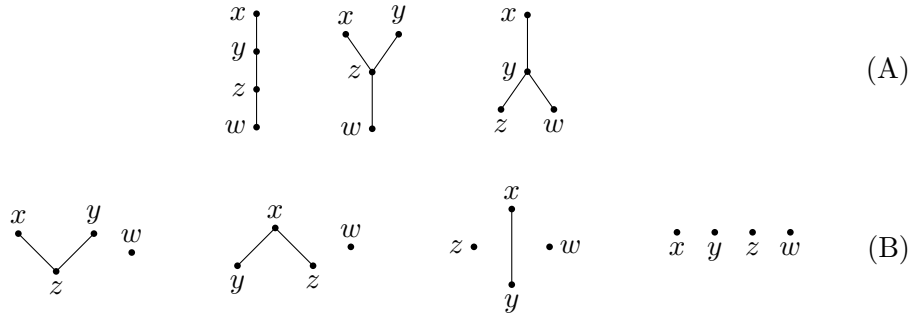
Note that by ruling out saws, we rule out K -crowns for $K \geq 6$ even.

J.2 Proof of the elaborated (im)possibility theorem

Proof that (1) implies (3). We prove the contra-positive. 4-crowns and diamonds are ruled out by Lemma 5 (p. 26) and the existence theorem. For the remainder, it suffices by Lemma 5 to provide, for each forbidden subposet, a pair of preferences $\pi = (\succeq_1, \succeq_2) \in \mathcal{P}^2$ which possess no minimum upper bound \succeq^* such that $x \succeq^*(\succ^*) y$ whenever $x \succeq_i(\succ_i) y$ for $i \in \{1, 2\}$.

Example 1 (pp. 6 and 26) does the job for a hook. For a ball and chain (x, y, z, w) , $z \succ_1 w \succ_1 x \succ_1 y$ and $y \succ_2 z \succ_2 w \succ_2 x$ is an example. For a pair of dumbbells (x, y, z, w) , $x \succ_1 y \succ_1 z \succ_1 w$ and $z \succ_2 w \succ_2 x \succ_2 y$ is an example. For a saw (x, y, z, w) , $w \succ_1 x \succ_1 y \succ_1 z$ and $y \succ_2 z \succ_2 w \succ_2 x$ is an example. ■

Proof that (3) implies (2). Suppose that (3) holds; we will argue that (\mathcal{X}, \succeq) must be a fork or a shattered fork. By inspection, the only four-element subposets permitted by (3) are (three-pronged) fork heads and the following:



It is tedious but straightforward to verify that (3) does not allow subposets of both types (A) and (B). It is clear that only upward (downward) fork

heads are compatible with the second (third) subposet in (A) and with the first (second) subposet in (B).

That leaves two possibilities. The first is that every four-element subposet is an upward (downward) three-pronged fork head, the first subposet in (A), or the second (third) subposet in (A). In that case, $(\mathcal{X}, \triangleright)$ is evidently an up-fork (down-fork).

The second possibility is that every subposet is an upward (downward) three-pronged fork head, the first (second) subposet in (B), or one of the third and fourth subposets in (B). In that case, $(\mathcal{X}, \triangleright)$ is a shattered up-fork (down-fork). ■

Proof that (2) implies (1). Suppose that $(\mathcal{X}, \triangleright)$ is a fork or shattered fork. By Lemma 5 (p. 26), what we must show is that for any $\pi = (\succeq_1, \dots, \succeq_n) \in \mathcal{P}^n$, there is a minimum upper bound $\succeq \in \mathcal{P}$ of π that satisfies the following *conditional Pareto property*: $x \succ^*(\succeq^*) y$ whenever $x \succ_i(\succeq_i) y$ for every $i \in \{1, \dots, n\}$ and there is no (strict) π -chain from y to x . To that end, fix an arbitrary preference profile $\pi \in \mathcal{P}^n$.

Fork: Suppose that $(\mathcal{X}, \triangleright)$ is a fork, wlog an up-fork. Then \mathcal{X} can be partitioned as $\mathcal{X} = X \cup \{x\} \cup Y$, where $X \cup \{x\}$ is an upward fork head with minimum element x and $\{x\} \cup Y$ is a chain with maximum element x .

Since \triangleright is crown- and diamond-free, π has minimum upper bounds by the existence theorem. Let \succeq^* be an arbitrary minimum upper bound. Partition X into sets $\{X_k\}_{k=1}^K$ such that $x', x'' \in X$ belong to the same X_k if and only if, for any z in the \triangleright -chain $\{x\} \cup Y$, $x' \succeq^*(\succ^*) z$ if and only if $x'' \succeq^*(\succ^*) z$.

Note that, since $X \cup \{x\}$ is an upward fork head with minimum element x , for all $1 \leq k \leq K$, the elements of X_k are mutually \triangleright -incomparable. Moreover, for any $x' \in X$, since $Y \cup \{x, x'\}$ is a \triangleright -chain, all minimum upper bounds of π agree on $Y \cup \{x, x'\}$ by the uniqueness proposition. It follows that, for any $(\succeq_k)_{k=1}^K$ where \succeq_k is a preference on X_k for all $1 \leq k \leq K$, there exists a minimum upper bound \succeq^{**} of π that agrees with \succeq_k on X_k for each $k \leq K$.

\succeq_k may clearly be chosen to satisfy the conditional Pareto property on every pair in X_k for each $1 \leq k \leq K$. It thus suffices to show that \succeq^{**} satisfies the conditional Pareto property on pairs x', y' that do not both lie in a single X_k . So suppose that $x' \succ_i(\succeq_i) y'$ for every i and that there is no (strict) π -chain from y' to x' ; we will deduce that $x' \succ^{**}(\succeq^{**}) y'$.

There are two cases. First, suppose that one of x' and y' belongs to the \triangleright -chain $\{x\} \cup Y$. Then x' and y' are \triangleright -comparable, so the result follows from properties (\star) – $(\star\star)$ in the characterisation theorem. Second, suppose that neither x' nor y' belongs to $\{x\} \cup Y$. Since they do not belong to the same

X_k by hypothesis, there is a $z \in \{x\} \cup Y$ such that either (i) $x' \succeq^*(\succ^*) z$ and $y' \not\prec^*(\not\succeq^*) z$, or (ii) vice-versa. Since $x' \succ_i(\succeq_i) y'$ for every i , it must be (i) that obtains. Therefore $x' \succ^{**} y'$, as desired.

Shattered fork: Suppose that (\mathcal{X}, \succeq) is a shattered fork. Then \mathcal{X} may be partitioned as $\mathcal{X} = X \cup Y$, where X is a fork head, wlog an upward fork head, and Y is a set of isolated elements. Write $x_0 \in X$ for the minimum element of the upward fork head X .

Let \succeq' be the binary relation on \mathcal{X} such that, for any \succeq -comparable $x, y \in \mathcal{X}$, wlog $x \succeq y$, \succeq' satisfies properties (\star) – $(\star\star)$ in the characterisation theorem and, for any \succeq -unranked x, y , $x \succeq' y$ if and only if $x \succeq_i y$ for every i . Let \succeq'' be the transitive closure of \succeq' . Since \succeq'' is transitive, it admits a complete and transitive extension \succeq^* by the Richter–Suzumura extension theorem (appendix B). We will show that \succeq^* is a minimum upper bound of π and that it satisfies the conditional Pareto property.

For the conditional Pareto property, it suffices to show that, for any $x, y \in \mathcal{X}$ such that there is no (strict) π -chain from y to x and further $x \succ_i(\succeq_i) y$ for every i , we have $x \succ''(\succeq'') y$. Note that, if there is no (strict) π -chain from y to x and $x \succ_i(\succeq_i) y$ for every i , then $x \succeq' y$. It remains only to show that if there is no π -chain from y to x and $x \succ_i y$ for every i , then $x \succ'' y$.

So suppose toward a contradiction that there is no π -chain from y to x and that $x \succ_i y$ for every i , and yet $y \succeq'' x$. Since \succeq'' is the transitive closure of \succeq' , there exists a \succeq' -decreasing sequence $(w_k)_{k=1}^K$ with $w_1 = y$ and $w_K = x$; we will show that this is impossible. We proceed by strong induction on the length $K \geq 2$ of the sequence. The base case $K = 2$ follows by definition of \succeq' . For the inductive step, take $K' > 2$, and suppose that there is no \succeq -decreasing sequence of length K from y to x for any $K < K'$. Toward a contradiction, suppose that there is a \succeq' -decreasing sequence $(w_k)_{k=1}^{K'}$ of length K' from y to x . There are two cases.

Case 1.1: There is at most one $k' < K'$ such that there is a π -chain from $w_{k'}$ to $w_{k'+1}$. Fix k' , and note that $w_k \succeq_i w_{k+1}$ for every i and $k \neq k'$. By hypothesis, we have $x \succ_i y$ for every i . It follows by transitivity that $w_{k'+1} \succ_i w_{k'}$. Since all π -chains have length at most 2, there is no π -chain from $w_{k'}$ to $w_{k'+1}$. But then, since $w_{k'+1} \succ_i w_{k'}$ for every i , it cannot be that $w_{k'} \succeq' w_{k'+1}$ —a contradiction.

Case 1.2: There are $k_1 < k_2 < K'$ such that there is a π -chain from w_{k_n} to w_{k_n+1} for $n = 1, 2$. Then, since all π -chains end at x_0 , $w_{k_1+1} = w_{k_2+1} = x_0$. Then $(w_1, \dots, w_{k_1}, w_{k_2+1}, \dots, w_{K'})$ is a \succeq' -chain and, since $k_1 < k_2$, it has length $< K'$. This contradicts the inductive hypothesis.

It remains to show that \succeq^* is a minimum upper bound of π . Since \succeq^* is

an extension of \geq'' , it suffices by the characterisation theorem to show that for any $x \triangleright y$, \geq'' satisfies properties (\star) – $(\star\star)$. Since \geq'' is the transitive closure of \geq' and \geq' satisfies (\star) – $(\star\star)$, we need only show that (a) $y \not\geq'' x$ if there is a strict π -chain from x to y , and that (b) $x \not\geq'' y$ if there is no π -chain from x to y .

Claim (b) is easy: suppose toward a contradiction that there is no π -chain from x to y , and yet $x \geq'' y$. Since $x \triangleright y$ by assumption, it follows that $y \succ_i x$ for every i . Then the argument used to show that \geq^* satisfies the conditional Pareto property applies, yielding a contradiction.

To prove (a), suppose toward a contradiction that there is a strict π -chain from x to y , yet $y \geq'' x$. Then there exists a \geq' -decreasing sequence $(w_k)_{k=1}^K$ from $w_1 = y$ to $w_K = x$. We show by strong induction on $K \geq 2$ that this is impossible. The base case $K = 2$ follows by definition of \geq' . For the inductive step, take any $K' > 2$, and suppose that for all $K < K'$, there is no \geq' -decreasing sequence of length K from y to x . Toward a contradiction, suppose that $(w_k)_{k=1}^{K'}$ is a \geq' -decreasing sequence of length K' . There are two cases.

Case 2.1: For some $k < K'$, there is a π -chain from w_k to w_{k+1} . Then, since all π -chains end at x_0 , $w_{k+1} = y = x_0$. Then, as above, the induction hypothesis produces a contradiction.

Case 2.2: For every $k < K'$, there is no π -chain from w_k to w_{k+1} . Then since $w_k \geq' w_{k+1}$, we have $w_k \succeq_i w_{k+1}$ for every i . It follows by transitivity that $y \succeq_i x$ for every i . Since all π -chains have length 2, this contradicts the existence of a strict π -chain from x to y . ■

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